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CORRECTIONS TO ERRORS IN THE PAPER "HADAMARD-TYPE INEQUALITIES FOR s-CONVEX FUNCTIONS I" AND NEW INTEGRAL INEQUALITIES OF s-CONVEX FUNCTIONS IN THE SECOND SENSE

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Abstract In the work, the authors correct some errors appeared in the paper "S. Hussain, M. I. Bhatti and M. Iqbal, *Hadamard-type inequalities for s-convex functions I*, Punjab Univ. J. Math. (Lahore), **41** (2009), 51–60" and establish some new integral inequalities of s-convex functions in the second sense.

Keywords Hermite–Hadamard type inequality, correction, s-convex function, concave function.

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1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. A real-valued function $f : I \to \mathbb{R}$ is said to be convex (or concave, respectivey) on I if the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval I such that $a, b \in I$ and $a < b$. Then the well-known Hermite–Hadamard integral inequality reads that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

In [1, 4], the concept of s-convex functions was innovated below.

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Definition 1.1 ([1,4]). Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0 = [0,\infty)$ is said to be s-convex in the second sense if the inequality

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

It is easy to see that for $s = 1$ the s-convexity reduces to the classical and ordinary convexity of functions defined on $\mathbb{R}_0$.

The Hermite–Hadamard type integral inequalities for s-convex functions in the second sense are a very active research topic. We now recall some of them as follows.

Theorem 1.1 ([9]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, the numbers $a, b \in I$ with $a < b$, and $f' \in L_1([a,b])$. If $|f'|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q + |f'(b)|^q\right]^{1/q}.\end{aligned}$$

Theorem 1.2 ([11]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, let $a, b \in I$ with $a < b$, and let $f' \in L_1([a,b])$. If $|f'|$ is s-convex on $[a,b]$ for some $s \in (0,1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\big(|f'(a)| + |f'(b)|\big).\end{aligned}$$

For some other related papers on Hermite–Hadamard type inequalities for convex functions and s-convex functions, please refer to [3,7,14,15].

In [5], Hussain and his two coauthors studied the Hermite–Hadamard type inequality of s-convex functions in the second sense, established several Hermite–Hadamard type inequalities for differentiable and twice differentiable functions based on concavity and s-convexity, and applied to construct some special means.

2. Hermite–Hadamard type inequalities by Hussain and his coauthors

Hussain and his two coauthors introduced in [5] the following lemma.

Lemma 2.1 ([5, Lemma 3]). *Let $I \subseteq \mathbb{R}$ denote an interval, $f : I \to \mathbb{R}$ be a differentiable function on I° (the interior of I), and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1([a,b])$, then*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\\ =&\frac{(b-a)^2}{4}\int_0^1 (1-t)\left[f'\left(ta + (1-t)\frac{a+b}{2}\right) + f'\left(tb + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t. \qquad (2.1)\end{aligned}$$

Using Lemma 2.1, Hussain and his coauthors established the following Theorems 2.1 to 2.4 in the paper [5].

Theorem 2.1 ([5, Theorem 4]). *Let $f : I \subset \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \left(\frac{1}{2}\right)^{1/p}\left\{\frac{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}} + \frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}.$$

Remark 2.1. If $q \ge 1$, the factor $\left(\frac{1}{2}\right)^{1/p}$ in [5, Theorem 4] should be modified to $\left(\frac{1}{2}\right)^{1-1/q}$. Otherwise, if $q=1$, the number $p=\frac{q}{q-1}$ is meaningless.

Theorem 2.2 ([5, Theorem 5]). *Let $f : I \subset \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is a concave function on $[a,b]$ for $q>1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{4(p+1)^{1/p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right],$$

where $p=\frac{q}{q-1}$.

Theorem 2.3 ([5, Theorem 6]). *Let $f : I \subset \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q>1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{s+1}\right)^{1/q}\\ &\times\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right],\end{aligned}$$

where $p=\frac{q}{q-1}$.

Theorem 2.4 ([5, Theorem 7]). *Let $f :\subset \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s \in (0,1]$ and $q>1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ \le&\frac{(b-a)^2}{4(p+1)^{1/p}}2^{(s-1)/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right],\end{aligned}$$

where $p=\frac{q}{q-1}$.

We note that many typos in the above lemma and theorems quoted from the paper [5] have been corrected.

In this article, we will modify and correct the conditions and errors in Theorems 2.1 to 2.4 about s-convex functions in the second sense.

3. Errors and two lemmas

We first give a counterexample of [5, Lemma 3], that is, Lemma 2.1 mentioned above in this paper.

Example 3.1. Letting $f(x) = x^2$ for $x \in [a, b]$ and taking $a = 0$ and $b = 1$ in Lemma 2.1, then

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{(b-a)^2}{4}\int_0^1 (1-t)\left[f'\left(ta + (1-t)\frac{a+b}{2}\right) + f'\left(tb + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t = -\frac{1}{3}.$$

Therefore, we can be sure that Lemma 2.1, that is, [5, Lemma 3], is not valid.

In [13, Remark 1], among other things, the invalidness of the integral identity in [5, Lemma 3] has been pointed out and alternatively corrected.

Making use of [8, Lemma 2.1], we correct [5, Lemma 3] as follows.

Lemma 3.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and let $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$, then*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\ =&\frac{b-a}{4}\int_0^1 (1-t)\left[f'\left(ta + (1-t)\frac{a+b}{2}\right) - f'\left(tb + (1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t. \end{aligned} \tag{3.1}$$

Example 3.2. Let $f(x) = x^2$ for $x \in [a, b]$. Then $|f'(x)|^q$ is an s-convex function in the second sense on $[a, b]$ for $s = 1$ and $q = 1$.

If setting $a = 0$, $b = 12.12$, and $s = q = 1$ in Theorem *2.1*, then

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &= \frac{1}{12}(b-a)^2 \\ &= 12.2412 \\ &> 12.12 \\ &= \frac{|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right|}{6} + \frac{2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|}{6}. \end{aligned}$$

If letting $a = 0$, $b = 6$, and $s = q = 1$ in Theorem *2.1*, then

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &= \frac{1}{12}(b-a)^2 \\ &= 3 \\ &< 6 \\ &= \frac{|f'(a)| + 2\left|f'\left(\frac{a+b}{2}\right)\right|}{6} + \frac{2\left|f'\left(\frac{a+b}{2}\right)\right| + |f'(b)|}{6}. \end{aligned}$$

These numerical computations reveal that Theorems 2.1 to 2.4 are not necessarily true.

Remark 3.1. Comparing the factors $\frac{(b-a)^2}{4}$ and $\frac{b-a}{4}$ on the right hand sides of the integral equalities (2.1) and (3.1), we can illustrate that Theorems 2.3 to 2.4 are not necessarily true.

Now we establish the Jensen type integral inequalities for s-concave functions.

Lemma 3.2. *Let $\varphi : [a,b] \to \mathbb{R}_0$ be continuous and $g,p : [a,b] \to \mathbb{R}$ be integrable functions with $g(x) \in [a,b]$, $p(x) \geq 0$ for $x \in [a,b]$, and $\int_a^b p(x)\,\mathrm{d}x > 0$. If φ is an s-concave function in the second sense for some $s \in (0,1]$, then the Jensen type integral inequality*

$$\varphi\left(\frac{\int_a^b p(x)g(x)\,\mathrm{d}x}{\int_a^b p(x)\,\mathrm{d}x}\right) \geq \frac{\int_a^b [p(x)]^s \varphi(g(x))\,\mathrm{d}x}{\left[\int_a^b p(x)\,\mathrm{d}x\right]^s} \tag{3.2}$$

is sound.

Proof. Let $x_0 < x_1 < \cdots < x_n$ be a partition of $[a,b]$ and denote $\Delta x_i = x_i - x_{i-1}$ for $i = 1,2,\ldots,n$ such that $\max_{1\leq i\leq n}\{\Delta x_i\} \leq 1$. In this way, we see that $(\Delta x_i)^s \geq \Delta x_i$ for $i = 1,2,\ldots,n$. By the s-concavity in the second sense of φ on $[a,b]$, see [16, Corollary 4], we obtain

$$\begin{aligned}\varphi\left(\frac{\sum_{i=1}^n p(x_i)g(x_i)\Delta x_i}{\sum_{i=1}^n p(x_i)\Delta x_i}\right) &\geq \frac{\sum_{i=1}^n [p(x_i)\Delta x_i]^s \varphi(g(x_i))}{\left[\sum_{i=1}^n p(x_i)\Delta x_i\right]^s}\\ &\geq \frac{\sum_{i=1}^n [p(x_i)]^s \varphi(g(x_i))\Delta x_i}{\left[\sum_{i=1}^n p(x_i)\Delta x_i\right]^s}.\end{aligned}$$

Further taking the limit of $n \to \infty$ on both sides of the above inequality leads to the inequality (3.2). The proof of Lemma 3.2 is completed. □

4. Modifications and corrections of integral inequalities of s-convex functions in the second sense

In this section, we modify and correct the conditions and errors in Theorems 2.1 to 2.4 about s-convex functions in the second sense.

Theorem 4.1 (Modifications and corrections of Theorem 2.1)**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ be a differentiable function on I° such that $f' \in L_1([a,b])$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \geq 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\\ &\quad\times\left\{\frac{\left[|f'(a)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}} + \frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}.\end{aligned} \tag{4.1}$$

Proof. Since $|f'|^q$ is s-convex in the second sense on $[a,b]$, using Lemma 3.1 and by the Hölder integral inequality, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\int_0^1(1-t)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t \\
&\le \frac{b-a}{4}\left[\int_0^1(1-t)\,\mathrm{d}t\right]^{1-1/q} \\
&\quad\times\left\{\left[\int_0^1(1-t)\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right. \\
&\quad\left.+\left[\int_0^1(1-t)\left(t^s|f'(b)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\frac{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right. \\
&\quad\left.+\frac{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right]^{1/q}}{[(s+1)(s+2)]^{1/q}}\right\}.
\end{aligned}$$

The proof of Theorem 4.1 is completed. □

Theorem 4.2 (Generalization of Theorem 2.3)**.** *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-convex function in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and for $q>1$ and $q\ge\ell\ge0$, then*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left\{\left[\frac{sB(s,\ell+1)|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+\ell+1}\right]^{1/q}\right. \\
&\quad\left.+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+sB(s,\ell+1)|f'(b)|^q}{s+\ell+1}\right]^{1/q}\right\},
\end{aligned}$$

where $B(u,v)$ denotes the classical beta function defined by

$$B(u,v)=\int_0^1 z^{u-1}(1-z)^{v-1}\,\mathrm{d}z,\quad \Re(u)>0,\Re(v)>0.$$

Proof. Similar to the proof of the inequality (4.1) in Theorem 4.1, using Lemma 3.1, employing the Hölder integral inequality, and utilizing the s-convexity in the second sense on $[a,b]$ of $|f'|^q$, we derive

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{b-a}{4}\left[\int_0^1(1-t)^{(q-\ell)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{\left[\int_0^1(1-t)^\ell\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1(1-t)^\ell\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}\\
&\le\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\\
&\times\left\{\left[\int_0^1(1-t)^\ell\left(t^s|f'(a)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1(1-t)^\ell\left(t^s|f'(b)|^q+(1-t)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\
&=\frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left\{\left[\frac{sB(s,\ell+1)|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+\ell+1}\right]^{1/q}\right.\\
&\quad\left.+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+sB(s,\ell+1)|f'(b)|^q}{s+\ell+1}\right]^{1/q}\right\}.
\end{aligned}
$$

The proof of Theorem 4.2 is completed. □

If $q>1$ and $\frac{1}{p}=1-\frac{1}{q}$, then $\frac{1}{(p+1)^{1/p}}=\left(\frac{q-1}{2q-1}\right)^{1-1/q}$. Therefore, putting $\ell=0$ in Theorem 4.2 yields

Corollary 4.1 (Modifications and corrections of Theorem 2.3)**.** *Under conditions of Theorem 4.2 applied to $\ell=0$, we have*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
\le&\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}\\
&\times\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1/q}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q\right)^{1/q}\right].
\end{aligned}
$$

Next, we will study the Hermite–Hadamard type integral inequalities of s-concave functions. We first establish an Hermite–Hadamard type integral inequality of s-concave functions in the case of $q\ge1$.

Theorem 4.3. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function in the second sense on $[a,b]$ for $q\ge1$ and some fixed $s\in(0,1]$, then*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
\le&\frac{b-a}{2^{3-1/q}}\left(\frac{s}{s+1}\right)^{s/q}\\
&\times\left[\left|f'\left(\frac{(3s+1)a+(s+1)b}{2(2s+1)}\right)\right|+\left|f'\left(\frac{(s+1)a+(3s+1)b}{2(2s+1)}\right)\right|\right]. \qquad (4.2)
\end{aligned}
$$

Proof. Using Lemma 3.1 and employing the Hölder integral inequality, we obtain

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|
$$

$$
\begin{aligned}
&\le \frac{b-a}{4}\int_0^1 (1-t)\left[\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| \mathrm{d}t + \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right] \mathrm{d}t\\
&\le \frac{b-a}{4}\left(\int_0^1 (1-t)\,\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right.\\
&\quad\left. + \left[\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}.
\end{aligned}
$$

Taking $p(t)=(1-t)^{1/s}$ for $[0,1]$ in Lemma 3.2, utilizing Lemma 3.2, and using the s-convexity of $|f'|^q$ in the second sense of $[a,b]$, we derive

$$
\begin{aligned}
&\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\\
&\le \left(\int_0^1 (1-t)^{1/s}\,\mathrm{d}t\right)^s \left|f'\left(\frac{\int_0^1 (1-t)^{1/s}\big(ta+(1-t)\frac{a+b}{2}\big)\,\mathrm{d}t}{\int_0^1 (1-t)^{1/s}\,\mathrm{d}t}\right)\right|^q\\
&= \left(\frac{s}{s+1}\right)^s \left|f'\left(\frac{(3s+1)a+(s+1)b}{2(2s+1)}\right)\right|^q
\end{aligned}
$$

and

$$
\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t \le \left(\frac{s}{s+1}\right)^s \left|f'\left(\frac{(s+1)a+(3s+1)b}{2(2s+1)}\right)\right|^q.
$$

Substituting these two inequalities into the first inequality in this proof yields the inequality (4.2). The proof of Theorem 4.3 is completed. □

If taking $s=1$ in Theorem 4.3, we have

Corollary 4.2. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is a concave function in the second sense on $[a,b]$ for $q\ge1$, then*

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right].
$$

Theorem 4.4 (Generalization of Theorems 2.2 and 2.4)**.** *Suppose $q>1$ and $q\ge\ell\ge0$. Let $f: I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s\in(0,1]$, then*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le \frac{b-a}{4}\left(\frac{q-1}{2q-(\ell+1)}\right)^{1-1/q}\left(\frac{s}{s+\ell}\right)^{s/q}\\
&\quad\times\left[\left|f'\left(\frac{(3s+\ell)a+(s+\ell)b}{2(2s+\ell)}\right)\right|+\left|f'\left(\frac{(s+\ell)a+(3s+\ell)b}{2(2s+\ell)}\right)\right|\right]. \qquad (4.3)
\end{aligned}
$$

Proof. Using Lemma 3.1 and utilizing the Hölder integral inequality, we obtain

$$
\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|
$$

$$
\begin{aligned}
&\le \frac{b-a}{4}\left(\int_0^1 (1-t)^{(q-\ell)/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1 (1-t)^{\ell}\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right.\\
&\quad+\left.\left[\int_0^1 (1-t)^{\ell}\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right\}. \qquad (4.4)
\end{aligned}
$$

Taking $p(t)=(1-t)^{\ell/s}$ for $t\in[0,1]$ in Lemma 3.2 and using the s-convexity of $|f'|^q$ in the second sense on $[a,b]$, we have

$$
\begin{aligned}
&\int_0^1 (1-t)^{\ell}\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\\
&\le\left[\int_0^1 (1-t)^{\ell/s}\,\mathrm{d}t\right]^s\left|f'\left(\frac{\int_0^1 (1-t)^{\ell/s}\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t}{\int_0^1 (1-t)^{\ell/s}\,\mathrm{d}t}\right)\right|^q\\
&=\left(\frac{s}{s+\ell}\right)^s\left|f'\left(\frac{(3s+\ell)a+(s+\ell)b}{2(2s+\ell)}\right)\right|^q
\end{aligned}
$$

and

$$
\int_0^1 (1-t)^{\ell}\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q \mathrm{d}t\le\left(\frac{s}{s+\ell}\right)^s\left|f'\left(\frac{(s+\ell)a+(3s+\ell)b}{2(2s+\ell)}\right)\right|^q.
$$

Substituting these two inequalities into the inequality (4.4) yields the inequality (4.3). The proof of Theorem 4.4 is completed. □

If putting $s=1$ and $\ell=0$ in Theorem 4.4, we acquire

Corollary 4.3. (Modifications and corrections of Theorem 2.2) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is a concave function on $[a,b]$ for $q>1$, then*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
\le&\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].
\end{aligned}
$$

If letting $\ell=0$ in Theorem 4.4, we obtain

Corollary 4.4. (Modifications and corrections of Theorem 2.4) *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}_0$ be a differentiable function on I° such that $f'\in L_1([a,b])$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is an s-concave function on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
\le&\frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right].
\end{aligned}
$$

5. Conclusions

In this paper, we pointed out many errors appeared in the article [5], corrected these errors, and established several new integral inequalities of s-convex functions in the second sense.

For more information on recent developments of this topic, please refer to the papers [2, 6, 10, 12, 13, 17] and closely-related references therien.

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