Journal of Applied Analysis and Computation
Volume 15, Number 5, October 2025, 2663–2694
Website:http://www.jaac-online.com
DOI:10.11948/20240410

THEORETICAL AND NUMERICAL STABILITY OF THE BRESSE SYSTEM: EXPLORING FRACTIONAL DAMPING THROUGH TRADITIONAL AND NEURAL NETWORK APPROACHES

Ahmed Bchatnia[1,†], Abderrahmane Beniani[2], Boumediene Boukhari[2] and Foued Mtiri[3]

Abstract This paper investigates the theoretical and numerical stability of the one-dimensional Bresse system with fractional damping terms in a bounded domain. We first establish the well-posedness of the system. Using the frequency domain approach and a theorem by Borichev and Tomilov, we derive the polynomial decay rate of the system. To validate these theoretical results, we propose a numerical scheme and compare its performance with the Fractional Physics-Informed Neural Network (fPINN). The comparative analysis highlights the effectiveness of traditional numerical methods and fPINNs in capturing the decay rate, offering new insights into the advancement of computational techniques for complex physical systems.

Keywords Bresse system, asymptotic stability, finite difference scheme, Caputo's fractional derivative.

MSC(2010) 35L05, 26A33.

1. Introduction

This paper is dedicated to the study of the one-dimensional linear Bresse system, given by:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) + \partial_t^{\alpha_1,\eta}\varphi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + \partial_t^{\alpha_2,\mu}\psi = 0, \\ \rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) + \partial_t^{\alpha_3,\nu} w = 0, \end{cases} \tag{1.1}$$

[†]The corresponding author.
[1]Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, LR Analyse Non-Linéaire et Géométrie, LR21ES08, El Manar 2, 2092, Tunis, Tunisia
[2]Department of Mathematics, University of Ain Temouchent, Faculty of Science and Technology, Engineering and Sustainable Development, Laboratory, Ain Témouchent, 46000, Algeria
[3]Department of Mathematics, Applied College in Mahayil, King Khalid University, Abha, Saudi Arabia
Email: ahmed.bchatnia@fst.utm.tn(A. Bchatnia), abderrahmane.beniani@univ-temouchent.edu.dz(A. Beniani), boumediene.boukhari@univ-temouchent.edu.dz(B. Boukhari), mtirifoued@yahoo.fr(F. Mtiri)

where $(x,t) \in (0,L) \times (0,+\infty)$ denote the space and time variables, respectively. The constants ρ_1, ρ_2, κ_0, κ, l, and b are positive, η is a non-negative constant, and $\alpha_i \in (0,1)$ for $i = 1,2,3$.

The initial conditions are given by:

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,L),\\ &\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,L),\\ &w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in (0,L). \end{aligned} \tag{1.2}$$

The boundary conditions are:

$$\begin{aligned} &\varphi_x(t,x) = \psi_x(t,x) = w_x(t,x) = 0, \quad \text{for } x = 0, L,\\ &\varphi(t,x) = \psi(t,x) = w(t,x) = 0, \quad \text{for } x = 0, L. \end{aligned} \tag{1.3}$$

The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo fractional derivative of order α, $0 < \alpha < 1$, with respect to the time t. It is defined as follows:

$$\partial_t^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)\, ds, \quad \eta \geq 0.$$

Here, x and t denote the space and time variables, respectively.

The Bresse system, or the curved beam [14], is modeled by the system:

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = 0,\\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = 0,\\ \rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = 0. \end{cases}$$

The terms $\kappa_0 l(w_x - l\varphi)$, $\kappa(\varphi_x + \psi + lw)$, and $b\psi_x$ denote the axial force, the shear force, and the bending moment, respectively. The functions φ, ψ, and w represent, respectively, the transverse displacement of a curved beam, the rotation angle of the filament, and the longitudinal displacement. We denote by $\kappa_0 = EH$, $\kappa = GH$, $b = EI$, where ρ_1, ρ_2, l, G, E, and H are positive constants characterizing the physical properties of the beam and the filament. Additionally, $l = \frac{1}{R}$, where R is the radius of curvature (see [8, 16] for more details).

In [19], B. Mbodje explored the asymptotic behavior of solutions for the system:

$$\begin{cases} \partial_t^2 u(x,t) - u_{xx}(x,t) = 0, \quad (x,t) \in (0,1) \times (0,+\infty),\\ u(0,t) = 0,\\ \partial_x u(1,t) = -k\partial_t^{\alpha,\eta} u(1,t), \quad \alpha \in (0,1), \eta \geq 0, k > 0,\\ u(x,0) = u_0(x),\\ \partial_t u(x,0) = v_0(x). \end{cases}$$

He proved strong asymptotic stability of solutions when $\eta = 0$, and a polynomial decay rate of t^{-1} as time tends to infinity when $\eta \neq 0$. The energy method was used to establish the polynomial decay rate. Akil et al., in [2] under the equal speed propagation condition, they established the optimal polynomial energy decay rate and they proved the indirect boundary exact controllability of the Timoshenko system with mixed Dirichlet–Neumann boundary conditions and boundary control. In [3,5] studied the stabilization for a coupled wave equations with fractional-damping. They proved the polynomial stability of the system. Recently, in [4] they proved the energy decay of hyperbolic systems of wave-wave, wave-Euler Bernoulli beam and beam-beam types. they established different types of polynomial energy decay rate which depends on the order of the fractional derivative and the type of the damped equation in the system.

In [9], the Bresse model for circular beams, with the addition of two frictional dissipations in the system, was analyzed. Exponential stability was found if and only if $\kappa = \kappa_0$, with polynomial decay in the general case. The problem of the optimality of the polynomial decay rate was also studied. In [21], the exponential decay of a dissipative Bresse system was demonstrated using techniques developed in [18], and numerical simulations were provided to support their results.

When thermal effects are considered, the asymptotic behavior of the Bresse system may become more complicated due to the coupling between elasticity and heat conduction. Currently, there are some theoretical and numerical results on the asymptotic behavior of thermoelastic Bresse systems [10, 17].

Recently, in [12], Beniani et al. examined a system comprising coupled wave equations featuring a diffusive internal control of a general nature:

$$\begin{cases} \partial_{tt}u - \Delta_x u + \zeta \displaystyle\int_{-\infty}^{+\infty} \varrho(\omega)\phi(x,\omega,t)\, d\omega + \beta v = 0, \\ \partial_{tt}v - \Delta_x v + \zeta \displaystyle\int_{-\infty}^{+\infty} \varrho(\omega)\varphi(x,\omega,t)\, d\omega + \beta u = 0, \\ u = v = 0, \text{ on } \partial\Omega, \\ \phi_t(x,\omega,t) + (\omega^2+\eta)\phi(x,\omega,t) - \partial_t u \varrho(\omega) = 0, \\ \varphi_t(x,\omega,t) + (\omega^2+\eta)\varphi(x,\omega,t) - \partial_t v \varrho(\omega) = 0, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x), \\ \phi(x,\omega,0) = \phi_0(x,\omega), \quad \text{and} \quad \varphi(x,\omega,0) = \varphi_0(x,\omega). \end{cases}$$

They demonstrated the absence of exponential stability and investigated the asymptotic stability of the model, establishing a general decay rate that is dependent on the density function ϱ.

Numerically, the finite element method has been widely used in many studies related to control systems (see [7,13,15,20]). However, to the best of my knowledge, no study has yet validated the decay rate using the Fractional Physics-Informed Neural Network (fPINN).

This paper is organized as follows: In Section 2, we prove the well-posedness of System (1.1) using arguments that combine semigroup theory. In Section 3, we establish the polynomial stability of the Bresse system (1.1) through a frequency domain approach and a theorem by Borichev and Tomilov. Section 4 is dedicated to the discretization of the energy using the finite element method, and Section 5 explores the fPINN approach.

2. Augmented model and well-posedness of the system

This section is concerned with the reformulation of the model (1.1) into an augmented system. We need the following theorem:

Theorem 2.1. [6] *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Consider the system governed by the equation

$$\partial_t \varphi(x,\xi,t) + \left(|\xi|^2 + \eta\right)\varphi(x,\xi,t) - U(x,t)\mu(\xi) = 0, \quad \xi \in \mathbb{R},\ \eta \geq 0,\ t > 0,$$

with the initial condition

$$\varphi(x,\xi,0) = 0,$$

and the output defined as

$$O(x,t) = \pi^{-1}\sin(\alpha\pi)\int_{\mathbb{R}} \mu(\xi)\varphi(x,\xi,t)\,d\xi.$$

The relationship between the 'input' U and the 'output' O is then given by

$$O(x,t) = I^{1-\alpha,\eta}U(x,t) = D^{\alpha,\eta}U(x,t),$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau)\,d\tau.$$

We also need the following lemma in the sequel:

Lemma 2.1. [1] *If $\lambda \in D = \{\lambda \in \mathbb{C} \mid \Re\lambda + \eta > 0\} \cup \{\Im\lambda \neq 0\}$, then*

$$\tau(\alpha)\int_{\mathbb{R}^n} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}\,d\xi = (\lambda+\eta)^{\alpha-1},$$

where $\tau(\alpha) = \pi^{-1}\sin(\alpha\pi)$.

Lemma 2.2. *If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty, -\eta]$, then*

$$\int_{-\infty}^{+\infty} \frac{|\xi|\mu(\xi)}{(i\lambda + \xi^2 + \eta)^2}\,d\xi = \frac{1-2\alpha}{4}\frac{\pi}{\sin\frac{(2\alpha+3)}{4}\pi}(i\lambda+\eta)^{\frac{(2\alpha-5)}{4}}, \tag{2.1}$$

$$\int_{-\infty}^{+\infty} \frac{1}{|i\lambda + \xi^2 + \eta|^2}\,d\xi \leq \frac{\pi}{2}|i\lambda+\eta|^{-\frac{3}{2}} \text{ and}$$

$$\int_{-\infty}^{+\infty} \frac{|\xi|^2}{|i\lambda + \xi^2 + \eta|^4}\,d\xi \leq \frac{\pi}{16}|i\lambda+\eta|^{-\frac{5}{2}}.$$

Using the previous theorem, the system (1.1) can be rewritten as the following augmented model:

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) + \zeta_1 \displaystyle\int_{\mathbb{R}} \mu(\xi)\,\phi_1(x,\xi,t)\; d\xi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + \zeta_2 \displaystyle\int_{\mathbb{R}} \mu(\xi)\,\phi_2(x,\xi,t)\; d\xi = 0, \\ \rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) + \zeta_3 \displaystyle\int_{\mathbb{R}} \mu(\xi)\,\phi_3(x,\xi,t)\; d\xi = 0, \\ \partial_t\phi_1(x,\xi,t) + \left(\xi^2+\eta\right)\phi_1(x,\xi,t) - \mu(\xi)\,\partial_t\varphi(x,t) = 0, \\ \partial_t\phi_2(x,\xi,t) + \left(\xi^2+\eta\right)\phi_2(x,\xi,t) - \mu(\xi)\,\partial_t\psi(x,t) = 0, \\ \partial_t\phi_3(x,\xi,t) + \left(\xi^2+\eta\right)\phi_3(x,\xi,t) - \mu(\xi)\,\partial_t w(x,t) = 0, \end{cases} \tag{2.2}$$

where $(x,\xi,t) \in (0,L)\times\mathbb{R}\times(0,+\infty)$ and with the following initial conditions:

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,L), \\ \psi(x,0) &= \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,L), \\ w(x,0) &= w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in (0,L). \end{aligned}$$

For a solution $U = (\varphi,\varphi_t,\psi,\psi_t,w,w_t,\phi_1,\phi_2,\phi_3)$ of (2.2), we define the energy by

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^L \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + b|\psi_x|^2 + \kappa_0|w_x - l\varphi|^2 \right. \\ & \left. + \kappa|\varphi_x + \psi + lw|^2\right) dx + \frac{1}{2}\sum_{i=1}^{3}\zeta_i \int_0^L \left(\int_{\mathbb{R}} |\phi_i|^2\, d\xi\right) dx, \end{aligned} \tag{2.3}$$

where $\zeta_i = \gamma_i \pi^{-1}\sin(\alpha_i\pi)$.

The following lemma characterizes the decay of the energy functional for the system described by (2.2).

Lemma 2.3. *Let $U = (\varphi,\varphi_t,\psi,\psi_t,w,w_t,\phi_1,\phi_2,\phi_3)$ be a regular solution of the problem* (2.2)*. Then, the functional energy defined in equation* (2.3) *satisfies*

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{3}\zeta_i \int_0^L \int_{\mathbb{R}} \left(|\xi|^2 + \eta\right)|\phi_i(x,\xi,t)|^2\, d\xi\, dx.$$

Proof. Multiplying the equations $(2.2)_1$, $(2.2)_2$, and $(2.2)_3$ by φ_t, ψ_t, and w_t respectively, using integration by parts over $(0,L)$, and adding the results, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\int_0^L \left(\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|w_t|^2 + b|\psi_x|^2 + \kappa_0|w_x - l\varphi|^2 \right.\right. \\ & \left.\left. + \kappa|\varphi_x + \psi + lw|^2\right) dx\right) + \int_0^L (\zeta_1\varphi_t \int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)d\xi \\ & + \zeta_2\psi_t \int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)d\xi + \zeta_3 w_t \int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)d\xi)dx = 0. \end{aligned} \tag{2.4}$$

Multiplying the equations $(2.2)_4$, $(2.2)_5$, and $(2.2)_6$ by $\xi_1\phi_1$, $\xi_2\phi_2$, and $\xi_3\phi_3$ respectively, integrating over $(0,L)\times\mathbb{R}$, and summing, we obtain

$$\begin{aligned}
&\int_0^L \left(\zeta_1\varphi_t \int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)d\xi + \zeta_2\psi_t \int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)d\xi \right.\\
&\left. +\zeta_3 w_t \int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)d\xi\right) dx\\
=&\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{3}\xi_i \int_0^L \int_{\mathbb{R}} |\phi_i|^2\, d\xi\, dx\right)\\
&+\sum_{i=1}^{3}\xi_i \int_0^L \int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi_i(x,\xi,t)|^2\, d\xi\, dx.
\end{aligned} \tag{2.5}$$

Combining the equations (2.4) and (2.5), we obtain

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{3}\zeta_i \int_0^L \int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi_i(x,\xi,t)|^2\, d\xi\, dx.$$

This completes the proof of the lemma. □

We now discuss the well-posedness of (2.2). To this end, we introduce the following Hilbert space, which serves as the energy space:

$$\mathcal{H} = \left(H^1(0,L)\times L^2(0,L)\right)^3 \times \left(L^2(\mathbb{R})\right)^3.$$

For $U = (\varphi,\varphi_t,\psi,\psi_t,w,w_t,\phi_1,\phi_2,\phi_3)^T$ and $\tilde{U} = (\tilde{\varphi},\tilde{\varphi}_t,\tilde{\psi},\tilde{\psi}_t,\tilde{w},\tilde{w}_t,\tilde{\phi}_1,\tilde{\phi}_2,\tilde{\phi}_3)^T$, we define the following inner product in $\mathcal{H}$:

$$\begin{aligned}
&\left\langle U,\tilde{U}\right\rangle_{\mathcal{H}}\\
=&\int_0^L \left(\rho_1\varphi_t\overline{\tilde{\varphi}_t} + \rho_2\psi_t\overline{\tilde{\psi}_t} + \rho_1 w_t\overline{\tilde{w}_t} + b\varphi_x\overline{\tilde{\varphi}_x}\right) dx\\
&+\int_0^L \kappa_0(\varphi_x - l\varphi)\overline{(\tilde{\varphi}_x - l\tilde{\varphi})}\, dx\\
&+\int_0^L \kappa(\varphi_x+\psi+lw)\overline{(\tilde{\varphi}_x+\tilde{\psi}+l\tilde{w})}\, dx + \sum_{i=1}^{3}\xi_i\int_{\mathbb{R}}\phi_i\overline{\tilde{\phi}_i}\, d\xi.
\end{aligned}$$

We then reformulate the system (2.2) in the context of semigroup theory.

Introducing the vector function $U = (u_1,u_2,u_3,u_4,u_5,u_6,\phi_1,\phi_2,\phi_3)^T$, the system (2.2) can be reformulated as:

$$\begin{cases} U' = \mathcal{A}U, \quad t>0,\\ U(0)=U_0,\end{cases}$$

where $U_0 := \left(\varphi_0,\varphi_1,\psi_0,\psi_1,w_0,w_1,\phi_1^0,\phi_2^0,\phi_3^0\right)^T$.

The operator $\mathcal{A}$ is linear and is defined by

$$\mathcal{A}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ \dfrac{\kappa}{\rho_1}(u_{1x}+u_3+lu_5)_x + \dfrac{\kappa_0 l}{\rho_1}(u_{5x}-lu_1) - \dfrac{\zeta_1}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\,d\xi \\ u_4 \\ \dfrac{b}{\rho_2}u_{3xx} - \dfrac{\kappa}{\rho_2}(u_{1x}+u_3+lu_5) - \dfrac{\zeta_2}{\rho_2}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\,d\xi \\ u_6 \\ \dfrac{\kappa_0}{\rho_1}(u_{5x}-lu_1)_x - \dfrac{\kappa l}{\rho_1}(u_{1x}+u_3+lu_5) - \dfrac{\zeta_3}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\,d\xi \\ -(\xi^2+\eta)\phi_1 + u_2(x)\mu(\xi) \\ -(\xi^2+\eta)\phi_2 + u_4(x)\mu(\xi) \\ -(\xi^2+\eta)\phi_3 + u_6(x)\mu(\xi) \end{pmatrix}.$$

The domain of the operator $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u_1,u_2,u_3,u_4,u_5,u_6,\phi_1,\phi_2,\phi_3)^T \in \mathcal{H} \ | \ u_1,u_3,u_5 \in H^2 \cap H^1, \\ \xi\phi_1, \xi\phi_2, \xi\phi_3 \in L^2(\mathbb{R}), \\ -\left(|\xi|^2+\eta\right)\phi_i + u_{2i}(x)\mu(\xi) \in L^2(\mathbb{R}), \quad i=1,2,3, \\ \varphi_x(t,x) = \psi_x(t,x) = w_x(t,x) = \varphi(t,x) = \psi(t,x) = w(t,x) = 0 \\ \text{for } x = 0, L. \end{array} \right\}.$$

Theorem 2.2. *1. If $U_0 \in D(\mathcal{A})$, then system* (2.2) *has a unique strong solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+, \mathcal{H}).$$

2. If $U_0 \in \mathcal{H}$, then system (2.2) *has a unique weak solution*

$$U \in \mathcal{C}^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative.

For any $U = (u_1,u_2,u_3,u_4,u_5,u_6,\phi_1,\phi_2,\phi_3) \in D(\mathcal{A})$, we have

$$\mathcal{R}e\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\sum_{i=1}^{3} \zeta_i \int_{\mathbb{R}} \left(\xi^2+\eta\right)|\phi_i(x,\xi,t)|^2\,d\xi \le 0.$$

Hence, $\mathcal{A}$ is dissipative.

We will now show that the operator $I - \mathcal{A}$ is surjective.

Given $F = (f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9) \in \mathcal{H}$, we need to prove that there exists $U = (u_1,u_2,u_3,u_4,u_5,u_6,\phi_1,\phi_2,\phi_3) \in D(\mathcal{A})$ satisfying

$$(I - \mathcal{A})\,U = F.$$

That is,

$$\begin{cases} u_1 - u_2 = f_1, \\ u_2 - \dfrac{\kappa}{\rho_1}(u_{1x} + u_3 + lu_5)_x - \dfrac{\kappa_0 l}{\rho_1}(u_{5x} - lu_1) + \dfrac{\zeta_1}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\,d\xi = f_2, \\ u_3 - u_4 = f_3, \\ u_4 - \dfrac{b}{\rho_2}u_{3xx} + \dfrac{\kappa}{\rho_2}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_2}{\rho_2}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\,d\xi = f_4, \\ u_5 - u_6 = f_5, \\ u_6 - \dfrac{\kappa_0}{\rho_1}(u_{5x} - lu_1)_x + \dfrac{\kappa l}{\rho_1}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_3}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\,d\xi = f_6, \\ \phi_1(1+\xi^2+\eta) - \mu(\xi)u_2(x,t) = f_7, \\ \phi_2(1+\xi^2+\eta) - \mu(\xi)u_4(x,t) = f_8, \\ \phi_3(1+\xi^2+\eta) - \mu(\xi)u_6(x,t) = f_9. \end{cases} \tag{2.6}$$

Then, from $(2.6)_7$, $(2.6)_8$, and $(2.6)_9$, we obtain:

$$\begin{cases} \phi_1 = \dfrac{f_7 + \mu(\xi)u_2(x,t)}{1+\xi^2+\eta}, \\ \phi_2 = \dfrac{f_8 + \mu(\xi)u_4(x,t)}{1+\xi^2+\eta}, \\ \phi_3 = \dfrac{f_9 + \mu(\xi)u_6(x,t)}{1+\xi^2+\eta}. \end{cases} \tag{2.7}$$

Inserting the equations $(2.6)_1$ into $(2.6)_2$, $(2.6)_3$ into $(2.6)_4$, and $(2.6)_5$ into $(2.6)_6$, we obtain:

$$\begin{cases} \quad \rho_1 u_1 - \kappa(u_{1x} + u_3 + lu_5)_x - \kappa_0 l(u_{5x} - lu_1) + \zeta_1 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\,d\xi \\ = \rho_1(f_1 + f_2), \\ \quad \rho_2 u_3 - bu_{3xx} + \kappa(u_{1x} + u_3 + lu_5) + \zeta_2 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\,d\xi \\ = \rho_2(f_3 + f_4), \\ \quad \rho_1 u_5 - \kappa_0(u_{5x} - lu_1)_x + \kappa l(u_{1x} + u_3 + lu_5) + \zeta_3 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\,d\xi \\ = \rho_1(f_5 + f_6). \end{cases} \tag{2.8}$$

Solving the system (2.8) is equivalent to finding $u_1, u_3, u_5 \in H^2(0,L) \cap H^1(0,L)$ such that:

$$\int_0^L \left[\rho_1 u_1 - \kappa(u_{1x} + u_3 + lu_5)_x - \kappa_0 l(u_{5x} - lu_1) + \zeta_1 \int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\,d\xi\right] \chi\,dx$$
$$= \int_0^L \rho_1\,[f_1 + f_2]\,\chi\,dx,$$
$$\int_0^L \left[\rho_2 u_3 - bu_{3xx} + \kappa(u_{1x} + u_3 + lu_5) + \zeta_2 \int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\,d\xi\right] \zeta\,dx$$

$$= \int_0^L [\rho_2(f_3 + f_4)]\, \zeta\, dx,$$

$$\int_0^L \left[\rho_1 u_5 - \kappa_0(u_{5x} - l u_1)_x + \kappa l(u_{1x} + u_3 + l u_5) + \zeta_3 \int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\, d\xi\right] W\, dx$$

$$= \int_0^L [\rho_1(f_5 + f_6)]\, W\, dx, \tag{2.9}$$

for all $\chi, \zeta, W \in H^1(0,L)$.

Inserting the equations $(2.7)_1$ into $(2.9)_1$, $(2.7)_2$ into $(2.9)_2$, and $(2.7)_3$ into $(2.9)_3$, we obtain:

$$\left\{\begin{aligned}
&\int_0^L [\rho_1 u_1 - \kappa(u_{1x} + u_3 + l u_5)_x - \kappa_0 l(u_{5x} - l u_1) \\
&\quad + \zeta_1 u_1(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\Bigg]\, \chi\, dx \\
&= \int_0^L \left[\rho_1 f_1 + \rho_1 f_2 + \zeta_1 \int_{\mathbb{R}} \frac{\mu(\xi)(\mu(\xi) f_1 - f_7)}{1+\xi^2+\eta}\, d\xi\right] \chi\, dx, \\
&\int_0^L \left[\rho_2 u_3 - b u_{3xx} + \kappa(u_{1x} + u_3 + l u_5) + \zeta_2 u_3(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\right] \zeta\, dx \\
&= \int_0^L \left[\rho_2(f_3 + f_4) + \zeta_2 \int_{\mathbb{R}} \frac{\mu(\xi)(\mu(\xi) f_3 - f_8)}{1+\xi^2+\eta}\, d\xi\right] \zeta\, dx, \\
&\int_0^L [\rho_1 u_5 - \kappa_0(u_{5x} - l u_1)_x + \kappa l(u_{1x} + u_3 + l u_5) \\
&\quad + \zeta_3 u_5(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\Bigg]\, W\, dx \\
&= \int_0^L \left[\rho_1(f_5 + f_6) + \zeta_3 \int_{\mathbb{R}} \frac{\mu(\xi)(\mu(\xi) f_5 - f_9)}{1+\xi^2+\eta}\, d\xi\right] W\, dx.
\end{aligned}\right. \tag{2.10}$$

Consequently, the problem (2.10) is equivalent to the problem:

$$a\left((u_1, u_3, u_5), (\chi, \zeta, W)\right) = \mathcal{L}\left(\chi, \zeta, W\right), \tag{2.11}$$

where

$$\begin{aligned}
a\left((u_1, u_3, u_5), (\chi, \zeta, W)\right) = &\int_0^L [\rho_1 u_1 - \kappa(u_{1x} + u_3 + l u_5)_x - \kappa_0 l(u_{5x} - l u_1) \\
&+ \zeta_1 u_1(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\Bigg]\, \chi\, dx \\
&+ \int_0^L [\rho_2 u_3 - b u_{3xx} + \kappa(u_{1x} + u_3 + l u_5) \\
&+ \zeta_2 u_3(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\Bigg]\, \zeta\, dx \\
&+ \int_0^L [\rho_1 u_5 - \kappa_0(u_{5x} - l u_1)_x + \kappa l(u_{1x} + u_3 + l u_5) \\
&+ \zeta_3 u_5(x,t) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{1+\xi^2+\eta}\, d\xi\Bigg]\, W\, dx,
\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\left(\chi,\zeta,W\right)=&\int_0^L\left[\rho_1 f_1+\rho_1 f_2+\zeta_1\int_{\mathbb{R}}\frac{\mu(\xi)(\mu(\xi)f_1-f_7)}{1+\xi^2+\eta}\,d\xi\right]\chi\,dx\\&+\int_0^L\left[\rho_2(f_3+f_4)+\zeta_2\int_{\mathbb{R}}\frac{\mu(\xi)(\mu(\xi)f_3-f_8)}{1+\xi^2+\eta}\,d\xi\right]\zeta\,dx\\&+\int_0^L\left[\rho_1(f_5+f_6)+\zeta_3\int_{\mathbb{R}}\frac{\mu(\xi)(\mu(\xi)f_5-f_9)}{1+\xi^2+\eta}\,d\xi\right]W\,dx.\end{aligned}$$

It is easy to verify that a is continuous and coercive, and $\mathcal{L}$ is continuous. Applying the Lax-Milgram theorem, we infer that for all $(\chi,\zeta,W)\in H^1(0,L)\times H^1(0,L)\times H^1(0,L)$, problem (2.11) has a unique solution $(u_1,u_3,u_5)\in H^1_L(0,L)\times H^1_L(0,L)\times H^1_L(0,L)$. Applying classical elliptic regularity, it follows from (2.11) that $(u_1,u_3,u_5)\in H^2(0,L)\times H^2(0,L)\times H^2(0,L)$. Therefore, the operator $I-\mathcal{A}$ is surjective. Finally, the result of Theorem 2.2 follows from the Lumer-Phillips theorem. $\square$

3. Polynomial stability

In this section, we will prove a polynomial decay rate for the system. It is important to note that, in the decoupled case, the system fails to exhibit exponential decay.

First, we need to prove the following lemmas:

Lemma 3.1. *$\mathcal{A}$ has no eigenvalues on $i\mathbb{R}$.*

Proof. We prove that the unique $U=(u_1,u_2,u_3,u_4,u_5,u_6,\phi_1,\phi_2,\phi_3)\in D(\mathcal{A})$ satisfying

$$\mathcal{A}U=i\lambda U, \tag{3.1}$$

is $U=0$.

Equation (3.1) is equivalent to

$$\begin{cases}u_2=i\lambda u_1,\\ \dfrac{\kappa}{\rho_1}(u_{1x}-u_3+lu_5)_x+\dfrac{\kappa_0 l}{\rho_1}(u_{5x}-lu_1)-\dfrac{\zeta_1}{\rho_1}\displaystyle\int_{\mathbb{R}}\mu\left(\xi\right)\phi_1\left(\xi,t\right)d\xi=i\lambda u_2,\\ u_4=i\lambda u_3,\\ \dfrac{b}{\rho_2}u_{3xx}-\dfrac{\kappa}{\rho_2}(u_{1x}+u_3+lu_5)-\dfrac{\zeta_2}{\rho_2}\displaystyle\int_{\mathbb{R}}\mu\left(\xi\right)\phi_2\left(\xi,t\right)d\xi=i\lambda u_4,\\ u_6=i\lambda u_5,\\ \dfrac{\kappa_0}{\rho_1}(u_{5x}-lu_1)_x-\dfrac{\kappa l}{\rho_1}(u_{1x}+u_3+lu_5)-\dfrac{\zeta_3}{\rho_1}\displaystyle\int_{\mathbb{R}}\mu\left(\xi\right)\phi_3\left(\xi,t\right)d\xi=i\lambda u_6,\\ \phi_1(i\lambda+\xi^2+\eta)-\mu(\xi)u_2(x,t)=0,\\ \phi_2(i\lambda+\xi^2+\eta)-\mu(\xi)u_4(x,t)=0,\\ \phi_3(i\lambda+\xi^2+\eta)-\mu(\xi)u_6(x,t)=0.\end{cases} \tag{3.2}$$

Then, from $(3.2)_1$, $(3.2)_3$, $(3.2)_5$, $(3.2)_7$, $(3.2)_8$, and $(3.2)_9$, we obtain for $k\in$

$\{1,\,2,\,3\}$:

$$\begin{cases} \phi_k &= \dfrac{i\lambda\mu(\xi)u_{2k-1}(x,t)}{i\lambda+\xi^2+\eta}, \\ u_{2k} &= i\lambda u_{2k-1}(x,t). \end{cases} \tag{3.3}$$

On the other hand, multiplying $(3.2)_2$ by u_1, $(3.2)_4$ by u_3, and $(3.2)_6$ by u_5 leads to:

$$\begin{cases} \displaystyle\int_0^L -\kappa(u_{1x}+u_3+lu_5)u_{1x}+\kappa_0 l(u_{5x}-lu_1)u_1-\zeta_1\int_{\mathbb{R}}\frac{\mu^2\left(\xi\right)i\lambda u_1^2\left(x,t\right)}{i\lambda+\xi^2+\eta}d\xi \\ \displaystyle= -\lambda^2\rho_1\int_0^L u_1^2dx, \\ \displaystyle\int_0^L -bu_{3x}^2-\kappa(u_{1x}+u_3+lu_5)u_3-\zeta_2\int_{\mathbb{R}}\frac{\mu^2\left(\xi\right)i\lambda u_3^2\left(x,t\right)}{i\lambda+\xi^2+\eta}d\xi \\ \displaystyle= -\lambda^2\rho_2\int_0^L u_3^2dx, \\ \displaystyle\int_0^L -\kappa_0(u_{5x}-lu_1)u_{5x}-\kappa l(u_{1x}+u_3+lu_5)u_5-\zeta_3\int_{\mathbb{R}}\frac{\mu^2\left(\xi\right)i\lambda u_5^2\left(x,t\right)}{i\lambda+\xi^2+\eta}d\xi \\ \displaystyle= -\lambda^2\rho_1\int_0^L u_5^2dx. \end{cases} \tag{3.4}$$

Adding $(3.4)_1-(3.4)_3$, one gets:

$$\begin{aligned} &-\int_0^L \Big(bu_{3x}^2+\kappa(u_{1x}+u_3+lu_5)^2+\kappa_0(u_{5x}-lu_1)^2 \\ &+i\lambda\sum_{k=1}^{3}\zeta_k u_{2k-1}^2\int_{\mathbb{R}}\frac{\mu^2\left(\xi\right)}{i\lambda+\xi^2+\eta}d\xi\Big)dx \\ =&-\lambda^2\int_0^L\big(\rho_1u_1^2+\rho_2u_3^2+\rho_1u_5^2\big)dx. \end{aligned} \tag{3.5}$$

Here we distinguish 2 cases:

Case 1. $\lambda\neq 0$.

Taking the imaginary part in (3.5), we obtain

$$\lambda\sum_{k=1}^{3}\zeta_k\int_{\mathbb{R}}\frac{\mu^2\left(\xi\right)}{i\lambda+\xi^2+\eta}d\xi\int_0^L u_{2k-1}^2dx=0,\ \text{for } k\in\{1,\,2,\,3\},$$

and we deduce that $u_1=u_3=u_5\equiv 0$. Using now (3.3), it follows that $U\equiv 0$.

Case 2. $\lambda=0$.

Coming back to (3.3), we have:

$$u_2=u_4=u_6\equiv 0\ \text{ and } \phi_1=\phi_2=\phi_3\equiv 0.$$

On the other hand, we deduce from (3.5):

$$\begin{cases} u_{3x}=0, \\ u_{1x}+u_3+lu_5=0, \\ u_{5x}-lu_1=0. \end{cases} \tag{3.6}$$

Using the fact that $U \in D(\mathcal{A})$ and (3.6), we find that $u_3 = 0$. Consequently, u_1 satisfies the equation

$$u_{1xx} + l^2 u_1 = 0.$$

Given that $U \in D(\mathcal{A})$, this implies $u_1 = 0$, which in turn leads to the conclusion that $U \equiv 0$. □

Lemma 3.2. *The operator* $\Big(i\lambda I - \mathcal{A}\Big)$ *is surjective.*

Proof. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$ we looking for $U = (u_1, u_2, u_3, u_4, \phi_1, \phi_2) \in D(\mathcal{A})$ such that

$$i\lambda U - \mathcal{A}U = F.$$

That is,

$$\begin{cases} i\lambda u_1 - u_2 = f_1, \\ i\lambda u_2 - \dfrac{\kappa}{\rho_1}(u_{1x} + u_3 + lu_5)_x - \dfrac{\kappa_0 l}{\rho_1}(u_{5x} - lu_1) + \dfrac{\zeta_1}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\,d\xi = f_2, \\ i\lambda u_3 - u_4 = f_3, \\ i\lambda u_4 - \dfrac{b}{\rho_2}u_{3xx} + \dfrac{\kappa}{\rho_2}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_2}{\rho_2}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\,d\xi = f_4, \\ i\lambda u_5 - u_6 = f_5, \\ i\lambda u_6 - \dfrac{\kappa_0}{\rho_1}(u_{5x} - lu_1)_x + \dfrac{\kappa l}{\rho_1}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_3}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\,d\xi = f_6, \\ \phi_1(i\lambda + \xi^2 + \eta) - \mu(\xi)u_2(x,t) = f_7, \\ \phi_2(i\lambda + \xi^2 + \eta) - \mu(\xi)u_4(x,t) = f_8, \\ \phi_3(i\lambda + \xi^2 + \eta) - \mu(\xi)u_6(x,t) = f_9. \end{cases} \tag{3.7}$$

By eliminating u_2, u_4 and u_6 from the above system, we get the following system

$$\begin{cases} \quad -\rho_1\lambda^2 u_1 - \kappa(u_{1x} + u_3 + lu_5)_x - \kappa_0 l(u_{5x} - lu_1) + i\lambda\zeta_1 u_1(x,t)I_2(\lambda,\eta) \\ = \rho_1(i\lambda f_1 + f_2) + \zeta_1(I_2(\lambda,\eta)f_1 - I_1(\lambda,\eta)f_7), \\ \quad -\rho_2\lambda^2 u_3 - bu_{3xx} + \kappa(u_{1x} + u_3 + lu_5) + i\lambda\zeta_2 u_3(x,t)I_2(\lambda,\eta) \\ = \rho_2(i\lambda f_3 + f_4) + \zeta_2(I_2(\lambda,\eta)f_3 - I_1(\lambda,\eta)f_8), \\ \quad -\rho_1\lambda^2 u_5 - \kappa_0(u_{5x} - lu_1)_x + \kappa l(u_{1x} + u_3 + lu_5) + i\lambda\zeta_3 u_5(x,t)I_2(\lambda,\eta) \\ = \rho_1(i\lambda f_5 + f_6) + \zeta_3(I_2(\lambda,\eta)f_5 - I_1(\lambda,\eta)f_9), \end{cases} \tag{3.8}$$

where $I_1(\lambda,\eta) = \displaystyle\int_{\mathbb{R}} \frac{\mu(\xi)}{i\lambda + \xi^2 + \eta}d\xi$ and $I_2(\lambda,\eta) = \displaystyle\int_{\mathbb{R}} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}d\xi$.

We now distinguish two cases.

Step 1. $\lambda = 0$ and $\eta > 0$: System (3.8) is equivalent to finding $u_1, u_3, u_5 \in$

$H^2(0,L) \cap H_0^1(0,L)$ such that

$$\begin{cases} \displaystyle\int_0^L \left(-\kappa(u_{1x}+u_3+lu_5)_x - \kappa_0 l(u_{5x}-lu_1)\right)\chi\,dx \\ \displaystyle= \int_0^L \left(\rho_1 f_2 + \zeta_1(I_2(0,\eta)f_1 - I_1(0,\eta)f_7)\right)\chi\,dx, \\ \displaystyle\int_0^L \left(-bu_{3xx} + \kappa(u_{1x}+u_3+lu_5)\right)\zeta\,dx \\ \displaystyle= \int_0^L \left(\rho_2 f_4 + \zeta_2(I_2(0,\eta)f_3 - I_1(0,\eta)f_8)\right)\zeta\,dx, \\ \displaystyle\int_0^L \left(-\kappa_0(u_{5x}-lu_1)_x + \kappa l(u_{1x}+u_3+lu_5)\right)W\,dx \\ \displaystyle= \int_0^L \left(\rho_1 f_6 + \zeta_3(I_2(0,\eta)f_5 - I_1(0,\eta)f_9)\right)W\,dx, \end{cases} \tag{3.9}$$

for all $\chi, \zeta, W \in H_0^1(0,L)$.

Using integration by parts in (3.9) we deduce that (3.8) is equivalent to:

$$b\left((u_1,u_3,u_3),(\chi,\zeta,W)\right) = \mathcal{M}(\chi,\zeta,W), \tag{3.10}$$

where

$$b\left((u_1,u_3,u_5),(\chi,\zeta,W)\right) = \int_0^L \left[\kappa(u_{1x}+u_3+lu_5)(\chi_x+\zeta+lW) + bu_{3x}\zeta_x + \kappa_0(u_{5x}-lu_1)(W_x-l\chi)\right] dx,$$

and

$$\begin{aligned}\mathcal{M}(\chi,\zeta,W) = &\int_0^L \left[\rho_1 f_2 + \zeta_1(I_2(0,\eta)f_1 - I_1(0,\eta)f_7)\right]\chi\,dx \\ &+ \int_0^L \left[\rho_2 f_4 + \zeta_2(I_2(0,\eta)f_3 - I_1(0,\eta)f_8)\right]\zeta\,dx \\ &+ \int_0^L \left[\rho_1 f_6 + \zeta_3(I_2(0,\eta)f_5 - I_1(0,\eta)f_9)\right]W\,dx.\end{aligned}$$

It is straightforward to verify that the bilinear form b is continuous and coercive, and the operator $\mathcal{M}$ is continuous. By applying the Lax-Milgram theorem, we conclude that for all $(\chi,\zeta,W) \in \left(H_0^1(0,L)\right)^3$, the problem (3.10) has a unique solution $(u_1,u_3,u_5) \in \left(H_0^1(0,L)\right)^3$. Utilizing classical elliptic regularity, it follows from (3.9) that $(u_1,u_3,u_5) \in \left(H^2(0,L)\right)^3$. Consequently, the operator $-\mathcal{A}$ is surjective.

Step 2. $\lambda \neq 0$ and $\eta \geq 0$:

Now, consider the system:

$$\begin{cases} -\kappa(u_{1x}+u_3+lu_5)_x - \kappa_0 l(u_{5x}-lu_1) = g_1, \\ -bu_{3xx} + \kappa(u_{1x}+u_3+lu_5) = g_2, \\ -\kappa_0(u_{5x}-lu_1)_x + \kappa l(u_{1x}+u_3+lu_5) = g_3, \end{cases} \tag{3.11}$$

with the conditions

$$u_1(t,x)=u_3(t,x)=u_5(t,x)=0, \quad \text{for } x=0,L,$$
$$u_{1x}(t,x)=u_{3x}(t,x)=u_{5x}(t,x)=0, \quad \text{for } x=0,L,$$

where $(g_1,g_2,g_3)\in\Big(L^2(0,L)\Big)^3$.

Let us note $\mathcal{L}:(u_1,u_2,u_3)\longrightarrow(-\kappa(u_{1x}+u_3+lu_5)_x-\kappa_0 l(u_{5x}-lu_1),-bu_{3xx}+\kappa(u_{1x}+u_3+lu_5),-\kappa_0(u_{5x}-lu_1)_x+\kappa l(u_{1x}+u_3+lu_5))$ with domain $D(\mathcal{L})=\{(u_1,u_2,u_3)\in\Big(H_0^1(0,L)\cap H^2(0,L)\Big)^3,\ u_{1x}(x)=u_{3x}(x)=u_{5x}(x)=0,\ \text{for } x=0,L\}$.

Multiplying $(3.11)_1$ by χ, $(3.11)_2$ by ζ and $(3.11)_3$ by W one gets:

$$\begin{aligned}&\int_0^L[\kappa(u_{1x}+u_3+lu_5)(\chi_x+\zeta+lW)+bu_{3x}\zeta_x+\kappa_0(u_{5x}-lu_1)(W_x-l\chi)]\,dx\\ &=\int_0^L\big(g_1\chi+g_2\zeta+g_3W\big)dx,\end{aligned} \tag{3.12}$$

for all $(\chi,\zeta,W)\in\Big(H_0^1(0,L)\Big)^3$.

By applying the Lax–Milgram theorem once more, we deduce that there exists a unique strong solution $(u_1,u_3,u_5)\in D(\mathcal{L})$ for the variational problem (3.12).

Consequently, it follows that $\mathcal{L}^{-1}$ is compact in $\Big(L^2(0,L)\Big)^3$ and therefore (3.8) is equivalent to:

$$\Big(\mathcal{L}^{-1}\circ B-I\Big)U=\Phi,$$

where $U=(u_1,u_3,u_5)$,

$$BU:=\big((\rho_1\lambda^2-i\lambda\zeta_1I_2(\lambda,\eta))u_1,(\rho_2\lambda^2-i\lambda\zeta_2I_2(\lambda,\eta))u_3,(\rho_1\lambda^2-i\lambda\zeta_3I_2(\lambda,\eta))u_5\big)$$

and $\Phi=-\Big(\rho_1(i\lambda f_1+f_2)+\zeta_1(I_2(\lambda,\eta)f_1-I_1(\lambda,\eta)f_7),\rho_2(i\lambda f_3+f_4)+\zeta_2(I_2(\lambda,\eta)f_3-I_1(\lambda,\eta)f_8),\rho_1(i\lambda f_5+f_6)+\zeta_3(I_2(\lambda,\eta)f_5-I_1(\lambda,\eta)f_9)\Big)$. Noting that the operator B is bounded, so $\mathcal{L}\circ B$ is compact, and applying Fredholm's alternative, it is sufficient to show that

$$\ker\big(\mathcal{L}^{-1}\circ B-I\big)=\{0\}.$$

For this purpose, let $(y_1,y_3,y_5)\in Ker\Big(\mathcal{L}^{-1}\circ B-I\Big)$ then we have:

$$\begin{cases}(\rho_1\lambda^2-i\lambda\zeta_1I_2(\lambda,\eta))y_1+\kappa(y_{1x}+y_3+ly_5)_x+\kappa_0l(y_{5x}-ly_1)=0,\\ (\rho_2\lambda^2-i\lambda\zeta_2I_2(\lambda,\eta))y_3+by_{3xx}-\kappa(y_{1x}+y_3+ly_5)=0,\\ (\rho_1\lambda^2-i\lambda\zeta_3I_2(\lambda,\eta))y_5+\kappa_0(y_{5x}-ly_1)_x-\kappa l(y_{1x}+y_3+ly_5)=0,\end{cases} \tag{3.13}$$

with the conditions

$$y_1(t,x)=y_3(t,x)=y_5(t,x)=0, \quad \text{for } x=0,L,$$
$$y_{1x}(t,x)=y_{3x}(t,x)=y_{5x}(t,x)=0, \quad \text{for } x=0,L.$$

Multiplying $(3.13)_1$ by $\overline{y_1}$, $(3.13)_2$ by $\overline{y_3}$ and $(3.13)_3$ by $\overline{y_5}$, integrating over $(0, L)$, one gets:

$$\int_0^L \Big((\rho_1\lambda^2 - i\lambda\zeta_1 I_2(\lambda,\eta))|y_1|^2 + (\rho_2\lambda^2 - i\lambda\zeta_2 I_2(\lambda,\eta))|y_3|^2 + (\rho_1\lambda^2 - i\lambda\zeta_3 I_2(\lambda,\eta))|y_5|^2 \Big) dx + \int_0^L \left[\kappa|y_{1x} + y_3 + ly_5|^2 + b|y_{3x}|^2 + \kappa_0|y_{5x} - ly_1|^2 \right] dx = 0.$$

Taking the real part, we deduce that $(y_1, y_3, y_5) = (0, 0, 0)$. This completes the proof of Lemma 3.2. □

We now recall the following result, which characterizes the polynomial decay of the energy.

Lemma 3.3 ([11]). *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{3.14}$$

then for a fixed $\delta > 0$, the following conditions are equivalent:

$$\lim_{s\in\mathbb{R}} \sup_{|s|\to\infty} \frac{1}{|s|^\delta} \|(isI - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{3.15}$$

$$\|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})},\ U_0 \in D(\mathcal{A}),\ \textit{for some } c > 0.$$

Our main result in the section is the following:

Theorem 3.1. *The semigroup $\{S(t)\}_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{1-\alpha}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Furthermore, the energy decay rate of $t^{2/1-\alpha}$ is optimal for general initial data in $D(\mathcal{A})$.

Proof. Based on Lemma 3.3, the proof of Theorem 3.1 requires verifying the validity of (3.14) and (3.15), where $\delta = 1 - \alpha$. Since (3.14) follows from Lemma 3.1 and Lemma 3.2, our focus shifts solely to proving (3.15).

Here, we employ a contradiction argument. Suppose that (3.15) is invalid; consequently, there exists a sequence $\lambda_n \in \mathbb{R}, n \in \mathbb{N}$ such that $\lambda_n \to +\infty$ as $n \to +\infty$, and a sequence $U^n = (u_1^n, u_2^n, u_3^n, u_4^n, u_5^n, u_6^n, \phi_1^n, \phi_2^n, \phi_3^n) \in D(\mathcal{A})$ such that

$$\|U_n\| = 1, \tag{3.16}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\delta} \|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = 0. \tag{3.17}$$

We have

$$F^n = \lambda_n^\delta (i\lambda_n I - \mathcal{A})U_n = (f_1^n, f_2^n, f_3^n, f_4^n, f_5^n, f_6^n, f_7^n, f_8^n, f_9^n).$$

For simplicity, we drop the index n in the sequel. From (3.17), we get

$$\begin{cases} i\lambda u_1 - u_2 = \dfrac{f_1}{\lambda^\delta} \longrightarrow 0, \\ \quad i\lambda u_2 - \dfrac{\kappa}{\rho_1}(u_{1x} + u_3 + lu_5)_x - \dfrac{\kappa_0 l}{\rho_1}(u_{5x} - lu_1) + \dfrac{\zeta_1}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\, \phi_1(x, \xi, t)\, d\xi \\ = \dfrac{f_2}{\lambda^\delta} \longrightarrow 0, \\ i\lambda u_3 - u_4 = \dfrac{f_3}{\lambda^\delta} \longrightarrow 0, \\ \quad i\lambda u_4 - \dfrac{b}{\rho_2}u_{3xx} + \dfrac{\kappa}{\rho_2}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_2}{\rho_2}\displaystyle\int_{\mathbb{R}} \mu(\xi)\, \phi_2(x, \xi, t)\, d\xi \\ = \dfrac{f_4}{\lambda^\delta} \longrightarrow 0, \\ i\lambda u_5 - u_6 = \dfrac{f_5}{\lambda^\delta} \longrightarrow 0, \\ \quad i\lambda u_6 - \dfrac{\kappa_0}{\rho_1}(u_{5x} - lu_1)_x + \dfrac{\kappa l}{\rho_1}(u_{1x} + u_3 + lu_5) + \dfrac{\zeta_3}{\rho_1}\displaystyle\int_{\mathbb{R}} \mu(\xi)\, \phi_3(x, \xi, t)\, d\xi \\ = \dfrac{f_6}{\lambda^\delta} \longrightarrow 0, \\ \phi_1(i\lambda + \xi^2 + \eta) - \mu(\xi)u_2(x, t) = \dfrac{f_7}{\lambda^\delta} \longrightarrow 0, \\ \phi_2(i\lambda + \xi^2 + \eta) - \mu(\xi)u_4(x, t) = \dfrac{f_8}{\lambda^\delta} \longrightarrow 0, \\ \phi_3(i\lambda + \xi^2 + \eta) - \mu(\xi)u_6(x, t) = \dfrac{f_9}{\lambda^\delta} \longrightarrow 0. \end{cases} \tag{3.18}$$

In the following, we will prove that, $\|U\|_{\mathcal{H}} = o(1)$, hence reaching the desired contradiction. For clarity, we divide the proof into several lemmas.

On the other hand, for all $\delta > 0$, taking the real part of the inner product of (3.17) with U in $\mathcal{H}$, then using the fact that U is uniformly bounded in $\mathcal{H}$, we have

$$\int_0^L \int_{\mathbb{R}} \left(\xi^2 + \eta\right) |\phi_i(x, \xi, t)|^2\, d\xi dx = \frac{o(1)}{\lambda^\delta}, \quad \text{for } i = 1, 2, 3. \tag{3.19}$$

Inserting the equations $(3.18)_1$ into $(3.18)_2$, $(3.18)_3$ into $(3.18)_4$, and $(3.18)_5$ into $(3.18)_6$, we obtain:

$$\begin{cases} \quad \rho_1\lambda^2 u_1 + \kappa(u_{1x} + u_3 + lu_5)_x + \kappa_0 l(u_{5x} - lu_1) - \zeta_1 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_1(x, \xi, t)\, d\xi \\ = -\rho_1(\dfrac{f_2}{\lambda^\delta} + \dfrac{if_1}{\lambda^{\delta-1}}), \\ \quad \rho_2\lambda^2 u_3 + bu_{3xx} + \kappa(u_{1x} + u_3 + lu_5) - \zeta_2 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_2(x, \xi, t)\, d\xi \\ = -\rho_2(\dfrac{f_4}{\lambda^\delta} + \dfrac{if_3}{\lambda^{\delta-1}}), \\ \quad \rho_1\lambda^2 u_5 + \kappa_0(u_{5x} - lu_1)_x - \kappa l(u_{1x} + u_3 + lu_5) - \zeta_3 \displaystyle\int_{\mathbb{R}} \mu(\xi)\phi_3(x, \xi, t)\, d\xi \\ = -\rho_1(\dfrac{f_6}{\lambda^\delta} + \dfrac{if_5}{\lambda^{\delta-1}}). \end{cases} \tag{3.20}$$

To complete the proof of the theorem, we require the following lemmas:

Lemma 3.4. *Let $\delta > 0$, we have*
$\int_0^L |u_2(x)|^2 dx = \frac{o(1)}{\lambda^{\delta+\alpha-1}}$, $\int_0^L |u_4(x)|^2 dx = \frac{o(1)}{\lambda^{\delta+\alpha-1}}$, *and* $\int_0^L |u_6(x)|^2 dx = \frac{o(1)}{\lambda^{\delta+\alpha-1}}$.

Proof. From $(3.18)_7$, we have

$$(i\lambda + \xi^2 + \eta)\phi_1 - \frac{f_7}{\lambda^\delta} = u_2(x)\mu(\xi), \quad \text{on} \quad (0, L).$$

Then

$$|u_2(x)|\mu(\xi) \le (|\lambda| + \xi^2 + \eta)|\phi_1| + \frac{|f_7|}{\lambda^\delta}, \quad \text{on} \quad (0, L).$$

By multiplying it by $(|\lambda| + \xi^2 + \eta)^{-2}|\xi|$, we obtain

$$(i\lambda+\xi^2+\eta)^{-2}|\xi|u_2(x)\mu(\xi) = (i\lambda+\xi^2+\eta)^{-1}|\xi|\phi_1 - (i\lambda+\xi^2+\eta)^{-2}|\xi|\frac{f_7}{\lambda^\delta}, \forall x \in (0, L). \tag{3.21}$$

Taking the absolute values of both sides of (3.21), integrating over$(-\infty, +\infty)$ with respect to the variable ξ, and applying Cauchy-Schwarz's inequality, we obtain

$$\mathcal{P}|u_2(x)| \le \mathcal{M}\left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1(x, \xi)|^2\, d\xi\right)^{\frac{1}{2}} + \mathcal{N}\left(\int_{-\infty}^{+\infty} \left|\frac{f_7}{\lambda^\delta}\right|^2 d\xi\right)^{\frac{1}{2}}, \tag{3.22}$$

where $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{N}$ are defined as:

$$\mathcal{P} := \left|\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2}|\xi|\mu(\xi)\, d\xi\right|, \ \mathcal{M} := \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2}\, d\xi\right)^{\frac{1}{2}},$$
$$\mathcal{N} := \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-4}|\xi|^2\, d\xi\right)^{\frac{1}{2}}.$$

By applying Young's inequality and integrating (3.22) over $(0, L)$, we obtain

$$\int_0^L |u_2(x)|^2 dx \le \frac{2\mathcal{M}^2}{\mathcal{P}^2}\int_0^L \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1(x, \xi)|^2\, d\xi dx + \frac{2\mathcal{N}^2}{\mathcal{P}^2}\int_0^L \int_{-\infty}^{+\infty} \left|\frac{f_7}{\lambda^\delta}\right|^2 d\xi dx.$$

Using lemma 2.2, we get

$$\mathcal{P} = \frac{|1 - 2\alpha|}{4}\frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}(|\lambda| + \eta)^{\frac{(2\alpha-5)}{4}}, \ \mathcal{M} \le \sqrt{\frac{\pi}{2}}(|\lambda| + \eta)^{-\frac{3}{4}},$$

and

$$\mathcal{N} \le \frac{\sqrt{\pi}}{4}(|\lambda| + \eta)^{-\frac{5}{4}}.$$

It is simple to check that

$$\mathcal{P}^2 = O(|\lambda|^{\frac{2\alpha-5}{2}}), \quad \mathcal{M}^2 = O(|\lambda|^{-\frac{3}{2}}) \ \text{and} \ \mathcal{N}^2 = O(|\lambda|^{-\frac{5}{2}}).$$

Then

$$\int_0^L |u_2(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}} + \frac{o(1)}{\lambda^{\alpha+2\delta}} = \frac{o(1)}{\lambda^{\alpha-1+\delta}}. \tag{3.23}$$

Using the same argument, we can prove

$$\int_0^L |u_4(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}} \quad \text{and} \quad \int_0^L |u_6(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}. \tag{3.24}$$

□

Lemma 3.5. *Let $\delta > 0$. Then the solution $(u_1, u_2, u_3, u_4, u_5, u_6, \phi_1, \phi_2, \phi_3) \in D(\mathcal{A})$ of* (3.18) *satisfies the following asymptotic behavior estimation:*

$$\int_0^L |\lambda u_1(x)|^2 dx = \int_0^L |\lambda u_3(x)|^2 dx = \int_0^L |\lambda u_5(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}. \tag{3.25}$$

Proof. From $(3.18)_1$, we obtain

$$\int_0^L |\lambda u_1|^2 dx \leq 2\int_0^L |u_2|^2 dx + \frac{2}{\lambda^{2\delta}} \int_0^L |f_1|^2 dx.$$

Hence, using Lemma 3.4 and the fact that $\|f_1\| = o(1)$, we get

$$\int_0^L |\lambda u_1(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}.$$

Using the same argument, we can prove

$$\int_0^L |\lambda u_3(x)|^2 dx = \int_0^L |\lambda u_5(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}.$$

□

Lemma 3.6. *Let $\delta > 0$. Then the solution $(u_1, u_2, u_3, u_4, u_5, u_6, \phi_1, \phi_2, \phi_3) \in D(\mathcal{A})$ of* (3.18) *satisfies the following asymptotic behavior estimation:*

$$\int_0^L |u_{1x}(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}},$$

$$\int_0^L |u_{3x}(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}} \quad \textit{and} \quad \int_0^L |u_{5x}(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}. \tag{3.26}$$

Proof. Multiplying $(3.20)_1$ by $\overline{u}_1$, $(3.20)_2$ by $\overline{u}_3$, and $(3.20)_3$ by $\overline{u}_5$ leads to:

$$\begin{cases} \int_0^L \Big(-\rho_1\lambda^2|u_1|^2 + \kappa(u_{1x} + u_3 + lu_5)\overline{u}_{1x} - \kappa_0 l(u_{5x} - lu_1)\overline{u}_1 \\ + \zeta_1 \int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\overline{u}_1 d\xi\Big)\, dx = \int_0^L \rho_1(\frac{f_2}{\lambda^\delta} + \frac{if_1}{\lambda^{\delta-1}})\overline{u}_1 dx, \\ \int_0^L \Big(-\rho_2\lambda^2|u_3|^2 + b|u_{3x}|^2 + \kappa(u_{1x} + u_3 + lu_5)\overline{u}_3 \\ + \zeta_2 \int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\overline{u}_3 d\xi\Big)\, dx = \int_0^L \rho_2(\frac{f_4}{\lambda^\delta} + \frac{if_3}{\lambda^{\delta-1}})\overline{u}_3 dx, \\ \int_0^L \Big(-\rho_1\lambda^2|u_5|^2 + \kappa_0(u_{5x} - lu_1)\overline{u}_{5x} + \kappa l(u_{1x} + u_3 + lu_5)\overline{u}_5 \\ + \zeta_3 \int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\overline{u}_5 d\xi\Big)\, dx = \int_0^L \rho_1(\frac{f_6}{\lambda^\delta} + \frac{if_5}{\lambda^{\delta-1}})\overline{u}_5 dx. \end{cases} \tag{3.27}$$

Adding $(3.27)_1 - (3.27)_3$, one gets

$$\begin{aligned}
&\int_0^L \left(-\rho_1\lambda^2|u_1|^2 - \rho_2\lambda^2|u_3|^2 - \rho_1\lambda^2|u_5|^2 + b|u_{3x}|^2 + \kappa|u_{1x} + u_3 + lu_5|^2\right.\\
&\left.+\kappa_0|u_{5x} - lu_1|^2\right) dx\\
&+ \int_0^L \left(\zeta_1 \int_{\mathbb{R}} \mu(\xi)\phi_1(x,\xi,t)\overline{u}_1 d\xi + \zeta_2 \int_{\mathbb{R}} \mu(\xi)\phi_2(x,\xi,t)\overline{u}_3 d\xi\right.\\
&\left.+\zeta_3 \int_{\mathbb{R}} \mu(\xi)\phi_3(x,\xi,t)\overline{u}_5 d\xi\right) dx\\
=& \int_0^L \left(\rho_1(\frac{f_2}{\lambda^\delta} + \frac{if_1}{\lambda^{\delta-1}})\overline{u}_1 + \rho_2(\frac{f_4}{\lambda^\delta} + \frac{if_3}{\lambda^{\delta-1}})\overline{u}_3 + \rho_1(\frac{f_6}{\lambda^\delta} + \frac{if_5}{\lambda^{\delta-1}})\overline{u}_5\right) dx.
\end{aligned} \tag{3.28}$$

From (3.28) and considering (3.16), (3.19), Lemma 3.4, and Lemma 3.5, we obtain

$$\int_0^L \left(b|u_{3x}|^2 + \kappa|u_{1x} + u_3 + lu_5|^2 + \kappa_0|u_{5x} - lu_1|^2\right) dx = \frac{o(1)}{\lambda^{\delta+\alpha-1}}.$$

Consequently, it follows that

$$\int_0^L |u_{3x}(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+\delta}}. \tag{3.29}$$

Using the fact that

$$\|u_{1x}\|_{L^2(0,L)} \le \|u_{1x} + u_3 + lu_5\|_{L^2(0,L)} + \|u_3 + lu_5\|_{L^2(0,L)}$$

and

$$\|u_{5x}\|_{L^2(0,L)} \le \|u_{5x} - lu_1\|_{L^2(0,L)} + \|lu_1\|_{L^2(0,L)},$$

we complete the proof of Lemma 3.6. □

Returning to the proof of Theorem 3.1, and taking into account Lemmas 3.4, 3.5, and 3.6, we establish that $\|U\| = o(1)$, which contradicts (3.16). Moreover, we confirm the optimality of the decay rate, which closely aligns with the asymptotic expansion of the eigenvalues. Specifically, it reveals a behavior in the real part resembling $k^{(1-\alpha)}$. This concludes the proof.

□

4. Discrete energy of the system

We will start by using the Finite Element Method (FEM) to obtain a discrete representation of the solution to equation (1.1)-(1.3). Before calculating the discrete energy, we employ the finite difference method to approximate the fractional derivative. The energy discrete $\hat{E}(t)$ will then be calculated using this method.

4.1. Discrete formulation by Finite Element Method

Let $\Omega = [0, L]$ be a finite domain. Let Ω_e be a uniform partition of Ω, with a uniform grid given by:

$$0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = L,$$

so $\Omega = \bigcup_{i=0}^{m-1} \Omega_i$, where $\Omega_i = [x_i, x_{i+1}]$. The time discretization of the interval $I = [0, T]$ is given by

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T,$$

where m and n are positive integers, $\Delta x = x_i - x_{i-1} = \frac{L}{m}$, so $x_i = i\Delta x$ for $i = 1, \ldots, m$, and $\Delta t = t_j - t_{j-1} = \frac{T}{n}$, so $t_j = j\Delta t$ for $j = 1, \ldots, n$.

Denote by $\varphi(x_i, t_j) = \varphi_i^j$, $\psi(x_i, t_j) = \psi_i^j$, and $w(x_i, t_j) = w_i^j$ the value of the functions φ, ψ, and w evaluated at the point x_i and the instant t_j. We also define the space S_k as the set of piecewise linear functions associated with this partition:

$$S_k = \{u; \, u|_{\Omega_i} \in P_1(\Omega_i), \, u \in \mathcal{C}(\Omega)\},$$

where $P_1(\Omega_i)$ is the space of linear polynomials defined on Ω_i.

The basis functions h_i of S_k for each Ω_i in changing from the real base to the reference base are given by:

$$\mathcal{B} = \left\{ h_1 = \frac{1}{x_2 - x_1}(x_2 - x), \; h_2 = \frac{1}{x_2 - x_1}(x - x_1) \right\}.$$

Denoting by φ^j, ψ^j, and w^j the approximations of $\varphi^j(t_j, x)$, $\psi^j(t_j, x)$, and $w^j(t_j, x)$, we have:

$$\varphi^j := \sum_{i=0}^{m} \varphi_i^j h_i(x), \qquad \psi^j := \sum_{i=0}^{m} \psi_i^j h_i(x), \quad \text{and} \quad w^j := \sum_{i=0}^{m} w_i^j h_i(x),$$

where

$$h_i(x) = \begin{cases} \dfrac{1}{\Delta x}(x - x_{i-1}), & \forall x \in [x_{i-1}, x_i], \\ \dfrac{1}{\Delta x}(x_{i+1} - x), & \forall x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere}, \end{cases} \quad \frac{\partial}{\partial x} h_i(x) = \begin{cases} \dfrac{1}{\Delta x}, & \forall x \in [x_{i-1}, x_i], \\ \dfrac{-1}{\Delta x}, & \forall x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere}, \end{cases}$$

$$h_0(x) = \begin{cases} \dfrac{1}{\Delta x}(x_1 - x), & \forall x \in [x_0, x_1], \\ 0, & \text{elsewhere}, \end{cases}$$

$$h_m(x) = \begin{cases} \dfrac{1}{\Delta x}(x - x_{m-1}), & \forall x \in [x_{m-1}, x_m], \\ 0, & \text{elsewhere}. \end{cases}$$

In Figure 1 below, we show the distribution of test functions across elements.

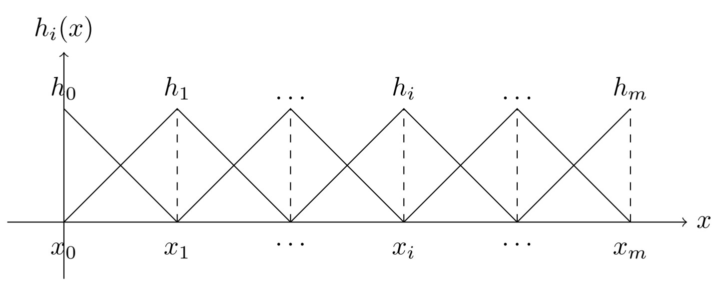

Figure 1. Piecewise Linear Interpolation Functions $h_i(x)$ over (x_0, x_m).

To summarize the principle of the finite element method, we multiply the equations $(1.1)_1$, $(1.1)_2$, and $(1.1)_3$ by h respectively, and integrate over Ω. We obtain:

$$\begin{cases} \Big(\rho_1\varphi_{tt},h\Big)_\Omega - \kappa\Big(\Big(\varphi_{xx},h\Big)_\Omega + \Big(\psi_x,h\Big)_\Omega + \Big(lw_x,h\Big)_\Omega\Big) \\ -\kappa_0 l\Big(\Big(w_x,h\Big)_\Omega - \Big(l\varphi,h\Big)_\Omega\Big) + \Big(\partial_t^{\alpha_1,\eta}\varphi,h\Big)_\Omega = 0, \\ \Big(\rho_2\psi_{tt},h\Big)_\Omega - \Big(b\psi_{xx},h\Big)_\Omega + \kappa\Big(\Big(\varphi_x,h\Big)_\Omega + \Big(\psi,h\Big)_\Omega + \Big(lw,h\Big)_\Omega\Big) \\ +\Big(\partial_t^{\alpha_2,\mu}\psi,h\Big)_\Omega = 0, \\ \Big(\rho_1 w_{tt},h\Big)_\Omega - \kappa_0\Big(\Big(w_{xx},h\Big)_\Omega - \Big(l\varphi_x,h\Big)_\Omega\Big) \\ +\kappa l\Big(\Big(\varphi_x,h\Big)_\Omega + \Big(\psi,h\Big)_\Omega + \Big(lw,h\Big)_\Omega\Big) + \Big(\partial_t^{\alpha_3,\nu}w,h\Big)_\Omega = 0. \end{cases}$$

The weak formulation of the problem can also be expressed by choosing each test function h as h_i, $i=\overline{0,m}$, and for $j=\overline{1,n-1}$, as follows:

$$\begin{cases} \Big(\rho_1\varphi^j_{tt},h_i\Big)_\Omega + \kappa\Big(\Big(\varphi^j_x,h_i\Big)_\Omega - \Big(\psi^j_x,h_i\Big)_\Omega - \Big(lw^j_x,h_i\Big)_\Omega\Big) \\ -\kappa_0 l\Big(\Big(w^j_x,h_i\Big)_\Omega - \Big(l\varphi^j,h_i\Big)_\Omega\Big) + \Big(\partial_t^{\alpha_1,\eta}\varphi^j,h_i\Big)_\Omega = 0, \\ \Big(\rho_2\psi^j_{tt},h_i\Big)_\Omega + \Big(b\psi^j_x,h_i\Big)_\Omega + \kappa\Big(\Big(\varphi^j_x,h_i\Big)_\Omega + \Big(\psi^j,h_i\Big)_\Omega + \Big(lw^j,h_i\Big)_\Omega\Big) \\ +\Big(\partial_t^{\alpha_2,\mu}\psi^j,h_i\Big)_\Omega = 0, \\ \Big(\rho_1 w^j_{tt},h_i\Big)_\Omega + \kappa_0\Big(\Big(w^j_x,h_i\Big)_\Omega + \Big(l\varphi^j_x,h_i\Big)_\Omega\Big) + \kappa l\Big(\Big(\varphi^j_x,h_i\Big)_\Omega + \Big(\psi^j,h_i\Big)_\Omega \\ +\Big(lw^j,h_i\Big)_\Omega\Big) + \Big(\partial_t^{\alpha_3,\nu}w^j,h_i\Big)_\Omega = 0. \end{cases}$$

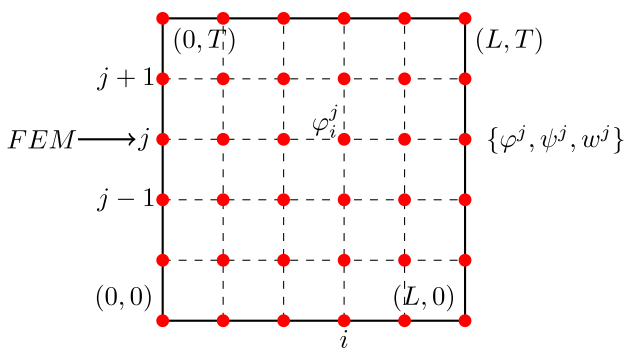

Figure 2. Mesh of the domain $[0,L]\times[0,T]$ with red points at each (x_i,t_j).

In Figure 2, we show the pattern mesh of φ, ψ, and w using the discretization of the intervals $(0,L)$ and $(0,T)$.

Now, using the finite difference method, we define the following approximations of the derivatives of φ, ψ, and w, respectively:

$$\varphi^j_{tt} = \frac{\varphi^{j+1}-2\varphi^j+\varphi^{j-1}}{\Delta t^2}, \quad \psi^j_{tt} = \frac{\psi^{j+1}-2\psi^j+\psi^{j-1}}{\Delta t^2}, \tag{4.1}$$

and

$$w_{tt}^{j} = \frac{w^{j+1} - 2w^{j} + w^{j-1}}{\Delta t^{2}}. \tag{4.2}$$

The following lemmas will be useful.

Lemma 4.1. *For $\alpha \in (0,1)$ we have:*

$$\begin{aligned}\partial_t^{\alpha,\eta}\varphi^j =& \frac{\eta^{\alpha-1}}{\Delta t\Gamma(1-\alpha)}\sum_{k=0}^{j-1}(\varphi^{k+1}-\varphi^k)\\ &\times\Big(\gamma\Big(1-\alpha,\eta(t_j-t_k)\Big)-\gamma\Big(1-\alpha,\eta(t_j-t_{k+1})\Big)\Big).\end{aligned} \tag{4.3}$$

Proof. Recall that

$$\partial_t^{\alpha,\eta}\varphi(t_j,x) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_j}(t_j-r)^{-\alpha}e^{-\eta(t_j-r)}\frac{\partial\varphi}{\partial r}(r,x)\,dr,$$

where $\eta \geq 0$. Using finite differences, we get

$$\partial_t^{\alpha,\eta}\varphi^j = \frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(t_j-r)^{-\alpha}e^{-\eta(t_j-r)}\frac{\varphi^{k+1}-\varphi^k}{\Delta t}\,dr.$$

We then have

$$\partial_t^{\alpha,\eta}\varphi^j = \frac{1}{\Delta t\Gamma(1-\alpha)}\sum_{k=0}^{j-1}(\varphi^{k+1}-\varphi^k)\int_{t_k}^{t_{k+1}}(t_j-r)^{-\alpha}e^{-\eta(t_j-r)}\,dr. \tag{4.4}$$

Changing variables with $u = \eta(t_j - r)$, we obtain

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}(t_j-r)^{-\alpha}e^{-\eta(t_j-r)}\,dr\\ =&\eta^{\alpha-1}\int_{\eta(t_j-t_{k+1})}^{\eta(t_j-t_k)}u^{1-\alpha-1}e^{-u}\,du\\ =&\eta^{\alpha-1}\Big(\gamma\Big((1-\alpha),\eta(t_j-t_k)\Big)-\gamma\Big((1-\alpha),\eta(t_j-t_{k+1})\Big)\Big),\end{aligned} \tag{4.5}$$

where γ is the lower incomplete gamma function defined by:

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt.$$

Substituting (4.5) into (4.4), we obtain (4.3). □

Then, using (4.1), (4.2), and Lemma 4.1, we obtain the fully discrete scheme

(1.1) as follows:

$$\begin{cases} p_1 M \dfrac{\varphi^{j+1} - 2\varphi^j + \varphi^{j-1}}{\Delta t^2} + \kappa \left(K\varphi^j - S\psi^j - lSw^j \right) - \kappa_0 l \left(Sw^j - lM\varphi^j \right) \\ +M \displaystyle\sum_{k=0}^{j-1} (\varphi^{k+1} - \varphi^k) \cdot C_k^{\alpha_1,\eta} = 0, \\ p_2 M \dfrac{\psi^{j+1} - 2\psi^j + \psi^{j-1}}{\Delta t^2} + bK\psi^j + \kappa \left(S\varphi^j + M\psi^j + lMw^j \right) \\ +M \displaystyle\sum_{k=0}^{j-1} (\psi^{k+1} - \psi^k) \cdot C_k^{\alpha_2,\mu} = 0, \\ p_1 M \dfrac{w^{j+1} - 2w^j + w^{j-1}}{\Delta t^2} + \kappa_0 \left(Kw^j + lS\varphi^j \right) + \kappa l \left(S\varphi^j + M\psi^j + lMw^j \right) \\ +M \displaystyle\sum_{k=0}^{j-1} (w^{k+1} - w^k) \cdot C_k^{\alpha_3,\nu} = 0, \end{cases} \tag{4.6}$$

for $j = \overline{1, n-1}$, where

$$\varphi^j = (\varphi_0^j, \varphi_1^j, \dots, \varphi_M^j)^t, \quad \psi^j = (\psi_0^j, \psi_1^j, \dots, \psi_M^j)^t, \quad w^j = (w_0^j, w_1^j, \dots, w_M^j)^t$$

with the following initial conditions:

$$\begin{cases} \varphi^0 = \varphi_0(x), & \psi^0 = \psi_0(x), & w^0 = w_0(x), \\ \dfrac{\varphi^1 - \varphi^0}{\Delta t} = \varphi_1(x), & \dfrac{\psi^1 - \psi^0}{\Delta t} = \psi_1(x), & \dfrac{w^1 - w^0}{\Delta t} = w_1(x), \end{cases}$$

and

$$C_k^{\alpha,\beta} = \frac{\alpha^{\beta-1}}{\Delta t \Gamma(1-\beta)} \left(\gamma\Big(1-\beta, \alpha(t_j - t_k)\Big) - \gamma\Big(1-\beta, \alpha(t_j - t_{k+1})\Big) \right).$$

The matrices M, K, and S are given as follows:

$$M_{ij} = \Big(h_j, h_i\Big)_\Omega = \begin{cases} \dfrac{\Delta x}{3}, & \text{if } i = 0 \text{ or } i = m \text{ and } i = j, \\ \dfrac{2\Delta x}{3}, & \text{if } i = j \text{ and } 1 \le i \le m-1, \\ \dfrac{\Delta x}{6}, & \text{if } |i-j| = 1 \text{ and } 0 \le i \le m, \\ 0, & \text{otherwise,} \end{cases}$$

$$K_{ij} = \Big(h'_j, h'_i\Big)_\Omega = \begin{cases} \dfrac{1}{\Delta x}, & \text{if } i = 0 \text{ or } i = m \text{ and } i = j, \\ \dfrac{2}{\Delta x}, & \text{if } i = j \text{ and } 1 \le i \le m-1, \\ \dfrac{-1}{\Delta x}, & \text{if } |i-j| = 1 \text{ and } 0 \le i \le m, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_{ij} = \left(h'_j, h_i\right)_\Omega = \begin{cases} \dfrac{-1}{2}, & \text{if } (i,j) \in \{(0,0),(m,m-1)\}, \\ \dfrac{(i-j)}{2}, & \text{if } |i-j| = 1 \text{ and } 1 \le i \le m-1, \\ \dfrac{1}{2}, & \text{if } (i,j) \in \{(m,m),(0,1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

4.2. Calculation of the discrete energy $\hat{E}(t)$

Recall that the energy $E(t)$ of the system (1.1)-(1.3) is defined as:

$$E(t) = \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + b\psi_x^2 + \kappa_0(w_x - l\varphi)^2 + \kappa(\varphi_x + \psi + lw)^2\right)\, dx.$$

Let (φ^j, ψ^j, w^j) be the solution of the scheme (4.6). To evaluate the energy $E(t)$ at t_{j+1}, we use the mass matrix M, the stiffness matrix K, and the skew-symmetric matrix S.

Given that M is a symmetric positive definite mass matrix, K is a symmetric positive definite stiffness matrix, and S is a skew-symmetric matrix, we have the following approximations:

$$\begin{aligned} &\int_0^L \varphi_t^2\, dx \approx \varphi_t^T M\varphi_t, \quad \int_0^L \varphi_x^2\, dx \approx \varphi^T K\varphi, \quad \int_0^L \varphi^2\, dx \approx \varphi^T M\varphi, \\ &\int_0^L (w_x - l\varphi)^2\, dx \approx w^T K w - 2l\varphi^T S w + l^2\varphi^T M\varphi, \\ &\quad \int_0^L (\varphi_x + \psi + lw)^2\, dx \\ &\approx \varphi^T K\varphi + 2\psi^T S\varphi + 2lw^T S\varphi + \psi^T M\psi + 2lw^T M\psi + l^2 w^T M w, \end{aligned}$$

where

$$\begin{cases} \varphi_t = \dfrac{\varphi^{j+1} - \varphi^j}{\Delta t}, \ \psi_t = \dfrac{\psi^{j+1} - \psi^j}{\Delta t}, \ w_t = \dfrac{w^{j+1} - w^j}{\Delta t}, \\ \varphi = \varphi^{j+1}, \qquad \psi = \psi^{j+1}, \qquad w = w^{j+1}. \end{cases}$$

Consequently, the discrete energy of the system (1.1)-(1.3) at time t_{j+1} is written as follows:

$$\begin{aligned} \hat{E}(t_{j+1}) = \frac{1}{2}\Big(&\rho_1\varphi_t^T M\varphi_t + \rho_2\psi_t^T M\psi_t + \rho_1 w_t^T M w_t + b\psi^T K\psi \\ &+\kappa_0\left(w^T K w - 2l\varphi^T S w + l^2\varphi^T M\varphi\right) \\ &+\kappa\left(\varphi^T K\varphi + 2\psi^T S\varphi + 2lw^T S\varphi + \psi^T M\psi + 2lw^T M\psi + l^2 w^T M w\right)\Big). \end{aligned}$$

Here is an algorithm that summarizes all the steps for calculating the discrete energy $\hat{E}(t)$:

Algorithm 1 Calculation of the solution and the energy discrete $\hat{E}(t)$

Require: Δt, Δx, $\{M, K, S\}$, $\{p_1, p_2, p_3, \kappa, \kappa_0, l, \cdots\}$, $\{\varphi^0, \psi^0, w^0\}$, $\{\varphi^1, \psi^1, w^1\}$, $\{\varphi_0, \psi_0, w_0\}$, $\{\varphi_m, \psi_m, w_m\}$

Ensure: $\varphi^j, \psi^j, w^j, E^j$

for $j = 1$ **to** $N-1$ **do**

$$\varphi^{j+1} \leftarrow -\tfrac{\Delta t^2}{p_1} M^{-1}\Big(\kappa(K\varphi^j - S\psi^j - lSw^j) - \kappa_0 l(Sw^j - lM\varphi^j) + M\sum_{k=0}^{j-1}(\varphi^{k+1} - \varphi^k).C_k^{\alpha_1,\eta}\Big) + 2\varphi^j - \varphi^{j-1},$$

$$\psi^{j+1} \leftarrow -\tfrac{\Delta t^2}{p_2} M^{-1}\Big(bK\psi^j + \kappa(S\varphi^j + M\psi^j + lMw^j) + M\sum_{k=0}^{j-1}(\psi^j - \psi^{j-1}).C_k^{\alpha_2,\mu}\Big) + 2\psi^j - \psi^{j-1},$$

$$w^{j+1} \leftarrow -\tfrac{\Delta t^2}{p_1} M^{-1}\Big(\kappa_0(Kw^j + lS\varphi^j) + \kappa l(S\varphi^j + M\psi^j + lMw^j) + M\sum_{k=0}^{j-1}(w^j - w^{j-1}).C_k^{\alpha_3,\nu}\Big) + 2w^j - w^{j-1}.$$

// Compute L^2 norms at time step $j+1$

$$norm\_phi\_t \leftarrow \Big(\frac{\varphi^{j+1} - \varphi^j}{\Delta t}\Big)^T M \Big(\frac{\varphi^{j+1} - \varphi^j}{\Delta t}\Big),$$

$$norm\_psi\_t \leftarrow \Big(\frac{\psi^{j+1} - \psi^j}{\Delta t}\Big)^T M \Big(\frac{\psi^{j+1} - \psi^j}{\Delta t}\Big),$$

$$norm\_w\_t \leftarrow \Big(\frac{w^{j+1} - w^j}{\Delta t}\Big)^T M \Big(\frac{w^{j+1} - w^j}{\Delta t}\Big),$$

$$norm\_phi\_x \leftarrow \Big(\varphi^{j+1}\Big)^T K \Big(\varphi^{j+1}\Big).$$

// Compute the discrete energy

$E^{j+1} \leftarrow$ Apply the Result (4.2)

end for

5. Fractional Physics-Informed Neural Networks (fPINN) approach

Physics-Informed Neural Networks (PINNs) represent a novel category of neural networks that integrate machine learning with physical laws. This innovative algorithmic technology emerged relatively recently, in 2019, from research laboratories.

To solve a system involving the Caputo fractional derivative, we employ both the Physics-Informed Neural Network (PINN) model and the Finite Difference Method (FDM) (see Figure 3). The PINN captures the complex behaviors of the studied system, while the FDM discretizes the differential or integral equations, enabling a numerical approach to problem resolution. By combining these two approaches, we obtain a scheme termed Fractional Physics-Informed Neural Networks (fPINNs), which is capable of efficiently solving a variety of mathematical and physical problems.

To the best of our knowledge, this is the first study to utilize this combined approach of fractional physics-informed neural networks to solve systems with the Caputo fractional derivative.

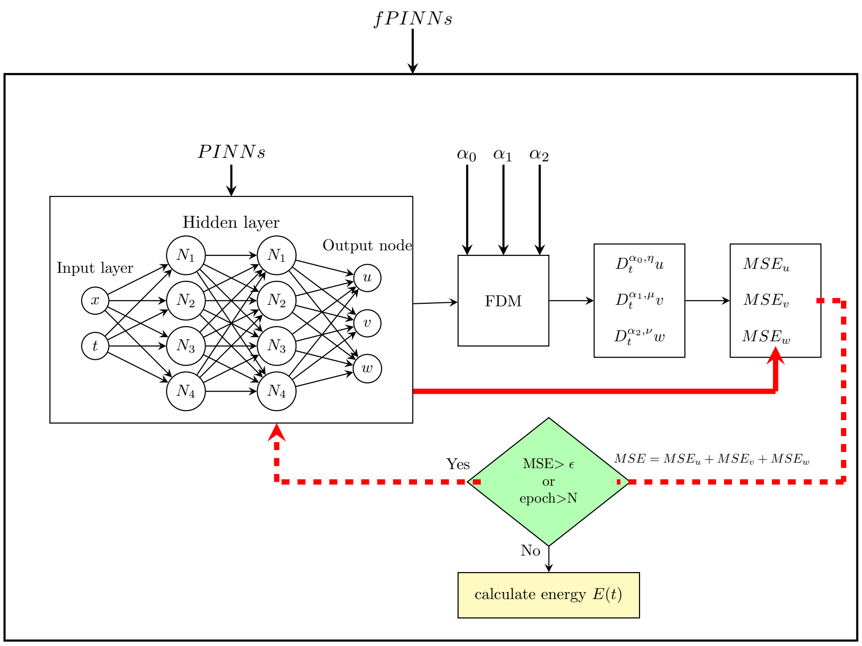

Figure 3. fPINNs to solve the problem (5) for calculate energy E(t).

The predicted function values, denoted as f^1_{pred}, f^2_{pred}, and f^3_{pred}, are defined as follows:

$$\begin{cases} f^1_{pred} = \rho_1 u_{tt} - \kappa(u_x + v + lw)_x - \kappa_0 l(w_x - lu) + \partial_t^{\alpha_1,\eta} u, \\ f^2_{pred} = \rho_2 v_{tt} - b v_{xx} + \kappa(u_x + v + lw) + \partial_t^{\alpha_2,\mu} v, \\ f^3_{pred} = \rho_1 w_{tt} - \kappa_0(w_x - lu)_x + \kappa l(u_x + v + lw) + \partial_t^{\alpha_3,\nu} w. \end{cases}$$

The boundary conditions are:

$$\begin{cases} g_1(x) = u_t(0,x), \quad g_2(x) = v_t(0,x), \quad g_3(x) = w_t(0,x), \\ h_1(t) = u_x(t,0), \quad h_2(t) = v_x(t,0), \quad h_3(t) = w_x(t,0). \end{cases}$$

In this context, $u(x,t)$, $v(x,t)$, and $w(x,t)$ will be approximated by a neural network. The objective of the network is to minimize the following loss function:

$$MSE = MSE_u + MSE_v + MSE_w,$$

where

$$\begin{cases} MSE_u = MSE^0_u + MSE^{0L}_u + MSE_{f_1}, \\ MSE_v = MSE^0_v + MSE^{0L}_v + MSE_{f_2}, \\ MSE_w = MSE^0_w + MSE^{0L}_w + MSE_{f_3}. \end{cases}$$

We calculate the right-hand sides of MSE^0_u, MSE^0_v, and MSE^0_w as follows:

$$\mathrm{MSE}^0_u = \frac{1}{N_0} \sum_{i=0}^{N_0} \Big(\big(u(x^0_i, 0) - u^0_i\big)^2 + \big(g_1(x^0_i) - u^1_i\big)^2 \Big),$$

$$\mathrm{MSE}_v^0 = \frac{1}{N_0}\sum_{i=0}^{N_0}\Big(\big(v(x_i^0,0)-v_i^0\big)^2 + \big(g_2(x_i^0)-v_i^1\big)^2\Big),$$

$$\mathrm{MSE}_w^0 = \frac{1}{N_0}\sum_{i=0}^{N_0}\Big(\big(w(x_i^0,0)-w_i^0\big)^2 + \big(g_3(x_i^0)-w_i^1\big)^2\Big),$$

$$\mathrm{MSE}_u^{0L} = \frac{1}{N_b}\sum_{j=0}^{N_b}\Big(\big(u(0,t_0^j)\big)^2 + \big(u(L,t_L^j)\big)^2 + \big(h_1(t_0^j)\big)^2 + \big(h_1(t_L^j)\big)^2\Big),$$

$$\mathrm{MSE}_v^{0L} = \frac{1}{N_b}\sum_{j=0}^{N_b}\Big(\big(v(0,t_0^j)\big)^2 + \big(v(L,t_L^j)\big)^2 + \big(h_2(t_0^j)\big)^2 + \big(h_2(t_L^j)\big)^2\Big),$$

$$\mathrm{MSE}_w^{0L} = \frac{1}{N_b}\sum_{j=0}^{N_b}\Big(\big(w(0,t_0^j)\big)^2 + \big(w(L,t_L^j)\big)^2 + \big(h_3(t_0^j)\big)^2 + \big(h_3(t_L^j)\big)^2\Big),$$

and

$$\mathrm{MSE}_{f^1} = \frac{1}{N_f}\sum_{i=0}^{N_f}\big(f_{pred}^1(x_i,t_i)-0\big)^2,$$
$$\mathrm{MSE}_{f^2} = \frac{1}{N_f}\sum_{i=0}^{N_f}\big(f_{pred}^2(x_i,t_i)-0\big)^2,$$
$$\mathrm{MSE}_{f^3} = \frac{1}{N_f}\sum_{i=0}^{N_f}\big(f_{pred}^3(x_i,t_i)-0\big)^2.$$

Here, $\{x_i^0, u_i^0\}$ denotes the initial data at $t = 0$, $\{t_0^j, t_L^j\}$ the boundary data, and $\{x_i, t_i\}$ corresponds to collocation points on $f_{pred}^1(x,t)$, $f_{pred}^2(x,t)$, and $f_{pred}^3(x,t)$, where N_0, N_b, and N_f are the number of available observations. Figure 4 shows the point cloud used for training the PINN and calculating the fractional derivative for each point (x_i, t_j).

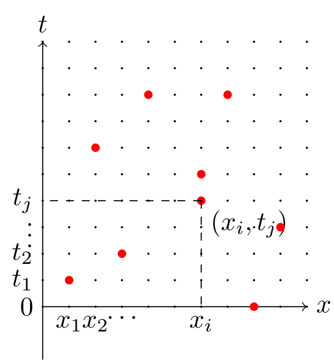

Figure 4. Point cloud used for training the PINN and calculating the fractional derivative for each point (x_i, t_j).

For $0 < \alpha < 1$ and the interval $[t_0, t_j]$ discretized into $j+1$ points, $0 = t_0 < t_1 < \cdots < t_j$, the Caputo fractional derivative of order α using the method of finite differences and Lemma 4.1 is approximated as:

$$D_t^{\alpha,\eta} u(x_i, t_j)$$

$$
\begin{aligned}
&= \frac{\eta^{\alpha-1}}{\Delta t \Gamma(1-\alpha)} \sum_{k=0}^{j-1} (u(x_i, t_{k+1}) - u(x_i, t_k)) \\
&\quad \times \left(\gamma\Big(1-\alpha, \eta(t_j - t_k)\Big) - \gamma\Big(1-\alpha, \eta(t_j - t_{k+1})\Big) \right) \\
&\approx \frac{\eta^{\alpha-1}}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{\partial}{\partial t} u(x_i, t_{k+1}) \Big(\gamma\Big(1-\alpha, \eta(t_j - t_k)\Big) - \gamma\Big(1-\alpha, \eta(t_j - t_{k+1})\Big) \Big).
\end{aligned}
$$

The Physics-Informed Neural Network (PINN) calculates the integer-order partial derivative $\frac{\partial^n u}{\partial^n x}$ using automatic differentiation to obtain the gradients of the model's predictions with respect to the inputs.

Recalling that the energy $E(t)$ is defined by:

$$
E(t) = \frac{1}{2} \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 + b \psi_x^2 + \kappa_0 (w_x - l\varphi)^2 + \kappa(\varphi_x + \psi + lw)^2 \right) dx.
$$

Now, we define the following approximation of the derivatives of φ and ψ, respectively. The L^2 norm of a discretized function is approximated by:

$$
\|f\|_2^2 \approx \Delta x \sum_{i=0}^{N} f(x_i, t_j)^2. \tag{5.1}
$$

Thus, the discrete energy of the system (1.1)-(1.3) at time t_{j+1} is approximated as follows:

$$
\begin{aligned}
&E(t_{j+1}) \\
&\approx \frac{\Delta x}{2} \sum_{i=0}^{M} \left(p1 \left(\frac{u_i^{j+1} - u_i^j}{\Delta t} \right)^2 + p2 \left(\frac{v_i^{j+1} - v_i^j}{\Delta t} \right)^2 \right. \\
&\quad + p3 \left(\frac{w_i^{j+1} - w_i^j}{\Delta t} \right)^2 + b \left(\frac{v_{i+1}^{j+1} - v_i^{j+1}}{\Delta x} \right)^2 \\
&\quad \left. + \kappa_0 \left(\frac{w_{i+1}^{j+1} - w_i^{j+1}}{\Delta x} - l v_i^{j+1} \right)^2 + \kappa \left(\frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x} + u_i^{j+1} + l w_i^{j+1} \right)^2 \right).
\end{aligned} \tag{5.2}
$$

The following algorithm summarizes all the steps for the calculation of the discrete energy $E(t)$ using the L^2 norm defined by (5.1).

6. Numerical test

To verify the asymptotic behavior of the solutions to the system (1.1), we use the following parameters: $\Delta t = 10^{-2}$, $\Delta x = 10^{-1}$, $L = 1$, and the initial conditions given by:

$$
\begin{aligned}
&\varphi(x,0) = x^2(x-1)^2, \quad \varphi_t(x,0) = 0, \quad x \in (0,L), \\
&\psi(x,0) = \frac{1}{2}x^3 - x^2, \quad \psi_t(x,0) = 0, \quad x \in (0,L), \\
&w(x,0) = x^2(x-1)^3, \quad w_t(x,0) = 0, \quad x \in (0,L).
\end{aligned}
$$

Algorithm 2 Calculation of Solution and Energy $E(t)$ by fPINN

Require: $\{x^0, u_i^0, v_i^0, w_i^0, u_i^1, v_i^1, w_i^1, N_0\}$,$\{t_0, t_L, N_b\}$, $\{x_i, t_i, N_f\}$,
Ensure: u_i^j-Matrix, v_i^j-Matrix, w_i^j-Matrix, E
  #Create the PINN and perform an initial training phase.
  MSE $\leftarrow \epsilon + 1$
  # Do the learning phase. Train PINNs
  while $MSE \geq \epsilon$ **do**
    uvw $\leftarrow$ Train(PINN)
    # Calculate MSE.
    $MSE = MSE_u + MSE_v + MSE_w$,
  end while
  // Compute energy
  $E^{j+1} \leftarrow$ Apply the Result (5.2)

Figures 5-6 illustrate the comparison between the numerical FEM approximations of energy and their corresponding fPINN approximations at different time steps. Figures 7 and 8 specifically illustrate the polynomial decay of the energy for both

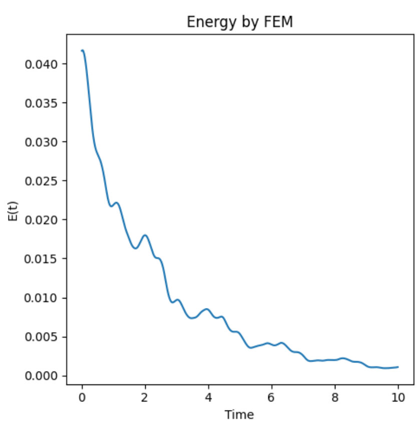

Figure 5. Energy by FEM for $T = 10$.

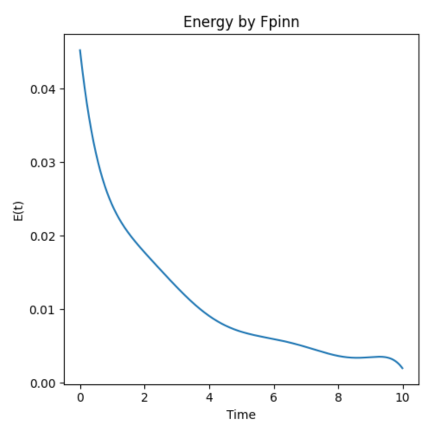

Figure 6. Energy by fPINN for $T = 10$.

the numerical FEM and fPINN approaches. The curves demonstrate that the decay cannot be exponential and, in other words, confirm the lack of exponential energy decay.

We calculate the Root Mean Square Error (RMSE) to quantify the accuracy of the fPINN solutions compared to FEM. The RMSE is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=0}^{m}\sum_{j=0}^{n}(u_i^j - \hat{u}_i^j)^2},$$

where $\hat{u}_i^j$ represents the values obtained by the finite element method, u_i^j represents the values obtained by the physics-informed neural network, and N is the number of observations.

Table 1 presents the RMSE between the fPINN solution and its numerical ap-

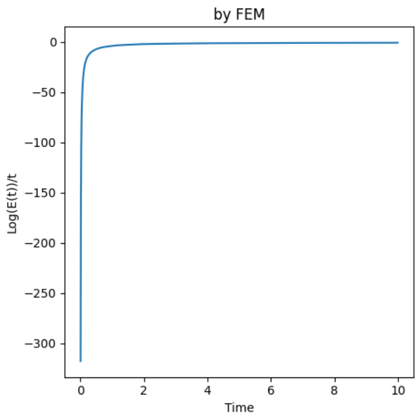

Figure 7. Log Energy/t by FEM for $T = 10$.

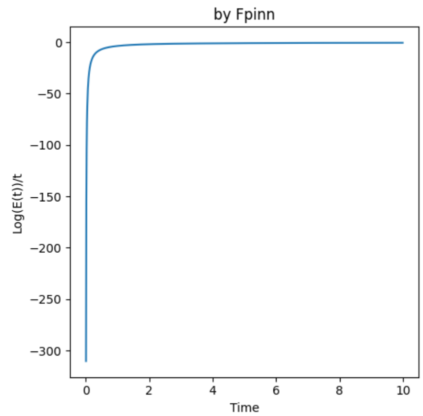

Figure 8. Log Energy/t by fPINN for $T = 10$.

Table 1. RMSE between the fPINN solution and its numerical approximation by FEM for φ, ψ and w.

	FEM		
	φ	Ψ	w
fPINN	0.012210	0.013177	0.006276

Table 2. RMSE between the energy calculated by fPINN and its numerical energy approximation by FEM.

	FEM		
	$T = 1$	$T = 5$	$T = 10$
fPINN	0.011884	0.027809	0.051233

proximation by FEM for φ, ψ, and w. Table 2 shows the RMSE for the energy computed by fPINN compared to its FEM approximation at different time instances.

These values indicate that the FPINN method closely approximates the FEM solutions with relatively low RMSE across all variables. The RMSE values being close to zero suggests that FPINN can effectively capture the behavior of the system as predicted by the traditional FEM approach.

The relatively small RMSE values in both tables highlight that fPINN is a robust method for approximating solutions and energy decay in complex systems like the one studied. However, the slightly higher RMSE at larger time intervals suggests that while fPINN is effective, it may not yet fully match the precision of FEM for long-term predictions without further refinement. The overall comparison indicates that fPINN provides a viable and promising alternative to traditional numerical methods like FEM, especially for problems involving fractional derivatives and complex coupled systems. The differences in RMSE are minimal, demonstrating that fPINN can achieve similar accuracy with potentially less computational cost and greater flexibility in handling complex problems.

This comparison underscores the potential of fPINN as a powerful tool for numerical analysis, while also highlighting areas where further optimization might enhance its performance relative to established methods like FEM.

7. Conclusion

In this work, we investigated the polynomial stabilization of the Bresse system with three types of fractional derivative dissipation. We began by analyzing the polynomial stability of the system (1.1)-(1.3). Then, we applied a finite difference scheme to compute numerical solutions, demonstrating the stability of the discrete energy.

This manuscript makes a significant contribution to numerical analysis and applied mathematics by enhancing the use of Physics-Informed Neural Networks (PINNs) in solving complex fractional and coupled PDEs. This advancement provides a powerful tool for researchers and practitioners facing sophisticated modeling challenges.

Our findings suggest that PINNs represent a robust and promising approach for addressing complex PDEs, potentially offering a transformative alternative in the field.

Acknowledgement

The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khaled University for funding this work through Large Research Project under grant number RGP2/368/45.

Declarations

Ethics approval and consent to participate. This was not required for the present study.

Competing interests. The authors declare no competing interests.

Funding. Not Applicable.

Availability of data and materials. The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] Z. Achouri, N. Amroun and A. Benaissa, *The Euler Bernoulli beam equation with boundary dissipation of fractional derivative*, Math. Methods Appl. Sci., 2017, 40(11), 3837–3854.

[2] M. Akil, Y. Chitour, M. Ghader and A. Wehbe, *Stability and exact controllability of a Timoshenko system with only one fractional damping on the boundary*, Asymptotic Analysis, 2020, 119(3), 221–280. DOI: 10.3233/ASY-191574.

[3] M. Akil, M. Ghader and A. Wehbe, *The influence of the coefficients of a system of wave equations coupled by velocities with fractional damping on its boundary stabilization*, Bull. Span. Soc. Appl. Math. SEMA, 2021, 78, 287–333. DOI: 10.1007/s40324-020-00233-y.

[4] M. Akil, I. Issa and A. Wehbe, *Energy decay rate of the Euler-Bernoulli beam and wave equations via boundary connection with one locally non-regular fractional Kelvin-Voigt damping*, Math. Control Relat. Fields, 2023, 13(1), 330–381.

[5] M. Akil and A. Wehbe, *Indirect stability of a multidimensional coupled wave equations with one locally boundary fractional damping*, Math. Nachr., 2022, 295(12), 2272–2300. DOI: 10.1002/mana.202100185.

[6] M. Akil and A. Wehbe, *Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions*, Math. Control Relat. Fields, 2019, 9(1), 97–116.

[7] S. Alfalqi, B. Boukhari, A. Bchatnia and A. Beniani, *Advanced neural network approaches for coupled equations with fractional derivatives*, Bound. Value Probl., 2024, 96, 1–18.

[8] D. S. Almeida Junior, J. E. Munoz Rivera and M. L. Santos, *Bresse system with Fourier law on shear force*, Adv. Differ. Equ., 2016, 21(1–2), 55–84.

[9] M. O. Alves, L. H. Fatori, M. A. Jorge Silva and R. N. Monteiro, *Stability and optimality of decay rate for a weakly dissipative Bresse system*, Math. Methods Appl. Sci., 2015, 38, 898–908.

[10] T. El Arwadi, M. I. M. Copetti and W. Youssef, *On the theoretical and numerical stability of the thermoviscoelastic Bresse system*, Z. Angew. Math. Mech., 2019, 99.

[11] C. J. K. Batty, R. Chill and Y. Tomilov, *Fine scales of decay of operator semigroups*, J. Eur. Math. Soc. (JEMS), 2016, 18, 853–929.

[12] A. Beniani, N. Bahri, R. Alharbi, K. Bouhali and Kh. Zennir, *Stability for weakly coupled wave equations with a general internal control of diffusive type*, Axioms, 2023, 12(1), 48. DOI: 10.3390/axioms12010048.

[13] C. Bernardi and M. I. M. Copetti, *Discretization of a nonlinear dynamic thermoviscoelastic Timoshenko beam model*, Z. Angew. Math. Mech., 2017, 97, 532–549.

[14] J. A. C. Bresse, *Cours de Mécaniques Appliquée*, Mallet-Bachelier, Paris, 1859.

[15] M. Bzeih, T. El Arwadi and M. Hindi, *Numerical analysis and simulation for Rayleigh beam equation with dynamical boundary controls*, Arab. J. Math., 2021, 10, 331–349.

[16] M. I. M. Copetti, T. El Arwadi, J. R. Fernández, M. G. Naso and W. Youssef, *Analysis of a contact problem for a viscoelastic Bresse system*, ESAIM Math. Model. Numer. Anal., 2021, 55, 887–911.

[17] L. H. Fatori and J. E. Muñoz Rivera, *Rates of decay to weak thermoelastic Bresse system*, IMA J. Appl. Math., 2010, 75, 881–904.

[18] Z. Liu and B. Rao, *Energy decay rate of the thermoelastic Bresse system*, Z. Angew. Math. Phys., 2009, 60, 54–69.

[19] B. Mbodje, *Wave energy decay under fractional derivative controls*, IAM J. Control Optim, 2006, 23, 237–257.

[20] M. A. Rincon and M. I. M. Copetti, *Numerical analysis for a locally damped wave equation*, J. Appl. Anal. Comput., 2013, 30, 169–182.

[21] M. L. Santos and D. S. Almeida Junior, *Numerical exponential decay to dissipative Bresse system*, J. Appl. Math., 2010.

Received September 2024; Accepted February 2025; Available online April 2025.