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NULL CONTROL OF CHAFEE-INFANTE EQUATIONS WITH SPATIALLY SCHEDULED ACTUATORS AND SENSORS*

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Abstract This paper addresses a switched sampled-data control design for stabilization of Chafee-Infante reaction-diffusion equation under Dirichlet boundary conditions with spatially scheduled actuators. The interval $[0, L]$ is divided into N subdomains. It is assumed that discrete-time point-like or average measurements are available and N sensors are placed in each subdomain and measure the average value of the state in the discrete time. The system is stabilized by switching sampled-data static output-feedback. A suitable control law for switching sampled-data is given. The proposed switching controller can be implemented either by placing N actuators and sensors in each subdomain or by using an actuator-sensor pair that can move to the active subdomain. Constructive conditions are derived to ensure that the resulting closed-loop system is exponentially stable by means of the Lyapunov approach. Numerical example verifies our results.

Keywords State/output-dependent switching, Chafee-Infante reaction-diffusion equation, sampled-data control, scheduled actuators.

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1. Introduction

In recent years, considerable efforts have been taken to develop switched control of partial differential equations (PDEs) [15,29]. In [29], the controllability of some classes of PDEs and the corresponding switching laws was given. In [4,6,7], guidance law of moving actuators and sensors were proposed for diffusion partial differential equations (PDEs). Note that the proposed method may not be effective for the case of unstable open-loop system. Intermittent control of reaction-diffusion equation by time-dependent switching between all working pairs of collocated mobile actuators and sensors and the rest (all not working) has been studied in [27]. Control of Kuramoto-Sivashinsky equation has been studied by many authors, see [1,5,17,18]. Wang & Zhang [25] considered the observability for fractional order parabolic equations, see [24] for nonlinear Schrödinger equations. Meanwhile, null controllability of

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stochastic reaction-diffusion equations has been studied by many authors [8,20,21]. Stabilization of unstable systems by switching is an interesting issue. The key idea of switching control design for PDEs is to schedule the position of the actuator and sensor in order to achieve the control aim. In paper [18], the authors introduced a switched sampled-data control design for stabilization of Kuramoto-Sivashinsky equation under the Dirichlet/periodic boundary conditions with spatially scheduled actuators. In this paper, we will use the method “a switched sampled-data control design” introduced by [18] to Chafee-Infante reaction-diffusion equation.

In addition, the stabilization problem of linear reaction-diffusion equation with time-varying delay [28] was dealt with by the projection modification algorithm using collocated mobile actuators and sensors. Compared with [28], as shown in simulations, static pairs stabilize the system, while mobile ones enhance the performance, and we can not stabilize the system with only one non-switching law of active pair.

Therefore, we only use one actuator (if it is moving) or reduce energy consumption to stabilize the system, where only one pair is active. In order to stabilize the reaction-diffusion equation, this paper proposes a new state-dependent switching control law that extends the sate-dependent switching of [14] to PDEs. We assume that placing N identical sensors and actuators in each domain can stabilize the system. Our goal is to design a control law to ensure that there is only one moving actuator allows stabilization of the systems by a switching output-feedback. Our goal is to provide the guidance of mobile (or active) actuator, thereby enhancing the closed-loop performance.

In the present work, a switched sampled-data control is proposed to stabilize 1-D nonlinear reaction-diffusion equation via the employment of moving actuating and sensing devices for stabilization. This work is organized as follows. We will give the preliminaries in Section 2. In Sections 3 and 4, the switching control strategy for Chafee-Infante reaction-diffusion equation is proposed under the point-like measurements and main theoretical results are presented, as well the extensions to the case of periodic boundary conditions and the case of averaged state measurements are presented too.

2. Problem formulation

In the section, we introduce the problem. Assume the sampling moments satisfy

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots , \quad \lim_{k\to\infty} t_k = \infty.$$

We consider the Chafee-Infante reaction-diffusion equation

$$\begin{cases} \quad z_t(x,t) - z_{xx}(x,t) \\ = \beta z(x,t) - z^3(x,t) + b_{\sigma_k}(x) u_{\sigma_k}(t), x \in (0,L), t \in [t_k, t_{k+1}), \\ z(x,0) = z_0(x), \end{cases} \tag{2.1}$$

with the Dirichlet boundary conditions

$$z(0,t) = z(L,t) = 0, \quad t > 0, \tag{2.2}$$

where $\beta > 0$, $k \in \mathbb{Z}_+$. Here $z(x,t)$ is the state of Chafee-Infante reaction-diffusion equation. Let $u_{\sigma_k}(t)$ be the control input. $\sigma_k : k \in \mathbb{Z}_+ \to \{1,...,N\}$ is the switching function, which selects one of the N available actuators at each sampling instant t_k. The shape function $b_{\sigma_k}(x)$ will be defined soon. The following hypotheses are needed

1. Inspired by [2, 10, 11, 22], we divide the interval $[0,L]$ into N equal-length subintervals $\Omega_j = [x_{j-1}, x_j)$ by the points $0 = x_0 < x_1 < \cdots < x_N = L$, which implies that $\cup_j \Omega_j = [0,L]$ and $|\Omega_j| = \frac{L}{N}$. The shape functions $b_j(x)$ are taken as the characteristic functions $b_j(x)$ of Ω_j as followings:

$$\begin{cases} b_j(x) = 0, & \text{if } x \notin \Omega_j, \\ b_j(x) = 1, & \text{otherwise}, \end{cases} \quad j = 1,...,N. \tag{2.3}$$

2. For simplicity, we assume that the length is uniformly bounded:

$$0 < h_0 \leq t_{k+1} - t_k \leq h, \quad \forall k \in \mathbb{Z}_+. \tag{2.4}$$

3. The moving time $\delta \in (0, h_0)$ for sensors and actuators to the appropriate domain Ω_{σ_k} is taken into account.

At first, the sensors offer discrete-time point-like measurements is considered:

$$y_j(t_k) = \int_{\Omega_j} c_j(x) z(x,t_k) dx, \quad k \in \mathbb{Z}_+, \tag{2.5}$$

with

$$0 \leq c_j \in L^2(\Omega_j), \quad \int_{\Omega_j} c_j(x) dx = 1, \tag{2.6}$$

$$c_j(x) = \begin{cases} \frac{1}{\varepsilon}, & \text{if} \quad \mathrm{x} \in \bar{\Omega}_\mathrm{j}, \\ 0, & \text{otherwise}, \end{cases} \quad j = 1,...,N, \tag{2.7}$$

where $\bar{\Omega}_j$ is the subinterval of Ω_j with the length ε which is independent of j.

We will take into account the averaged state measurements as well

$$y_j(t_k) = \frac{\int_{\Omega_j} z(x,t_k) dx}{|\Omega_j|} = \frac{N}{L} \int_{\Omega_j} z(x,t_k) dx, \quad j = 1,...,N, \quad k \in \mathbb{Z}_+. \tag{2.8}$$

Unlike point-like measurements, in the case of averaged measurements, the sensors cover the entire subdomain. Nonetheless, under the averaged measurements our method results in fewer actuators and sensors or permits larger sampling in time. We observe that the proposed method under the averaged measurements can be extended to $N-D$ PDEs for any N (based on the static output-feedback without switching proposed in [3], while this extension under the point-like measurements is problematic (see [23], where non-switched static output-feedback for heat equation under point-like measurements is limited to $N \leq 2$).

For these two measurements, our goal is to find a sampled-data switching law and a sampled-data regionally exponentially stabilizing controller for Chafee-Infante reaction-diffusion equation (2.1) implemented by zero-order hold device. As mentioned earlier, in this article, we will consider the moving time of sensors and actuators δ. For the actuators moving time, we consider the additional switching which is between the open-loop system (when the actuator is moving) during the part of the sampling interval and the closed-loop switched system during the remaining part of the interval, where

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta), \\ -Ky_{\sigma_k}(t_k), & t \in [t_k + \delta, t_{k+1}) \end{cases} \tag{2.9}$$

with some $K > 0$. The switching signal σ_k is calculated at time t_k, whereas it takes δ seconds for actuators and sensors to move to the domain Ω_{σ_k}.

Our main goal is to find an appropriate output-depending switching law. Using $\chi_{[t_k,t_k+\delta]}(t)$ represents the characteristic function of the time interval $[t_k, t_k + \delta]$. Firstly, consider the case of the averaged state measurements (2.8), where the closed-loop system (2.1) and (2.9) has the following form, for $x \in (0, L)$, $\quad t \in [t_k, t_{k+1})$,

$$\begin{aligned} &z_t(x,t) - z_{xx}(x,t) \\ &= \beta z(x,t) - z^3(x,t) - \frac{KN}{L}(1 - \chi_{[t_k,t_k+\delta]}(t))b_{\sigma_k}(x)\int_{\Omega_{\sigma_k}} z(x,t_k)dx \end{aligned} \tag{2.10}$$

obey (2.2). Note that if $b_{\sigma_k}(x)u_{\sigma_k}(t)$ in (2.1) is changed by $\sum_{j=1}^{N} b_j(x)u_j(t)$, then there exists $K > 0$, which is exponentially stabilizes the system in the region through $u_j(t) = -Ky_j(t)$ (Kang and Fridman [16]). The latter means that the the average of systems (2.1) with $b_{\sigma_k}(x)u_{\sigma_k}(t)$ which is changed by $b_j(x)u_j(t)$ can be stabilized through the static output-feedback (2.9). Similar to state-dependent switching for ODEs in the case of stable convex combination of systems [14], we will define a min-type switching function by using the corresponding Lyapunov function $V(t)$ according to

$$\sigma_k \approx \arg\min \dot{V}(t)$$

for $t \in [t_k + \delta, t_{k+1})$ along the closed-loop system. As a result, for $V(t) = \int_0^L z^2(x,t)dx$ we obtain

$$\begin{aligned} \dot{V}(t) &= \int_0^L 2z(x,t)z_t(x,t)dx \\ &= 2\int_0^L z(x,t)[z_{xx}(x,t) + \beta z(x,t) - z^3(x,t)]dx \\ &\quad -\frac{2KN}{L}\int_{\Omega_j} z(x,t)dx \int_{\Omega_j} z(x,t_k)dx \end{aligned}$$

that results (for small enough h) in

$$\begin{aligned} \arg\min \dot{V}(t) &= \arg_j \min\left[-\int_{\Omega_j} z(x,t)dx \int_{\Omega_j} z(x,t_k)dx\right] \\ &\approx \arg_j \max\left[\int_{\Omega_j} z(x,t_k)dx\right]^2 \end{aligned}$$

i.e. to the below discrete-time switching law:

$$\sigma_k = \arg_j \max\left[\int_{\Omega_j} z(x,t_k)dx\right]^2. \tag{2.11}$$

Similar to (2.11) for the point-like measurements we select

$$\sigma_k = \arg_j \max \left[\int_{\Omega_j} c_j(x) z(x, t_k) dx \right]^2 . \tag{2.12}$$

Our sampled-data switching law (2.12) with (2.4) and $\lim_{k\to\infty} t_k = \infty$ rules out the possibility of Zeno behavior. Note that (2.12) is calculated at time t_k. The law (2.12) means the σ_k-th mode is active if

$$\left[\int_{\Omega_j} c_j(x) z(x, t_k) dx \right]^2 \leq \left[\int_{\Omega_{\sigma_k}} c_{\sigma_k}(x) z(x, t_k) dx \right]^2 , \quad \forall j = 1, ..., N. \tag{2.13}$$

Let *suppg* be the support of a function g, $conv(suppg)$ be the convex hull of *suppg*, and $L^2(0, L)$ be the Hilbert space of the whole set of square-integrable functions. Similarly, the Sobolev space $H^k(0, L)$ and $H_0^k(0, L)$ with $k \in \mathbb{Z}$ is defined as in [12]. Throughout this paper, the matrix $P > 0$ ($P < 0$) means the matrix P is positive-definite matrix (negative-definite matrix).

3. Main results

In this section, we will analyze the well-posedness and regional exponential stability of system (2.1) under the static output-feedback (2.9) and the switching law (2.9) (where $c_j = 1$ in the case of averaged measurements).

3.1. Well-posedness of the cotrolled system

We demonstrate the existence, uniqueness, and regularity of the system (2.1) under the switching control laws (2.9), (2.12) and Dirichlet boundary conditions (2.2) by using the step method (see e.g. Section 1.2 in [9]). We suppose that σ_k-th mode is active because of the switching laws (2.9), (2.12). Firstly, we take into account $t \in [0, \delta]$. Subsequently, (2.1) and (2.2) turn into

$$\begin{cases} z_t(x,t) - z_{xx}(x,t) = \beta z(x,t) - z^3(x,t), \quad x \in (0, L), \quad t \in [0, \delta], \\ z(0,t) = z(L,t) = 0, \\ z(x,0) = z_0(x). \end{cases} \tag{3.1}$$

Define the system operator $A : D(A) \subset L^2(0, L) \to L^2(0, L)$ as below:

$$\begin{cases} Af = -\dfrac{\partial^2 f}{\partial x^2}, \\ D(A) = H^2(0, L) \cap H_0^1(0, L). \end{cases}$$

As is well known, A is a dissipative operator that generates an analytical semigroup. It follows from $\int_0^L f(x) Af(x) dx = \int_0^L |\nabla f(x)|^2 dx \geq 0$ that operator A is positive, which implies that its square root $(A)^{\frac{1}{2}}$ is also positive. In addition, $D((A)^{\frac{1}{2}}) =$

$H_0^2(0,L)$ with the norm $\|f\|_{D((A)^{\frac{1}{2}})} = \|f''\|_{L^2(0,L)}$. Then the system (3.1) can be rewritten into an evolution equation£º

$$\begin{cases} \dfrac{d}{dt}z(\cdot,t) = Az(\cdot,t) + F(z(\cdot,t)), \\ z(\cdot,0) = z_0(\cdot), \end{cases} \tag{3.2}$$

where the nonlinear term F is defined on the function $z(\cdot,t)$, which conforms to

$$F(z(\cdot,t)) = \beta z(\cdot,t) - z^3(\cdot,t), \quad t \in [0,\delta].$$

It should be highlighted that the nonlinear term F is locally Lipschitz continuous, which means there exists a positive constant $l(M)$ such that

$$\|F(z_1) - F(z_2)\|_{L^2(0,L)} \le l(M)\|z_1 - z_2\|_{H_0^1(0,L)}$$

holds for $z_1, z_2 \in H_0^1(0,L)$ with $\|z_1\|_{H_0^1(0,L)} \le M, \|z_2\|_{H_0^1(0,L)} \le M$. Therefore, Theorem 3.3.3 of [13] applies to (3.2). For any initial condition $z_0 \in H_0^1(0,L)$, on some interval $[0,T] \subset [0,\delta]$, there exists a unique local strong solution of (3.2), where $T = T(z_0) > 0$:

$$\begin{aligned} &z \in C([0,T]; H_0^1(0,L)) \cap L^2([0,T]; D(A)), \\ &\dot{z} \in L^2([0,T]; L^2(0,T)). \end{aligned}$$

According to Theorem 6.23.5 of [19], we know that if the solution allows a prior estimate (i.e. bounded), then it exists on the entire interval $[0,\delta]$. The conditions we provide will ensure a prior estimate on the solutions (see Theorem 3.1).

For $t \in [\delta, t_1]$, the system (2.1) under the switching control laws (2.9) and (2.12) can be written in the form of (3.2) as well with the below nonlinearity

$$F(z(x,t)) = \beta z(x,t) - z^3(x,t) - Kb_{\sigma_k}(x)\int_{\Omega_{\sigma_k}} c_{\sigma_k}(x)z_0(x)dx, \quad t \in [\delta, t_1].$$

Due to F is locally Lipschitz continuous, we apply reasoning method to the time interval $[\delta, t_1]$. Because of a prior estimation on the solutions starting from the domain of attraction, there exists a strong solution on $[\delta, t_1]$, which is ensured by the stability conditions of Theorem 3.1.

3.2. Stability analysis of the switched system

By the mean-value theorem, from (2.6) it follows that there exists $\bar{x}_j^t \in \mathrm{conv}(\mathrm{supp c_j})$ such that

$$\int_{\Omega_j} c_j(x)z(x,t)dx = z(\bar{x}_j^t, t), \quad t \in [t_k, t_{k+1}).$$

Denote

$$f_j(x,t) \triangleq z(x,t) - z(\bar{x}_j^t, t), \quad t \in [t_k, t_{k+1}), \tag{3.3}$$

$$\rho_j(t) \triangleq \int_{\Omega_j}\int_{t_k}^{t} c_j(x)z_s(x,s)dsdx, \quad t \in [t_k, t_{k+1}). \tag{3.4}$$

Then the switching controller (2.9) can be represented as

$$u_{\sigma_k}(t)=\begin{cases}0, & t\in[t_k,t_k+\delta),\\ -K[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)], & t\in[t_k+\delta,t_{k+1}),\end{cases}\tag{3.5}$$

while the switching law selects σ_k which satisfies

$$\int_{\Omega_j}[z(x,t)-f_j(x,t)-\rho_j(t)]^2\,dx\le\int_{\Omega_{\sigma_k}}[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]^2\,dx,\tag{3.6}$$

where $j=1,2,...,N$. Hence, under the controller (3.5), the closed-loop system turns into

$$\begin{aligned}&z_t(x,t)-z_{xx}(x,t)\\ &=\beta z(x,t)-z^3(x,t)-Kb_{\sigma_k}(x)(1-\chi_{[t_k,t_k+\delta]}(t))\\ &\quad\times[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]\,,\ x\in(0,L),t\in[t_k,t_{k+1}),\end{aligned}\tag{3.7}$$

subject to (2.2) and (2.12).

Observe that (2.1) may not be stable with a expected decay rate under the non-switching control. The challenge in the stability analysis is to effectively consider the switching conditions (2.12) to derive feasible stability conditions (see (3.24) below and the resulting expressions in (3.25)).

We pay attention to the stability of the closed-loop system that switches at times t_k and $t_k+\delta$. The following Lyapunov-Krasovskii functional is taken into account:

$$V(t)=V_{P_1}(t)+V_{P_2}(t)+V_R(t),\quad t\in[t_k,t_{k+1}),\tag{3.8}$$

where

$$\begin{aligned}V_{P_1}(t)&=P_1\int_0^L z^2(x,t)dx,\\ V_{P_2}(t)&=P_2\int_0^L z_x^2(x,t)dx,\\ V_R(t)&=R\frac{4h^2}{\pi^2}\sum_{j=1}^N\int_{\Omega_j}\int_{t_k}^t e^{-2\alpha(t-s)}[\rho_{js}(s)]^2dsdx\\ &\qquad -Re^{-2\alpha h}\sum_{j=1}^N\int_{\Omega_j}\int_{t_k}^t e^{-2\alpha(t-s)}[\rho_j(s)]^2dsdx.\end{aligned}$$

Among them, $P_1>0$, $P_2>0$, and $R>0$. Here $\rho_{js}(s)$ is the derivative of $\rho_j(s)$ with respect to s. According to the Wirtinger's inequality, $V_R(t)$ is non- negative (see Lemma 1 in [16]), and it does not grow in the switching time t_k, while it is continuous in the switching time $t_k+\delta$. In addition, V_R extends the corresponding terms in [23] to the Wirtinger-based Lyapunov functional.

For $z(\cdot,t)\in H_0^2(0,L)$ we define

$$\|z(\cdot,t)\|_V^2=P_1\|z(\cdot,t)\|^2_{L^2(0,L)}+P_2\|z_x(\cdot,t)\|^2_{L^2(0,L)}$$

with $P_1>0$, $P_2>0$.

Remark 3.1. To find a bound on the domain of attraction for closed-loop system (3.7) subject to (2.2), we use positive invariance principle in Theorem 3.1: we prove that if $\Psi_0 < 0$, $\Psi_1 < 0$ and $\Psi_2 < 0$, where Ψ_0, Ψ_1, Ψ_2 are given by (3.12) - (3.14), for all $t \geq 0$, $V(t) \leq V(0)$. Matrices Ψ_0, Ψ_1, Ψ_2 are affine in z. Let $C > 0$ be the upper bound of z, i.e. $\max_{x\in[0,L]} |z(x,t)| \leq C$ for all $t \geq 0$. So it is enough to verify in the vertices $z = \pm C$ the matrix inequalities $\Psi_0 < 0$, $\Psi_1 < 0$, $\Psi_2 < 0$ (see(3.9) - (3.11)).

The following result provides sufficient stability conditions for the closed-loop systems (3.7), (2.2) and (2.12) in the form of linear matrix inequalities (LMIs).

Theorem 3.1. *Consider the closed-loop system (3.7) constrained by (2.2) and the switching law (2.12). Given positive scalars* h, α, K *and tuning parameter* $C > 0, \alpha_0 > 0$*, which leads to* $\alpha h_0 > (\alpha_0 + \alpha)\delta$*. Assuming that there are scalars* $R > 0, P_n > 0, \lambda_n \geq 0 \quad (n = 1, 2, 3)$ *that satisfy the following inequalities:*

$$\Psi_1|_{z=\pm C} < 0, \tag{3.9}$$

$$\Psi_2|_{z=\pm C} < 0, \tag{3.10}$$

$$\Psi_0|_{z=\pm C} < 0, \tag{3.11}$$

where

$$\Psi_1 = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 & \frac{\lambda_1}{N-1} & \frac{\lambda_1}{N-1} \\ * & \psi_{22} & 0 & 0 & 0 \\ * & * & \psi_{33} & 0 & 0 \\ * & * & * & -\lambda_2 - \frac{\lambda_1}{N-1} & -\frac{\lambda_1}{N-1} \\ * & * & * & * & -Re^{-2\alpha h} - \frac{\lambda_1}{N-1} \end{pmatrix}, \tag{3.12}$$

$$\Psi_2 = \begin{pmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} & 0 & P_3K - \lambda_1 & P_3K - \lambda_1 \\ * & \psi_{22} & 0 & P_2K & P_2K \\ * & * & \psi_{33} & 0 & 0 \\ * & * & * & \lambda_1 - \lambda_2 & \lambda_1 \\ * & * & * & * & \lambda_1 - Re^{-2\alpha h} \end{pmatrix}, \tag{3.13}$$

$$\Psi_0 = \begin{pmatrix} -2\alpha_0 P_1 + 2\beta P_4 & P_1 - P_4 + \beta P_2 - P_2 z^2 & 0 \\ * & -2P_2 + R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon} & 0 \\ * & * & -2\alpha_0 P_2 - 2P_4 \end{pmatrix}, \tag{3.14}$$

$$\psi_{11} = 2\alpha P_1 + 2\beta P_3 - \frac{\lambda_1}{N-1} - \lambda_3 \frac{\pi^2}{L^2},$$

$$\psi_{12} = P_1 - P_3 + \beta P_2 - P_2 z^2,$$

$$\psi_{33} = -2P_3 + 2\alpha P_2 + \frac{\lambda_2(\frac{L}{N} + \varepsilon)^2}{\pi^2} + \lambda_3,$$

$$\psi_{22} = R\frac{4h^2}{\pi^2} \cdot \frac{L}{N\varepsilon} - 2P_2,$$
$$\tilde{\psi}_{11} = 2\alpha P_1 + 2\beta P_3 - 2P_3 K + \lambda_1 - \lambda_3 \frac{\pi^2}{L^2},$$
$$\tilde{\psi}_{12} = P_1 - P_3 + \beta P_2 - P_2 z^2 - P_2 K.$$

Let α_1 be subject to

$$0 < \alpha_1 h_0 \le \alpha h_0 - (\alpha_0 + \alpha)\delta. \tag{3.15}$$

Hence, for any initial function $z_0 \in H_0^2(0,L)$ that satisfies the bound $\|z_0\|_V < \sqrt{\frac{P_2}{L}}C$, the closed-loop system (3.7) that satisfies (2.2) and (2.12) is exponentially stable with a decay rate α_1, i.e. the following holds

$$\|z(\cdot,t)\|_V^2 \le V(t) \le e^{-2\alpha_1(t-h)+2\alpha_0\delta}V(0).$$

Proof. Step 1. Let us only emphasize that on interval $[0,T] \subset [0,\delta]$, there exists a unique local strong solution of (3.1), where $T = T(z_0)$. Due to [19, Theorem 6.23.5], if the solution is bounded, then the solution exists on the interval $[0,\delta]$. As a result, it can be concluded that for all $t \ge 0$ there exists a strong solution by using the same arguments at $[\delta, t_1]$ and any step $k \in N$.

Step 2. Formally assume that the strong solution of (3.7) follows (2.2), starting from $\|z_0\|_V < \sqrt{\frac{P_2}{L}}C$ exists for all $t \ge 0$. We first derive sufficient LMI- based conditions to ensure that $\dot{V}(t) + 2\alpha V(t) \le 0$ for $[t_k + \delta, t_{k+1})$. Differentiating $V(t)$ along the solution of the closed-loop system and partially integrating, we get

$$\begin{aligned}\dot{V}(t) + 2\alpha V(t) = {} & 2P_1 \int_0^L z(x,t)z_t(x,t)dx + 2\alpha P_1 \int_0^L z^2(x,t)dx \\ & +2P_2 \int_0^L z_x(x,t)z_{xt}(x,t)dx + 2\alpha P_2 \int_0^L z_x^2(x,t)dx \\ & +R\frac{4h^2}{\pi^2}\sum_{j=1}^N \int_{\Omega_j} [\rho_{jt}(t)]^2 dx - Re^{-2\alpha h}\sum_{j=1}^N \int_{\Omega_j} [\rho_j(t)]^2 dx.\end{aligned} \tag{3.16}$$

Jensen's inequality yields that

$$\begin{aligned}\int_{\Omega_j} [\rho_{jt}(t)]^2 dx &= \frac{L}{N}\left(\int_{\Omega_j} c_j(x) z_t(x,t)dx\right)^2 \\ &\le \frac{L}{N}\int_{\Omega_j} c_j(x)dx \int_{\Omega_j} c_j(x) z_t^2(x,t)dx \\ &\le \frac{L}{N\varepsilon}\int_{\Omega_j} z_t^2(x,t)dx.\end{aligned} \tag{3.17}$$

Observe that $f_j(x,t) \triangleq z(x,t) - z(\bar{x}_j^t, t)$ and $f_{jx}(x,t) = z_x(x,t)$. Applying Wirtinger's inequality obtains

$$\int_{\Omega_j} f_j^2(x,t)dx = \int_{x_{j-1}}^{\bar{x}_j^t} [z(x,t) - z(\bar{x}_j^t,t)]^2 dx + \int_{\bar{x}_j^t}^{x_j} [z(x,t) - z(\bar{x}_j^t,t)]^2 dx$$

$$\leq \frac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\int_{\Omega_j} z_x^2(x,t)dx. \tag{3.18}$$

In addition, we have

$$\|z(\cdot,t)\|^2_{L^2(0,L)} \leq (\frac{L}{\pi})^2\|z_x(\cdot,t)\|^2_{L^2(0,L)}. \tag{3.19}$$

Thus, combining (3.6), (3.18) and (3.19), we get

$$\begin{aligned}
&-\frac{\lambda_1}{N-1}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}[z(x,t)-f_j(x,t)-\rho_j(t)]^2dx \\
&+\lambda_1\int_{\Omega_{\sigma_k}}[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]^2dx \geq 0,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\lambda_2\left[\frac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\|z_x(\cdot,t)\|^2_{L^2(0,L)} - \sum_{j=1}^{N}\|f_j(\cdot,t)\|^2_{L^2(\Omega_j)}\right] \\
&= \lambda_2\left[\frac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j} z_x^2(x,t)dx - \sum_{j\neq\sigma_k}^{N}\int_{\Omega_j} f_j^2(x,t)dx\right] \\
&\quad+\lambda_2\left[\frac{(\frac{L}{N}+\varepsilon)^2}{\pi^2}\int_{\Omega_{\sigma_k}} z_x^2(x,t)dx - \int_{\Omega_{\sigma_k}} f_{\sigma_k}^2(x,t)dx\right] \\
&\geq 0,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\lambda_3\left[\|z_x(\cdot,t)\|^2_{L^2(0,L)} - (\frac{\pi}{L})^2\|z(\cdot,t)\|^2_{L^2(0,L)}\right] \\
&= \lambda_3\left[\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j} z_x^2(x,t)dx - \frac{\pi^2}{L^2}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j} z^2(x,t)dx\right] \\
&\quad+\lambda_3\left[\int_{\Omega_{\sigma_k}} z_x^2(x,t)dx - \frac{\pi^2}{L^2}\int_{\Omega_{\sigma_k}} z^2(x,t)dx\right] \\
&\geq 0.
\end{aligned} \tag{3.22}$$

And then, we apply the descriptor method ([9, Section 3.5]), where the left-hand side of the following equation

$$\begin{aligned}
&2\int_0^L[P_3z(x,t)+P_2z_t(x,t)]\{-z_t(x,t)+z_{xx}(x,t)+\beta z(x,t)-z^3(x,t) \\
&-Kb_{\sigma_k}(x)[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]\}dx = 0
\end{aligned} \tag{3.23}$$

with some $P_3 > 0$ is added to $\dot{V}$. Then adding the left-hand sides of (3.20), (3.21) and (3.22) to (3.16) and consider (3.17), we have

$$\begin{aligned}
&\dot{V}(t)+2\alpha V(t) \\
&\leq (2P_1-2P_3)\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j} z(x,t)z_t(x,t)dx
\end{aligned}$$

$$
\begin{aligned}
&+(2\alpha P_1+2\beta P_3-\lambda_3\frac{\pi^2}{L^2})\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z^2(x,t)dx\\
&-\left[2P_3-2\alpha P_2-\frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{\pi^2}-\lambda_3\right]\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_x^2(x,t)dx\\
&+\left(R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon}-2P_2\right)\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_t^2(x,t)dx\\
&+2P_2\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_t(x,t)[\beta z(x,t)-z^3(x,t)]dx\\
&+(2P_1-2P_3)\int_{\Omega_{\sigma_k}}z(x,t)z_t(x,t)dx+(2\alpha P_1+2\beta P_3-\lambda_3\frac{\pi^2}{L^2})\int_{\Omega_{\sigma_k}}z^2(x,t)dx\\
&-\left[2P_3-2\alpha P_2-\frac{\lambda_2(\frac{L}{N}+\varepsilon)^2}{\pi^2}-\lambda_3\right]\int_{\Omega_{\sigma_k}}z_x^2(x,t)dx\\
&+\left(R\frac{4h^2}{\pi^2}\frac{L}{N\varepsilon}-2P_2\right)\int_{\Omega_{\sigma_k}}z_t^2(x,t)dx+2P_2\int_{\Omega_{\sigma_k}}z_t(x,t)[\beta z(x,t)-z^3(x,t)]dx\\
&-2P_3K\int_{\Omega_{\sigma_k}}z(x,t)[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]dx\\
&-2P_2K\int_{\Omega_{\sigma_k}}z_t(x,t)[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]dx\\
&-Re^{-2\alpha h}\sum_{j\neq\sigma_k}^{N}\rho_j^2(t)dx-Re^{-2\alpha h}\int_{\Omega_{\sigma_k}}\rho_{\sigma_k}^2(t)dx\\
&-\lambda_2\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}f_j^2(x,t)dx-\lambda_2\int_{\sigma_k}f_{\sigma_k}^2(x,t)dx\\
&-\frac{\lambda_1}{N-1}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}[z(x,t)-f_j(x,t)-\rho_j(t)]^2dx\\
&+\lambda_1\int_{\Omega_{\sigma_k}}[z(x,t)-f_{\sigma_k}(x,t)-\rho_{\sigma_k}(t)]^2dx.
\end{aligned}
\tag{3.24}
$$

From (3.24), we get

$$
\begin{aligned}
&\dot V(t)+2\alpha V(t)\\
\leq\ &\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}\eta_1^T\Psi_1\eta_1dx+\int_{\Omega_{\sigma_k}}\eta_2^T\Psi_2\eta_2dx,\ \forall t\in[t_k+\delta,t_{k+1}),
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\eta_1&=\mathrm{col}\{\mathrm{z(x,t),z_t(x,t),z_x(x,t),f_j(x,t)},\rho_{\mathrm{j}}\mathrm{(x,t)}\},\\
\eta_2&=\mathrm{col}\{\mathrm{z(x,t),z_t(x,t),z_x(x,t),f_{\sigma_k}(x,t)},\rho_{\sigma_{\mathrm{k}}}\mathrm{(x,t)}\},
\end{aligned}
$$

and $\Psi_i (i = 1, 2)$ are given by (3.12), (3.13) respectively.

Then we assume that

$$\max_{x\in[0,L]} |z(x,t)| < C, \quad \forall t \geq 0. \tag{3.26}$$

Under the assumption (3.26), we obtain

$$\dot{V}(t) + 2\alpha V(t) \leq 0, \tag{3.27}$$

if $\Psi_1 < 0$ and $\Psi_2 < 0$ hold for all $-C \leq z \leq C$.

Matrices $\Psi_i (i = 1, 2)$ given by (3.12) and (3.13) are affine in z. Therefore, if these inequalities hold in the vertices $z = \pm C$, i.e. if LMIs (3.9) and (3.10) are feasible, then $\Psi_1 < 0$ and $\Psi_2 < 0$ for all $-C \leq z \leq C$.

Step 3. Now we derive sufficient LMI-based conditions to ensure that $\dot{V}(t) - 2\alpha_0 V(t) \leq 0$ for $[t_k, t_k + \delta)$.

Differentiating $V(t)$ along (3.7) subject to (2.2), we obtain

$$\begin{aligned}\dot{V}(t) - 2\alpha_0 V(t) = {} & 2P_1 \int_0^L z(x,t) z_t(x,t) dx - 2\alpha_0 P_1 \int_0^L z^2(x,t) dx \\ & +2P_2 \int_0^L z_x(x,t) z_{xt}(x,t) dx - 2\alpha_0 P_2 \int_0^L z_x^2(x,t) dx \\ & +R\frac{4h^2}{\pi^2} \sum_{j=1}^{N} \int_{\Omega_j} [\rho_{jt}(t)]^2 dx - Re^{-2\alpha h} \sum_{j=1}^{N} \int_{\Omega_j} [\rho_j(t)]^2 dx \\ & -2(\alpha + \alpha_0) V_R(t).\end{aligned}$$

We further apply the descriptor method, where the left-hand side of the below equation

$$2\int_0^L [P_4 z(x,t) + P_2 z_t(x,t)][-z_t(x,t) + z_{xx}(x,t) + \beta z(x,t) - z^3(x,t)] dx = 0$$

with some $P_4 > 0$ is added to $\dot{V}$.

Taking into account (3.17), we have

$$\begin{aligned} & \dot{V}(t) - 2\alpha_0 V(t) \\ \leq {} & (-2\alpha_0 P_1 + 2\beta P_4) \int_0^L z^2(x,t) dx \\ & + \int_0^L [2P_1 - 2P_4 + 2\beta P_2 - 2P_2 z^2(x,t)] z(x,t) z_t(x,t) dx \\ & +(-2\alpha_0 P_2 - 2P_4) \int_0^L z_x^2(x,t) dx + \left(R\frac{4h^2}{\pi^2} \frac{L}{N\varepsilon} - 2P_2 \right) \int_0^L z_t^2(x,t) dx.\end{aligned}$$

Thus, we have

$$\dot{V}(t) - 2\alpha_0 V(t) \leq \int_0^L \eta_0^T \Psi_0 \eta_0 dx, \quad \forall t \in [t_k, t_k + \delta),$$

where $\eta_0 = \mathrm{col}\{\mathrm{z(x,t), z_t(x,t), z_x(x,t)}\}$.

Step 4. From Step 1 to Step 3, we obtain that if $\|z_0\|_V < \sqrt{\frac{P_2}{L}} C$, then the feasibility of LMIs (3.9) - (3.11) means that any strong solution of (3.7) and (2.2) initialized with z_0 allows a prior estimate

$$V(t) \le e^{2\alpha_0(t-t_k)}V(t_k), \qquad \forall t \in [t_k, t_k+\delta),$$
$$V(t) \le e^{-2\alpha(t-t_k-\delta)}V(t_k+\delta), \ \forall t \in [t_k+\delta, t_{k+1}). \tag{3.28}$$

Since $\alpha_1 < \alpha$ and $t_{k+1} - t_k \ge h_0$, (3.15) implies

$$(\alpha_1 - \alpha)(t_{k+1} - t_k) \le (\alpha_1 - \alpha)h_0 \le -(\alpha_0 + \alpha)\delta,$$

which together with (3.28) leads to

$$V(t_{k+1}) \le e^{2\alpha_0\delta}e^{-2\alpha(t_{k+1}-t_k-\delta)}V(t_k) \le e^{-2\alpha_1(t_{k+1}-t_k)}V(t_k). \tag{3.29}$$

From (3.28) it follows

$$V(t) \le e^{2\alpha_0\delta}V(t_k), \qquad \forall t \in [t_k, t_k+\delta);$$
$$V(t) \le V(t_k+\delta) \le e^{2\alpha_0\delta}V(t_k), \ \forall t \in [t_k+\delta, t_{k+1}).$$

Hence, for $t \in [t_k+\delta, t_{k+1})$

$$\begin{aligned} V(t) &\le e^{2\alpha_0\delta}V(t_k) \\ &\le e^{2\alpha_0\delta-2\alpha_1(t_k-t_{k-1})}V(t_{k-1}) \\ &\le e^{2\alpha_0\delta-2\alpha_1(t-t_{k-1}-h)}V(t_{k-1}) \\ &\le e^{2\alpha_0\delta-2\alpha_1(t-t_{k-2}-h)}V(t_{k-2}) \\ &\le \cdots \\ &\le e^{2\alpha_0\delta-2\alpha_1(t-h)}V(0). \end{aligned}$$

Therefore,

$$V(t) \le e^{2\alpha_0\delta-2\alpha_1(t-h)}V(0), \quad \forall t \ge 0.$$

The latter bound ensures the existence of these strong solutions for all $t \in [0, t_1]$. Then using step method [9], we conclude that for all $t \ge 0$, there exists a strong solution.

Next, we will prove that (3.26) holds. On one hand, for $t = 0$, inequality (3.26) holds through the assumptions in Theorem 3.1. On the other hand, for some $t > 0$, assume (3.26) be false, and let t^* be the smallest moment such that $V(t^*) \ge \frac{P_2}{L}C^2$. Since V is continuous in time, for $t \in [0, t^*)$ we have $V(t^*) = \frac{P_2}{L}C^2$ and $V(t) < \frac{P_2}{L}C^2$. Due to $z(0,t) = 0$, the Sobolev inequality implies that $\max_{x\in[0,L]} |z(x,t)|^2 \le L\|z_x(\cdot,t)\|^2_{L^2(0,L)} \le \frac{L}{P_2}V(t) \le \frac{L}{P_2}V(0) = \frac{L}{P_2}\|z_0\|^2_V < C^2$ for $t \in [0, t^*)$. Therefore, the feasible of LMIs (3.9) - (3.11) ensure that (3.27) is true for all $t \in [0, t^*)$. Thus, by continuity, $V(t) \le V(0) < \frac{P_2}{L}C^2$ for all $t \in [0, t^*]$, which contradicts the definition of t^*. Hence, (3.26) holds. □

Theorem 3.2. *Consider the closed-loop system (2.10) constrained by (2.2) and the switching law (2.12) with $c_j = 1$. Given positive scalars h, α, K and tuning parameter $C > 0, \alpha_0 > 0$ leads to $\alpha h_0 > (\alpha_0 + \alpha)\delta$. Assuming that there are scalars $R > 0, P_n > 0, \lambda_n \ge 0 \quad (n = 1, 2, 3)$ that satisfy the LMIs:*

$$\Theta_1|_{z=\pm C} < 0, \tag{3.30}$$

$$\Theta_2|_{z=\pm C} < 0, \tag{3.31}$$
$$\Psi_0|_{z=\pm C} < 0, \tag{3.32}$$

where

$$\Theta_1 = \Psi_1 + \Pi, \tag{3.33}$$
$$\Theta_2 = \Psi_2 + \Pi, \tag{3.34}$$

and Ψ_0, Ψ_1, Ψ_2 *are given by (3.12), (3.13), (3.14) respectively, and*

$$\Pi = \begin{pmatrix} 0\ 0 & \frac{\lambda_2}{2\pi^2}\left(\frac{2L\varepsilon}{N}+\varepsilon^2\right) & 0\ 0 \\ *\ R\frac{4h^2}{\pi^2}\left(1-\frac{L}{N\varepsilon}\right) & 0 & 0\ 0 \\ *\ * & 0 & 0\ 0 \\ *\ * & * & 0\ 0 \\ *\ * & * & *\ 0 \end{pmatrix}. \tag{3.35}$$

Then, for any initial function $z_0 \in H_0^2(0,L)$ *that satisfies the bound* $\|z_0\|_V < \sqrt{\frac{p_2}{L}}C$, *the closed-loop system (2.10) that satisfies (2.2) is exponentially stable with a decay rate* $\alpha_1 > 0$, *i.e. the following holds*

$$\|z(\cdot,t)\|_V^2 \le V(t) \le e^{-2\alpha_1(t-h)+2\alpha_0\delta}V(0).$$

Proof. For the case of switched controller under the averaged measurements, by arguments of Theorem 3.1, the well-posedness of (2.10) subject to (2.2) can be established via the step method.

Denote

$$\tilde{f}_j(x,t) \triangleq z(x,t) - \frac{\int_{\Omega_j} z(x,t)dx}{|\Omega_j|},$$
$$\tilde{\rho}_j(t) \triangleq \frac{\int_{\Omega_j}\int_{t_k}^t z_s(x,s)dsdx}{|\Omega_j|},$$

where $|\Omega_j| = \frac{L}{N}$.

Then the switching controller (2.9) via the switching law (2.12) with $c_j = 1$ can be rewritten as

$$u_{\sigma_k}(t) = -K[z(x,t) - \tilde{f}_{\sigma_k}(x,t) - \tilde{\rho}_{\sigma_k}(t)]. \tag{3.36}$$

We select the Lyapunov function V with $\tilde{\rho}_j$ replace ρ_j. Differentiating V along the solution of the closed-loop system (2.10) subject to (2.2), we get (3.16) with $\tilde{\rho}_j$ replace ρ_j. The substitution $f_j \to \tilde{f}_j$ and $\rho_j \to \tilde{\rho}_j$ in Theorem 3.1 leads to the following changes:

$$\int_{\Omega_j}[\tilde{\rho}_{jt}(t)]^2dx = \frac{1}{|\Omega_j|}\left(\int_{\Omega_j} z_t(x,t)dx\right)^2 \le \int_{\Omega_j} z_t^2(x,t)dx,$$

and

$$-\frac{\lambda_1}{N-1}\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}[z(x,t)-\tilde{f}_j(x,t)-\tilde{\rho}_j(t)]^2dx$$
$$+\lambda_1\int_{\Omega_{\sigma_k}}[z(x,t)-\tilde{f}_{\sigma_k}(x,t)-\tilde{\rho}_{\sigma_k}(t)]^2dx\geq 0, \tag{3.37}$$
$$\frac{\lambda_2 L^2}{N^2\pi^2}\|z_x(\cdot,t)\|^2_{L^2(0,L)}-\lambda_2\sum_{j=1}^{N}\|\tilde{f}_j(\cdot,t)\|^2_{L^2(\Omega_j)}\geq 0 \tag{3.38}$$

for any $\lambda_1\geq 0, \lambda_2\geq 0$.

Set

$$\begin{aligned}\tilde{\eta}_1 &= \mathrm{col}\{\mathrm{z(x,t), z_t(x,t), z_x(x,t), \tilde{f}_j(x,t)}, \tilde{\rho}_{\mathrm{j}}\mathrm{(x,t)}\},\\ \tilde{\eta}_2 &= \mathrm{col}\{\mathrm{z(x,t), z_t(x,t), z_x(x,t), \tilde{f}_{\sigma_k}(x,t)}, \tilde{\rho}_{\sigma_{\mathrm{k}}}\mathrm{(x,t)}\},\\ \eta_0 &= \mathrm{col}\{\mathrm{z(x,t), z_t(x,t), z_x(x,t)}\}.\end{aligned}$$

Applying descriptor method and adding the left-hand sides of (3.22), (3.37) and (3.38) to $\dot{V}$, we can get

$$\begin{aligned}&\dot{V}(t)+2\alpha V(t)\\
\leq\ &(2P_1-2P_3)\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z(x,t)z_t(x,t)dx\\
&+(2\alpha P_1+2\beta P_3-\lambda_3\frac{\pi^2}{L^2})\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z^2(x,t)dx\\
&-\left[2P_3-2\alpha P_2-\frac{\lambda_2 L^2}{N^2\pi^2}-\lambda_3\right]\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_x^2(x,t)dx\\
&+\left(R\frac{4h^2}{\pi^2}-2P_2\right)\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_t^2(x,t)dx+2P_2\sum_{j\neq\sigma_k}^{N}\int_{\Omega_j}z_t(x,t)[\beta z(x,t)-z^3(x,t)]dx\\
&+(2P_1-2P_3)\int_{\Omega_{\sigma_k}}z(x,t)z_t(x,t)dx+(2\alpha P_1+2\beta P_3-\lambda_3\frac{\pi^2}{L^2})\int_{\Omega_{\sigma_k}}z^2(x,t)dx\\
&-\left[2P_3-2\alpha P_2-\frac{\lambda_2 L^2}{N^2\pi^2}-\lambda_3\right]\int_{\Omega_{\sigma_k}}z_x^2(x,t)dx\\
&+\left(R\frac{4h^2}{\pi^2}-2P_2\right)\int_{\Omega_{\sigma_k}}z_t^2(x,t)dx+2P_2\int_{\Omega_{\sigma_k}}z_t(x,t)[\beta z(x,t)-z^3(x,t)]dx\\
&-2P_3K\int_{\Omega_{\sigma_k}}z(x,t)[z(x,t)-\tilde{f}_{\sigma_k}(x,t)-\tilde{\rho}_{\sigma_k}(t)]dx\\
&-2P_2K\int_{\Omega_{\sigma_k}}z_t(x,t)[z(x,t)-\tilde{f}_{\sigma_k}(x,t)-\tilde{\rho}_{\sigma_k}(t)]dx\\
&-Re^{-2\alpha h}\sum_{j\neq\sigma_k}^{N}\tilde{\rho}_j^2(t)dx-Re^{-2\alpha h}\int_{\Omega_{\sigma_k}}\tilde{\rho}_{\sigma_k}^2(t)dx\end{aligned}$$

$$-\lambda_2 \sum_{j\neq\sigma_k}^{N} \int_{\Omega_j} \tilde{f}_j^2(x,t)dx - \lambda_2 \int_{\sigma_k} \tilde{f}_{\sigma_k}^2(x,t)dx$$
$$-\frac{\lambda_1}{N-1} \sum_{j\neq\sigma_k}^{N} \int_{\Omega_j} [z(x,t) - \tilde{f}_j(x,t) - \tilde{\rho}_j(t)]^2 dx$$
$$+\lambda_1 \int_{\Omega_{\sigma_k}} [z(x,t) - \tilde{f}_{\sigma_k}(x,t) - \tilde{\rho}_{\sigma_k}(t)]^2 dx.$$

Hence,

$$\dot{V}(t) + 2\alpha V(t) \leq \sum_{j\neq\sigma_k}^{N} \int_{\Omega_j} \tilde{\eta}_1^T \Theta_1 \tilde{\eta}_1 dx + \int_{\Omega_{\sigma_k}} \tilde{\eta}_2^T \Theta_2 \tilde{\eta}_2 dx, \quad if\ t \in [t_k + \delta, t_{k+1}),$$

and

$$\dot{V}(t) - 2\alpha_0 V(t) \leq \int_0^L \eta_0^T \Psi_0 \eta_0 dx, \quad if\ t \in [t_k, t_k + \delta),$$

where $\Theta_l (l = 1, 2)$ and Ψ_0 are given by (3.33), (3.34) and (3.14).

Thus, $\dot{V}(t)+2\alpha V(t) \leq 0, \dot{V}(t)-2\alpha_0 V(t) \leq 0$, if $\Theta_l < 0 (l = 1, 2)$ and $\Psi_0 < 0$ hold for all $-C \leq z \leq C$. Matrices $\Theta_l (l = 1, 2)$ and Ψ_0 are given by (3.33), (3.34) and (3.14) are affine in z. Thus, $\Theta_l < 0 (l = 1, 2)$ and $\Psi_0 < 0$ hold for all $-C \leq z \leq C$ if these inequalities hold in the vertices $z = \pm C$, i.e. if LIMs (3.30) - (3.32) are feasible. The proof is complete. □

4. Numerical example

In this section, we will present a numerical example which verifies our result. Consider the equation (2.1) with $L = 6$ and initial data $z(x, 0)$. For simplicity, we take $z(x, 0) = 0.8$. Figure 1 demonstrates the profile of the open-loop system initialized by $z(x, 0)$. Then we Let $N = 60$, $K = 30$, $\alpha_0 = 0.2$, $\alpha = 0.02$, $C = 1$, $h = 0.005$, $\delta = 0.001$ and $\varepsilon = \pi/30$. Then equation (2.10) with

$$u_{\sigma_k}(t) = \begin{cases} 0, & t \in [t_k, t_k + \delta), \\ -\dfrac{225}{\pi} \displaystyle\int_{\Omega_{\sigma_k}} z(x, t_k)dx, & t \in [t_k + \delta, t_{k+1}), \end{cases}$$

and $t_{k+1} - t_k = 0.03$, $|\Omega_{\sigma_k}| = \frac{2\pi}{15}$. Set the steps in time and space as $dx = \pi/30$ and $dt = 10^{-3}$. Figure 2 shows the profile of closed-loop system. The locations of sensor/actuator under the switching control law are given in Figure 3. The results also hold for stochastic case and in our further paper, we will solve it, see [26].

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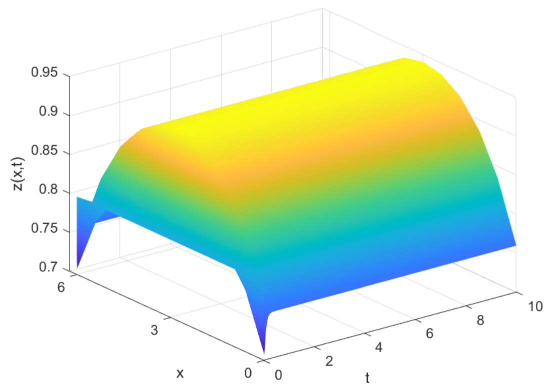

Figure 1. State of unforced system.

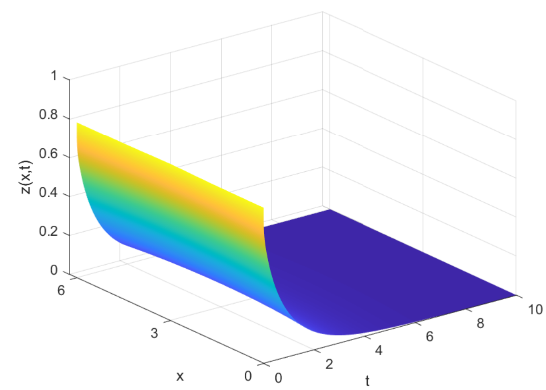

Figure 2. State response of closed-loop system.

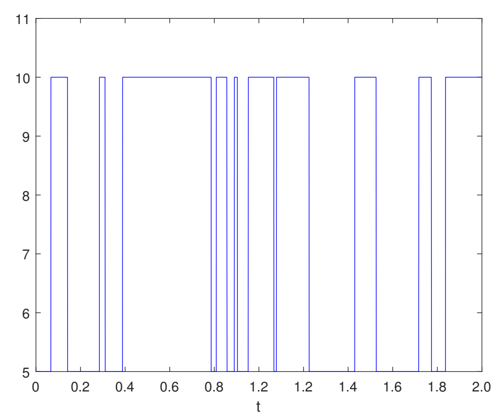

Figure 3. Sensor/actuator locations: $N = 60$, $t_{k+1} - t_k = 0.03$. The ordinate denotes σ_k.

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