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A NEW PARAMETERIZED MATRIX SPLITTING PRECONDITIONER FOR THE SADDLE POINT PROBLEMS*

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Abstract Recently, Zheng and Lu [*International Journal of Computer Mathematics*, 96(1): 1-17, DOI: 10.1080/00207160.2017.1420179] constructed a parameterized matrix splitting (PMS) preconditioner for the large sparse saddle point problems, and gave the corresponding theoretical analysis and numerical experiments. In this paper, based on the parameterized matrix splitting, we generalize the PMS algorithms and further present the new parameterized matrix splitting (NPMS) preconditioner for the saddle point problems. Moreover, by similar theoretical analysis, we analyze the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the NPMS preconditioned saddle point matrices. Finally, one example is provided to confirm the effectiveness.

Keywords Matrix splittings, saddle point problem, convergence, preconditioner, eigenvalue.

MSC(2010) 65F10, 65F15, 65F50.

1. Note to practitioners

This paper was motivated by different applications of scientific computing, such as constrained optimization, the finite element method for solving the Navier-Stokes equation, and constrained least squares problems and generalized least squares problems. In recent years, there has been a surge of interest in the saddle point problem

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of the form (2.1), and a large number of stationary iterative methods have been proposed. In this paper, we will generalize the existing algorithms and further present the new parameterized matrix splitting preconditioner for the saddle point problems. Moreover, by similar theoretical analysis, we will analyze the convergence conditions of the corresponding matrix splitting iterative methods and preconditioning properties of the new preconditioned saddle point matrices. Finally, one example is provided to confirm the effectiveness.

2. Introduction

Consider the following 2×2 block saddle point problems

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} = b, \tag{2.1}$$

where $A \in R^{m,m}$ is a symmetric positive definite matrix, $B \in R^{m,n}, m \geq n$, is a matrix of full column rank, $B^T \in R^{n,m}$ is the conjugate transpose of B, and $f \in C^m, g \in C^n$ are two given vectors. It appear in many different applications of scientific computing, such as constrained optimization [55], the finite element method for solving the Navier-Stokes equation [1, 27, 28, 30], and constrained least squares problems and generalized least squares problems [2,31,35,44,45] and so on; see [9-17, 20,21,30,38,41-45,49-54] and references therein.

In recent years, there has been a surge of interest in the saddle point problem of the form (2.1), and a large number of stationary iterative methods have been proposed. For example, Santos et al. [35] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Golub et al. [32] presented SOR-like algorithms for solving linear systems (2.1). Darvishi et al. [26] studied SSOR method for solving the augmented systems. Bai et al. [16,17,25,55] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving linear systems (2.1). Zhang and Lu [46] showed the generalized symmetric SOR method for augmented systems. Peng and Li [34] studied the unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [3,7–9,18,33,38] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on.

Recently, based on a parameterized matrix splitting, Zheng and Lu [56] constructed a parameterized matrix splitting (PMS) preconditioner for the large sparse saddle point problems, and gave the corresponding theoretical analysis and numerical experiments.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [37].

In this paper, based on the parameterized matrix splitting by Zheng and Lu [56], we generalize the PMS algorithms and further present the new parameterized matrix splitting (NPMS) preconditioner for the saddle point problems. Moreover, by similar theoretical analysis, we analyze the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the NPMS preconditioned saddle point matrices. Finally, one example is provided to confirm the effectiveness.

3. New parameterized matrix splitting (NPMS) preconditioner

Recently, for the coefficient matrix of the augmented system (2.1), Zheng and Lu [56] made the following splitting

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix} - \begin{pmatrix} \beta Q_1 - \beta A & -\beta B \\ \beta B^T & \beta Q_2 \end{pmatrix} \\ &= (\beta \Omega + \alpha \mathcal{A}) - (\beta \Omega - \beta \mathcal{A}) \\ &= \mathcal{P}_{PMS} - \mathcal{R}_{PMS},\end{aligned} \tag{3.1}$$

where $\Omega = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, $\alpha, \beta > 0, \alpha + \beta = 1$. Here, $Q_1 \in R^{m,m}, Q_2 \in R^{n,n}$ are two symmetric positive definite matrices. Based on the iteration methods studied in [56], we establish the new parameterized matrix splitting (NPMS) of the saddle point matrix $\mathcal{A}$, which is as follows:

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix} - \begin{pmatrix} \beta Q_1 - (1-\alpha) A & -(1-\alpha) B \\ (1-\alpha) B^T & \beta Q_2 \end{pmatrix} \\ &= (\beta \Omega + \alpha \mathcal{A}) - (\beta \Omega - (1-\alpha) \mathcal{A}) \\ &= \mathcal{P}_{NPMS} - \mathcal{R}_{NPMS},\end{aligned} \tag{3.2}$$

where $\Omega = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$, $\alpha, \beta > 0$. Here, $Q_1 \in R^{m,m}, Q_2 \in R^{n,n}$ are two symmetric positive definite matrices.

Remark 3.1. Since $\alpha, \beta > 0$ and Q_1, Q_2 are two symmetric positive definite matrices, we can see that $\mathcal{P}_{NPMS}$ is a nonsingular matri x. Moreover, α, β are two unrestricted parameters in the new parameterized matrix splitting (NPMS).

By this special splitting, the new parameterized matrix splitting (NPMS) method can be defined for solving the saddle point problem (2.1):

New parameterized matrix splitting (NPMS) method: Let Q_1 and Q_2 be two symmetric positive definite matrices. Give initial vectors $x^0 \in R^m, y^0 \in R^n$, and two relaxation factors α, β which satisfy $\alpha, \beta > 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{[(x^k)^T, (y^k)^T]^T\}$ converges to the exact solution of the saddle point problem(2.1), compute

$$\begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix} \begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} \beta Q_1 - (1-\alpha)A & -(1-\alpha)B \\ (1-\alpha)B^T & \beta Q_2 \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix}. \tag{3.3}$$

It is easy to see that the iteration matrix of the NPMS iteration is

$$\Gamma = \begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix}^{-1} \begin{pmatrix} \beta Q_1 - (1-\alpha)A & -(1-\alpha)B \\ (1-\alpha)B^T & \beta Q_2 \end{pmatrix}. \tag{3.4}$$

If we use a Krylov subspace method such as GMRES (Generalized Minimal Residual) method or its restarted variant to approximate the solution of this system of linear equations, then

$$\mathcal{P}_{NPMS} = \begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix}, \tag{3.5}$$

can be served as a preconditioner. We call the preconditioner $\mathcal{P}_{NPMS}$ the NPMS preconditioner for the generalized saddle point matrix $\mathcal{A}$.

In every iteration of the NPMS iteration (3.3) or the preconditioned Krylov subspace method, we need solve a residual equation

$$\begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix} z = r \tag{3.6}$$

needs to be solved for a given vector r at each step. Here, we may refer to Algorithm 2.1 in [56] about the corresponding algorithmic version of the NPMS iteration method.

Remark 3.2. On the new parameterized matrix splitting (NPMS) method method, when $\alpha = \frac{1}{2}, Q_1 = Q_2 = \gamma I(\gamma > 0), \alpha + \beta = 1$, the NPMS method reduces to the shift-splitting (SS) method [21]; When $\alpha = \frac{1}{2}, Q_1 \gamma I, Q_2 = \xi I(\gamma, \xi > 0), \alpha + \beta = 1$, the NPMS method reduces to the generalized shift-splitting (GS) method [24]; When $\alpha = \beta = \frac{1}{2}$, the NPMS method reduces to the extended shift-splitting (ESS) method [57]; When $\alpha + \beta = 1$, the NPMS method reduces to the parameterized matrix splitting (PMS) method [56]. So, the NPMS method is the generalization of existing iterative algorithm.

4. Covergence of NPMS method

Now, we turn to study the convergence of the NPMS iteration for solving saddle point problems (2.1). It is well known that the iteration method (3.3) is convergent for every initial guess if and only if $\rho(\Gamma) < 1$, where $\rho(\Gamma)$ denotes the spectral radius of Γ. In [56], based PMS method, Zheng and Lu established the spectral properties of the iteration matrix and the preconditioned matrix $\mathcal{P}_{PMS}^{-1}\mathcal{A}$. In this section, by similar theoretical analysis, we will analyze the convergence conditions of the corresponding matrix splitting iteration methods and preconditioning properties of the NPMS preconditioned saddle point matrices.

Lemma 4.1. *Let $A \in R^{m,m}$ be symmetric positive definite, and $B \in R^{m,n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices. If λ is an eigenvalue of the iteration matrix Γ of the NPMS iteration method, then $\lambda \neq 1$ and $\lambda \neq 1 - \frac{1}{\alpha}$.*

Proof. Let λ be a nonzero eigenvalue of the iteration matrix and $[u^*, v^*]^*$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} \beta Q_1 - (1-\alpha)A & -(1-\alpha)B \\ (1-\alpha)B^T & \beta Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \beta Q_1 + \alpha A & \alpha B \\ -\alpha B^T & \beta Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.1}$$

or equivalently

$$(1-\alpha+\alpha\lambda)Au + (\beta\lambda - \beta)Q_1 u + (1-\alpha+\alpha\lambda)Bv = 0, \tag{4.2}$$

and

$$(1-\alpha+\alpha\lambda)B^T u + (\beta - \beta\lambda)Q_2 v = 0. \tag{4.3}$$

If $\lambda = 1$, then from (4.2) and (4.3) we can obtain

$$\begin{cases} Au + Bv = 0, \\ B^T u = 0. \end{cases} \tag{4.4}$$

By assumptions, we know that the coefficient matrix of (4.4) is nonsingular. Hence $u = 0$ and $v = 0$, which contradicts with the assumption that $[u^*, v^*]^*$ is an eigenvector. So $\lambda \neq 1$.

If $\lambda = 1 - \frac{1}{\alpha}$, then from (4.2) and (4.3) we have

$$-\frac{\beta}{\alpha} Q_1 u = 0 \quad \text{and} \quad \frac{\beta}{\alpha} Q_2 v = 0.$$

Hence, we have $u = v = 0$ because Q_1 and Q_2 are symmetric positive definite matrices. This also contradicts that $[u^*, v^*]^*$ is an eigenvector of Γ. So $\lambda \neq 1 - \frac{1}{\alpha}$. □

Lemma 4.2. [56] *Let $A \in R^{m,m}$ be symmetric positive definite, and $B \in R^{m,n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices. Assume λ is an eigenvalue of the iteration matrix Γ of the NPMS method and $z = [u^*, v^*]^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, is the corresponding eigenvector. Then $u \neq o$. Moreover, if $v = 0$, then $u \in \mathrm{null}(B^T)$.*

Lemma 4.3. *Let $A \in R^{m,m}$ be symmetric positive definite, and $B \in R^{m,n}$ be of full column rank, with $m \geq n$. Q_1 and Q_2 are two symmetric positive definite matrices. Assume λ is an eigenvalue of the iteration matrix Γ of the NPMS method* *and* $z = [u^*, v^*]^* \in C^{m+n}$, *with* $u \in C^m$ *and* $v \in C^n$ *being two complex vectors, is the corresponding eigenvector. Denote*

$$a = \frac{u^* A u}{u^* u}, \ b = \frac{u^* Q_1 u}{u^* u} \text{ and } c = \frac{u^* B Q_2^{-1} B^T u}{u^* u}, \tag{4.5}$$

where a, b, c *are real numbers. Then* λ *satisfies the following quadratic equation:*

$$\lambda^2 - \left[1 - \frac{a\beta(1-\alpha) - b\beta^2 - c(1-\alpha)^2 + c}{a\alpha\beta + b\beta^2 + c\alpha^2}\right]\lambda + \frac{b\beta^2 + c(1-\alpha)^2 - a\beta(1-\alpha)}{a\alpha\beta + b\beta^2 + c\alpha^2}. \tag{4.6}$$

Proof. From Lemma 4.1 we can obtain that $\lambda \neq 1$. Solving v from (4.3) and substituting $v = \frac{1-\lambda+\alpha\lambda}{\beta\lambda-\beta} Q_2^{-1} B^T u$ into (4.2), we have

$$(1-\alpha+\alpha\lambda)Au + (\beta\lambda - \beta)Q_1 u + \frac{(1-\alpha+\alpha\lambda)^2}{\beta\lambda-\beta} B Q_2^{-1} B^T u = 0. \tag{4.7}$$

From Lemma 4.2, we also know that $u \neq 0$. Multiplying $(\beta\lambda - \beta)\frac{u^*}{u^* u}$ on both sides of (4.7) and using the notation (4.5), we obtain the following complex quadratic equation of λ

$$(\beta\lambda-\beta)(1-\alpha+\alpha\lambda)\frac{u^* A u}{u^* u} + (\beta\lambda-\beta)^2 \frac{u^* Q_1 u}{u^* u} + (1-\alpha+\alpha\lambda)^2 \frac{u^* B Q_2^{-1} B^T u}{u^* u} = 0, \tag{4.8}$$

which can be rewritten as

$$a(\beta\lambda - \beta)(1-\alpha+\alpha\lambda) + b(\beta\lambda-\beta)^2 + c(1-\alpha+\alpha\lambda)^2 = 0 \tag{4.9}$$

$$\Longleftrightarrow \lambda^2 - \left[1 - \frac{a\beta(1-\alpha) - b\beta^2 - c(1-\alpha)^2 + c}{a\alpha\beta + b\beta^2 + c\alpha^2}\right]\lambda + \frac{b\beta^2 + c(1-\alpha)^2 - a\beta(1-\alpha)}{a\alpha\beta + b\beta^2 + c\alpha^2}. \tag{4.10}$$

The above two lemmas characterize the property of the eigenvalues and the eigenvectors of the iteration matrix T of the NPMS method. Moreover, from Lemma 4.3, we can get the following result. □

Corollary 4.1. *From Equation (4.6) in Lemma 4.3, we can give the specific expression of the eigenvalue* λ *for the iteration matrix* Γ *of the NPMS method when the conditions of Lemma 4.3 are satisfied. That is*

$$\lambda = \frac{b\beta^2 + c(1-\alpha)^2 - a\beta(1-\alpha) - c \pm \sqrt{a^2\beta(1-\alpha) - 4bc\beta(1-\alpha)}}{a\alpha\beta + b\beta^2 + c\alpha^2}. \tag{4.11}$$

Lemma 4.4. [36,58] *Both roots of the real quadratic equation* $x^2 - px + q = 0$ *are less than 1 in modulus if and only if* $|q| < 1$ *and* $|p| < 1 + q$.

With Lemmas 4.3 and 4.4, we can get the following important theorem which shows the convergence of the NPMS iteration method.

Theorem 4.1. *Assume* $A \in R^{m,m}$ *be symmetric positive definite, and* $B \in R^{m,n}$ *be of full column rank, with* $m \geq n$. Q_1 *and* Q_2 *are two symmetric positive definite matrices. Then the NPMS method is convergent if the following conditions are satisfied:*

$$\lambda_{\min}(Q_1) \geq \frac{1-\alpha}{\beta}\lambda_{\max}(A), \alpha \geq \frac{1}{2}, \alpha + \beta \geq 1. \tag{4.12}$$

Here, $\lambda_{\max}(A)$ *and* $\lambda_{\min}(Q_1)$ *are the largest and the smallest eigenvalues of* A *and* Q_1, *respectively.*

Proof. Assume λ is an eigenvalue of the iteration matrix Γ of the NPMS method and $z = [u^*, v^*]^* \in C^{m+n}$, with $u \in C^m$ and $v \in C^n$ being two complex vectors, is the corresponding eigenvector. Then from Lemma 4.3, we know that λ satisfies the real quadratic equation (4.6).

By making use of Lemma 4.4, both roots λ of the real quadratic equation (4.4) satisfy $|\lambda| < 1$ if and only if

$$\left| \frac{b\beta^2 + c(1-\alpha)^2 - a\beta(1-\alpha)}{a\alpha\beta + b\beta^2 + c\alpha^2} \right| < 1 \tag{4.13}$$

and

$$\left| 1 - \frac{a\beta(1-\alpha) - b\beta^2 - c(1-\alpha)^2 + c}{a\alpha\beta + b\beta^2 + c\alpha^2} \right| < 1 + \frac{b\beta^2 + c(1-\alpha)^2 - a\beta(1-\alpha)}{a\alpha\beta + b\beta^2 + c\alpha^2}. \tag{4.14}$$

When $\alpha > 0, \beta > 0, c > 0$, inequalities (4.13) and (4.14) hold true if and only if the following conditions are satisfied:

$$\begin{aligned} &a\beta + (2\alpha - 1)c > 0, \\ &a(2\alpha-1)\beta + 2b\beta^2 + c[(1-\alpha)^2 + \alpha^2] > 0, \\ &a(2\alpha-1)\beta + 4b\beta^2 + c(2\alpha-1)^2 > 0. \end{aligned} \tag{4.15}$$

If $\alpha \geq \frac{1}{2}$, then we can obtain that Equation (4.15) holds true. So we have $|\lambda| < 1$.

For the case $\alpha > 0, \beta > 0, c = 0$, from the result of Corollary 4.1, we can obtain

$$\lambda_1 = \frac{b\beta^2 - a\beta(1-\alpha) + a\sqrt{\beta(1-\alpha)}}{a\alpha\beta + b\beta^2}, \ \lambda_1 = \frac{b\beta^2 - a\beta(1-\alpha) - a\sqrt{\beta(1-\alpha)}}{a\alpha\beta + b\beta^2}. \tag{4.16}$$

First, $|\lambda_1| < 1$ if and only if

$$\left| \frac{b\beta^2 - a\beta(1-\alpha) + a\sqrt{\beta(1-\alpha)}}{a\alpha\beta + b\beta^2} \right| < 1 \tag{4.17}$$

$$\Longleftrightarrow -a\alpha\beta - b\beta^2 < b\beta^2 - a\beta(1-\alpha) + a\sqrt{\beta(1-\alpha)} < a\alpha\beta + b\beta^2. \tag{4.18}$$

So, the following conditions are satisfied

$$\begin{cases} -a\alpha\beta - b\beta^2 < b\beta^2 - a\beta(1-\alpha) \Rightarrow a(1-2\alpha) < 2b\beta, \\ -a\beta(1-\alpha) + a\sqrt{\beta(1-\alpha)} \leq a\alpha\beta \Rightarrow \alpha + \beta \geq 1. \end{cases} \tag{4.19}$$

Since $\alpha \geq \frac{1}{2}$, then we have $1 - 2\alpha \leq 0$, so the first equation of formula (4.19) is valid.

Next, $|\lambda_2| < 1$ if and only if

$$\left| \frac{b\beta^2 - a\beta(1-\alpha) - a\sqrt{\beta(1-\alpha)}}{a\alpha\beta + b\beta^2} \right| < 1 \tag{4.20}$$

$$\Longleftrightarrow -a\alpha\beta - b\beta^2 < b\beta^2 - a\beta(1-\alpha) - a\sqrt{\beta(1-\alpha)} < a\alpha\beta + b\beta^2. \tag{4.21}$$

So, the following conditions are satisfied

$$\begin{cases} \quad -a\alpha\beta - b\beta^2 < b\beta^2 - a\beta(1-\alpha) - a\sqrt{\beta(1-\alpha)} \\ \Rightarrow a\beta(1-2\alpha) < 2b\beta^2 - a\sqrt{\beta(1-\alpha)}, \\ \quad b\beta^2 - a\beta(1-\alpha) - a\sqrt{\beta(1-\alpha)} < a\alpha\beta + b\beta^2 \\ \Rightarrow -a\sqrt{\beta(1-\alpha)} < a\alpha\beta. \end{cases} \tag{4.22}$$

Obviously, the second equation of formula (4.22) is valid. On the first equation of formula (4.22), since $\alpha + \beta \geq 1$, then we have $-a\sqrt{\beta(1-\alpha)} \geq -a\beta$. So, the following conditions are satisfied

$$a\beta(1-2\alpha) \leq 2b\beta^2 - a\beta. \tag{4.23}$$

This implies

$$a(1-\alpha) \leq b\beta \Rightarrow b \geq \frac{1-\alpha}{\beta}a. \tag{4.24}$$

If $\lambda_{\min}(Q_1) \geq \frac{1-\alpha}{\beta}\lambda_{\max}(A)$, then $b \geq b_{\min} \geq \frac{1-\alpha}{\beta}a_{\max} \geq \frac{1-\alpha}{\beta}a$. Hence, the first equation of formula (4.22) holds true. So $|\lambda| < 1$. □

Remark 4.1. On the one hand, the NPMS method is the generalization of the PMS method. On the other hand, when the appropriate parameters are selected, the NPMS method will have better convergence than the PMS method.

5. Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated based on a two-dimensional time-harmonic Maxwell equations in mixed form, respectively. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-6}\|b\|_2$).

Example 5.1. In this section, to further assess the effectiveness of the iterative matrix $\mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$, we present a sample of numerical examples which are based on a two-dimensional time-harmonic Maxwell equations in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$. For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where $\mathrm{nz}(A)$ denote the nonzero elements of matrix A.

Since the new preconditioners have two parameters, in numerical experiments we will test different values. Numerical experiments show the spectrum of the iterative matrix $\mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when choosing different parameters, which coincides with theoretical analysis.

In Figures 2, 3 and 4 we display the eigenvalues of the iteration matrix $\mathcal{P}_{NPMS}^{-1}$ $\mathcal{R}_{NPMS}$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters.

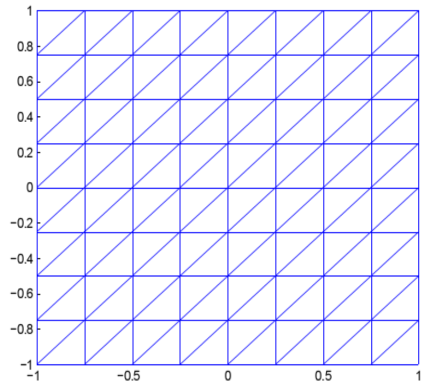

Figure 1. A uniform mesh with $h = \frac{\sqrt{2}}{4}$.

In Table 2, we show the Spectral radius of iterative matrix $\mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when choosing different parameters.

Remark 5.1. Figures 2 $\sim$ 4 show that the distribution of eigenvalues of the iteration matrix confirm our above theoretical analysis.

Table 1. Datasheet for different grids.

Grid	m	n	nz(A)	nz(B)	nz(W)	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

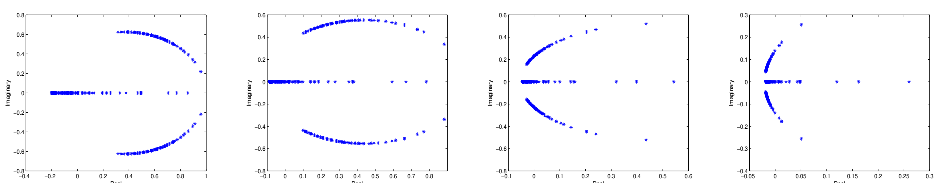

Figure 2. The eigenvalue distribution for the NPMS iteration matrix $\Gamma = \mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when $\alpha = 0.8, \beta = 0.5$(the first), $\alpha = 0.9, \beta = 0.3$(the second),$\alpha = 0.95, \beta = 0.1$(the third) and $\alpha = 0.98, \beta = 0.03$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

6. Conclusion

In this study, we introduced a new parameterized matrix splitting (NPMS) preconditioner for addressing large sparse saddle point problems. This method generalizes existing parameterized matrix splitting (PMS) approaches by incorporating additional flexibility through unrestricted parameters, which enhance the convergence

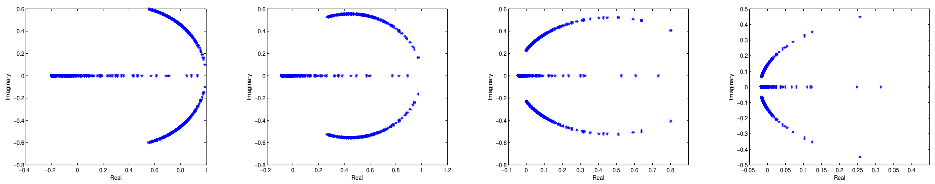

Figure 3. The eigenvalue distribution for the NPMS iteration matrix $\Gamma = \mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when $\alpha = 0.8, \beta = 0.5$(the first), $\alpha = 0.9, \beta = 0.3$(the second),$\alpha = 0.95, \beta = 0.1$(the third) and $\alpha = 0.98, \beta = 0.03$(the fourth), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

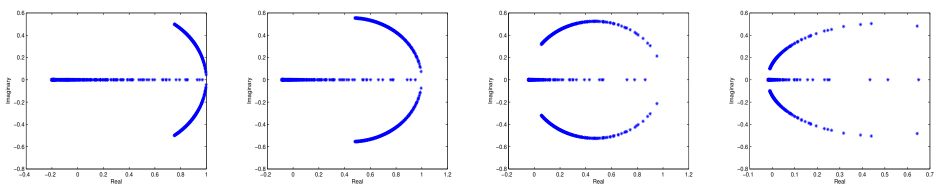

Figure 4. The eigenvalue distribution for the NPMS iteration matrix $\Gamma = \mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when $\alpha = 0.8, \beta = 0.5$(the first), $\alpha = 0.9, \beta = 0.3$(the second), $\alpha = 0.95, \beta = 0.1$(the third) and $\alpha = 0.98, \beta = 0.03$ (the fourth), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

Table 2. Spectral radius of iterative matrix $\mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$ when choosing different parameters. Here, $h = \frac{\sqrt{2}}{8}, h = \frac{\sqrt{2}}{12}, h = \frac{\sqrt{2}}{18}$ denote the size of the corresponding grid, respectively. ρ denotes spectral radius of iterative matrix $\mathcal{P}_{NPMS}^{-1}\mathcal{R}_{NPMS}$.

α	β	$\rho(h = \frac{\sqrt{2}}{8})$	$\rho(h = \frac{\sqrt{2}}{12})$	$\rho(h = \frac{\sqrt{2}}{18})$
0.8	0.5	0.9832	0.9961	0.9990
0.9	0.3	0.9449	0.9875	0.9971
0.95	0.1	0.6786	0.8992	0.9764
0.98	0.03	0.2608	0.5176	0.8039

properties of the iterative methods. Through rigorous theoretical analysis, we established the convergence conditions and eigenvalue distribution for the NPMS method, demonstrating its superiority over traditional PMS techniques under specific parameter settings. Numerical experiments validated the theoretical findings, showing that the NPMS preconditioner achieves a significant reduction in spectral radius compared to other preconditioners, leading to faster convergence in solving saddle point problems. The results highlight the NPMS method's potential for applications in constrained optimization, finite element methods, and other computational problems involving saddle point structures.

Conflict of Interest

I'm sure there's no conflict of interest.

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