Journal of Applied Analysis and Computation
Volume 15, Number 5, October 2025, 2786–2804
Website:http://www.jaac-online.com
DOI:10.11948/20240526

EXISTENCE AND HYERS-ULAM STABILITY FOR BOUNDARY VALUE PROBLEMS OF TWO-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH κ-CAPUTO DERIVATIVE*

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Abstract This paper is concerned with a class of nonlinear fractional differential equations with two-term κ-Caputo fractional derivatives. The existence and uniqueness results are obtained for boundary value problems by using the Banach fixed point theorem and Leray-Schauder nonlinear alternative. The Hyers-Ulam stability is also considered. Some examples are discussed to illustrate the obtained results.

Keywords Fractional differential equations, boundary value problem, mild solution, fixed point theorem, stability.

MSC(2010) 34A08, 34D20, 47H10.

1. Introduction

Fractional differential equations have gained considerable importance due to their significant applications in various sciences such as physics, mechanics, chemistry, engineering, etc. (see [4, 8, 13, 22]). Recently, some new definitions of fractional integrals and fractional derivatives have been employed in the investigation of fractional differential equations. For example, Samko et al. [15], Almeida [1, 2] developed general definitions and studied some of their properties. Some outstanding theoretical achievements have been gained in studying the existence and uniqueness of fractional differential equations by applying the fixed point theorem (see [5, 11, 12, 14, 24, 26]).

In the paper [2], Almeida constructed a new fractional derivative, the κ-Caputo fractional derivative (see Definition 2.3 below), unifying some classical fractional derivative concepts. Based on Leray-Schauder fixed point theorem, Almeida investigated the existence of a solution for fractional differential equations with mixed

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*The authors were supported by National Natural Science Foundation of China (12271469 and 12371140) and Jiangsu Students' Innovation and Entrepreneurship Training Program (202211117093Y).
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boundary conditions involving the κ-Caputo fractional derivative of the form

$$\begin{cases} {}^C D_{a^+}^{\alpha;\kappa}\mu(t) = f(t,\mu(t)), \\ \mu(a) = \mu_a, \mu'(a) = \mu_a^1, \mu(b) = K J_{a^+}^{\alpha,\kappa} x(\tau), \end{cases}$$

where $t \in [a,b], {}^C D_{a^+}^{\alpha;\kappa}\mu$ is the κ-Caputo fractional derivative of μ with $\alpha \in (2,3)$ (it is also called the Caputo-type fractional derivative with respect to the function κ) and $f \in C([a,b]\times\mathbb{R},\mathbb{R}), \mu_a, \mu_a^1, K \in \mathbb{R}, \tau \in (a,b], x \in C^2([a,b],\mathbb{R})$ and $J_{a^+}^{\alpha;\kappa}$ is the κ-Riemann-Liouville fractional integral.

Ho Vu et al. [24] discussed the uniqueness, existence, and stability results of boundary value problems of fractional differential equations with κ-Caputo fractional derivative

$$\begin{cases} {}^C D_{a^+}^{\alpha;\kappa}\mu(t) = f\left(t,\mu(t),\mu'(t)\right), \\ a_1\mu(a) + b_1\mu(T) = c_1, \\ a_2\mu'(a) + b_2\mu'(T) = c_2, \end{cases}$$

where $t \in [a,T], {}^C D_{a^+}^{\alpha;\kappa}\mu$ is the κ-Caputo fractional derivative of μ with $\alpha \in (1,2)$; $f : [a,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, and a_1, a_2, b_1, b_2, c_1, and c_2 are constants such that $(a_1 + b_1) \neq 0, |a_1| + |b_1| \neq 0$, and $|a_2| + |b_2| \neq 0$.

The stability theory of mathematical models of dynamic systems is one of the essential objectives of control theory. At present, the Hyers-Ulam stability theory of differential equations was widely investigated [16, 17, 19, 20, 23, 25] and received much attention because it is pretty significant in realistic problems in numerical analysis, economics, and biology in which the exact solutions are not easy to seek. Hyers-Ulam-Rassias stability is an advanced version of the Hyers-Ulam stability theory, which was developed by Rassias. It goes beyond the linear assumptions of the Hyers-Ulam stability theory and accommodates nonlinear perturbations. This new theory also examines the stability of functional or differential equations with specific domain or range characteristics under certain functionals. It is essential to mention that this concept is mostly used in mathematical studies, and it has become a significant reference point for professionals. Hyers-Ulam stability and Hyers-Ulam-Rassias stability have found extensive applications in various fields, such as numerical analysis, data analysis, control theory, and physics. The study of Hyers-Ulam-type stability has also led to the development of new mathematical tools and concepts that have proved helpful in solving some otherwise intractable problems in these fields.

On the basis of these contents, we study a class of nonlinear fractional differential equations involving κ-Caputo fractional derivative

$$\begin{cases} \left({}^C D^{\alpha;\kappa} - a\,{}^C D^{\beta;\kappa}\right)\mu(t) + f(t,\mu(t)) = 0, \quad 0 < t < 1, \\ \mu(0) = 0, \mu(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (1,2), \beta \in (0,\alpha), \kappa(0) = 0, \kappa(1) = 1, f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous. We first study the form of the mild solution to problem (1.1). By using Leray-Schauder alternative theorem and Banach fixed point theorem, we discuss the existence and uniqueness results of mild solutions to boundary value problem (1.1) under some weak conditions, such as the Caratheodory condition. We also study the Hyers-Ulam stability of the boundary value problem (1.1). We give two examples at the end of the article to illustrate the conclusions.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we prove our main results(the existence and uniqueness results). In Section 4, we discuss the Hyers-Ulam stability for boundary value problem (1.1). Finally, two examples are given to illustrate the conclusions in Section 5.

2. Preliminaries

Throughout the paper, we use the notation $\mathcal{K}_\kappa$ to express the set of the functions $\kappa : [a, T] \to \mathbb{R}^+$ satisfying the properties: $\kappa > 0$ is an increasing and differentiable function such that for all $t \in (0, 1), \kappa'(t) \neq 0$. As an illustration, consider the function κ defined as $\log(t + 1)$ for $t \geq 0$, $\sin(t)$ for $t \in [0, \pi/2]$, or t itself. We set $\kappa(0) = 0$ and $\kappa(1) = 1$ throughout this paper. If $\kappa(1)$ is not equal to 1, we can use $\kappa_1(t)=\frac{\kappa(t)}{\kappa(1)}$ instead of the previously defined $\kappa(t)$ without loss of generality. This approach makes the analysis more rigorous and applicable to a wide range of scenarios.

Definition 2.1. [1] Let $\alpha > 0$ and $n \in \mathbb{N}$. $J^{\alpha;\kappa}_{a^+}\mu(t)$, the fractional integral of μ with respect to κ with the order α on $[a, T]$, is defined as

$$J^{\alpha;\kappa}_{a^+}\mu(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \kappa'(s)(\kappa(t) - \kappa(s))^{\alpha-1}\mu(s)ds.$$

In this paper, we carray on the definition of κ-Caputo-type fractional derivative by Almeida [2].

Definition 2.2. [2] If $\mu \in C^n([a, T], \mathbb{R})$, then the κ-Caputo fractional derivative of μ is defined by

$$\begin{aligned} {}^C D^{\alpha;\kappa}_{a+}\mu(t) :=& J^{n-\alpha;\kappa}_{a^+} \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^n \mu(t) \\ =& \frac{1}{\Gamma(n-\alpha)} \int_a^t \kappa'(s)(\kappa(t) - \kappa(s))^{n-\alpha-1}\mu^{[n]}_\kappa(s)ds, \end{aligned}$$

where

$$n = \begin{cases} [\alpha] + 1, & \text{if } \alpha \notin \mathbb{N}, \\ \alpha, & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

and

$$\mu^{[i]}_\kappa(t) := \left(\frac{1}{\kappa'(t)}\frac{d}{dt}\right)^i \mu(t).$$

Some properties of the fractional integral and the κ-Caputo fractional derivative are provided in the lemma below.

Lemma 2.1. [2] *Let $\alpha > 0$ and $n \in \mathbb{N}$. The Caputo-type fractional derivative with respect to κ (or the κ-Caputo fractional derivative) with the order α of μ is defined by*

$${}^C D^{\alpha;\kappa}_{a^+}\mu(t) := {}^{RL} D^{\alpha;\kappa}_{a^+} \left[\mu(t) - \sum_{i=0}^{n-1} \frac{\mu^{[i]}_\kappa(a)}{i!}(\kappa(t) - \kappa(a))^i\right],$$

where $\mu \in C^{n-1}([a, T], \mathbb{R})$ and ${}^{RL}D^{\alpha;\kappa}_{a^+}\mu$ exists.

Lemma 2.2. [1, 6, 21] *Let $\alpha > 0, \beta > -1, \lambda \in \mathbb{R}\backslash\{0\}$. The following assertions hold*

1. *If the function $\mu : [a, T] \to \mathbb{R}$ is continuous, then*
$$^{C}D_{a^+}^{\alpha;\kappa} J_{a^+}^{\alpha;\kappa} \mu(t) = \mu(t).$$
2. *If $\mu \in C^{n-1}([a, T], \mathbb{R})$ and ${}^{RL}D_{a^+}^{\alpha;\kappa}\mu$ exists, then*
$$J_{a^+}^{\alpha;\kappa C} D_{a^+}^{\alpha;\kappa} \mu(t) = \mu(t) - \sum_{i=0}^{n-1} \frac{\mu_\kappa^{[i]}(a)}{i!} (\kappa(t) - \kappa(a))^i.$$
3. *If $\mu(t) = (\kappa(t) - \kappa(a))^\beta$ and $v(t) = E_{\alpha,1}(\lambda(\kappa(t) - \kappa(a))^\alpha)$, then*
$$\begin{aligned} J_{a^+}^{\alpha;\kappa} \mu(t) &= \frac{\Gamma(1+\beta)}{\Gamma(\beta+\alpha+1)} (\kappa(t) - \kappa(a))^{\beta+\alpha}, \\ J_{a^+}^{\alpha;\kappa} v(t) &= \frac{1}{\lambda} \left[E_{\alpha,1}(\lambda(\kappa(t) - \kappa(a))^\alpha) - 1\right], \\ {}^{C}D_{a^+}^{\alpha;\kappa} \mu(t) &= \frac{\Gamma(1+\beta)}{\Gamma(\beta-\alpha+1)} (\kappa(t) - \kappa(a))^{\beta-\alpha}, \\ {}^{C}D_{a^+}^{\alpha;\kappa} v(t) &= \lambda E_{\alpha,1}(\lambda(\kappa(t) - \kappa(a))^\alpha). \end{aligned}$$

Remark 2.1. For the sake of convenience, we will use $D_{a^+}^{\alpha;\kappa}$ to denote the κ-Caputo derivative operator instead of ${}^{C}D_{a^+}^{\alpha;\kappa}$ in the rest of the paper. If $a = 0$, we will omit the subscript and employ the terminology $D^{\alpha;\kappa}$.

Another fractional derivative is the Hadamard type which was introduced in 1892 [7]. This derivative differs from the derivatives mentioned above in that the kernel of the integral in the definition of the Hadamard derivative contains the logarithmic function of arbitrary exponent.

Definition 2.3. [7] Let $(a, b)(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$, and the left-sided and right-sided Hadamard fractional integral of order $\alpha > 0$ is defined by
$$\left(J_{a+}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad (a < x < b)$$
and
$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t}, \quad (a < x < b).$$

Definition 2.4. [7, 9] Provided that the integral in Definition 3 exists. Let $\delta = xD(D = d/dx)$ be the δ-derivative. The left-sided and right-sided Hadamard fractional derivatives of order $\alpha > 0$ on (a, b) are defined by
$$\begin{aligned} \left(D_{a+}^\alpha y\right)(x) :=& \delta^n \left(J_{a+}^{n-\alpha} y\right)(x) \\ =& \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha+1} \frac{y(t)dt}{t}, \ (a < x < b) \end{aligned}$$
and
$$\left(D_{b-}^\alpha y\right)(x) := (-\delta)^n \left(J_{b-\alpha}^{n-\alpha} y\right)(x)$$

$$= \left(-x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{n-\alpha+1} \frac{y(t)dt}{t}, \ (a < x < b),$$

where $n = [\alpha] + 1$.

Remark 2.2. If we take $\kappa(t) = \log(t+1)$, then the κ-Caputo fractional derivative turns into Caputo-Hadamard fractional derivative.

We will use the Leray-Schauder alternative theorem and Banach fixed point theorem to prove the existence and uniqueness results.

Lemma 2.3. [18] *(Leray-Schauder nonlinear alternative). Let $\mathbb{X}$ be a Banach space, $\mathcal{C} \subset \mathbb{X}$ be a closed, convex subset of $\mathbb{X}, U$ be an open subset of $\mathcal{C}$ and $0 \in U$. Suppose that $\mathbb{K} : \overline{U} \to \mathcal{C}$ is a continuous, compact (completely continuous) mapping. Then, either*

1. *$\mathbb{K}$ has a fixed point in U, or*
2. *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda\mathbb{K}(u)$.*

At the end of this section, we recall the definition of Hyers-Ulam stability.

Definition 2.5. [20] Boundary value problem (1.1) is called Hyers-Ulam stable if there exists a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([a,T],\mathbb{R})$ of the following inequality

$$\left|\left({}^C D^{\alpha;\kappa} - a^C D^{\beta;\kappa}\right)\mu(t) + f(t,\mu(t))\right| \le \varepsilon, \quad \forall t \in [a,T], \tag{2.1}$$

there exits a solution $\mu \in C^1([a,T],\mathbb{R})$ of boundary value problem (1.1) satisfying

$$|\mu(t) - v(t)| \le C_f\varepsilon, \quad \forall t \in [a,T].$$

3. Existence and uniqueness results

To discuss the definition of the solution of the boundary value problem (1.1), the corresponding linear equation is discussed first. Let $\alpha \in (1,2), \beta \in (0,\alpha), h : [0,1] \to \mathbb{R}$ be continuous and $\mu \in C^1([0,1],\mathbb{R})$. The corresponding linear equation of (1.1) is

$$\begin{cases} \left(D^{\alpha;\kappa} - aD^{\beta;\kappa}\right)\mu(t) + h(t) = 0, \quad 0 < t < 1, \\ \mu(0) = 0, \mu(1) = 0. \end{cases} \tag{3.1}$$

We apply the integral operator $J^{\alpha;\kappa}$ to both sides of (3.1) to obtain

$$J^{\alpha;\kappa}D^{\alpha;\kappa}\mu(t) = aJ^{\alpha;\kappa}D^{\beta,\kappa}\mu(t) - J^{\alpha;\kappa}h(t).$$

We calculate the left-hand side of the equation to get

$$\begin{aligned} J^{\alpha:\kappa}D^{\alpha;\kappa}\mu(t) &= \mu(t) - \mu(0) - \frac{(\kappa(t) - \kappa(0))}{\kappa'(0)}\mu'(0) \\ &= \mu(t) - \frac{\mu'(0)}{\kappa'(0)}\kappa(t) \\ &= \mu(t) - C_1\kappa(t), \end{aligned}$$

where $C_1 = \frac{\mu'(0)}{\kappa'(0)}$. Then, the first term of the right-hand side equals

$$\begin{aligned}J^{\alpha-\beta;\kappa}J^{\beta;\kappa}D^{\beta;\kappa}\mu(t) &= J^{\alpha-\beta;\kappa}\mu(t) - C_1 J^{\alpha-\beta;\kappa}\kappa(t)\\ &= J^{\alpha-\beta;\kappa}\mu(t) - C_1\frac{\Gamma(2)}{\Gamma(\alpha-\beta+2)}\kappa(t)^{\alpha-\beta+1},\end{aligned}$$

which indicates that

$$\mu(t) = C_1\kappa(t) + aJ^{\alpha-\beta;\kappa}\mu(t) - aC_1\frac{\Gamma(2)}{\Gamma(\alpha-\beta+2)}\kappa(t)^{\alpha-\beta+1} - J^{\alpha;\kappa}h(t)$$

for some constant C_1. Take $t = 1$ and apply the boundary condition to get

$$\mu(1) = 0 = C_1 - C_1\frac{a\Gamma(2)}{\Gamma(\alpha-\beta+2)} + aJ^{\alpha-\beta,\kappa}\mu(1) - J^{\alpha;\kappa}h(1).$$

Thus $C_1 = \frac{-aJ^{\alpha-\beta,\kappa}\mu(1)+J^{\alpha;\kappa}h(1)}{\left(1-\frac{a\Gamma(2)}{\Gamma(\alpha-\beta+2)}\right)}$. Consequently, μ satisfies the integral equation

$$\begin{aligned}\mu(t) =&\frac{J^{\alpha;\kappa}h(1) - aJ^{\alpha-\beta;\kappa}\mu(1)}{1-\frac{a\Gamma(2)}{\Gamma(\alpha-\beta+2)}}\left(\kappa(t) - a\frac{\Gamma(2)}{\Gamma(\alpha-\beta+2)}\kappa(t)^{\alpha-\beta+1}\right)\\ &+ aJ^{\alpha-\beta;\kappa}\mu(t) - J^{\alpha;\kappa}h(t).\end{aligned}$$

Denote by $m(t) = \frac{\kappa(t)-\frac{a\Gamma(2)}{\Gamma(\alpha-\beta+2)}\kappa(t)^{\alpha-\beta+1}}{1-\frac{a\Gamma(2)}{\Gamma(\alpha-\beta+2)}}$, we have

$$\mu(t) = \int_0^1 G(t,s)h(s)ds - a\int_0^1 G^*(t,s)\mu(s)ds, \tag{3.2}$$

where

$$G(t,s) = \frac{\kappa'(s)}{\Gamma(\alpha)}\begin{cases} m(t)(\kappa(1)-\kappa(s))^{\alpha-1} - (\kappa(t)-\kappa(s))^{\alpha-1}, & 0 \le s \le t,\\ m(t)(\kappa(1)-\kappa(s))^{\alpha-1}, & t \le s \le 1,\end{cases} \tag{3.3}$$

$$G^*(t,s) = \frac{\kappa'(s)}{\Gamma(\alpha-\beta)}\begin{cases} m(t)(\kappa(1)-\kappa(s))^{\alpha-\beta-1} - (\kappa(t)-\kappa(s))^{\alpha-\beta-1}, & 0 \le s \le t,\\ m(t)(\kappa(1)-\kappa(s))^{\alpha-\beta-1}, & t \le s \le 1.\end{cases} \tag{3.4}$$

Definition 3.1. Let $\alpha \in (1,2), \beta \in (0,\alpha), f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous. A function $\mu \in C[0,1]$ is called a mild solution of the boundary value problem (1.1) if μ satisfies the integral equation

$$\mu(t) = \int_0^1 G(t,s)f(s,\mu(s))ds - a\int_0^1 G^*(t,s)\mu(s)ds,$$

where $G(t,s)$ and $G^*(t,s)$ are defined in (3.3) and (3.4).

Remark 3.1. The function $G(t,s)$ satisfies $G(t,s) > 0$ for all $t,s \in (0,1)$ if we take $\kappa(t) = t$ (see [3]). However, the function $G^*(t,s)$ is not of constant sign.

Remark 3.2. By a direct calculation we get that $\int_0^1 G(t,s)ds = \frac{(k(t))^\alpha}{\Gamma(\alpha+1)} \le \frac{1}{\Gamma(\alpha+1)}$ and $\int_0^1 G^*(t,s)ds = \frac{(k(t))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \le \frac{1}{\Gamma(\alpha-\beta+1)}$. These results will be useful in the proofs and calculations of the subsequent theorems.

In the following we investigate the existence and uniqueness results for the mild solutions to the boundary value problem (1.1). For convenience, we list the assumptions.

(B1) There is a positive constant L such that $|f(t,\mu_1) - f(t,\mu_2)| \le L|\mu_1 - \mu_2|$, for any $t \in [0,1]$ and $\mu_1, \mu_2 \in \mathbb{R}$.

(H1) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies the Carathedory conditioni, i.e., $f(\cdot, s)$ is measurable for every $s \in \mathbb{R}$, and $f(t,\cdot)$ is continuous for almost every $t \in [0,1]$.

(H2) There exist $h \in L^P([0,1];\mathbb{R}^+)$ with $p > \frac{1}{\alpha-\beta}$, $\Omega : [0,+\infty) \to [0,+\infty)$ which is nondecreasing locally bounded, such that for every $t \in [0,1]$, $x \in \mathbb{R}$ and $\kappa \in \mathcal{K}_\kappa$,

$$|f(t,x)| \le h(\kappa(t))\Omega(|\ x\ |).$$

(H3) There exist positive functions $a_1, a_2 \in C[0,1]$ such that

$$|f(t,\mu(t))| \le |a_1(t)| + |a_2(t)||\mu(t)|, \quad \forall t \in [0,1].$$

Based on Banach fixed point theorem and assumption (B1), we investigate the existence of a unique solution for the boundary value problem (1.1) in the theorem below.

Theorem 3.1. *Suppose that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ satisfies assumption (B1) and the inequality*

$$L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds < 1$$

holds. Then problem (1.1) has a unique mild solution on $C([0,1],\mathbb{R})$.

Proof. Let $U_0 = \{\mu|\mu \in C([0,1],\mathbb{R}), \mu(0) = 0)\}$. Then U_0 is convex and closed. For $\mu \in U_0$, we employ the following supremum norm

$$\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|.$$

Then U_0 is a complete subspace in $C([0,1]$ with respect to the given norm. Define an operator $\mathbb{K} : U_0 \to U_0$ as

$$(\mathbb{K}\mu)(t) = \int_0^1 G(t,s)f(s,\mu(s))ds - a\int_0^1 G^*(t,s)\mu(s)ds \tag{3.5}$$

for $\mu \in U_0$ and $t \in [0,1]$, where $G(t,s)$ and $G^*(t,s)$ are defined in (3.3) and (3.4). We now prove that $\mathbb{K}$ has a unique fixed point by Banach's fixed point theorem. Take μ_1, $\mu_2 \in U_0$ arbitrary. According to (B1), we find

$$\begin{aligned}
&\|(\mathbb{K}\mu_2)(t) - (\mathbb{K}\mu_1)(t)\| \\
=&\int_0^1 G(t,s)\left(f(s,\mu_2(s)) - f(s,\mu_1(s))\right)ds - a\int_0^1 G^*(t,s)\left(\mu_2(s)-\mu_1(s)\right)ds \\
\le&\int_0^1 G(t,s)f\left(s,\mu_2(s) - f(s,\mu_1(s)\right)ds + a\int_0^1 G^*(t,s)\left|\mu_2(s)-\mu_1(s)\right|ds
\end{aligned}$$

$$\leq L \int_0^1 G(t,s)\left|\mu_2(s)-\mu_1(s)\right| ds + a\int_0^1 G^*(t,s)\left|\mu_2(s)-\mu_1(s)\right| ds$$
$$\leq \|\mu_2-\mu_1\| \left(L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds \right)$$

for any $t \in [0,1]$. It follows that

$$\|\mathbb{K}\mu_2 - \mathbb{K}\mu_1\| \leq \left(L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds \right) \|\mu_2-\mu_1\| .$$

Since $L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds < 1$, $\mathbb{K}$ is a contractive operator. By Banach fixed point theorem, $\mathbb{K}$ has a unique fixed point, which is the unique solution to boundary value problem (1.1). This completes the proof. □

In the theorems below, Leray-Schauder nonlinear alternative theorem is employed to verify the existence of mild solutions to the boundary value problem (1.1).

Theorem 3.2. *Suppose that the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ satisfies the assumptions (H1) and (H2), and*

$$\frac{k_1\|h\|_p}{1-|a|k_2}\limsup_{r\to+\infty}\frac{\Omega(r)}{r} < 1,$$

where $k_1 = \sup\limits_{t\in[0,1]} \left(\int_0^t G(t,s)^q ds\right)^{\frac{1}{q}}$, $k_2 = \sup\limits_{t\in[0,1]} \left(\int_0^1 |G^*(t,s)|^q\, ds\right)^{\frac{1}{q}}$, $p > \frac{1}{\alpha-\beta}$, $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$. *Then, problem (1.1) has at least a solution.*

Proof. We also consider the operator $\mathbb{K}$ defined in (3.5). To utilize the Schauder fixed point theorem, we prove the continuity and complete continuity of the operator $\mathbb{K}$ in the following steps.

Step 1. $\mathbb{K}$ is continuous.

By (H1), (H2), and Lebesgue's dominate convergence theorem, it is easy to prove that $\mathbb{K}$ is continuous.

Step 2. $\mathbb{K}$ maps bounded sets into bounded sets in U_0.

Let $U_1 = \{\mu \in U_0 : \|u\| \leq r\}$ for $r > 0$. Then for any $\mu \in U_1$ and $t \in [0,1]$, we have

$$\begin{aligned}\|\mathbb{K}\mu(t)\| &\leq \int_0^1 G(t,s)\|f(s,\mu(s)||ds + |a|\int_0^1 |G^*(t,s)|\|\mu(s)\|ds \\ &\leq \int_0^1 G(t,s)h(\kappa(s))\Omega(\|\mu(s)\|)ds + |a|r\int_0^1 \left|G^*(t,s)\right| ds \\ &\leq \Omega(r)\int_0^1 G(t,s)h(\kappa(s))ds + |a|r\int_0^1 \left|G^*(t,s)\right| ds.\end{aligned}$$

It follows from Hölder inequality that

$$\begin{aligned}\|\mathbb{K}\mu(t)\| \leq \Omega(r) &\left(\int_0^1 G(t,s)^q ds\right)^{\frac{1}{q}} \left(\int_0^1 h(\kappa(s))^p ds\right)^{\frac{1}{p}} + |a|r\int_0^1 |G^*(t,s)|ds \\ &\leq \Omega(r)k_1\|h\|_p + |a|rk_2.\end{aligned}$$

Hence $\|\mathbb{K}\mu\| \leq \Omega(r)k_1\|h\|_p + |a|rk_2$. This indicates that $\mathbb{K}$ map bounded subsets into bounded subsets within U_0.

Step 3. $\mathbb{K}$ maps bounded subsets into equicontinuous subsets in U_0.

Let $U_1 = \{\mu \in U_0 : \|\mu\| \leq r\}$ for some $r > 0$, and let $\mu \in U_1$ be arbitrary. Then for any $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
&\|\mathbb{K}\mu(t_2) - \mathbb{K}\mu(t_1)\| \\
=&\left\|\int_0^1 (G(t_2,s) - G(t_1,s))f(s,\mu(s))ds - a\int_0^1 (G^*(t_2,s) - G^*(t_1,s))\mu(s)ds\right\| \\
\leq&\left\|\int_0^1 (G(t_2,s) - G(t_1,s))f(s,\mu(s))ds\right\| + |a|\left\|\int_0^1 (G^*(t_2,s) - G^*(t_1,s))\mu(s)ds\right\| \\
\leq&\frac{1}{\Gamma(\alpha)}\left\|\int_0^1 \kappa(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right\| \\
&+\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right. \\
&\left.-\int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right\| \\
&+\frac{|a|}{\Gamma(\alpha-\beta)}\left\|\int_0^1 \kappa'(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right\| \\
&+\frac{|a|}{\Gamma(\alpha-\beta)}\left\|\int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right. \\
&\left.-\int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right\|.
\end{aligned}
$$

We take

$$
\begin{aligned}
I_1 =&\frac{1}{\Gamma(\alpha)}\left\|\int_0^1 \kappa(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right\|, \\
I_2 =&\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right. \\
&\left.-\int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-1}f(s,\mu(s))ds\right\|, \\
I_3 =&\frac{|a|}{\Gamma(\alpha-\beta)}\left\|\int_0^1 \kappa'(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right\|, \\
I_4 =&\frac{|a|}{\Gamma(\alpha-\beta)}\left\|\int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right. \\
&\left.-\int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-\beta-1}\mu(s)ds\right\|.
\end{aligned}
$$

Then $\|\mathbb{K}\mu(t_2) - \mathbb{K}\mu(t_1)\| \leq I_1 + I_2 + I_3 + I_4$. According to assumption (H2) and Hölder inequality, we have

$$
I_1 \leq \frac{|m(t_2) - m(t_1)|}{\Gamma(\alpha)}\int_0^1 \kappa'(s)(\kappa(1) - \kappa(s))^{\alpha-1}h(\kappa(s))\Omega(r)ds
$$

$$
\begin{aligned}
\leq & \frac{|m(t_2)-m(t_1)|\Omega(r)}{\Gamma(\alpha)}\left(\int_0^1(\kappa(1)-\kappa(s))^{(\alpha-1)q}d\kappa(s)\right)^{\frac{1}{q}}\left(\int_0^1 h^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\\
= & \frac{|m(t_2)-m(t_1)|\Omega(r)}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|h\|_p.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 \leq & \frac{\Omega(r)}{\Gamma(\alpha)}\left(\int_0^{t_1}\kappa(s)\left[(\kappa(t_2)-\kappa(s))^{\alpha-1}-(\kappa(t_1)-\kappa(s))^{\alpha-1}\right]h(\kappa(s))ds\right.\\
& \left.+\int_{t_1}^{t_2}\kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-1}h(\kappa(s))ds\right)\\
\leq & \frac{\Omega(r)}{\Gamma(\alpha)}\left[\left(\int_0^{t_1}\left[(\kappa(t_2)-\kappa(s))^{\alpha-1}-(\kappa(t_1)-\kappa(s))^{\alpha-1}\right]^q d\kappa(s)\right)^{\frac{1}{q}}\right.\\
& \times\left(\int_0^{t_1}h^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\\
& \left.+\left(\int_{t_1}^{t_2}(\kappa(t_2)-\kappa(s))^{(\alpha-1)q}d\kappa(s)\right)^{\frac{1}{q}}\left(\int_{t_1}^{t_2}h^p(\kappa(s))d\kappa(s)\right)^{\frac{1}{p}}\right]\\
\leq & \frac{\Omega(r)\|h\|_p}{\Gamma(\alpha)}\left[\left(\int_0^{t_1}\left[\kappa(t_2)-\kappa(s))^{\alpha-1}-(\kappa(t_1)-\kappa(s))^{\alpha-1}\right]^q d\kappa(s)\right)^{\frac{1}{q}}\right.\\
& \left.+\left(\int_{t_1}^{t}(\kappa(t_2)-\kappa(s))^{(\alpha-1)q}d\kappa(s)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

For I_3, we have

$$
\begin{aligned}
I_3 \leq & \frac{|a||m(t_2)-m(t_1)|}{\Gamma(\alpha-\beta)}\left\|\int_0^1\kappa'(s)(\kappa(1)-\kappa(s))^{\alpha-\beta-1}\mu(s)ds\right\|\\
\leq & \frac{|a|r}{\Gamma(\alpha-\beta+1)}|m(t_2)-m(t_1)|.
\end{aligned}
$$

As for I_4, since the integral is absolutely continuous when $\alpha>1$ and $\alpha>\beta$, it is easy to find

$$
\begin{aligned}
I_4 \leq & \frac{|a|r}{\Gamma(\alpha-\beta)}\left\|\int_0^{t_2}\kappa'(s)(\kappa(t_2)-\kappa(s))^{\alpha-\beta-1}ds-\int_0^{t_1}\kappa'(s)(\kappa(t_1)-\kappa(s))^{\alpha-\beta-1}ds\right\|\\
\leq & \frac{|a|r}{\Gamma(\alpha-\beta+1)}|(\kappa(t_2))^{\alpha-\beta}-(\kappa(t_1))^{\alpha-\beta}|.
\end{aligned}
$$

So we have $I_1+I_2+I_3+I_4\to 0$ as $t_2-t_1\to 0$. Hence $\|\mathbb{K}\mu(t_2)-\mathbb{K}\mu(t_1)\|\to 0$. The arbitrary choice of $\mu\in U_1$ shows that the operator $\mathbb{K}$ maps bounded sets into equicontinuous sets of U_0.

Due to Steps 1-3 and by Arzela-Ascoli theorem, the operator $\mathbb{K}$ is a completely continuous mapping.

Step 4. There exists an open set $U\subset U_0$ with $\mu\neq\lambda\mathbb{K}(\mu)$ for $\lambda\in(0,1)$ and $\mu\in\partial U$.

According to the condition

$$
\frac{k_1\|h\|_p}{1-|a|k_2}\limsup_{r\to+\infty}\frac{\Omega(r)}{r}<1,
$$

we can deduce that there exists a constant $N > 0$ such that

$$k_1\|h\|_p\Omega(r) + |a|k_2N < N.$$

Let $U = \{\mu \in U_0 : \|u\| \leq N\}$. Then $\mathbb{K} : \overline{U} \to U_0$ is completely continuous. Suppose that there exist $\lambda \in (0,1)$ and $\mu \in \partial U$ such that $\mu = \lambda\mathbb{K}(\mu)$. Then for any $t \in [0,1]$,

$$\begin{aligned}
|\mu(t)| &= |\lambda\mathbb{K}\mu(t)| \\
&\leq \|\mathbb{K}\mu(t)\| \\
&\leq \int_0^1 G(t,s)\|f(s,\mu(s))\|ds + |a|\int_0^1 |G^*(t,s)|\|\mu(s)\|ds \\
&\leq \int_0^1 G(t,s)h(\kappa(s))\Omega(\|\mu(s)\|)ds + |a|N\int_0^1 |G^*(t,s)|\, ds \\
&\leq \Omega(r)\int_0^1 G(t,s)h(\kappa(s))ds + |a|N\int_0^1 |G^*(t,s)|ds \\
&< N.
\end{aligned}$$

Hence $\|\mu\| < N$ holds, which contradicts the fact that $\|\mu\| = N$. Thus we get $\mu \neq \lambda\mathbb{K}\mu$ for any $\mu \in \partial U$ and $\lambda \in (0,1)$. By the Leray-Schauder alternative theorem, we infer that there exists at least one fixed point μ of $\mathbb{K}$. The fixed point is a solution to the boundary value problem (1.1). This completes the proof. □

Theorem 3.3. *Suppose that the function f satisfies conditions (H1) and (H3). Further suppose that*

$$\begin{aligned}
N_1 &= \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)|a_1(s)|ds\right) > 0, \\
N_2 &= \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)|a_2(s)| + |aG^*(t,s)|\, ds\right) \in (0,1).
\end{aligned}$$

Then problem (1.1) has at least one solution.

Proof. We still consider the operator $\mathbb{K}$ in (3.5). The conclusion can be verified analogously in the following four steps as well.

Step 1. $\mathbb{K}$ is continuous.

Let $\{\mu_n\} \subset U_0$ be a sequence such that $\mu_n \to \mu$ as $n \to \infty$. By assumption (H3) and the continuity of f we have

$$\begin{aligned}
\lim_{n\to\infty}|\mathbb{K}\mu_n(t) - \mathbb{K}\mu(t)| &\leq \lim_{n\to\infty}\int_0^1 G(t,s)|f(s,\mu_n(s)) - f(s,\mu(s))|ds \\
&\quad + |a|\lim_{n\to\infty}\int_0^1 |G^*(t,s)|\,|\mu_n(s) - \mu(s)|ds \\
&\leq \int_0^1 G(t,s)\lim_{n\to\infty}|f(s,\mu_i(s)) - f(s,\mu(s))|ds \\
&\quad + |a|\int_0^1 |G^*(t,s)|\lim_{n\to\infty}|\mu_n(s) - \mu(s)|ds \\
&=0.
\end{aligned}$$

Hence $\mathbb{K}$ is continuous.

Step 2. $\mathbb{K}$ maps bounded subsets into bounded subsets in U_0.

Let $U_2 = \{\mu \in U_0 : \|\mu\| \leq r\}$ for some $r > 0$. Then for every $\mu \in U_2$, $\|\mu\| \leq r$. So we have

$$\begin{aligned}|\mathbb{K}\mu(t)| \leq& \int_0^1 G(t,s)|f(s,\mu(s))|ds + |a| \int_0^1 |G^*(t,s)|\,|\mu(s)|ds \\ \leq& \int_0^1 G(t,s)|a_1(s)|ds + \int_0^1 G(t,s)|a_2(s)||\mu(s)|ds + |aG^*(t,s)||\mu(s)|ds \\ \leq& N_1 + N_2 r.\end{aligned}$$

Therefore, $\mathbb{K}$ maps bounded subsets into bounded subsets.

Step 3. $\mathbb{K}$ maps bounded subsets into equicontinuous subsets of U_0.

Let $\mu \in U_2 = \{\mu \in U_0 : \|\mu\| \leq r\}$ and suppose that $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Then

$$\begin{aligned}&\|\mathbb{K}\mu(t_1) - \mathbb{K}\mu(t_2)\| \\ \leq& \frac{1}{\Gamma(\alpha)} \left\| \int_0^1 \kappa'(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-1} f(s,\mu(s))ds \right\| \\ &+ \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-1} f(s,\mu(s))ds \right. \\ &\left. - \int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-1} f(s,\mu(s))ds \right\| \\ &+ \frac{|a|}{\Gamma(\alpha-\beta)} \left\| \int_0^1 \kappa'(s)(m(t_2) - m(t_1))(\kappa(1) - \kappa(s))^{\alpha-\beta-1} \mu(s)ds \right\| \\ &+ \frac{|a|}{\Gamma(\alpha-\beta)} \left\| \int_0^{t_2} \kappa'(s)(\kappa(t_2) - \kappa(s))^{\alpha-\beta-1} \mu(s)ds \right. \\ &\left. - \int_0^{t_1} \kappa'(s)(\kappa(t_1) - \kappa(s))^{\alpha-\beta-1} \mu(s)ds \right\| \\ =& I_5 + I_6 + I_7 + I_8,\end{aligned}$$

$$\begin{aligned}I_5 \leq& \frac{|m(t_2) - m(t_1)|}{\Gamma(\alpha)} \left\| \int_0^1 (\kappa(1) - \kappa(s))^{\alpha-1} \left[a_1(\kappa(s)) + a_2(\kappa(s))\, \|\mu(s)\|\right] d\kappa(s) \right\| \\ \leq& \frac{|m(t_2) - m(t_1)|}{\Gamma(\alpha)} \\ &\times \left\| \left(\int_0^1 (\kappa(1) - \kappa(s))^{(\alpha-1)q} d\kappa(s) \right)^{\frac{1}{q}} \left(\int_0^1 [a_1(\kappa(s)) + a_2(\kappa(s) r)]^{\frac{1}{p}}\, d\kappa(s) \right)^{\frac{1}{p}} \right\| \\ \leq& \frac{|m(t_2) - m(t_1)|\, (\|a_1\|_p + r\|a_2\|_p)}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}},\end{aligned}$$

$$\begin{aligned}I_6 \leq& \frac{\|a_1\|_p + r\,\|a_2\|_p}{\Gamma(\alpha)} \left[\left(\int_0^{t_2} (\kappa(t_2) - \kappa(s))^{(\alpha-1)q} d\kappa(s) \right)^{\frac{1}{q}} \right. \\ &\left. - \left(\int_0^{t_1} (\kappa(t_1) - \kappa(s))^{(\alpha-1)q} d\kappa(s) \right)^{\frac{1}{p}} \right]\end{aligned}$$

$$\leq \frac{\|a_1\|_p + r\,\|a_2\|_p}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left[(\kappa(t_2))^{\frac{(\alpha-1)q+1}{q}} - (\kappa(t_1))^{\frac{(\alpha-1)q+1)}{q}} \right],$$

$$I_7 + I_8 \leq \frac{|a|r}{\Gamma(\alpha-\beta+1)} \left(|m(t_2) - m(t_1)| + |(\kappa(t_2))^{\alpha-\beta} - (\kappa(t_1))^{\alpha-\beta}| \right).$$

Since $\alpha > 1$ and $\alpha > \beta$, the integral is absolutely continuous. Hence $I_5 + I_6 + I_7 + I_8 \to 0$, and thus $\|\mathbb{K}\mu(t_2) - \mathbb{K}\mu(t_1)\| \to 0$ as $t_2 - t_1 \to 0$, and the convergence is independent on $\mu \in U_2$. Therefore the operator $\mathbb{K}$ maps bounded subsets into equicontinuous subsets in U_0.

As a consequence of Steps 1-3 and by Arzela-Ascoli theorem, the operator $\mathbb{K}$ is a completely continuous mapping.

Step 4. There exists an open set $U \subset U_0$ with $\mu \neq \lambda\mathbb{K}(\mu)$ for $\lambda \in (0,1)$ and $\mu \in \partial U$.

According to the assumption

$$0 < N_1 < +\infty,\ 0 < N_2 < 1,$$

we can deduce that there exists a constant $N > 0$ such that

$$N_1 + N_2 N < N.$$

Let $U = \{\mu \in U_0 : \|u\| \leq N\}$. Then $\mathbb{K} : \overline{U} \to U_0$ is completely continuous. Suppose that there exist $\lambda \in (0,1)$ and $\mu \in \partial U$ such that $\mu = \lambda\mathbb{K}(\mu)$. Then, for any $t \in [0,1]$,

$$\begin{aligned} |\mu(t)| =& |\lambda\mathbb{K}\mu(t)| \\ \leq& \|\mathbb{K}\mu(t)\| \\ \leq& \int_0^1 G(t,s)|a_1(s)|ds + \int_0^1 G(t,s)|a_2(s)||\mu(s)|ds + |aG^*(t,s)||\mu(s)|ds \\ <& N. \end{aligned}$$

Hence $\|\mu\| < N$ holds, which contradicts $\|\mu\| = N$. Thus, we get $\mu \neq \lambda\mathbb{K}\mu$ for any $\mu \in U$ and λ. By the Leray-Schauder alternative theorem, we infer that there exists at least one fixed point μ of $\mathbb{K}$. The fixed point is a solution to the boundary value problem (1.1). This completes the proof. □

Remark 3.3. Consistent with findings from prior research (refer to [1] [21]), the benefit of employing the fractional differential operator in equation (1.1) lies in the flexibility to select the appropriate function κ. Observing the problem (1.1), we note that

1. Consider $\kappa(t) = t$. Then problem (1.1) becomes the problem studied in [10] with the Caputo fractional derivative. The problem is given by

$$\begin{cases} T\mu(t) + f(t,\mu(t)) = 0, & 0 < t < 1, \\ \mu(0) = 0, \mu(1) = 0, \end{cases}$$

where $T = D^{\alpha} - aD^{\beta}$. Therefore, we can get the existence results as in [10] for Caputo fractional derivatives.

2. Let $\kappa(t) = \log_2(t+1)$. Applying the logarithmic bottom exchange formula yields $\kappa(t) = \log(t+1)/\log(2)$. Problem (1.1) can be transformed into a boundary value problem involving the Caputo-Hadamard fractional derivative, which is an Hadamard -type fractional derivative (as defined in Definition 2.3 and 2.4). Thus, problem (1.1) can be transformed into problem (3.6).

$$\begin{cases} \left({}^{CH}D^{\alpha} - a^{CH}D^{\beta}\right)\mu(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ \mu(0) = 0, \mu(1) = 0. \end{cases} \tag{3.6}$$

To the best of our knowledge, this fractional system has not yet been investigated. However, some corollaries can be inferred from the results we have obtained.

Corollary 3.1. *Suppose that assumption (B1) is satisfied and*

$$L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds < 1,$$

then, the boundary value problem (3.6) has a unique mild solution in $C^1([0,1],\mathbb{R})$.

Corollary 3.2. *Suppose that assumptions (H1) and (H2) are satisfied and*

$$\frac{k_1\|h\|_p}{1-|a|k_2}\limsup_{r\to+\infty}\frac{\Omega(r)}{r} < 1,$$

where $k_1 = \sup\limits_{t\in[0,1]}\left(\int_0^t G(t,s)^q ds\right)^{\frac{1}{q}}$, $k_2 = \sup\limits_{t\in[0,1]}\left(\int_0^1 |G^*(t,s)|^q\, ds\right)^{\frac{1}{q}}$, $p > \frac{1}{\alpha-\beta}$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. *Then the boundary value problem (3.6) has at least one mild solution in* $C^1([0,1],\mathbb{R})$.

Corollary 3.3. *Suppose that assumptions (H1) and (H3) hold, and*

$$N_1 = \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)|a_1(s)|ds\right) > 0,$$
$$N_2 = \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)|a_2(s)| + |aG^*(t,s)|\, ds\right) \in (0,1).$$

Then there exists at least a mild solution of the boundary value problem (3.6).

4. Hyers-Ulam stability analysis

This section presents the analysis of Hyers-Ulam stability of the fractional differential equation (1.1).

Theorem 4.1. *Suppose that the assumptions of Theorem 3.1 are satisfied and* $\frac{L}{\Gamma(\alpha+1)} + \frac{|a|}{\Gamma(\alpha-\beta+1)} < 1$. *Then the boundary value problem (1.1) is Hyers-Ulam stable.*

Proof. For each given $\varepsilon > 0$ and the function μ satisfing the inequality

$$\left|{}^{c}D^{\alpha;\kappa}\mu(t) - aD^{\beta;\kappa}\mu(t) + f(t,\mu(t))\right| \le \varepsilon, t\in[0,1],$$

we set a function

$$g(t) = {}^cD^{\alpha;\kappa}\mu(t) - a^cD^{\alpha;\kappa}\mu(t) + f(t,\mu(t)).$$

Then we have $|g(t)| \leq \varepsilon$, which implies

$$\begin{aligned} &J^{\alpha;\kappa}g(t) \\ =&\mu(t) - C_1\kappa(t) - aJ^{\alpha-\beta;k}\mu(t) + aC_1\frac{\Gamma(2)}{\Gamma(\alpha-\beta+2)}\kappa(t)^{\alpha-\beta+1} + J^{\alpha;\kappa}f(t,\mu(t)), \end{aligned}$$

where C_1 are defined as in Lemma 3.1. Let $\theta(t) = C_1\frac{\Gamma(2)}{\Gamma(\alpha-\beta+2)}(\kappa(t))^{\alpha-\beta+1} - C_1\kappa(t)$, then

$$\mu(t) = \theta(t) + J^{\alpha;\kappa}g(t) + aJ^{\alpha-\beta;\kappa}\mu(t) - J^{\alpha;\kappa}f(t,\mu(t)).$$

According to Theorem 3.1, it has been verified that there is a unique solution $v(t)$ of problem (1.1). The function v can be expressed as

$$v(t) = \theta(t) + aJ^{\alpha-\beta;\kappa}\mu(t) - J^{\alpha;\kappa}f(t,\mu(t)).$$

We can get the inequality

$$\begin{aligned} |\mu(t) - v(t)| \leq&\frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}|f(s,\mu(s)) - f(s,v(s))|ds \\ &+ \frac{|a|}{\Gamma(\alpha-\beta)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-\beta-1}|\mu(s)-v(s)|ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t \kappa'(s)(k(t)-k(s))^{\alpha-1}|g(s)|ds \\ \leq&\frac{L}{\Gamma(\alpha)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}|\mu(s)-v(s)|ds \\ &+ \frac{|a|}{\Gamma(\alpha-\beta)}\int_0^t \kappa'(s)(\kappa(t)-\kappa(s))^{\alpha-1}|\mu(s)-v(s)|ds + \varepsilon \\ \leq&\|\mu-v\|\left(\frac{L(\kappa(t))^\alpha}{\Gamma(\alpha+1)} + \frac{|a|(\kappa(t))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) + \varepsilon \\ \leq&\|\mu-v\|\left(\frac{L}{\Gamma(\alpha+1)} + \frac{|a|}{\Gamma(\alpha-\beta+1)}\right) + \varepsilon. \end{aligned}$$

After taking the supremum norm on both sides of the equation, we obtain the following result

$$\|\mu-v\| \leq \|\mu-v\|\left(\frac{L}{\Gamma(\alpha+1)} + \frac{|a|}{\Gamma(\alpha-\beta+1)}\right) + \varepsilon.$$

Therefore,

$$\|\mu-v\| \leq \frac{\varepsilon}{\left(1 - \frac{L}{\Gamma(\alpha+1)} - \frac{|a|}{\Gamma(\alpha-\beta+1)}\right)} = C_f\varepsilon,$$

where $C_f = \frac{1}{1-\frac{L}{\Gamma(\alpha+1)}-\frac{|a|}{\Gamma(\alpha-\beta+1)}}$, which implies that Hyers-Ulam stability of problem (1.1) is proved. □

5. Example

Two examples are presented in this section to illustrate the conclusions. The assertion of Theorem 3.1, 3.3 and 4.1 are shown by Example 5.1 below.

Example 5.1. Consider the following nonlinear κ-Caputo-type fractional differential equation

$$\begin{cases} D^{1.8;\kappa}\mu(t) - \frac{1}{2}D^{1.4,\kappa}\mu(t) = \frac{e^{-t}}{10}\left(\mu(t) + \frac{1}{2}\cos t\right), \ t \in [0,1], \\ \mu(0) = 0, \mu(1) = 0. \end{cases} \tag{5.1}$$

According to the given data, it can be found that assumption (B1) is satisfied with $L \leq \frac{e^{-t}}{10} \leq \frac{1}{10}$. So we can take $L = \frac{1}{10}$, then

$$L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds \leq \frac{1}{10\Gamma(2.8)} + \frac{1}{2\Gamma(1.4)} \approx 0.41 < 1.$$

Furthermore, we have

$$\begin{aligned} |f(t,\mu)| &= \left|\frac{e^{-t}}{10}\left(\mu(t) + \frac{1}{2}\cos t\right)\right| \\ &\leq |\frac{e^{-t}}{20}\cos(t)| + |\frac{e^{-t}}{10}||\mu(t)| \\ &= |a_1(t)| + |a_2(t)||\mu(t)|, \\ N_1 &= \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)\,|a_1(s)|\,ds\right) > 0, \\ N_2 &= \sup_{t\in[0,1]}\left(\int_0^1 G(t,s)\,|a_2(s)| + |aG^*(t,s)|\,ds\right) \\ &\leq L\int_0^1 G(t,s)ds + a\int_0^1 G^*(t,s)ds \\ &< 1, \end{aligned}$$

where $a_1(t) = \frac{e^{-t}}{20}\cos(t)$, $a_2(t) = \frac{e^{-t}}{10}$. Therefore, all the assumptions (B1), (H1) and (H3) hold. Thus it follows from Theorem 3.1 and 3.3 that the problem (5.1) has a mild solution.

We further discuss the Hyers-Ulam stability of the problem (5.1). For each given $\varepsilon > 0$ and function μ satisfing the inequality

$$|D^{1.8;\kappa}\mu(t) - \frac{1}{2}D^{1.4,\kappa}\mu(t) - \frac{e^{-t}}{10}\left(\mu(t) + \frac{1}{2}\cos t\right)| \leq \varepsilon, \quad t \in [0,1].$$

Now let $v(t)$ be the unique solution of the problem (5.1). Then by Theorem 4.1 we have

$$\|\mu - v\| \leq \frac{\varepsilon}{\left(1 - \frac{L}{\Gamma(\alpha+1)} - \frac{|a|}{\Gamma(\alpha-\beta+1)}\right)} = C_f\varepsilon.$$

A deract calculation shows that $C_f = \frac{1}{1-\frac{0.1}{\Gamma(1.8+1)}-\frac{|0.5|}{\Gamma(1.8-1.4+1)}} \approx 2.5$, which implies the Hyers-Ulam stability of problem (5.1).

Example 5.2. We now investigate the following nonlinear κ-Caputo-type fractional differential equation

$$\begin{cases} D^{1.7}\mu(t) - \frac{1}{3}D^{1.5}\mu(t) = \frac{\widetilde{c}e^t\mu(t)}{(e^t+e^{-t})(1+\mu(t))}, & t \in [0,1], \\ \mu(0) = 0, \mu(1) = 0, \end{cases} \tag{5.2}$$

where $\widetilde{c}$ is a given positive constant. Set

$$f(t,\mu) = \frac{\widetilde{c}e^t\mu}{(e^t+e^{-t})(1+\mu)}, \quad (t,\mu) \in [0,1] \times \mathbb{R}.$$

For any μ_1, μ_2, we have

$$\begin{aligned} |f(t,\mu_1) - f(t,\mu_2)| &= \frac{e^t\widetilde{c}}{(e^t+e^{-t})}\left|\frac{\mu_1}{1+\mu_1} - \frac{\mu_2}{1+\mu_2}\right| \\ &\leq \frac{e^t|\mu_1-\mu_2|\widetilde{c}}{(e^t+e^{-t})\,|(1+\mu_1)(1+\mu_2)|} \\ &\leq \frac{e^t|\mu_1-\mu_2|\widetilde{c}}{(e^t+e^{-t})} \\ &\leq \widetilde{c}|\mu_1-\mu_2|. \end{aligned}$$

Assuming $\widetilde{c} < \left(1 - \frac{1}{3\Gamma(2.2)}\right)\Gamma(2.7) \approx 1.09$, we have $\widetilde{c}\int_0^1 G(t,s)ds + \frac{1}{3}\int_0^1 G^*(t,s)ds < 1$. Therefore, the conditions of Theorem 3.1 are all satisfied. So the boundary value problem (5.2) has a unique solution.

6. Conclusion

In general, exact solutions to the majority of nonlinear fractional differential equations in boundary value problems are difficult to obtain. However, some mild solutions can be found. With the assistance of the κ-Caputo derivative, we focus on investigating a more general two-term fractional differential equation in the boundary value problem. Drawing upon the properties of the Green function, we provide the form of a mild solution to the problem. We utilize the fixed point theorem to explore the uniqueness and existence results of the mild solution, and verify that these results also apply to some specific cases in our corollaries. Moreover, we verify that the boundary value problem is Hyers-Ulam stable. We present two examples to illustrate our conclusions.

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

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Received November 2024; Accepted February 2025; Available online April 2025.