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SHEAF IMPULSIVE FUZZY CONTROL PROBLEM UNDER THE SECOND TYPE HUKUHARA DERIVATIVE*

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Abstract The concept of sheaf-solution to impulsive fuzzy control differential equations under the second-type Hukuhara derivative is developed in this paper. The continuous dependence of the sheaf-solution of such equations on the initial value is investigated using the Gronwall inequality. In addition, the comparison theorem and the criteria for the stability for the impulsive fuzzy control differential equations are provided.

Keywords Impulsive fuzzy control differential equation, sheaf-solution, second-type Hukuhara derivative, Gronwall inequality, stability.

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1. Introduction

It is well known that numerous problems concerning uncertainty and processing vague or subjective information can be modeled by the fuzzy differential equations in practice.

In 1983, Puri and Ralescu [20] proposed a definition of the Hukuhara derivative (short for H-derivative) of a fuzzy function, which extended the differential of a set-valued function, while some authors investigated the fuzzy differential equations with the H-derivative. For example, Seikkala [21], Balasubramaniam and Muralisankar [1], Song and Wu [23], Ding, Ma and Kandel [5] studied the initial value problem of fuzzy differential equations. Nieto, Rodríguez López, and Rosana investigated the existence and uniqueness of the boundary value problem of fuzzy differential equations in [14]. In addition, Lakshmikantham and Leela [9], Pan [16], and Vatsala [28] analyzed the stability of fuzzy differential equations by utilizing the Lyapunov function method. However, the classical H-derivative defined as in [20] also has its inherent disadvantage, that is, the diameter of the solution is nondecreasing with increasing time, which is inconvenient from the viewpoint of application. Bede and Gal [2, 3] presented strongly generalized differentiability of a fuzzy valued function to overcome this drawback. Some authors obtained a wide range of results in this direction according to the generalized differentiability (see [4, 6, 7, 11–13, 17, 22, 24–27, 29, 30]).

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Recently, Ovsanikov introduced the sheaf-solution notion of the classical control differential equation in reference [15]. Under the framework of this notion, the study of the qualitative and stability theories of each solution can be transformed into the study of a sheaf-solution, that is, a set of solutions. The sheaf-solution problems of fuzzy control differential equations and fuzzy functional differential equations have been recently studied, and some results of existence, uniqueness, and stability were obtained, e.g. [18, 19, 26].

Inspired by the above existing research work, the concept of sheaf-solution is extended to the impulsive fuzzy control differential equations under the second type Hukuhara derivative, and the uncertainty represented by the diameter of the solution is assumed to be monotonically decreasing considering time. Simultaneously, the continuous dependence of the sheaf-solution on the initial values is investigated by using the Gronwall inequality. Finally, the comparison theorems of the sheaf-solutionare provided, and the stability of sheaf impulsive fuzzy control equations is examined.

2. Preliminaries

For the convenience of discussion, some notations, definitions and propositions are briefly introduced in the following .

The notation $K_c(\mathbb{R}^n)$ is used to denote a set, that is the set of the collection of non-empty, compact, and convex subsets of $\mathbb{R}^n$. $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$. Here are the addition and multiplication operations defined on $K_c(\mathbb{R}^n)$:

$$\mathcal{X}+\mathcal{Y}=\{x+y : x\in\mathcal{X},\ y\in\mathcal{Y}\},\quad \mu\mathcal{X}=\{\mu x:\ x\in\mathcal{X}\},$$

where $\mathcal{X},\mathcal{Y}\in K_c(\mathbb{R}^n)$ and $\mu\in\mathbb{R}$.

The Hausdorff metric $\mathbf{D}$ is defined on $K_c(\mathbb{R}^n)$ as follows:

$$\mathbf{D}[\mathcal{X},\mathcal{Y}]=\max\{\sup_{x\in\mathcal{X}}\inf_{y\in\mathcal{Y}}\|x-y\|,\ \sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}}\|y-x\|\}, \tag{2.1}$$

where $\mathcal{X},\mathcal{Y}$ are any two elements in $K_c(\mathbb{R}^n)$. Considering the above metric, the $(K_c(\mathbb{R}^n),\mathbf{D})$ is a complete metric space.

Let $\mathbb{E}^n$ be a set, that is

$\mathbb{E}^n=\{\vartheta:\mathbb{R}^n\to[0,1]\big|\ \vartheta$ satisfies the requirements (i)-(iv)$\}$.

(i) ϑ is normal, that is, there exists a $u_0\in\mathbb{R}^n$ satisfying $\vartheta(u_0)=1$;

(ii) ϑ is fuzzy convex, that is, for any $u,v\in\mathbb{R}^n$ and $0\le\mu\le1$,

$$\vartheta(\mu u+(1-\mu)v)\ge\min\{\vartheta(u),\vartheta(v)\}; \tag{2.2}$$

(iii) ϑ is upper semicontinuous;

(iv) $[\vartheta]^0=\mathrm{cl}\{u\in\mathbb{R}^n:\vartheta(u)>0\}$ is compact, in which cl denote the closure.

The λ-level set is denoted as $[\vartheta]^\lambda=\{u\in\mathbb{R}^n:\vartheta(u)\ge\lambda\}$ for any $\lambda\in(0,1]$. From (i)-(iv), we know that $[\vartheta]^\lambda\in K_c(\mathbb{R}^n)$ for $0\le\lambda\le1$. For later applications, the fuzzy set θ is denoted by

$$\theta(u)=\begin{cases}1, & \text{if } u=\mathbf{0},\\ 0, & \text{if } u\ne\mathbf{0},\end{cases} \tag{2.3}$$

where $\mathbf{0} = (0, 0, \cdots, 0)^T$ is the element of $\mathbb{R}^n$.

For $\vartheta, \varrho \in \mathbb{E}^n$ and $\mu \in \mathbb{R}$, the addition $\vartheta + \varrho$ and multiplication $\mu\vartheta$ are given in terms of λ level sets as follows:

$$[\vartheta + \varrho]^\lambda = [\vartheta]^\lambda + [\varrho]^\lambda, \quad [\mu\vartheta]^\lambda = \mu[\vartheta]^\lambda, \quad \text{for all } \lambda \in [0, 1],$$

and the metric on $\mathbb{E}^n$ is defined as shown below:

$$\mathbf{d}[\vartheta, \varrho] = \sup_{0 \le \lambda \le 1} \mathbf{D}[[\vartheta]^\lambda, [\varrho]^\lambda].$$

Considering the above metric $\mathbf{d}$, $\mathbb{E}^n$ is a complete metric space [10], and the metric $\mathbf{d}[\vartheta, \varrho]$ meets some common properties:
$\mathbf{d}[\vartheta + \omega, \varrho + \omega] = \mathbf{d}[\vartheta, \varrho]$,
$\mathbf{d}[\vartheta, \varrho] = \mathbf{d}[\varrho, \vartheta]$,
$\mathbf{d}[\mu\vartheta, \mu\varrho] = |\mu|\mathbf{d}[\vartheta, \varrho]$,
$\mathbf{d}[\vartheta, \varrho] \le \mathbf{d}[\vartheta, \omega] + \mathbf{d}[\omega, \varrho]$,
$\mathbf{d}[\vartheta + \varrho, \phi + \omega] \le \mathbf{d}[\vartheta, \phi] + \mathbf{d}[\varrho, \omega]$,
for all $\vartheta, \varrho, \omega, \phi \in \mathbb{E}^n$ and $\mu \in \mathbb{R}$.

Let $\vartheta, \varrho \in \mathbb{E}^n$. If $\omega \in \mathbb{E}^n$ is found to satisfy $\vartheta = \varrho + \omega$, then ω is said to be the Hukuhara difference (H-difference) of ϑ, ϱ, and denoted by $\omega = \vartheta \ominus_H \varrho$.

Remark 2.1. If $\vartheta, \varrho, \omega, \phi \in \mathbb{E}^n$, there exist H-difference $\vartheta \ominus_H \varrho$, $\vartheta \ominus_H \phi$ and $\phi \ominus_H \omega$, then $\mathbf{d}[\vartheta \ominus_H \varrho, \theta] = \mathbf{d}[\vartheta, \varrho]$, $\mathbf{d}[\vartheta \ominus_H \varrho, \vartheta \ominus_H \phi] = \mathbf{d}[\varrho, \phi]$, $\mathbf{d}[\vartheta \ominus_H \varrho, \phi \ominus_H \omega] \le \mathbf{d}[\vartheta, \phi] + \mathbf{d}[\varrho, \omega]$.

Definition 2.1 ([4], p114)**.** The mapping $\vartheta : I \to \mathbb{E}^n$ is called to be the second type Hukuhara differentiable (II-differentiable) at $\xi \in I$, if there exists $D_H\vartheta(\xi) \in \mathbb{E}^n$, such that for any $h > 0$ sufficiently small, $\vartheta(\xi) \ominus_H \vartheta(\xi + h)$, $\vartheta(\xi - h) \ominus_H \vartheta(\xi)$ exists and the limits are presented as follows

$$\lim_{h \to 0^+} \frac{\vartheta(\xi) \ominus_H \vartheta(\xi + h)}{-h} = D_H\vartheta(\xi) \text{ and } \lim_{h \to 0^+} \frac{\vartheta(\xi - h) \ominus_H \vartheta(\xi)}{-h} = D_H\vartheta(\xi). \quad (2.4)$$

Remark 2.2. If the numerator of limit (2.4) is replaced with the Hukuhara difference $\vartheta(\xi + h) \ominus_H \vartheta(\xi)$ and $\vartheta(\xi) \ominus_H \vartheta(\xi - h)$, and a limit $D_H\vartheta(\xi) \in \mathbb{E}^n$ exists, then the $\vartheta(\xi)$ is said to be classical Hukuhara differentiable (I-differentiable).

Proposition 2.1. *If ϑ is II-differentiable, then it is continuous.*

Proposition 2.2. *If ϑ, ϱ are II-differentiable at the point $\xi \in [a, b]$ and $\mu, \nu \in \mathbb{R}$, then*

$$D_H(\mu\vartheta + \nu\varrho) = \mu D_H\vartheta + \nu D_H\varrho.$$

Proposition 2.3. *Let $\vartheta \in C[[a, b], \mathbb{E}^n]$. Then $\varrho(\xi) = \int_a^\xi \vartheta(s)ds$ is II-differentiable and $D_H\varrho(\xi) = \vartheta(\xi)$.*

The integral of ϑ over $[a, b]$ is defined levelwise by the equation

$$\begin{aligned}
&\left[\int_a^b \vartheta(\xi)d\xi\right]^\lambda \\
=&\int_a^b [\vartheta(\xi)]^\lambda d\xi \\
=&\left\{\int_a^b \tilde{\vartheta}(\xi)d\xi \,\middle|\, \tilde{\vartheta} : [a, b] \to \mathbb{R}^n \text{ is continuous selection for } [\vartheta(\xi)]^\lambda\right\}.
\end{aligned}$$

Proposition 2.4. *Let ϑ be II-differentiable on $[a, b]$ and suppose that $D_H\vartheta(\xi)$ is integrable on $[a, b]$. Then*

$$\vartheta(a) = \vartheta(\xi) + (-1)\int_a^\xi D_H\vartheta(s)ds,\ \textit{for } \xi \in [a, b].$$

3. Main results

The following impulsive fuzzy control differential equations with Hukuhara derivative of the second type are considered:

$$\begin{cases} D_H\vartheta(\xi) = f(\xi, \vartheta(\xi), c(\xi)),\ \xi \in [0, T],\ \xi \neq \xi_i, \\ \vartheta(\xi_i^+) = \vartheta(\xi_i) + I_i(\xi_i, \vartheta(\xi_i)),\ i = 1, 2, \cdots, m, \\ \vartheta(\xi_0) = \vartheta_0, \end{cases} \tag{3.1}$$

where $\vartheta \in \mathbb{E}^n$ is the state, and $c : [\xi_0, T] \to \mathbb{E}^q$ is the fuzzy control, $0 \leq \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < \xi_m < \xi_{m+1} = T$. The mapping $f : [\xi_0, T] \times \mathbb{E}^n \times \mathbb{E}^q \to \mathbb{E}^n$ is continuous on $(\xi_i, \xi_{i+1}]$, $f(\xi, \theta, \theta) \equiv \theta$ and $I_i(\xi_i, \theta) = \theta$, $I_i : [\xi_0, T] \times \mathbb{E}^n \to \mathbb{E}^n$ are continuous, $i = 1, 2, \cdots, m$.

The function c is called an admissible control, if it is integrable. Suppose that $\Omega \subset \mathbb{E}^q$ is the set of all admissible fuzzy controls.

Lemma 3.1. *Suppose that the fuzzy mapping $\vartheta : [\xi_0, T] \to \mathbb{E}^n$ is II-differentiable. Then, ϑ is a solution of the equations (3.1) on $[\xi_0, T]$ if and only if it is piecewise continuous and fulfils the following integral equations.*

$$\vartheta(\xi) = \begin{cases} \vartheta_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s, \vartheta(s), c(s))ds,\ \xi \in [\xi_0, \xi_1], \\ \vartheta_0 \ominus_H (-1)\Big[\sum_{l=1}^{i}\int_{\xi_{l-1}}^{\xi_l} f(s, \vartheta(s), c(s))ds + \int_{\xi_i}^{\xi} f(s, \vartheta(s), c(s))ds\Big] \\ \quad + \sum_{l=1}^{i} I_l(\xi_l, \vartheta(\xi_l)),\ \xi \in (\xi_i, \xi_{i+1}],\ i = 1, 2, \cdots, m. \end{cases} \tag{3.2}$$

Proof. This conclusion will be proven in two steps.

Step I. If $\vartheta(\xi)$ is the solution of equations (3.1), then it can be written as (3.2).

For $\xi \in [\xi_0, \xi_1]$, suppose that $\vartheta(\xi)$ is a solution of (3.1). According to the second type differentiability of $\vartheta(\xi)$ and continuity of $D_H\vartheta(\xi)$ on $[\xi_0, \xi_1]$, as well as from Propositions 2.3 and 2.4,

$$\vartheta(\xi_0) = \vartheta(\xi) + (-1)\int_{\xi_0}^{\xi} D_H\ \vartheta(s)ds,\ \xi \in [\xi_0, \xi_1],$$

and

$$\vartheta(\xi) = \vartheta_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s, \vartheta(s), c(s))ds,\ \xi \in [\xi_0, \xi_1]. \tag{3.3}$$

For $\xi \in (\xi_1, \xi_2]$, considering (3.3) can obtain the expression as below:

$$\vartheta(\xi) = \vartheta(\xi_1^+) \ominus_H (-1)\int_{\xi_1}^{\xi} f(s, \vartheta(s), c(s))ds$$

$$= (\vartheta(\xi_1) + I_1(\xi_1, \vartheta(\xi_1))) \ominus_H (-1) \int_{\xi_1}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$= \vartheta_0 \ominus_H (-1) \int_{\xi_0}^{\xi_1} f(s, \vartheta(s), c(s))ds \ominus_H (-1) \int_{\xi_1}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$+ I_1(\xi_1, \vartheta(\xi_1)).$$

Mathematical induction reveals

$$\vartheta(\xi_i) = \vartheta_0 \ominus_H (-1) \sum_{l=1}^{i} \int_{\xi_{l-1}}^{\xi_l} f(s, \vartheta(s), c(s))ds + \sum_{l=1}^{i-1} I_l(\xi_l, \vartheta(\xi_l)), \ i = 1, 2, \cdots, m.$$

Furthermore, for $\xi \in (\xi_i, \xi_{i+1}]$, $i = 1, 2, \cdots, m$, the general expression for $\vartheta(\xi)$ is obtained:

$$\vartheta(\xi) = (\vartheta(\xi_i) + I_i(\xi_i, \vartheta(\xi_i))) \ominus_H (-1) \int_{\xi_i}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$= \vartheta_0 \ominus_H (-1) \sum_{l=1}^{i} \int_{\xi_{l-1}}^{\xi_l} f(s, \vartheta(s), c(s))ds \ominus_H (-1) \int_{\xi_i}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$+ \sum_{l=1}^{i} I_l(\xi_l, \vartheta(\xi_l)),$$

Step II. If $\vartheta(\xi)$ satisfies (3.2), then it is the solution of equations (3.1).

For $\xi \in [\xi_0, \xi_1]$, $\vartheta(\xi_0) = \vartheta_0$ can be easily obtained from (3.2), and the Hukuhara difference $\vartheta_0 \ominus_H (-1) \int_{\xi_0}^{\xi} f(s, \vartheta(s), c(s))ds$ exists.

Let $\xi \in [\xi_0, \xi_1)$ and given a positive number h such that $\xi + h \in [\xi_0, \xi_1]$. Then, from the operational properties of H-difference, it follows that:

$$\vartheta(\xi + h) + (-1) \int_{\xi}^{\xi+h} f(s, \vartheta(s), c(s))ds$$
$$= \vartheta_0 \ominus_H (-1) \int_{\xi_0}^{\xi+h} f(s, \vartheta(s), c(s))ds$$
$$+ (-1) \int_{\xi_0}^{\xi+h} f(s, \vartheta(s), c(s))ds \ominus_H (-1) \int_{\xi_0}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$= \vartheta_0 \ominus_H (-1) \int_{\xi_0}^{\xi} f(s, \vartheta(s), c(s))ds$$
$$= \vartheta(\xi).$$

Namely,

$$\vartheta(\xi) \ominus_H \vartheta(\xi + h) = (-1) \int_{\xi}^{\xi+h} f(s, \vartheta(s), c(s))ds. \tag{3.4}$$

Under (3.4), it can be easily get:

$$\lim_{h \to 0^+} \frac{\vartheta(\xi) \ominus_H \vartheta(\xi + h)}{-h} = \lim_{h \to 0^+} \frac{1}{h} \int_{\xi}^{\xi+h} f(s, \vartheta(s), c(s))ds.$$

Furthermore, the following observation is obtained

$$\begin{aligned}
&\mathbf{d}\Big[\frac{1}{h}\int_{\xi}^{\xi+h} f(s,\vartheta(s),c(s))ds, f(\xi,\vartheta(\xi),c(\xi))\Big]\\
&=\mathbf{d}\Big[\frac{1}{h}\int_{\xi}^{\xi+h} f(s,\vartheta(s),c(s))ds, \frac{1}{h}\int_{\xi}^{\xi+h} f(\xi,\vartheta(\xi),c(\xi))ds\Big]\\
&\leq \frac{1}{h}\int_{\xi}^{\xi+h} \mathbf{d}[f(s,\vartheta(s),c(s)), f(\xi,\vartheta(\xi),c(\xi))]ds\\
&\leq \sup_{0\leq s-\xi\leq h} \mathbf{d}[f(s,\vartheta(s),c(s)), f(\xi,\vartheta(\xi),c(\xi))].
\end{aligned} \tag{3.5}$$

According to (3.5) and the continuity of f, the limit

$$\lim_{h\to 0^+}\frac{\vartheta(\xi)\ominus_H \vartheta(\xi+h)}{-h} = f(\xi,\vartheta(\xi),c(\xi)),\ \xi\in[\xi_0,\xi_1). \tag{3.6}$$

As with the derivation of (3.6), the limit can also be obtained:

$$\lim_{h\to 0^+}\frac{\vartheta(\xi-h)\ominus_H \vartheta(\xi)}{-h} = f(\xi,\vartheta(\xi),c(\xi)),\ \xi\in(\xi_0,\xi_1]. \tag{3.7}$$

From (3.6) and (3.7), $\vartheta(\xi)$ is II-differentiable and

$$D_H\vartheta(\xi) = f(\xi,\vartheta(\xi),c(\xi)),\ \xi\in[\xi_0,\xi_1].$$

By mathematical induction, resulting in the expression for $\xi\in(\xi_i,\xi_{i+1}]$ as follows:

$$D_H\vartheta(\xi) = f(\xi,\vartheta(\xi),c(\xi)),\ \xi\in(\xi_i,\xi_{i+1}],\ i=1,2,\cdots,m.$$

Meanwhile, direct computation obtains the relation:

$$\vartheta(\xi_i^+) = \vartheta(\xi_i) + I_i(\xi_i,\vartheta(\xi_i)),\ i=1,2,\cdots,m.$$

□

The subsequent result is given if the mapping ϑ is I-differentiable.

Remark 3.1. Suppose that the fuzzy mapping $\vartheta:[\xi_0,T]\to\mathbb{E}^n$ is I-differentiable. Then, ϑ is a solution of the equations (3.1) on $[\xi_0,T]$ if and only if it is piecewise continuous and fulfils the following integral equations.

$$\vartheta(\xi) = \begin{cases} \vartheta_0 + \displaystyle\int_{\xi_0}^{\xi} f(s,\vartheta(s),c(s))ds,\ \xi\in[\xi_0,\xi_1],\\ \vartheta_0 + \displaystyle\sum_{l=1}^{i}\int_{\xi_{l-1}}^{\xi_l} f(s,\vartheta(s),c(s))ds + \int_{\xi_i}^{\xi} f(s,\vartheta(s),c(s))ds\\ +\displaystyle\sum_{l=1}^{i} I_l(\xi_l,\vartheta(\xi_l)),\ \xi\in(\xi_i,\xi_{i+1}],\ i=1,2,\cdots,m. \end{cases} \tag{3.8}$$

To extend the concept of sheaf-solution, the solution to the impulsive fuzzy control differential equations (3.1) is given by:

$$\vartheta(\xi;\xi_0,\vartheta_0)=\begin{cases}\vartheta(\xi;\xi_0,\vartheta_0),\ \xi\in[\xi_0,\xi_1],\\ \vartheta(\xi;\xi_1,\vartheta(\xi_1^+)),\ \xi\in(\xi_1,\xi_2],\\ \quad\vdots\\ \vartheta(\xi;\xi_i,\vartheta(\xi_i^+)),\ \xi\in(\xi_i,\xi_{i+1}],\\ \quad\vdots\\ \vartheta(\xi;\xi_m,\vartheta(\xi_m^+)),\ \xi\in(\xi_m,\xi_{m+1}],\end{cases}\tag{3.9}$$

where $\vartheta(\xi;\xi_i,\vartheta(\xi_i^+))$ is the solution of the following fuzzy control differential equations with the initial condition

$$\begin{aligned}&D_H\vartheta(\xi)=f(\xi,\vartheta(\xi),c(\xi)),\ \xi\in(\xi_i,\xi_{i+1}],\\ &\vartheta(\xi_i^+)=\vartheta(\xi_i)+I_i(\xi_i,\vartheta(\xi_i)),\ i=1,2,\cdots,m.\end{aligned}\tag{3.10}$$

The concept of sheaf-solution to the equations (3.1) is then provided.

Definition 3.1. The sheaf-solution(or sheaf-trajectory) of (3.1) provides the following set at time ξ:

$$\Xi_{\xi,c}=\begin{cases}\vartheta(\xi;\vartheta_0,c(\xi))-\text{solution of (3.1): }\vartheta_0\in\Xi_0,\ \xi\in[\xi_0,\xi_1],\\ \vartheta(\xi;\vartheta(\xi_1^+),c(\xi))-\text{solution of (3.10): }\vartheta(\xi_1^+)\in\Xi_1,\ \xi\in(\xi_1,\xi_2],\\ \quad\vdots\\ \vartheta(\xi;\vartheta(\xi_i^+),c(\xi))-\text{solution of (3.10): }\vartheta(\xi_i^+)\in\Xi_i,\ \xi\in(\xi_i,\xi_{i+1}],\\ \quad\vdots\\ \vartheta(\xi;\vartheta(\xi_m^+),c(\xi))-\text{solution of (3.10): }\vartheta(\xi_m^+)\in\Xi_m,\ \xi\in(\xi_m,\xi_{m+1}],\end{cases}\tag{3.11}$$

where $c(\xi)\in\Omega\subset\mathbb{E}^q$, $\Xi_0\subset\mathbb{E}^n$,

$$\Xi_i=\{\vartheta(\xi_i^+)=\vartheta(\xi_i;\xi_{i-1},\vartheta(\xi_{i-1}^+))+I_i(\xi_i,\vartheta(\xi_i;\xi_{i-1},\vartheta(\xi_{i-1}^+))):\vartheta(\xi_{i-1}^+)\in\Xi_{i-1}\},$$

$i=1,2,\cdots,m$. In particular, $\vartheta(\xi_0^+)=\vartheta_0\in\Xi_0$.

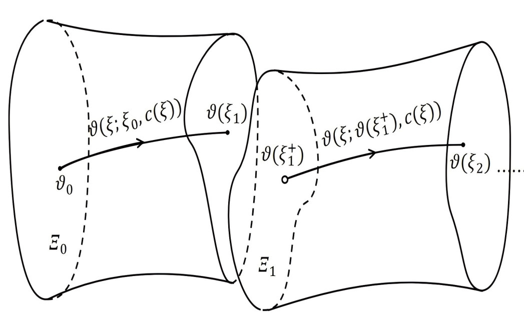

Figure 1. Sheaf-solution $\Xi_{\xi,c}$ of the impulsive fuzzy control differential equations (3.1).

The cross-section of the aforementioned sheaf-solution at (ξ, c)(or $(\xi, c)-$section) is denoted by $\Xi_{\xi,c}$. Two sheaf-solution have different admissible controls $c(\xi)$ and $\bar{c}(\xi)$, and their respective solutions have cross-sections (3.11) and

$$\bar{\Xi}_{\xi,\bar{c}} = \begin{cases} \bar{\vartheta}(\xi; \bar{\vartheta}_0, \bar{c}(\xi)) - \text{solution of (3.1)}: \bar{\vartheta}_0 \in \bar{\Xi}_0,\ \xi \in [\xi_0, \xi_1], \\ \bar{\vartheta}(\xi; \bar{\vartheta}(\xi_1^+), \bar{c}(\xi)) - \text{solution of (3.10)}: \bar{\vartheta}(\xi_1^+) \in \bar{\Xi}_1,\ \xi \in (\xi_1, \xi_2], \\ \quad \vdots \\ \bar{\vartheta}(\xi; \bar{\vartheta}(\xi_i^+), \bar{c}(\xi)) - \text{solution of (3.10)}: \bar{\vartheta}(\xi_i^+) \in \bar{\Xi}_i,\ \xi \in (\xi_i, \xi_{i+1}], \\ \quad \vdots \\ \bar{\vartheta}(\xi; \bar{\vartheta}(\xi_m^+), \bar{c}(\xi)) - \text{solution of (3.10)}: \bar{\vartheta}(\xi_m^+) \in \bar{\Xi}_m,\ \xi \in (\xi_m, \xi_{m+1}], \end{cases}$$

where $\bar{c}(\xi) \in \Omega \subset \mathbb{E}^q$, $\bar{\Xi}_0 \subset \mathbb{E}^n$,

$$\bar{\Xi}_i = \{\bar{\vartheta}(\xi_i^+) = \bar{\vartheta}(\xi_i; \xi_{i-1}, \bar{\vartheta}(\xi_{i-1}^+)) + I_i(\xi_i, \bar{\vartheta}(\xi_i; \xi_{i-1}, \bar{\vartheta}(\xi_{i-1}^+))) : \bar{\vartheta}(\xi_{i-1}^+) \in \bar{\Xi}_{i-1}\},$$

$i = 1, 2, \cdots, m$. In particular, $\bar{\vartheta}(\xi_0^+) = \bar{\vartheta}_0 \in \bar{\Xi}_0$.

Let $\mathfrak{A},\ \mathfrak{B} \subset \mathbb{E}^n$, the notations below are provided:

$\tilde{\mathbf{d}}[\mathfrak{A}, \mathfrak{B}] = \sup\{\mathbf{d}[\mathfrak{a}, \mathfrak{b}] : \mathfrak{a} \in \mathfrak{A},\ \mathfrak{b} \in \mathfrak{B}\}$.

$\mathrm{diam}[\mathfrak{A}] = \sup\{\mathbf{d}[\mathfrak{a}, \bar{\mathfrak{a}}] : \mathfrak{a},\ \bar{\mathfrak{a}} \in \mathfrak{A}\}$, diam denotes the diameter of $\mathfrak{A}$.

The discussion of the properties of solution of equations (3.1) can be transformed into the analysis of a set of solutions by introducing the definition of sheaf-solution. In the following, the continuous dependence of the sheaf-solution on the initial values is investigated.

Theorem 3.1. *Suppose that*
$(A_{3.1})$ $f(\xi, \vartheta(\xi), c(\xi))$ *satisfies the Lipschitz condition, that is,*

$$\mathbf{d}[f(\xi, \vartheta(\xi), c(\xi)), f(\xi, \bar{\vartheta}(\xi), \bar{c}(\xi))] \leq p_i(\xi)(\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] + \mathbf{d}[c(\xi), \bar{c}(\xi)]),$$

where $\xi \in (\xi_i, \xi_{i+1}]$, $\vartheta, \bar{\vartheta} \in \mathbb{E}^n$, $c, \bar{c} \in \Omega$, $p_i(\xi)$ *are positive, bounded, and integral function on* $(\xi_i, \xi_{i+1}]$, $i = 0, 1, 2, \cdots, m$;
$(A_{3.2})$ $\mathbf{d}[I_i(\xi_i, \vartheta(\xi_i)), I_i(\xi_i, \bar{\vartheta}(\xi_i))] \leq L_i \mathbf{d}[\vartheta(\xi_i), \bar{\vartheta}(\xi_i)]$, *where* L_i *are positive constants,* $i = 1, 2, \cdots, m$.

Then, for any given $\varepsilon > 0$, *there exists a number* $\delta(\varepsilon) > 0$, *such that*

$$\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] \leq \varepsilon,\ \xi \in [\xi_0, T].$$

Whenever $\mathbf{d}[\vartheta_0, \bar{\vartheta}_0] \leq \delta(\varepsilon)$ *and* $\mathbf{d}[c(\xi), \bar{c}(\xi)] \leq \delta(\varepsilon), \xi \in [\xi_0, T]$, *where* $\vartheta(\xi), \bar{\vartheta}(\xi)$ *are two solutions of equations* (3.1) *with different initial values* $\vartheta_0,\ \bar{\vartheta}_0$ *and control terms* $c(\xi),\ \bar{c}(\xi) \in \Omega$, *respectively.*

The following notation is introduced to facilitate the proof of Theorem 3.1.

Let $\mathcal{I}_i = \xi_{i+1} - \xi_i$, $P_i = \int_{\xi_i}^{\xi_{i+1}} p_i(\xi) d\xi$, $i = 0, 1, 2, \cdots, m$. By $p_i(\xi)$ being bounded, let $p_i(\xi) \leq M_i$, M_i is a positive constant, where $\xi \in (\xi_i, \xi_{i+1}]$, $i = 0, 1, 2, \cdots, m$.

Proof. Let $\xi \in [\xi_0, \xi_1]$. Due to Lemma 3.1, $\mathbf{d}[\vartheta_0, \bar{\vartheta}_0] \leq \delta(\varepsilon)$ and $\mathbf{d}[c(\xi), \bar{c}(\xi)] \leq \delta(\varepsilon)$, therefore, the inequality below can be obtained:

$$\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)]$$

$$
\begin{aligned}
&= \mathbf{d}\Big[\vartheta_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s,\vartheta(s),c(s))ds, \bar{\vartheta}_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s,\bar{\vartheta}(s),\bar{c}(s))ds\Big] \\
&\le \mathbf{d}[\vartheta_0,\bar{\vartheta}_0] + \int_{\xi_0}^{\xi} p_0(s)(\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)] + \mathbf{d}[c(s),\bar{c}(s)])ds \qquad (3.12) \\
&\le \mathbf{d}[\vartheta_0,\bar{\vartheta}_0] + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)]ds + M_0\int_{\xi_0}^{\xi}\mathbf{d}[c(s),\bar{c}(s)]ds \\
&\le (1+M_0\mathcal{I}_0)\delta(\varepsilon) + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)]ds.
\end{aligned}
$$

By using Gronwall inequality, the following conclusion is obtained:

$$
\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \le (1+M_0\mathcal{I}_0)\exp(P_0)\delta(\varepsilon),\ \ \xi\in[\xi_0,\xi_1]. \tag{3.13}
$$

From the condition $(A_{3.2})$,

$$
\begin{aligned}
\mathbf{d}[\vartheta(\xi_1^+),\bar{\vartheta}(\xi_1^+)] &\le \mathbf{d}[\vartheta(\xi_1),\bar{\vartheta}(\xi_1)] + \mathbf{d}[I_1(\xi_1,\vartheta(\xi_1)), I_1(\xi_1,\bar{\vartheta}(\xi_1))] \qquad (3.14) \\
&\le (1+L_1)\mathbf{d}[\vartheta(\xi_1),\bar{\vartheta}(\xi_1)] \\
&\le (1+L_1)(1+M_0\mathcal{I}_0)\exp(P_0)\delta(\varepsilon).
\end{aligned}
$$

If $\xi\in(\xi_1,\xi_2]$, then we can derive the inequatilty by using (3.2), (3.9), (3.13), and (3.14):

$$
\begin{aligned}
&\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \\
&\le (1+L_1)(1+M_0\mathcal{I}_0)\exp(P_0)\delta(\varepsilon) + \int_{\xi_1}^{\xi} p_1(s)\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)]ds \\
&\quad + M_1\int_{\xi_1}^{\xi}\mathbf{d}[c(s),\bar{c}(s)]ds \\
&\le [(1+L_1)(1+M_0\mathcal{I}_0)\exp(P_0) + M_1\mathcal{I}_1]\delta(\varepsilon) + \int_{\xi_1}^{\xi} p_1(s)\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)]ds.
\end{aligned}
$$

Applying Gronwall inequality again, the inequality can be obtained:

$$
\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \le [(1+L_1)(1+M_0\mathcal{I}_0)\exp(P_0+P_1) + M_1\mathcal{I}_1\exp(P_1)]\delta(\varepsilon),\ \ \xi\in(\xi_1,\xi_2].
$$

Repeating the above process, for $\xi\in(\xi_i,\xi_{i+1}]$, the estimate can be derived:

$$
\begin{aligned}
\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \le \Big\{&\sum_{l=1}^{i}\Big[\prod_{j=l}^{i}(1+L_j)\Big]M_{l-1}\mathcal{I}_{l-1}\exp\Big(\sum_{j=l-1}^{i}P_j\Big) \\
&+\prod_{l=1}^{i}(1+L_l)\exp\Big(\sum_{l=0}^{i}P_l\Big) + M_i\mathcal{I}_i\exp(P_i)\Big\}\delta(\varepsilon).
\end{aligned}
$$

Let

$$
\begin{aligned}
K = \Big\{&\sum_{l=1}^{m}\Big[\prod_{j=l}^{m}(1+L_j)\Big]M_{l-1}\mathcal{I}_{l-1}\exp\Big(\sum_{j=l-1}^{m}P_j\Big) \\
&+\prod_{l=1}^{m}(1+L_l)\exp\Big(\sum_{l=0}^{m}P_l\Big) + M_m\mathcal{I}_m\exp(P_m)\Big\}.
\end{aligned}
$$

For every $\xi \in [\xi_0, T]$, the inequality

$$\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] \leq K\delta(\varepsilon)$$

holds. For a given $\varepsilon > 0$, choosing $0 < \delta(\varepsilon) \leq \frac{\varepsilon}{K}$, then we have

$$\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] \leq \varepsilon,\ \xi \in [\xi_0, T].$$

□

Based on Theorem 3.1, the corollary can be obtained as follows.

Corollary 3.1. *Suppose that the conditions given in Theorem 3.1 hold, and* $\Xi_{\xi,v}$, $\bar{\xi}_{\xi,\bar{v}}$ *are two cross-sections of sheaf-solutions of (3.1) which correspond to* ξ_0 *and* $\bar{\xi}_0$, *the control terms* $c(\xi), \bar{c}(\xi) \in \Omega$. *Then, for arbitrary given* $\varepsilon > 0$, *there is a number* $\delta(\varepsilon) > 0$, *such that the below is satisfied:*

$$\tilde{\mathbf{d}}[\Xi_{\xi,c}, \bar{\Xi}_{\xi,\bar{c}}] \leq \varepsilon,\ \xi \in [\xi_0, T],$$

whenever $\tilde{\mathbf{d}}[\Xi_0, \bar{\Xi}_0] \leq \delta(\varepsilon)$ *and* $\mathbf{d}[c(\xi), \bar{c}(\xi)] \leq \delta(\varepsilon)$, $\xi \in [\xi_0, T]$.

If $\Xi_0,\ \bar{\Xi}_0 \subset \mathbb{E}^n,\ \Omega \subset \mathbb{E}^q$ are bounded, then the following theorem can be obtained.

Theorem 3.2. *Suppose that the conditions given in Theorem 3.1 hold, and* $\Xi_0, \bar{\Xi}_0$, Ω *are bounded subsets.* $\Xi_{\xi,c},\ \bar{\Xi}_{\xi,\bar{c}}$ *are two cross-sections of sheaf-solutions of* (3.1) *which correspond to* $\Xi_0, \bar{\Xi}_0 \subset \mathbb{E}^n$, *and the control terms* $c(\xi), \bar{c}(\xi) \in \Omega$. *Then,*

$$\begin{aligned}\tilde{\mathbf{d}}[\Xi_{\xi,c}, \bar{\Xi}_{\xi,\bar{c}}] \leq \operatorname{diam}[\Omega] \sum_{l=0}^{i} \Big[\prod_{j=l+1}^{i} (1+L_j)\Big] P_l \exp\Big(\sum_{j=l}^{i} P_j\Big) \\ + \tilde{\mathbf{d}}[\Xi_0, \bar{\Xi}_0] \prod_{l=1}^{i} (1+L_l) \exp\Big(\sum_{j=l}^{i} P_j\Big),\end{aligned}$$

where $\xi \in (\xi_i, \xi_{i+1}],\ i = 0, 1, 2, \cdots, m$.

Proof. Let $\xi \in [\xi_0, \xi_1]$. In the same proof process to inequality (3.12), the inequality below is obtained.

$$\begin{aligned}\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] &\leq \mathbf{d}[\vartheta_0, \bar{\vartheta}_0] + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[\vartheta(s), \bar{\vartheta}(s)]ds + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[c(s), \bar{c}(s)]ds \\ &\leq \tilde{\mathbf{d}}[\Xi_0, \bar{\Xi}_0] + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[\vartheta(s), \bar{\vartheta}(s)]ds + \operatorname{diam}[\Omega] \int_{\xi_0}^{\xi} p_0(s)ds \\ &\leq \tilde{\mathbf{d}}[\Xi_0, \bar{\Xi}_0] + P_0\operatorname{diam}[\Omega] + \int_{\xi_0}^{\xi} p_0(s)\mathbf{d}[\vartheta(s), \bar{\vartheta}(s)]ds.\end{aligned}$$

Applying Gronwall inequality yields the following result:

$$\mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)] \leq (\tilde{\mathbf{d}}[\Xi_0, \bar{\Xi}_0] + P_0\operatorname{diam}[\Omega]) \exp(P_0).$$

Similarly, the result of Theorem 3.2 can be taken, so its details are omitted herein. □

In Theorem 3.2, if Ξ_0 is equal to $\bar{\Xi}_0$, then the result

$$\tilde{\mathbf{d}}[\Xi_{\xi,c}, \bar{\Xi}_{\xi,\bar{c}}] \leq \text{diam}[\Omega] \sum_{l=0}^{i} \Big[\prod_{j=l+1}^{i} (1+L_j) \Big] P_l \exp\Big(\sum_{j=l}^{i} P_j \Big) + \text{diam}[\Xi_0] \prod_{l=1}^{i} (1+L_l) \exp\Big(\sum_{j=l}^{i} P_j \Big),$$

holds, where $\xi \in (\xi_i, \xi_{i+1}],\ i = 0, 1, 2, \cdots, m$.

For the purpose of talking about differential equation stability, the comparison is important. The comparison theorem of (3.1) is therefore given, followed by the introduction of the following lemma.

Lemma 3.2 (Theorem 3.1, [19])**.** *Assume that* $f \in C[[\xi_0, T] \times \mathbb{E}^n \times \mathbb{E}^q, \mathbb{E}^n]$ *and*

$$\mathbf{d}[f(\xi, \vartheta(\xi), c(\xi)), \theta] \leq z(\xi, \mathbf{d}[\vartheta(\xi), \theta]),\ \textit{for}\ (\xi, \vartheta, c) \in [\xi_0, T] \times \mathbb{E}^n \times \Omega,$$

where $z : [\xi_0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $z(\xi, \alpha)$ *is nondecreasing in* α *for* $\xi \in [\xi_0, T]$*. The maximal right-hand solution* $u(\xi; \xi_0, \alpha_0)$ *of the scalar differential equation*

$$\alpha' = z(\xi, \alpha),\ \alpha(\xi_0) = \alpha_0 \geq 0,$$

which exists on $[\xi_0, T]$*. Then, a solution* $\vartheta(\xi) = \vartheta(\xi; \xi_0, \vartheta_0, c(\xi))$ *of fuzzy control equations*

$$D_H \vartheta(\xi) = f(\xi, \vartheta(\xi), c(\xi)),\ \vartheta(\xi_0) = \vartheta_0,$$

which satisfies

$$\mathbf{d}[\vartheta(\xi), \vartheta_0] \leq u(\xi, \alpha_0) - \alpha_0,\ \xi \in [\xi_0, T],$$

where $\alpha_0 = \mathbf{d}[\vartheta_0, \theta]$*,* $c(\xi) \in \Omega$*.*

Corollary 3.2. *In particular, the following comparison result can be obtained by Lemma 3.2,*

$$\mathbf{d}[\vartheta(\xi), \theta] \leq \mathbf{d}[\vartheta(\xi), \vartheta_0] + \mathbf{d}[\vartheta_0, \theta] \leq u(\xi, \alpha_0),\ \xi \in [\xi_0, T].$$

Theorem 3.3. *Suppose that*
$(A_{3.3})$ $\mathbf{d}[f(\xi, \vartheta(\xi), c(\xi)), \theta] \leq z(\xi, \mathbf{d}[\vartheta(\xi), \theta])$ *for* $(\xi, \vartheta, c) \in (\xi_i, \xi_{i+1}] \times \mathbb{E}^n \times \Omega$*, where* $z \in C[(\xi_i, \xi_{i+1}] \times \mathbb{R}_+, \mathbb{R}_+]$*, and* $z(\xi, \alpha)$ *is nondecreasing in* α *for* $\xi \in (\xi_i, \xi_{i+1}]$*,* $i = 0, 1, 2, \cdots, m$*;*
$(A_{3.4})$ *the maximal right-hand solution* $u(\xi; \xi_0, \alpha_0)$ *of the differential equations*

$$\begin{cases} \alpha' = z(\xi, \alpha),\ \xi \in [\xi_0, T],\ \xi \neq \xi_i, \\ \alpha(\xi_i^+) = \alpha(\xi_i) + J_i(\xi_i, \alpha(\xi_i)),\ i = 1, 2, \cdots, m, \\ \alpha(\xi_0) = \alpha_0 \end{cases} \tag{3.15}$$

exists and $\alpha_0 \geq 0$*, where* $J_i(\xi, \alpha)$ *is nondecreasing in* α *for* $\xi \in [\xi_0, T]$*;*
$(A_{3.5})$ $\mathbf{d}[I_i(\xi_i, \vartheta(\xi_i)), \theta] \leq J_i(\xi_i, \alpha(\xi_i))$*,* $i = 1, 2, \cdots, m$*.*

Then, there exists a solution $\vartheta(\xi) = \vartheta(\xi; \xi_0, \vartheta_0, c(\xi))$ *of* (3.1) *which satisfies*

$$\mathbf{d}[\vartheta(\xi), \vartheta_0] \leq u(\xi, \alpha_0) - \alpha_0,\ \xi \in [\xi_0, T],$$

where $\alpha_0 = \mathbf{d}[\vartheta_0, \theta]$*.*

Proof. Let $\xi \in [\xi_0, \xi_1]$. From the conditions of Theorem 3.3 and Lemma 3.2, the inequality

$$\mathbf{d}[\vartheta(\xi), \vartheta_0] \leq u(\xi, \alpha_0) - \alpha_0,$$

can be obtained, where $\alpha_0 = \mathbf{d}[\vartheta_0, \theta]$.

The properties of metric $\mathbf{d}$ and condition $(A_{3.4})$ yield the following estimate:

$$\begin{aligned} \mathbf{d}[\vartheta(\xi_1^+), \vartheta_0] &\leq \mathbf{d}[\vartheta(\xi_1), \vartheta_0] + \mathbf{d}[I_1(\xi_1, \vartheta(\xi_1)), \theta] \\ &\leq u(\xi_1, \alpha_0) - \alpha_0 + J_1(\xi_1, u(\xi_1)) \\ &\leq u(\xi_1^+) - \alpha_0 \\ &\leq u(\xi_1^+). \end{aligned} \tag{3.16}$$

Corollary 3.3 can be used to obtain

$$\mathbf{d}[\vartheta(\xi; \xi_1, \vartheta(\xi_1^+)), \theta] \leq u(\xi; \xi_1, u(\xi_1^+)), \quad \xi \in (\xi_1, \xi_2]. \tag{3.17}$$

If $\xi \in (\xi_1, \xi_2]$, then the following can be obtained by using Lemma 3.2, $(A_{3.5})$, and (3.17):

$$\begin{aligned} &\mathbf{d}[\vartheta(\xi), \vartheta_0] \\ &= \mathbf{d}\Big[\vartheta_0 \ominus_H (-1)\Big[\int_{\xi_0}^{\xi_1} f(s, \vartheta(s), c(s))ds + \int_{\xi_1}^{\xi} f(s, \vartheta(s), c(s))ds\Big] \\ &\quad + I_1(\xi_1, \vartheta(\xi_1)), \vartheta_0\Big] \\ &\leq \int_{\xi_0}^{\xi_1} \mathbf{d}[f(s, \vartheta(s), c(s)), \theta]ds + \int_{\xi_1}^{\xi} \mathbf{d}[f(s, \vartheta(s), c(s)), \theta]ds + \mathbf{d}[I_1(\xi_1, \vartheta(\xi_1)), \theta] \\ &\leq \int_{\xi_0}^{\xi_1} z(s, \mathbf{d}[\vartheta(s), \theta])ds + \int_{\xi_1}^{\xi} z(s, \mathbf{d}[\vartheta(s), \theta])ds + J_1(\xi_1, \alpha(\xi_1)) \\ &\leq \int_{\xi_0}^{\xi_1} z(s, u(s))ds + \int_{\xi_1}^{\xi} z(s, u(s))ds + J_1(\xi_1, u(\xi_1)) \\ &= u(\xi, \alpha_0) - \alpha_0. \end{aligned}$$

Mathematical induction reveals the relation

$$\mathbf{d}[\vartheta(\xi), \vartheta_0] \leq u(\xi, \alpha_0) - \alpha_0, \ \xi \in [\xi_0, T].$$

□

Corollary 3.3. *Suppose that the conditions given in Theorem 3.3 hold, then*

$$\tilde{\mathbf{d}}[\Xi_{\xi,c}, \Xi_0] \leq u(\xi, \alpha_0) - \rho_0,$$

where $\rho_0 = \inf\{\mathbf{d}[\vartheta_0, \theta] : \vartheta_0 \in H_0\}$ *for* $c(\xi) \in \Omega$.

Theorem 3.4. *Suppose that the condition* $(A_{3.4})$ *holds and*
$(A_{3.6})$

$$\mathbf{d}[f(\xi, \vartheta(\xi), c(\xi)), f(\xi, \bar{\vartheta}(\xi), \bar{c}(\xi))] \leq z(\xi, \mathbf{d}[\vartheta(\xi), \bar{\vartheta}(\xi)]), \tag{3.18}$$

for (ξ, ϑ, c), $(\xi, \bar{\vartheta}, \bar{c}) \in (\xi_i, \xi_{i+1}] \times \mathbb{E}^n \times \Omega$, *where* $z \in C[(\xi_i, \xi_{i+1}] \times \mathbb{R}_+, \mathbb{R}_+]$, $z(\xi, \alpha)$ *is nondecreasing in* α *for* $\xi \in (\xi_i, \xi_{i+1}]$, $i = 0, 1, 2, \cdots, m$;
$(A_{3.7})$ $\mathbf{d}[I_i(\xi_i, \vartheta(\xi_i)), I_i(\xi_i, \bar{\vartheta}(\xi_i))] \leq J_i(\xi_i, \mathbf{d}[\vartheta(\xi_i), \bar{\vartheta}(\xi_i)])$, $i = 1, 2, \cdots, m$.

Then,

$$\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \leq u(\xi;\xi_0,\alpha_0),\ \xi \in [\xi_0,T],$$

when $\mathbf{d}[\vartheta_0,\bar{\vartheta}_0] \leq \alpha_0$, $\vartheta(\xi),\bar{\vartheta}(\xi)$ *are solutions of* (3.1) *with initial values* $\vartheta_0,\bar{\vartheta}_0$, *and control terms* $c(\xi),\bar{c}(\xi) \in \Omega$, *respectively.*

Proof. Let $m(\xi) = \mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)]$, such that $m(\xi_0) = \mathbf{d}[\vartheta_0,\bar{\vartheta}_0] \leq \alpha_0$. Firstly, the conclusion is proven on the interval $[\xi_0,\xi_1]$. Considering (3.18), it can be obtained as follows

$$\begin{aligned} m(\xi) &= \mathbf{d}\Big[\vartheta_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s,\vartheta(s),c(s))ds, \bar{\vartheta}_0 \ominus_H (-1)\int_{\xi_0}^{\xi} f(s,\bar{\vartheta}(s),\bar{c}(s))ds\Big] \\ &\leq \mathbf{d}[\vartheta_0,\bar{\vartheta}_0] + \int_{\xi_0}^{\xi} \mathbf{d}[f(s,\vartheta(s),c(s)), f(s,\bar{\vartheta}(s),\bar{c}(s))]ds \\ &\leq \mathbf{d}[\vartheta_0,\bar{\vartheta}_0] + \int_{\xi_0}^{\xi} z(s,\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)])ds \\ &= m(\xi_0) + \int_{\xi_0}^{\xi} z(s,m(s))ds. \end{aligned}$$

Then, from Theorem 1.9.2 in [8], the estimate can be given by:

$$m(\xi) \leq u(\xi;\xi_0,\alpha_0),\ \xi \in [\xi_0,\xi_1]. \tag{3.19}$$

Combining (3.19) with the condition $(A_{3.4})$,

$$\begin{aligned} m(\xi_1^+) &= \mathbf{d}[\vartheta(\xi_1) + I_1(\xi_1,\vartheta(\xi_1)), \bar{\vartheta}(\xi_1) + I_1(\xi_1,\bar{\vartheta}(\xi_1))] \\ &\leq \mathbf{d}[\vartheta(\xi_1),\bar{\vartheta}(\xi_1)] + \mathbf{d}[I_1(\xi_1,\vartheta(\xi_1)), I_1(\xi_1,\bar{\vartheta}(\xi_1))] \\ &\leq m(\xi_1) + J_1(\xi_1,\mathbf{d}[\vartheta(\xi_1),\bar{\vartheta}(\xi_1)]) \\ &\leq u(\xi_1;\xi_0,\alpha_0) + J_1(\xi_1,u(\xi_1)) \\ &= u(\xi_1^+). \end{aligned} \tag{3.20}$$

Subsequent, the conclusion is proven in the interval $(\xi_1,\xi_2]$. Thus,

$$\begin{aligned} & m(\xi) \\ &= \mathbf{d}\Big[\vartheta(\xi_1^+) \ominus_H (-1)\int_{\xi_1}^{\xi} f(s,\vartheta(s),c(s))ds, \bar{\vartheta}(\xi_1^+) \ominus_H (-1)\int_{\xi_1}^{\xi} f(s,\bar{\vartheta}(s),\bar{c}(s))ds\Big] \\ &\leq m(\xi_1^+) + \int_{\xi_1}^{\xi} z(s,\mathbf{d}[\vartheta(s),\bar{\vartheta}(s)])ds \\ &= m(\xi_1^+) + \int_{\xi_1}^{\xi} z(s,m(s))ds,\ \xi \in (\xi_1,\xi_2]. \end{aligned} \tag{3.21}$$

Using Theorem 1.9.2 in [8], (3.20) and (3.21) yields the inequality

$$m(\xi) \leq u(\xi;\xi_1,u(\xi_1^+)),\ \xi \in (\xi_1,\xi_2].$$

Mathematical induction reveals

$$\mathbf{d}[\vartheta(\xi),\bar{\vartheta}(\xi)] \leq u(\xi;\xi_0,\alpha_0),\ \text{for } \xi \in [\xi_0,T].$$

□

Corollary 3.4. *Suppose that the conditions given in Theorem 3.4 hold. Then,*

$$\tilde{\mathbf{d}}[\Xi_{\xi,c},\bar{\Xi}_{\xi,\bar{c}}] \le u(\xi;\xi_0,\alpha_0),\ \xi\in[\xi_0,T], \tag{3.22}$$

where $\tilde{\mathbf{d}}[\Xi_0,\bar{\Xi}_0]\le\alpha_0$*,* $c(\xi),\bar{c}(\xi)\in\Omega$*.*

4. Stability criteria

This section assumes that the solution of equations (3.1) exists on the interval $[\xi_0,+\infty)$. The stability of (3.1) will be discussed on basis of the comparison theorem.

Some notations and stability definitions are first provided for convenience.

$PC=\{\vartheta(\xi):\vartheta(\xi)\in C[(\xi_i,\xi_{i+1}],\mathbb{E}^n]$ and $\lim\limits_{\xi\to\xi_i^+}\vartheta(\xi)=\vartheta(\xi_i^+)$ exists, $i=0,1,2,\cdots,\lim\limits_{i\to\infty}\xi_i=+\infty\}$.

$\mathcal{K}=\{a(\xi)\in C[\mathbb{R}_+,\mathbb{R}_+]:a(\xi)$ is strictly increasing and $a(0)=0\}$.

$S(\rho)=\{\vartheta\in\mathbb{E}^n:\mathbf{d}[\vartheta,\theta]\le\rho\}$.

Definition 4.1. The Lyapunov-like function $V:\mathbb{R}_+\times S(\rho)\to\mathbb{R}_+$ for (3.1) belongs to the class $\mathcal{V}_0$ if $V\in C[(\xi_i,\xi_{i+1}]\times S(\rho),\mathbb{R}_+]$, $\lim\limits_{(\xi,\vartheta)\to(\xi_i^+,\bar{\vartheta})}V(\xi,\vartheta)=V(\xi_i^+,\bar{\vartheta})$, $i=1,2,\cdots$, and satisfies $|V(\xi,\bar{\vartheta})-V(\xi,\vartheta)|\le L\mathbf{d}[\bar{\vartheta},\vartheta]$, where L is a positive constant.

Definition 4.2. If the solution $\vartheta(\xi)$ of (3.1) is II-differentiable, then the upper right derivative $V(\xi,\vartheta)\in\mathcal{V}_0$ corresponding to (3.1) is given as

$${}_{(3.1)}D_{II}^+V(\xi,\vartheta(\xi))=\lim_{h\to0^+}\sup\frac{1}{h}[V(\xi+h,\vartheta(\xi)\ominus_H(-1)hf(\xi,\vartheta(\xi),c(\xi)))-V(\xi,\vartheta(\xi))].$$

Remark 4.1. If the solution $\vartheta(\xi)$ of (3.1) is I-differentiable, then the upper right derivative $V(\xi,\vartheta)\in\mathcal{V}_0$ corresponding to (3.1) is

$${}_{(3.1)}D_I^+V(\xi,\vartheta(\xi))=\lim_{h\to0^+}\sup\frac{1}{h}[V(\xi+h,\vartheta(\xi)+hf(\xi,\vartheta(\xi),c(\xi)))-V(\xi,\vartheta(\xi))].$$

Definition 4.3. The trivial sheaf-solutions of (3.1) is said to be
(SS_1) equi-stable if for arbitrary $\varepsilon>0$ and $\xi_0\ge0$, there exists a $\delta=\delta(\xi_0,\varepsilon)>0$ such that for arbitrary $c(\xi)\in\Omega$,

$$\tilde{\mathbf{d}}[\Xi_0,\theta]<\delta \text{ implies } \tilde{\mathbf{d}}[\Xi_{\xi,c},\theta]<\varepsilon,\ \xi\ge\xi_0.$$

(SS_2) uniformly stable if in the (SS_1) δ is independent of ξ_0.
(SS_3) equi-attractive, if for arbitrary $\varepsilon>0$ and $\xi_0\ge0$, there exist $\delta=\delta(\xi_0)>0$ and $\Gamma=\Gamma(\xi_0,\varepsilon)>0$ such that for arbitrary $c(\xi)\in\Omega$,

$$\tilde{\mathbf{d}}[\Xi_0,\theta]<\delta \text{ implies } \tilde{\mathbf{d}}[\Xi_{\xi,c},\theta]<\varepsilon,\ \xi\ge\xi_0+\Gamma.$$

(SS_4) uniformly attractive, if δ and Γ in (SS_3) are independent of ξ_0.
(SS_5) equi-asymptotically stable, if (SS_1) and (SS_3) hold.
(SS_6) uniformly asymptotically stable, if (SS_2) and (SS_4) hold.

The stability definition of the solution of (3.1) can be modelled according to the one given above and omitted here.

Theorem 4.1. *Suppose that the solution $\vartheta(\xi)$ of* (3.1) *with control terms $c(\xi) \in \Omega$ is II-differentiable and*
$(A_{4.1})$ $V \in \mathcal{V}_0$ *and* ${}_{(3.1)}D^+_{II}V(\xi, \vartheta(\xi)) \le z(\xi, V(\xi, \vartheta(\xi)))$, $\xi \ne \xi_i$, *in which* $z(\xi, \alpha) \in C[(\xi_i, +\xi_{i+1}] \times \mathbb{R}_+, \mathbb{R}_+]$, $i = 1, 2, \cdots$;
$(A_{4.2})$ $V(\xi_i^+, \vartheta(\xi_i^+))) \le V(\xi_i, \vartheta(\xi_i)) + J_i(\xi_i, V(\xi_i, \vartheta(\xi_i)))$, $J_i(\xi, \alpha)$ *is non-decreasing in* α *for arbitrary* $\xi \ge \xi_0$, $i = 1, 2, \cdots$.

Then, $V(\xi, \vartheta(\xi)) \le u(\xi; \xi_0, \alpha_0)$, $\xi \ge \xi_0$, *when* $V(\xi_0, \vartheta_0) \le \alpha_0$, *where* $u(\xi; \xi_0, \alpha_0)$ *is the maximal right-hand solution of* (3.15).

Proof. Let $\vartheta(\xi) = \vartheta(\xi; \xi_0, \vartheta_0)$ be any solution of (3.1) and $\mathbf{d}[\vartheta(\xi), \theta] < \rho$. Define $m(\xi) = V(\xi, \vartheta(\xi))$ such that $m(\xi_0) = V(\xi_0, \vartheta_0) \le \alpha_0$.

If $\xi \in [\xi_0, \xi_1]$, for arbitrary $h > 0$, there is

$$\begin{aligned} m(\xi+h) - m(\xi) &= V(\xi+h, \vartheta(\xi+h)) - V(\xi+h, \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))) \\ &\quad + V(\xi+h, \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))) - V(\xi, \vartheta(\xi)) \\ &\le L\mathbf{d}[\vartheta(\xi+h), \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))] \\ &\quad + V(\xi+h, \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))) - V(\xi, \vartheta(\xi)). \end{aligned} \tag{4.1}$$

From (4.1) and Definition 4.2,

$$\begin{aligned} &D^+ m(\xi) \\ &= \lim_{h \to 0^+} \sup \frac{1}{h}[m(\xi+h) - m(\xi)] \\ &\le {}_{(3.1)}D^+_{II}V(\xi, \vartheta(\xi)) + L \lim_{h\to 0^+} \sup \frac{1}{h}\mathbf{d}[\vartheta(\xi+h), \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))]. \end{aligned}$$

Combining

$$\frac{1}{h}\mathbf{d}[\vartheta(\xi+h), \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))] = \mathbf{d}\Big[\frac{\vartheta(\xi) \ominus_H \vartheta(\xi+h)}{-h}, f(\xi, \vartheta(\xi), c(\xi))\Big],$$

with $\vartheta(\xi)$ is II-differentiable any solution of (3.1),

$$\begin{aligned} \lim_{h\to 0^+} \sup \frac{1}{h}\mathbf{d}[\vartheta(\xi+h), \vartheta(\xi) \ominus_H (-1)hf(\xi, \vartheta(\xi), c(\xi))] &= \mathbf{d}[D_H\vartheta(\xi), f(\xi, \vartheta(\xi), c(\xi))] \\ &= 0. \end{aligned}$$

Therefore,

$$D^+ m(\xi) \le z(\xi, m(\xi)).$$

Furthermore, $m(\xi) \le u(\xi; \xi_0, \alpha_0)$, $\xi \in [\xi_0, \xi_1]$, that is, $V(\xi, \vartheta(\xi)) \le u(\xi; \xi_0, \alpha_0)$, $\xi \in [\xi_0, \xi_1]$, which implies that $V(\xi_1, \vartheta(\xi_1)) \le u(\xi_1; \xi_0, \alpha_0)$. Using condition $(A_{4.2})$ easily helps verify

$$\begin{aligned} V(\xi_1^+, \vartheta(\xi_1^+)) &\le V(\xi_1, \vartheta(\xi_1)) + J_1(\xi_1, V(\xi_1, \vartheta(\xi_1))) \\ &\le u(\xi_1; \xi_0, \alpha_0) + J_1(\xi_1, u(\xi_1; \xi_0, \alpha_0)) \\ &= u(\xi_1^+). \end{aligned} \tag{4.2}$$

If $\xi \in (\xi_1, \xi_2]$, then using conditions $(A_{4.1})$ and (4.2) yields the following conclusion:

$$V(\xi, \vartheta(\xi)) \le u(\xi; \xi_0, \alpha_0), \ \xi \in (\xi_1, \xi_2].$$

By going through the procedure again, the validity of the theorem's conclusion is established. □

When the solution $\vartheta(\xi)$ of the equations (3.1) is I-differentiable, the following conclusion can be obtained by changing the conditions of the differentiability of the solution $\vartheta(\xi)$ and the notation upper right derivative of $V(\xi, \vartheta)$ in Theorem 4.1.

Theorem 4.2. *Suppose that the solution $\vartheta(\xi) = \vartheta(\xi; \xi_0, \vartheta_0)$ of (3.1) with control terms $c(\xi) \in \Omega$ is I-differentiable and condition $(A_{4.2})$ holds, the condition $(A_{4.1})$ is replaced by*
$(A_{4.1'})$ $V \in \mathcal{V}_0$ and ${}_{(3.1)}D_I^+V(\xi, \vartheta(\xi)) \le z(\xi, V(\xi, \vartheta(\xi)))$, $\xi \ne \xi_i$, $z(\xi, \alpha) : [\xi_0, +\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous in $(\xi_i, \xi_{i+1}]$, $i = 1, 2, \cdots$.

Then, $V(\xi, \vartheta(\xi)) \le u(\xi; \xi_0, \alpha_0)$, $\xi \ge \xi_0$, when $V(\xi_0, \vartheta_0) \le \alpha_0$, where $u(\xi; \xi_0, \alpha_0)$ is the maximal right-hand solution of (3.15).

Corollary 4.1. *In Theorem 4.1 or 4.2, the functions $z(\xi, \alpha) \equiv 0$ and $J_i(\xi, 0) = 0$ are admissible, $i = 1, 2, \cdots$. Then, the following inequality is proven:*

$$V(\xi, \vartheta(\xi)) \le V(\xi_0, \vartheta_0), \ \xi \ge \xi_0.$$

Let $\mathfrak{V}(\xi, \Xi_{\xi,c}) = \sup\{V(\xi, \vartheta(\xi)) : \vartheta(\xi) \in \Xi_{\xi,c}\}$, $\mathfrak{V}(\xi_0, \Xi_0) = \sup\{V(\xi_0, \vartheta_0) : \vartheta_0 \in \Xi_0\}$, where $c(\xi) \in \Omega$. Then, the following corollaries are provided.

Corollary 4.2. *Suppose that the conditions of Theorem 4.1 and 4.2 hold, except that $V(\xi_0, \vartheta_0) \le \alpha_0$ is replaced by $\mathfrak{V}(\xi_0, \Xi_0) \le \alpha_0$. Then, the inequality below is valid:*

$$\mathfrak{V}(\xi, \Xi_{\xi,c}) \le u(\xi; \xi_0, \alpha_0), \ \xi \ge \xi_0.$$

where $c(\xi) \in \Omega$.

Corollary 4.3. *Let the assumptions of Corollary 4.1 hold, then, we have*

$$\mathfrak{V}(\xi, H_{\xi,c}) \le \mathfrak{V}(\xi_0, \Xi_0), \ \xi \ge \xi_0,$$

where $c(\xi) \in \Omega$.

Theorem 4.3. *Suppose that the solution $\vartheta(\xi)$ of (3.1) with control terms $c(\xi) \in \Omega$ is II-differentiable:*
$(A_{4.3})$ $V \in \mathcal{V}_0$ and ${}_{(3.1)}D_{II}^+V(\xi, \vartheta(\xi)) \le 0$, $\xi \ne \xi_i$, $i = 1, 2, \cdots$;
$(A_{4.4})$ there is a $\bar{\rho} > 0$ such that $\vartheta(\xi_i) \in S(\bar{\rho})$ implies that $\vartheta(\xi_i^+) \in S(\rho)$, $i = 1, 2, \cdots$;
$(A_{4.5})$ $V(\xi_i^+, \vartheta(\xi_i^+)) \le V(\xi_i, \vartheta(\xi_i))$, $i = 1, 2, \cdots$;
$(A_{4.6})$ $a(\mathbf{d}[\vartheta(\xi), \theta]) \le V(\xi, \vartheta(\xi))$, $\xi \ne \xi_i$, $a \in \mathcal{K}$, $i = 1, 2, \cdots$.

Then, the trivial solution of (3.1) is equi-stable.

Proof. For arbitrary given $0 < \varepsilon < \min\{\rho, \bar{\rho}\}$ and $\xi \ge \xi_0$, a number $\delta = \delta(\xi_0, \varepsilon)$ is identified, satisfying

$$V(\xi_0, \vartheta_0) < a(\varepsilon), \tag{4.3}$$

when $\mathbf{d}[\vartheta_0, \theta] < \delta$. The solution $\vartheta(\xi)$ of (3.1) is verified to be equi-stable, therefore,

$$\mathbf{d}[\vartheta_0, \theta] < \delta \text{ implies } \mathbf{d}[\vartheta(\xi), \theta] < \varepsilon, \ \xi \ge \xi_0,$$

for arbitrary $c(\xi) \in \Omega$.

Suppose that the aforementioned is untrue, then a $\xi^* \in (\xi_i, \xi_{i+1}]$, where i is a certain integer, and a $c(\xi) \in \Omega$ exist, such that

$$\mathbf{d}[\vartheta(\xi^*), \theta] \ge \varepsilon \text{ and } \mathbf{d}[\vartheta(\xi), \theta] < \varepsilon, \ \xi \in [\xi_0, \xi_i]. \tag{4.4}$$

Since $0 < \varepsilon < \bar{\rho}$, $\mathbf{d}[\vartheta(\xi_i^+), \theta] < \rho$ can be obtained by using condition $(A_{4.4})$ and (4.4). Therefore, $\hat{\xi} \in (\xi_i, \xi^*]$ which makes $\varepsilon \le \mathbf{d}[\vartheta(\hat{\xi}), \theta] < \rho$ true. The conditions of Corollary 4.1 for $\xi \in [\xi_0, \hat{\xi}]$ are determined to be tenable, then,

$$V(\xi, \vartheta(\xi)) \le V(\xi_0, \vartheta_0),\ \xi \in [\xi_0, \hat{\xi}]. \tag{4.5}$$

(4.3) and (4.5) yield

$$a(\varepsilon) \le a(\mathbf{d}[\vartheta(\hat{\xi}), \theta]) \le V(\hat{\xi}, \vartheta(\hat{\xi})) \le V(\xi_0, \vartheta_0) < a(\varepsilon). \tag{4.6}$$

Inequality (4.6) is a contradiction, which shows that

$$\mathbf{d}[\vartheta(\xi), \theta] < \varepsilon,\ \xi \ge \xi_0,$$

for $\mathbf{d}[\vartheta_0, \theta] < \delta$ and any $c(\xi) \in \Omega$. It follows that the trivial solution of (3.1) is equi-stable. □

Theorem 4.4. *Suppose that the conditions $(A_{4.4}) - (A_{4.6})$ given in Theorem 4.3 hold, the solution $\vartheta(\xi)$ of (3.1) with control terms $c(\xi) \in \Omega$ is I-differentiable, and*
$(A_{4.3'})$ $V \in \mathcal{V}_0$ and ${}_{(3.1)}D_I^+ V(\xi, \vartheta(\xi)) \le 0$, $\xi \ne \xi_i$, $i = 1, 2, \cdots$.
Then the trivial solution of (3.1) is equi-stable.

Theorem 4.5. *Suppose that the conditions given in Theorem 4.3 hold and $(A_{4.6})$ is replaced by*
$(A_{4.7})$ $a(\mathbf{d}[\vartheta(\xi), \theta]) \le V(\xi, \vartheta(\xi)) \le b(\mathbf{d}[\vartheta(\xi), \theta])$, $\xi \ne \xi_i$, $a, b \in \mathcal{K}$, $i = 1, 2, \cdots$.
Then the trivial solution of (3.1) is uniformly stable.

Proof. A positive number $\delta = \delta(\varepsilon)$ can be found for arbitrary $0 < \varepsilon < \min\{\rho, \bar{\rho}\}$ and $\xi_0 \ge 0$ such that $b(\delta) < a(\varepsilon)$. Then, for arbitrary $c(\xi) \in \Omega$, $\mathbf{d}[\vartheta_0, \theta] < \delta$ implies

$$V(\xi_0, \vartheta_0) < a(\varepsilon).$$

It is assume that the solution $\vartheta(\xi)$ of (3.1) is uniformly stable, that is, for arbitrary $c(\xi) \in \Omega$,

$$\mathbf{d}[\vartheta_0, \theta] < \delta \text{ implies } \mathbf{d}[\vartheta(\xi), \theta] < \varepsilon,\ \xi \ge \xi_0.$$

The proof that follows is ignored because it is standard. □

Theorem 4.6. *In addition to the conditions of Theorem 4.4, $(A_{4.6})$ is replaced with $(A_{4.7})$. Then, the trivial solution of (3.1) is uniformly stable.*

Theorem 4.7. *Suppose that the conditions of Theorem 4.3 hold and the condition $(A_{4.3})$ is strengthened to*
$(A_{4.8})$ $V \in \mathcal{V}_0$ and ${}_{(3.1)}D_{II}^+ V(\xi, \vartheta(\xi)) \le -k(\mathbf{d}[\vartheta(\xi), \theta])$, $\xi \ne \xi_i$, $i = 1, 2, \cdots$, $k \in \mathcal{K}$.
Then, the trivial solution of (3.1) is equi-asymptotically stable.

Proof. For the trivial solution of (3.1), the explanation of Theorem 4.3 indicates that it is equi-stable. Then, it is shown that the trivial solution of (3.1) is equi-attractive.

Considering the equi-stability of (3.1), for given $\varepsilon_1 = \min\{\rho, \bar{\rho}\}$, there is a $\delta_1 = \delta_1(t_0, \varepsilon_1)$, satisfying

$$\mathbf{d}[\vartheta(\xi), \theta] < \varepsilon_1,\ \xi \ge \xi_0,$$

when $\mathbf{d}[\vartheta_0, \theta] < \delta_1$.

By contrast, $\xi^* \in [\xi_0, \xi_0+\Gamma]$ exists for arbitrary given $0 < \varepsilon < \min\{\rho, \bar{\rho}\}$, thereby satisfying $\mathbf{d}[\vartheta(\xi^*), \theta] < \delta$ for any solution $\vartheta(\xi)$ of (3.1), when $\mathbf{d}[\vartheta_0, \theta] < \delta_1$, where $\delta = \delta(\xi_0, \varepsilon) > 0$ is a number corresponding to ε in equi-stability, and $\Gamma = \Gamma(\xi_0, \varepsilon) = 1 + \frac{a(\rho)}{k(\delta)}$. Assume that this condition is untrue, then, $\mathbf{d}[\vartheta(\xi), \theta] \geq \delta$, $\xi \in [\xi_0, \xi_0 + \Gamma]$. Let $m = \max\{i : \ \xi_i \in [\xi_0, \xi_0 + \Gamma]\}$. Conditions $(A_{4.5})$, $(A_{4.8})$ and (4.3) reveal

$$\begin{aligned} a(\delta) &\leq a(\mathbf{d}[\vartheta(\xi_0 + \Gamma), \theta]) \\ &\leq V(\xi_0 + \Gamma, \vartheta(\xi_0 + \Gamma)) \\ &\leq V(\xi_0, \vartheta_0) + \sum_{i=1}^{m} [V(\xi_i^+, \vartheta(\xi_i^+)) - V(\xi_i, \vartheta(\xi_i))] - \int_{\xi_0}^{\xi_0+\Gamma} k(\mathbf{d}[\vartheta(s), \theta]) ds \\ &< a(\rho) - k(\delta)\frac{a(\rho)}{k(\delta)} \\ &= 0, \end{aligned}$$

which creates a contradiction. Hence, a $\xi^* \in [\xi_0, \xi_0+\Gamma]$ exists such that $\mathbf{d}[\vartheta(\xi^*), \theta] < \delta$. On the basis of equi-stability of the trivial solution of (3.1),

$$\mathbf{d}[\vartheta(\xi), \theta] < \varepsilon, \ \xi \geq \xi_0 + \Gamma,$$

when $\mathbf{d}[\vartheta_0, \theta] < \delta_1$, $c(\xi) \in \Omega$. Then, the trivial solution of (3.1) is equi-asymptotically stable. Theorem 4.7 is now fully proved. □

Theorem 4.8. *Suppose that the conditions given in Theorem 4.4 hold, and the condition $(A_{4.3'})$ is strengthened to*
$(A_{4.8'})$ $V \in \mathcal{V}_0$ *and* ${}_{(3.1)}D_I^+ V(\xi, \vartheta(\xi)) \leq -k(\mathbf{d}[\vartheta(\xi), \theta])$, $\xi \neq \xi_i$, $i = 1, 2, \cdots$, $c \in \mathcal{K}$.
Then the trivial solution of (3.1) *is equi-asymptotically stable.*

Theorem 4.9. *Suppose that the conditions given in Theorem 4.7 or 4.8 hold and the condition $(A_{4.6})$ is strengthened to $(A_{4.7})$. Then the trivial solution of* (3.1) *is uniformly asymptotically stable.*

Theorem 4.10. *Suppose that the conditions given in Theorem 4.1, $(A_{4.4})$, $(A_{4.7})$ hold. Then the stable properties of the trivial solution of* (3.15) *imply the corresponding stable properties of the trivial solution of* (3.1).

Proof. Assume that the trivial solution of (3.15) is equi-stable. Based on Definition 4.3, for arbitrary given $0 < \varepsilon < \min(\rho, \bar{\rho})$, and $\xi_0 \geq 0$, there is a $\delta_1 = \delta_1(\xi_0, \varepsilon) > 0$, fulfilling

$$\alpha(\xi; \xi_0, \alpha_0) < a(\varepsilon), \ \xi \geq \xi_0,$$

when $\alpha_0 < \delta_1$, where $\alpha(\xi; \xi_0, \alpha_0)$ is any solution of (3.15).

Choosing $\delta = \delta(\xi_0, \varepsilon)$ that verifies $b(\delta) < \delta_1$ to be true. For arbitrary $c(\xi) \in \Omega$, $\mathbf{d}[\vartheta_0, \theta] < \delta$ implies $\mathbf{d}[\vartheta(\xi), \theta] < \varepsilon$, $\xi \geq \xi_0$. Assuming that this condition is untrue, $c(\xi) \in \Omega$ and $\xi^* \in (\xi_i, \xi_{i+1}]$ would exist for some i, satisfying

$$\mathbf{d}[\vartheta(\xi^*), \theta] \geq \varepsilon \text{ and } \mathbf{d}[\vartheta(\xi), \theta] < \varepsilon, \ \xi \in [\xi_0, \xi_i] \tag{4.7}$$

when $\mathbf{d}[\vartheta_0, \theta] < \delta$.

Since $0 < \varepsilon < \bar{\rho}$, $\mathbf{d}[\vartheta(\xi_i^+), \theta] < \rho$ can be obtained by condition $(A_{4.4})$ and (4.7). Therefore, a $\bar{\xi} \in (\xi_i, \xi^*]$ exists such that $\varepsilon \leq \mathbf{d}[\vartheta(\bar{\xi}), \theta] < \rho$. For $\xi \in [\xi_0, \bar{\xi}]$, the conditions of Theorem 4.1 hold. Then,

$$V(\xi, \vartheta(\xi)) \leq u(\xi; \xi_0, \alpha_0), \ \xi \in [\xi_0, \bar{\xi}], \tag{4.8}$$

where $u(\xi;\xi_0,\alpha_0)$ is the maximal solution of (3.15).

Applying $(A_{4.7})$ and (4.8), the contradiction is given by:

$$a(\varepsilon) \le a(\mathbf{d}[\vartheta(\bar{\xi}),\theta]) \le V(\bar{\xi},\vartheta(\bar{\xi})) \le u(\bar{\xi};\xi_0,\alpha_0) < a(\varepsilon),$$

which is a contradiction. Hence the trivial solution of (3.1) is equi-stable. Other proofs of stability were omitted herein. □

Theorem 4.11. *Suppose that the conditions given in Theorem 4.2 and $(A_{4.7})$ hold. Then the stable properties of the trivial solution of (*3.15*) imply the corresponding stable properties of (*3.1*).*

Theorem 4.12. (I) *Suppose that the conditions given in Theorem 4.3 or 4.4 hold. Then, the trivial sheaf-solution of (3.1) is equi-stable;*
(II) *Suppose that the conditions given in Theorem 4.5 or 4.6 hold. Then, the trivial sheaf-solution of (3.1) is uniformly stable.*
(III) *Suppose that the conditions given in Theorem 4.7 or 4.8 hold. Then, the trivial sheaf-solution of (3.1) is equi-asymptotically stable.*
(IV) *Suppose that the conditions given in Theorem 4.9 hold. Then, the trivial sheaf-solution of (3.1) is uniformly asymptotically stable.*

5. Conclusion

The concept of sheaf-solution to impulsive fuzzy control differential equations is proposed in this paper, and the continuous dependence of sheaf-solution to the initial state is investigated by using the Gronwall inequality. Simultaneously, the comparison theorems of sheaf-solution are also provided and the stability of the impulsive fuzzy control differential equation is investigated.

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