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SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH PARAMETRIC TYPE ANTI-PERIODIC BOUNDARY CONDITIONS

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Abstract We discuss the existence and uniqueness of solutions for sequential fractional differential equations supplemented with parametric type anti-periodic boundary conditions. We make use of fixed point theorems due to Krasnosel'skiĭ and Banach to obtain the desired results. Examples illustrating the obtained results are presented. Moreover, an interesting feature concerning the solutions of parametric type anti-periodic boundary value problems of lower and higher order sequential fractional differential equations is presented (see the conclusions section). Our results are novel in the given configuration and generalize the literature on anti-periodic boundary value problems.

Keywords Sequential fractional differential equation, parametric, anti-periodic boundary conditions, existence, fixed point.

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1. Introduction

During the last few decades, fractional order boundary value problems have been extensively investigated by many researchers. For some recent results on boundary value problems involving different kinds of fractional derivatives such as, Riemann-Liouville, Caputo, Hilfer, Hadamard, etc., for instance, see the text [7] and the research articles [1–3,6,9,13,16,18–20]. Fractional boundary value problems with anti-periodic boundary conditions also received considerable attention as such boundary conditions appear in a variety of practical situations. In the study of modes, many numerical problems converge faster with anti-periodic boundary conditions in contrast to the periodic boundary conditions [17]. One can find some recent work on anti-periodic boundary value problems in the papers [8, 11, 12]. In [5], the authors introduced the concept of dual anti-periodic boundary conditions and solved a fractional integro-differential equation equipped with these conditions. In [4], fractional differential equations complemented with nonlocal (parametric type) anti-periodic boundary conditions were studied.

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In this paper, we apply the idea used in [4] to investigate a parametric type anti-periodic boundary value problem for a sequential fractional differential equation given by

$$({}^CD^{\alpha}+k\,{}^CD^{\alpha-1})x(t)=g(t,x(t)),\ k>0,\quad 2<\alpha\le 3,\ t\in[0,T], \tag{1.1}$$

$$x(a)=-x(T),\quad x'(a)=-x'(T),\quad x''(a)=-x''(T),\ 0<a<<T, \tag{1.2}$$

where ${}^CD^{\alpha}$ denotes the Caputo (Liouville-Caputo) fractional derivative operator of order α and $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. Here, we mention that the condition $0<a<<T$ means that the boundary data is initially perturbed and thus the problem (1.1)-(1.2) is an initially perturbed anti-periodic boundary value problem. This work is motivated by the idea to examine the solution of a sequential fractional differential equation subject to the initially perturbed anti-periodic boundary conditions. Also, a relationship is developed between solutions of lower and higher order perturbed anti-periodic boundary value problems for sequential fractional differential equations (for details, see the Conclusions section).

In the rest of the paper, we set the material as follows. In Section 2, we prove a preliminary result that is used to convert the problem (1.1)-(1.2) into a fixed point problem. Section 3 contains the main results for the problem (1.1)-(1.2). In Section 4, we outline the results for a sequential fractional differential equation of order $\alpha\in(3,4]$ equipped with parametric type anti-periodic boundary conditions. The paper concludes with some interesting observations.

2. A preliminary result

Before proceeding for a preliminary lemma, we recall some definitions related to our work.

Definition 2.1. [14] Let υ be a locally integrable real-valued function on $-\infty\le a<t<b\le+\infty$. The Riemann-Liouville fractional integral I^{ω}_{a+} of order $\omega\in\mathbb{R}$ ($\omega>0$) for the function υ is defined as

$$I^{\omega}_{a+}\upsilon(t)=\int_a^t\frac{\upsilon(\vartheta)}{\Gamma(\omega)(t-\vartheta)^{1-\omega}}d\vartheta,$$

where Γ denotes the Euler gamma function.

Definition 2.2. [14] For $(p-1)$-times absolutely continuous differentiable function $\upsilon:[a,\infty)\longrightarrow\mathbb{R}$, the Caputo derivative of fractional order ω for the function υ is defined as

$${}^CD^{\omega}_{a+}\upsilon(t)=\int_a^t\frac{\upsilon^{(p)}(\vartheta)}{\Gamma(p-\omega)(t-\vartheta)^{1+\omega-p}}d\vartheta,\ p-1<\omega\le p,\ p=[\omega]+1,$$

where $[\omega]$ denotes the integer part of the real number ω.

In the present work, we use ${}^CD^{\omega}$ instead of ${}^CD^{\omega}_{0+}$ and I^{ω} instead of I^{ω}_{0+}.

Next, we solve a linear variant of the problem (1.1)-(1.2).

Lemma 2.1. *Let $h\in C([0,T],\mathbb{R})$. Then, the unique solution of the linear sequential fractional differential equation*

$$({}^CD^{\alpha}+k\,{}^CD^{\alpha-1})x(t)=h(t),\quad 2<\alpha\le 3, \tag{2.1}$$

subject to the boundary conditions (1.2), *is given by*

$$
\begin{aligned}
x(t) = &\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\right)ds \\
&+ v_1(t)\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)}h(p)dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)}h(p)dp\right] \\
&+ v_2(t)\left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\right] \\
&+ v_3(t)\left[\int_0^T e^{-k(T-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\right)ds\right. \\
&+ \left.\int_0^a e^{-k(a-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}h(p)dp\right)ds\right],
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
v_1(t) &= \frac{k(T+a)+2}{4k^2} - \frac{t}{2k} - \frac{e^{-kt}}{k^2(e^{-ka}+e^{-kT})},\ v_2(t) = -\frac{1}{2k} + \frac{e^{-kt}}{k(e^{-ka}+e^{-kT})}, \\
v_3(t) &= \frac{-e^{-kt}}{e^{-ka}+e^{-kT}}.
\end{aligned}
\tag{2.3}
$$

Proof. Applying the Riemann-Liouville operator $I^{\alpha-1}$ on both sides of (2.1) and then solving the resulting equation, we obtain

$$
x(t) = A_1 + A_2 t + A_3 e^{-kt} + \int_0^t e^{-k(t-s)} I^{\alpha-1}h(s)ds, \tag{2.4}
$$

where $A_i, i = 1,2,3$, are unknown arbitrary constants. From (2.4), we have

$$
x'(t) = A_2 - kA_3e^{-kt} + I^{\alpha-1}h(t) - k\int_0^t e^{-k(t-s)} I^{\alpha-1}h(s)ds, \tag{2.5}
$$

$$
x''(t) = k^2A_3e^{-kt} + I^{\alpha-2}h(t) - kI^{\alpha-1}h(t) + k^2\int_0^t e^{-k(t-s)} I^{\alpha-1}h(s)ds. \tag{2.6}
$$

Using (2.6) in the condition $x''(a) = -x''(T)$, we get

$$
\begin{aligned}
A_3 = \frac{-1}{(e^{-ka}+e^{-kT})}&\left\{\int_0^T e^{-k(T-s)} I^{\alpha-1}h(s)ds + \int_0^a e^{-k(a-s)} I^{\alpha-1}h(s)ds\right. \\
&\left. - \frac{1}{k}\Big(I^{\alpha-1}h(T) + I^{\alpha-1}h(a)\Big) + \frac{1}{k^2}\Big(I^{\alpha-2}h(T) + I^{\alpha-2}h(a)\Big)\right\}.
\end{aligned}
\tag{2.7}
$$

Making use of (2.5) in the condition $x'(a) = -x'(T)$, we find that

$$
\begin{aligned}
2A_2 = &kA_3(e^{-ka} + ^{-kT}) - I^{\alpha-1}h(a) + k\int_0^a e^{-k(a-s)} I^{\alpha-1}h(s)ds - I^{\alpha-1}h(T) \\
&+ k\int_0^T e^{-k(T-s)} I^{\alpha-1}h(s)ds,
\end{aligned}
\tag{2.8}
$$

which, on inserting the value A_3 from equation (2.7), yields

$$A_2 = -\frac{1}{2k}\Big[I^{\alpha-2}h(T) + I^{\alpha-2}h(a)\Big]. \tag{2.9}$$

Finally, using (2.4) in the condition $x(a) = -x(T)$, we obtain

$$\begin{aligned} 2A_1 =& - A_2(T+a) - A_3(e^{-kT} + e^{-ka}) - \int_0^T e^{-k(T-s)} I^{\alpha-1}h(s)ds \\ & - \int_0^a e^{-k(a-s)} I^{\alpha-1}h(s)ds. \end{aligned} \tag{2.10}$$

Inserting the values A_2 and A_3 from (2.7) and (2.9) respectively into (2.10), we find that

$$A_1 = \left(\frac{k(T+a)+2}{4k^2}\right)\Big[I^{\alpha-2}h(T) + I^{\alpha-2}h(a)\Big] - \left(\frac{1}{2k}\right)\Big[I^{\alpha-1}h(T) + I^{\alpha-1}h(a)\Big]. \tag{2.11}$$

Substituting the values of A_1, A_2 and A_3 from (2.11), (2.9) and (2.7), respectively into (2.4), we get (2.2). The converse follows by direct computation. □

3. Main results

Let $\mathcal{X} = C([0,T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ endowed with the usual supremum norm.

By Lemma 2.1, we can transform the nonlinear problem (1.1)-(1.2) into a fixed point problem as

$$x = \mathcal{H}(x), \tag{3.1}$$

where $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned} (\mathcal{H}x)(t) =& \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,x(p))dp\right) ds \\ & + v_1(t)\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} g(p,x(p))dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} g(p,x(p))dp\right] \\ & + v_2(t)\left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,x(p))dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,x(p))dp\right] \\ & + v_3(t)\left[\int_0^T e^{-k(T-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,x(p))dp\right) ds \right. \\ & \left. + \int_0^a e^{-k(a-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,x(p))dp\right) ds\right], \ t \in [0,T]. \end{aligned} \tag{3.2}$$

Observe that the problem (1.1)-(1.2) has a solution if only the operator $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ has a fixed point. In the sequel, we set

$$\sigma = \frac{1}{k\Gamma(\alpha)}\Big\{T^{\alpha-1}(1-e^{-kT})(1+\overline{v}_3) + k\overline{v}_2(T^{\alpha-1} + a^{\alpha-1}) + \overline{v}_3 a^{\alpha-1}(1-e^{-ka})\Big\}$$

$$+\frac{\overline{v}_1}{\Gamma(\alpha-1)}(T^{\alpha-2}+a^{\alpha-2}), \tag{3.3}$$

and

$$\begin{aligned}\sigma_1 =& \frac{\overline{v}_1}{\Gamma(\alpha-1)}(T^{\alpha-2}+a^{\alpha-2})\\ &+\frac{1}{k\Gamma(\alpha)}\Big\{k\overline{v}_2(T^{\alpha-1}+a^{\alpha-1})+\overline{v}_3[T^{\alpha-1}(1-e^{-kT})+a^{\alpha-1}(1-e^{-ka})]\Big\},\end{aligned} \tag{3.4}$$

where $\overline{v}_i = \sup_{t\in[0,T]} |v_i(t)|, i = 1, 2, 3$ (v_i are given in (2.3)).

3.1. Uniqueness result

In this subsection, we prove the existence of a unique solution to the problem (1.1)-(1.2) with the aid of Banach's fixed point theorem [10].

Theorem 3.1. *Let $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition:*

(N_1) $|g(t, x) - g(t, y)| \leq \theta|x - y|, \quad \forall t \in [0, T], \quad x, y \in \mathbb{R}$, *where $\theta > 0$ is the Lipschitz constant.*

Then, the boundary value problem (1.1)-(1.2) *has a unique solution on* $[0, T]$ *provided that $\theta\sigma < 1$, where σ is given by* (3.3).

Proof. In the first step, we show that $\mathcal{H}\mathfrak{B}_\lambda \subset \mathfrak{B}_\lambda$, where the operator $\mathcal{H}$ is defined by (3.1) and $\mathfrak{B}_\lambda = \{x \in \mathcal{X} : \quad \|x\| \leq \lambda\}$, with

$$\lambda \geq \frac{\upsilon\sigma}{1-\theta\sigma}, \;\; \upsilon = \sup_{t\in[0,T]} |g(t, 0)|. \tag{3.5}$$

For $x \in \mathfrak{B}_\lambda$, in view of (N_1), we have

$$\begin{aligned}|g(t, x(t))| &= |g(t, x(t)) - g(t, 0) + g(t, 0)|\\ &\leq |g(t, x(t)) - g(t, 0)| + |g(t, 0)|\\ &\leq \theta|x(t)| + |g(t, 0)|\\ &\leq \theta\lambda + \upsilon.\end{aligned} \tag{3.6}$$

In view of (3.5) and (3.6), we obtain

$$\begin{aligned}\|(\mathcal{H}x)\| = \sup_{t\in[0,T]} \Bigg\{ &\int_0^t e^{-k(t-s)}\bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}|g(p, x(p))|dp\bigg)ds\\ &+|v_1(t)|\Big[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)}|g(p, x(p))|dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)}|g(p, x(p))|dp\Big]\\ &+|v_2(t)|\Big[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)}|g(p, x(p))|dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)}|g(p, x(p))|dp\Big]\\ &+|v_3(t)|\Big[\int_0^T e^{-k(T-s)}\bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}|g(p, x(p))|dp\bigg)ds\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^a e^{-k(a-s)} \Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} |g(p,x(p))| dp \Big) ds \Bigg\} \\
&\leq (\theta\lambda + \upsilon) \sup_{t\in[0,T]} \Bigg\{ \int_0^t e^{-k(t-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} dp \Bigg) ds \\
&\quad + |v_1(t)| \Bigg[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} dp \Bigg] \\
&\quad + |v_2(t)| \Bigg[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} dp \Bigg] \\
&\quad + |v_3(t)| \Bigg[\int_0^T e^{-k(T-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} dp \Bigg) ds \\
&\quad + \int_0^a e^{-k(a-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} dp \Bigg) ds \Bigg] \Bigg\} \\
&\leq (\theta\lambda + \upsilon)\sigma \\
&\leq \lambda,
\end{aligned}
$$

which implies that $\mathcal{H}\mathfrak{B}_\lambda \subset \mathfrak{B}_\lambda$ as $x \in \mathfrak{B}_\lambda$ is an arbitrary element.

Now, we show that the operator $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ defined by (3.1) is a contraction. For that, let $x, y \in \mathbb{R}$. Then, for each $t \in [0,T]$, it follows by the condition (N_1) that

$$
\begin{aligned}
&\|(\mathcal{H}x) - (\mathcal{H}y)\| \\
&= \sup_{t\in[0,T]} |(\mathcal{H}x)(t) - (\mathcal{H}y)(t)| \\
&\leq \sup_{t\in[0,T]} \Bigg\{ \int_0^t e^{-k(t-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \Bigg) ds \\
&\quad + |v_1(t)| \Bigg[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \\
&\quad + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \Bigg] \\
&\quad + |v_2(t)| \Bigg[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \\
&\quad + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \Bigg] \\
&\quad + |v_3(t)| \Bigg[\int_0^T e^{-k(t-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \Bigg) ds \\
&\quad + \int_0^a e^{-k(t-s)} \Bigg(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big| g(p,x(p)) - g(p,y(p)) \Big| dp \Bigg) ds \Bigg] \Bigg\} \\
&\leq \theta\sigma \|x - y\|,
\end{aligned}
$$

where σ is given by (3.3).

From the above inequality, it follows that the operator $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ is a contraction according to the given condition $\theta\sigma < 1$. Therefore, the operator $\mathcal{H} : \mathcal{X} \to \mathcal{X}$ has a unique fixed point by the Banach's fixed point theorem. In consequence, the problem (1.1)-(1.2) has a unique solution on $[0, T]$. □

3.2. Existence result

Here, we prove an existence result for the problem (1.1)-(1.2). This result is based on a fixed point theorem due to Krasnosel'skiĭ [15], which is stated below.

Theorem 3.2. *Let M be a closed convex and nonempty subset of a Banach space X. Let $\mathbb{A}, \mathbb{B} : M \to X$ be the operators such that (i) $\mathbb{A}x + \mathbb{B}y \in M$ whenever $x, y \in M$; (ii) $\mathbb{A}$ is compact and continuous and (iii) $\mathbb{B}$ is a contraction mapping. Then, $\mathbb{A}x + \mathbb{B}x = x$.*

Theorem 3.3. *Let $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (N_1). In addition, we assume that the following condition holds:*

(N_2) $|g(t, x(t))| \leq \varpi(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}$, *and* $\varpi \in C([0, T], \mathbb{R}^+)$.

Then, there exists at least one solution for the problem (1.1)-(1.2) on $[0, T]$ if $\theta\sigma_1 < 1$, where σ_1 is given by (3.4) and θ is defined in (N_1).

Proof. Let us introduce a closed and bounded ball as $\mathfrak{B}_\varepsilon = \{x \in \mathcal{X} : \|x\| \leq \varepsilon\}$ with $\varepsilon \geq \|\varpi\|\sigma$, where $\sup_{t\in[0.T]} |\varpi(t)| = \|\varpi\|$ and σ is given by (3.3). Now, we verify the hypothesis of Krasnosel'skiĭ's fixed point theorem in three steps by introducing two operators $\mathcal{H}_1, \mathcal{H}_2 : \mathfrak{B}_\varepsilon \to \mathcal{X}$ as

$$
\begin{aligned}
(\mathcal{H}_1 x)(t) &= \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p, x(p)) dp \right) ds, \ t \in [0, T], \\
(\mathcal{H}_2 x)(t) &= v_1(t) \left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} h(p) dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} h(p) dp \right] \\
&\quad + v_2(t) \left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} h(p) dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} h(p) dp \right] \\
&\quad + v_3(t) \left[\int_0^T e^{-k(t-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} h(p) dp \right) ds \right. \\
&\quad \left. + \int_0^a e^{-k(t-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} h(p) dp \right) ds \right], \ t \in [0, T].
\end{aligned}
$$

Notice that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, where the operator $\mathcal{H}$ is defined in (3.2).

For $x, y \in \mathfrak{B}_\varepsilon$, we have

$$
\begin{aligned}
\|\mathcal{H}_1 x + \mathcal{H}_2 y\| \leq \sup_{t\in[0,T]} \Bigg| \Bigg\{ & \int_0^t e^{-k(t-s)} \left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p, x(p)) dp \right) ds \\
& + v_1(t) \left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p, y(p)) dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p, y(p)) dp \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ v_2(t)\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} g(p,y(p))dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} g(p,y(p))dp\right] \\
&+ v_3(t)\left[\int_0^T e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,y(p))dp\right)ds \right.\\
&\left.+ \int_0^a e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} g(p,y(p))dp\right)ds\right]\Bigg| \\
\le& \|\varpi\|\sigma \\
\le& \varepsilon.
\end{aligned}
$$

Thus, $\mathcal{H}_1 x + \mathcal{H}_2 y \in \mathfrak{B}_\varepsilon$. Next, it will be shown that the operator $\mathcal{H}_2$ is a contraction. For that, let $x, y \in \mathbb{R}$. Then, for each $t \in [0,T]$, we obtain by using the condition (N_1) that

$$
\begin{aligned}
&\|(\mathcal{H}_2 x) - (\mathcal{H}_2 y)\| \\
=& \sup_{t\in[0,T]} |(\mathcal{H}_2 x)(t) - (\mathcal{H}_2 y)(t)| \\
\le& \sup_{t\in[0,T]} \Bigg\{ |v_1(t)| \left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp \right.\\
&\left.+ \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp\right] \\
&+ |v_2(t)| \left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp \right.\\
&\left.+ \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp\right] \\
&+ |v_3(t)| \left[\int_0^T e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp\right)ds \right.\\
&\left.+ \int_0^a e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} \Big|g(p,x(p)) - g(p,y(p))\Big| dp\right)ds\right]\Bigg\} \\
\le& \theta\sigma_1 \|x-y\|,
\end{aligned}
$$

which implies that the operator $\mathcal{H}_2$ is a contraction as $\theta\sigma_1 < 1$, where σ_1 is given by (3.4).

Finally, it will be verified that $\mathcal{H}_1$ is compact on $\mathcal{B}_\varepsilon$. Notice that continuity of the operator $\mathcal{H}_1$ follows from that of the nonlinear function $g(t, x(t))$. Also, $\mathcal{H}_1$ is uniformly bounded on $\mathfrak{B}_\varepsilon$ as

$$
\|H_1 x\| \quad \le \quad \frac{T^{\alpha-1}(1-e^{-kT})}{k\Gamma(\alpha)}\|\varpi\|.
$$

Now we show that the operator $\mathcal{H}_1$ is equicontinuous. Letting $t_1, t_2 \in [0,T]$ with $t_2 < t_1$, we have

$$
|(\mathcal{H}_1 x)(t_1) - (\mathcal{H}_1)(t_2)|
$$

$$
\begin{aligned}
=&\Big|\int_0^{t_1} e^{-k(t_1-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}g(p,x(p))dp\Big)ds \\
&-\int_0^{t_2} e^{-k(t_2-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}g(p,x(p))dp\Big)ds\Big| \\
=&\Big|\int_0^{t_2}\Big(e^{-k(t_1-s)}-e^{-k(t_2-s)}\Big)\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}g(p,x(p))dp\Big)ds \\
&+\int_{t_2}^{t_1} e^{-k(t_1-s)}\Big(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}g(p,x(p))dp\Big)ds\Big| \\
\le&\frac{\|\varpi\|}{k\Gamma(\alpha)}\Big(1-e^{-k(t_1-t_2)}\Big)\Big(t_1^{\alpha-1}+t_2^{\alpha-1}(1-e^{-kt_2})\Big),
\end{aligned}
$$

which tends to 0 as $t_1-t_2\to 0$, independent of $x\in\mathfrak{B}_\varepsilon$. So $\mathcal{H}_1$ is equicontinuous. In consequence, it follows by an application of the Arzelá–Ascoli theorem that $\mathcal{H}_1$ is completely continuous. Thus, all the assumptions of Theorem 3.2 are satisfied. Hence, the problem (1.1)-(1.2) has at least one solution on $[0,T]$. □

Example 3.1. Consider the problem

$$
\begin{cases}
({}^CD^{2.5}+\ {}^CD^{1.5})x(t)=g(t,x), \quad t\in[0,2],\\
x(0.1)=-x(2), \quad x'(0.1)=-x'(2), \quad x''(0.1)=-x''(2),
\end{cases}
$$

where $\alpha=2.5, T=2, a=0.1, k=1$ and $g(t,x)=\dfrac{1}{t^2+10}(\sqrt{x^2+25}+\sin t)$.

Notice that $\theta=1/10$ as

$$
|g(t,x)-g(t,y)|\le\frac{1}{10}|x-y|.
$$

Using the given data, we find that $\overline{v}_1=0.19812$ (at $t=0.65376$), $\overline{v}_2=0.46138$, $\overline{v}_3=0.96138$ and $\sigma=4.9901$ (σ is given by (3.3)). Moreover, $\theta\sigma=0.49901<1$. Thus, the hypothesis of Theorem (3.1) holds true and hence it follows by its conclusion that the problem (3.1) has a unique solution on $[0,2]$.

Example 3.2. Consider the problem (3.1) with

$$
g(t,x)=\frac{e^{-t}}{\sqrt{t^4+121}}\Big(\frac{|x|}{1+|x|}+\cos x\Big)+\frac{1}{5}. \tag{3.7}
$$

Using the data given in Example 3.1, it is found that $\sigma_1=3.15036$ and $g(t,x)$ satisfies the assumptions (N_1) with $\theta=2/11$ and (N_2) with

$$
\varpi(t)=\frac{2e^{-t}}{\sqrt{t^4+121}}+\frac{1}{5}.
$$

Furthermore, $\theta\sigma_1=0.57279$. Clearly, all the assumptions of Theorem 3.3 are satisfied. Therefore, by the conclusion of Theorem 3.3, the the problem (3.1) with $g(t,x)$ given by (3.7) has at least one solution on $[0,2]$.

4. Sequential fractional differential equation of order $3 < \alpha \leq 4$

In this section, we consider a parametric type anti-periodic boundary value problem for a sequential fractional differential equation of order $3 < \alpha \leq 4$ given by

$$({}^cD^\alpha + k^cD^{\alpha-1})x(t) = f(t, x(t)),\ 3 < \alpha \leq 4,\ t \in [0, T], \tag{4.1}$$

$$x(a) = -x(T),\ x'(a) = -x'(T),\ x''(a) = -x''(T),\ x'''(a) = -x'''(T),\ 0 < a << T, \tag{4.2}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. The problem (4.1)-(4.2) can by regarded as is an initially perturbed anti-periodic boundary value problem in presence of the condition $0 < a << T$.

Now, we present a lemma for the linear variant of the problem (4.1)-(4.2).

Lemma 4.1. *For $\psi \in C([0, T], \mathbb{R})$, the unique solution of the linear sequential fractional differential equation*

$$({}^cD^\alpha + k^cD^{\alpha-1})x(t) = \psi(t),\ 3 < \alpha \leq 4, \tag{4.3}$$

subject to the boundary conditions (4.2) *is given by*

$$\begin{aligned} x(t) = &\int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}\psi(p)dp\right)ds \\ &+ w_1(t)\left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)}\psi(p)dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)}\psi(p)dp\right] \\ &+ w_2(t)\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)}\psi(p)dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)}\psi(p)dp\right] \\ &+ w_3(t)\left[\int_0^T \frac{(T-p)^{\alpha-4}}{\Gamma(\alpha-3)}\psi(p)dp + \int_0^a \frac{(a-p)^{\alpha-4}}{\Gamma(\alpha-3)}\psi(p)dp\right] \\ &+ w_4(t)\left[\int_0^a e^{-k(a-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}\psi(p)dp\right)ds \right. \\ &\left. + \int_0^T e^{-k(T-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)}\psi(p)dp\right)ds\right], \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} w_1(t) &= \frac{-1}{2k} + \frac{e^{-kt}}{k(e^{-ka} + e^{-kT})},\ w_2(t) = \frac{k(a+T)+2}{4k^2} - \frac{t}{2k} - \frac{e^{-kt}}{k^2(e^{-ka} + e^{-kT})}, \\ w_3(t) &= -\frac{aTk^2 + k(a+T) + 4}{4k^3} + \frac{kt(a+T)+2t}{4k^2} - \frac{t^2}{4k} + \frac{e^{-kt}}{k^3(e^{-ka} + e^{-kT})}, \\ w_4(t) &= \frac{-e^{-kt}}{e^{-ka} + e^{-kT}}. \end{aligned} \tag{4.5}$$

Proof. As argued in the proof of Lemma 2.1, the solution of (4.3) can be written as

$$x(t) = \mathcal{E}_0 + \mathcal{E}_1 t + \mathcal{E}_2 t^2 + \mathcal{E}_3 e^{-kt} + \int_0^t e^{-k(t-s)} I^{\alpha-1}\psi(s)ds, \tag{4.6}$$

where $\mathcal{E}_i, i = 0, 1, 2, 3$, are unknown arbitrary constants. Using (4.6) in the boundary conditions (4.2), we find that

$$\begin{aligned}
\mathcal{E}_0 &= \frac{1}{2}\Big(\frac{-aT}{2k} - \frac{a+T}{2k^2} - \frac{1}{k^3}\Big)\big[I^{\alpha-3}\psi(T) + I^{\alpha-3}\psi(a)\big] \\
&\quad + \frac{1}{2}\Big(\frac{a+T}{2k} + \frac{1}{k^2}\Big)\big[I^{\alpha-2}\psi(T) + I^{\alpha-2}\psi(a)\big] - \frac{1}{2k}\big[I^{\alpha-1}\psi(T) + I^{\alpha-1}\psi(a)\big], \\
\mathcal{E}_1 &= \left[\frac{a+T}{4k} + \frac{1}{2k^2}\right]\left[I^{\alpha-3}\psi(T) + I^{\alpha-3}\psi(a)\right] - \frac{1}{2k}\left[I^{\alpha-2}\psi(T) + I^{\alpha-2}\psi(a)\right], \\
\mathcal{E}_2 &= -\frac{1}{4k}\left(I^{\alpha-3}\psi(T) + I^{\alpha-3}\psi(a)\right), \\
\mathcal{E}_3 &= \frac{-1}{(e^{-ka} + e^{-kT})}\Big[\int_0^T e^{-k(T-s)} I^{\alpha-1}\psi(s)ds + \int_0^a e^{-k(a-s)} I^{\alpha-1}\psi(s)ds \\
&\quad - \frac{1}{k}\big(I^{\alpha-1}\psi(T) + I^{\alpha-1}\psi(a)\big) + \frac{1}{k^2}\big(I^{\alpha-2}\psi(T) + I^{\alpha-2}\psi(a)\big) \\
&\quad - \frac{1}{k^3}\big(I^{\alpha-3}\psi(T) + I^{\alpha-3}\psi(a)\big)\Big].
\end{aligned}$$

Inserting the above values in (4.6) together with the notation (4.5), we obtain the solution (4.4). □

In view of Lemma 4.1, the nonlinear problem (4.1)-(4.2) can be transformed into a fixed point problem as

$$x = \mathcal{Q}(x), \tag{4.7}$$

where $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned}
(\mathcal{Q}x) &= \int_0^t e^{-k(t-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p, p(x))dp\right)ds \\
&\quad + w_1(t)\left[\int_0^T \frac{(T-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p, p(x))dp + \int_0^a \frac{(a-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p, p(x))dp\right] \\
&\quad + w_2(t)\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)} f(p, p(x))dp + \int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)} f(p, p(x))dp\right] \\
&\quad + w_3(t)\left[\int_0^T \frac{(T-p)^{\alpha-4}}{\Gamma(\alpha-3)} f(p, p(x))dp + \int_0^a \frac{(a-p)^{\alpha-4}}{\Gamma(\alpha-3)} f(p, p(x))dp\right] \\
&\quad + w_4(t)\Bigg[\int_0^a e^{-k(a-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p, p(x))dp\right)ds \\
&\quad + \int_0^T e^{-k(T-s)}\left(\int_0^s \frac{(s-p)^{\alpha-2}}{\Gamma(\alpha-1)} f(p, p(x))dp\right)ds\Bigg], \quad t \in [0, T].
\end{aligned} \tag{4.8}$$

Next, we set the notation:

$$\begin{aligned}
\delta &= \frac{1}{k\Gamma(\alpha)}\Big\{T^{\alpha-1}(1 - e^{-kT})(1 + \overline{w}_4) + \overline{w}_4 a^{\alpha-1}(1 - e^{-ka}) + k\overline{w}_1(T^{\alpha-1} + a^{\alpha-1})\Big\} \\
&\quad + \frac{\overline{w}_2}{\Gamma(\alpha-1)}\Big[T^{\alpha-2} + a^{\alpha-2}\Big] + \frac{\overline{w}_3}{\Gamma(\alpha-2)}\Big[T^{\alpha-3} + a^{\alpha-3}\Big],
\end{aligned} \tag{4.9}$$

$$\delta_1 = \frac{1}{k\Gamma(\alpha)}\Big\{\overline{w}_4\Big(T^{\alpha-1}(1-e^{-kT}) + a^{\alpha-1}(1-e^{-ka}\Big) + k\overline{w}_1(T^{\alpha-1}+a^{\alpha-1})\Big\} + \frac{\overline{w}_2}{\Gamma(\alpha-1)}\Big[T^{\alpha-2}+a^{\alpha-2}\Big] + \frac{\overline{w}_3}{\Gamma(\alpha-2)}\Big[T^{\alpha-3}+a^{\alpha-3}\Big], \tag{4.10}$$

where $\overline{w}_i = \sup_{t\in[0,T]} |w_i(t)|, i = 1,2,3,4$ (w_i are given in (4.5)).

Now, we present the existence and uniqueness results for the problem (4.1)-(4.2). We do not provide the proofs for these results as the method of proof is similar to the one used in obtaining the results of the previous section.

Theorem 4.1. *Let $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the condition*

(M_1) $|f(t,x)-f(t,y)| \leq L|x-y|$, $\quad\forall t\in[0,T]$, $\quad x,y\in\mathbb{R}$, *where $L>0$ is the Lipschitz constant.*

Then, the problem (4.1)-(4.2) *has a unique solution on $[0,T]$, provided that $L\delta<1$, where δ is given in* (4.9).

Theorem 4.2. *Assume that $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying (M_1) and $|f(t,x(t))| \leq \varrho(t)$, $\forall (t,x)\in[0,T]\times\mathbb{R}$, where $\varrho\in C([0,T],\mathbb{R}^+)$. Then, the problem* (4.1)-(4.2) *has at least one solution on $[0,T]$ if $L\delta_1<1$, where δ_1 is given in* (4.10).

5. Conclusions

We established the existence criteria for solutions of sequential fractional differential equations of orders $\alpha\in(2,3]$ and $\alpha\in(3,4]$ supplemented with parametric type anti-periodic boundary conditions. Our results are useful when the anti-periodic phenomenon starts from a position after the initial value of the given domain.

It has been observed that the solution to the problem (4.1)-(4.2) for a sequential fractional differential equation of order $\alpha\in(3,4]$ (given by (4.7)) contains the solution to the problem (1.1)-(1.2) for a sequential fractional differential equation of order $\alpha\in(2,3]$ given in (3.1). In fact, if we extend the order α of a sequential fractional differential equation from $(2,3]$ to $(3,4]$ in the parametric type anti-periodic boundary value problem, then we have the following additional term in the solution of the parametric type anti-periodic boundary value problem for a sequential fractional differential equation of order $\alpha\in(2,3]$:

$$w_3(t)\Bigg[\int_0^T \frac{(T-p)^{\alpha-4}}{\Gamma(\alpha-3)}g(p,p(x))dp + \int_0^a \frac{(a-p)^{\alpha-4}}{\Gamma(\alpha-3)}g(p,p(x))dp\Bigg],$$

where

$$w_3(t) = -\frac{aTk^2+k(a+T)+4}{4k^3} + \frac{kt(a+T)+2t}{4k^2} - \frac{t^2}{4k} + \frac{e^{-kt}}{k^3(e^{-ka}+e^{-kT})}.$$

Moreover, the following term

$$\Big(\frac{k(T+a)+2}{4k^2} - \frac{t}{2k} - \frac{e^{-kt}}{k^2(e^{-ka}+e^{-kT})}\Big)$$

$$\times\left[\int_0^T \frac{(T-p)^{\alpha-3}}{\Gamma(\alpha-2)}g(p,x(p))dp+\int_0^a \frac{(a-p)^{\alpha-3}}{\Gamma(\alpha-2)}g(p,x(p))dp\right]$$

complemented to the solution of the parametric type anti-periodic boundary value problem for a sequential fractional differential equation of order $\alpha \in (1,2]$ discussed in Section 4 of [4] yields the solution to the problem (1.1)-(1.2). Thus, there exists a relationship between solutions of lower and higher orders sequential fractional differential equations with parametric type anti-periodic boundary conditions.

It is imperative to point out that the results obtained in this paper become the ones associated with anti-periodic boundary conditions when $a \to 0^+$. Hence, our results are novel in the given configuration and enrich the literature on fractional order anti-periodic boundary value problems.

Conflict of interest

The authors declare that they have no conflict of interest.

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