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GLOBAL STABILITY, BIFURCATIONS AND CHAOS CONTROL IN A DISCRETE AMENSALISM MODEL WITH COVER AND SATURATION EFFECT

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Abstract In this paper, we propose a discrete amensalism model incorporating cover and saturation effect. The discrete model is derived from its continuous counterpart using the piecewise constant parameter method. First, we determine the equilibria, outline the conditions for their existence, and analyze their local stability properties. Subsequently, by employing discrete equation deflation and iteration methods, we establish a set of sufficient conditions that ensure the global attractivity of the system. Our analysis demonstrates that the size of the cover and the natural growth rate of the second population significantly influence the global attractivity of the system. Next, we investigate the existence of both flip bifurcation and transcritical bifurcation near the positive and boundary equilibria under certain conditions, utilizing the central manifold theorem and bifurcation theory. To control chaos arising from flip bifurcation, we implement feedback control strategies, specifically the hybrid control method. Finally, numerical simulations verify the feasibility of our theoretical results. Our research reveals that both the presence of cover and saturation effect enhance the density of the first population, while an increase in the amensalism coefficient c_1 ultimately leads to the extinction of the first population.

Keywords Saturation effect, cover, global attractivity, flip bifurcation, transcritical bifurcation, chaos control.

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1. Introduction

In the natural world, the complex tapestry of life is woven through the intricate web of interactions between organisms, which shape the ecosystems they inhabit. These interactions encompass a variety of relationships, including but not limited to predation, competition, parasitism, and mutualism. Moreover, amensalism stands out as a unique interaction where one organism inflicts harm on another without being affected in return. For instance, the penicillium produces penicillin to inhibit the growth of gram-positive bacteria [40]. Similarly, juglone, a chemical found in the

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green husk of walnuts, is known for its potent inhibitory effects against Escherichia coli, highlighting the defensive properties that plants can possess [25]. Rodigiosin, a substance known for its stability and non-toxicity under normal environmental conditions, exhibits a significant inhibitory effect on the fungus Penicillium. This characteristic makes it particularly useful in the preservation of citrus fruits [9]. In the alpine meadows of Tibet, an interesting example of amensalism can be observed in the behavior of dominant herbivorous grasshoppers. Their unconscious natural hopping behavior reduces the feeding time and amount for caterpillar, thereby slowing their developmental rate. However, this behavior does not impact the performance of the grasshoppers themselves [38]. This observation underscores the subtle and often overlooked ways in which organisms can influence one another within their shared ecosystems.

In 2003, Sun [33] introduced a seminal model that pioneered the mathematical representation of amensalism interactions:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = r_1 x \left(\dfrac{k_1 - x - \alpha \mathrm{y}}{k_1} \right), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = r_2 y \left(\dfrac{k_2 - y}{k_2} \right), \end{cases} \tag{1.1}$$

where r_1, r_2, k_1, k_2, α are positive constants. The local stability of each equilibrium point was analyzed using the eigenvalue method as described in model (1.1). Subsequently, the permanence and extinction of the two populations were examined in [50]. Wang [35] demonstrated that under certain appropriate assumptions, all four non-negative equilibria in the amensalism model with density dependent birth rate may exhibit global attractivity. More relevant studies have been carried out to consider the impact of environmental factors, such as the effect of cover [5,37,39,47], the effect of harvesting [16,20,31,45], Allee effect [1,12,13,26,46,48], the effect of function response [46,49], the effect of delays [22,43], and fear effect [3,8,10,19,49].

Specifically, numerous nature reserves have been designated to safeguard endangered species and promote their proliferation. In biological terms, these conservation zones are often referred to as cover areas, which play a crucial role in mitigating inter-population interactions. Sita Rambabu, Narayan, and Bathul [32] investigated the effect of cover based on model (1.1) and established a mathematical framework that can be expressed as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_1 x - b_1 x^2 - c_1 (1-k) x y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = a_2 y - b_2 y^2, \end{cases} \tag{1.2}$$

where $k \in (0,1)$ represents the proportion of the first population x that is protected by the cover, and kx denotes the number of individuals from the first population remaining within the cover, unaffected by the second population. The authors derived approximate solutions using the Homotopy Analysis Method (HAM). More in-depth study of the above system was carried out by Xie, Chen, and He [39], they showed that the cover plays a decisive role in the permanence and extinction of populations and that when cover is large enough, the influence of the second population on the first is reduced, and the two populations can coexist stably in amensalism mode with cover for the first species.

In ecology and biomathematics, inter-species competition is a pivotal force in shaping community composition and maintaining species diversity. The study of competition models has become a widely pursued avenue among scholars investigating population dynamics [2, 11, 21, 24, 44]. Mathematically, these models elucidate the dynamic processes through which two or more species compete for limited resources. A quintessential example can be observed in a meadow, where sheep and rabbits engage in a competitive struggle over the shared resource of grass. The classical two-dimensional competition model, as proposed by Lotka-Volterra, takes the following form:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(r_1 - b_1 x - c_1 y\right), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(r_2 - b_2 y - c_2 x\right), \end{cases} \tag{1.3}$$

where, x and y represent the population densities of two distinct species at time t, respectively. Each parameter within the model (1.3) is a positive constant, ensuring the biological meaningfulness of the equations. The coefficients r_1 and r_2 denote the inherent growth rates of species one and two, respectively. b_1 and b_2 are associated with the intra-specific competition, respectively. c_1 and c_2 represent the inter-specific interaction coefficients, respectively. The terms $c_1 y$ and $c_2 x$ quantify the competitive impact exerted by one species upon another within the ecological community. The existence of a unique positive equilibrium point in system (1.3) was analyzed in [34]. This model, along with its various extensions, has garnered extensive attention and has been thoroughly examined in numerous scholarly works, as cited in the literature [2, 18, 44]. For instance, Chen and Teng [2] applied an iteration scheme and the comparison principle of difference equations to establish the global asymptotic stability of the positive equilibrium in a discrete (1.3). Jin and Li [18] applied the center manifold theorem and bifurcation theory to prove the occurrence of transcritical, period-doubling, and pitchfork bifurcations in a discrete model (1.3) with Michaelis-Menten type harvesting. Zhang, Zhen and Yan et al. [44] investigated stability of system and existence of Hopf bifurcation in a delayed competition system.

Noting that the influence of competition in equation (1.3) is linear, meaning that as the density of competitors increases, their competitiveness also increases without bound. However, in real ecological environments, it is more reasonable to consider that the effects of competition are nonlinear. Gopalsamy [11] proposed delay differential equations for population dynamics, which can be expressed in the following form

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x\left(t\right)\left(r_1 - b_1 x\left(t\right) - \dfrac{c_1 y\left(t-\tau\right)}{1+y\left(t-\tau\right)}\right), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y\left(t\right)\left(r_2 - b_2 y\left(t\right) - \dfrac{c_2 x\left(t-\tau\right)}{1+x\left(t-\tau\right)}\right), \end{cases} \tag{1.4}$$

where r_i, a_i and c_i, $(i = 1, 2)$ are positive constants, τ is a nonnegative constant. The authors investigated both the stability of the system and the existence of oscillations. Liu, Tan and Chen [24] further explored permanence and the existence of positive periodic solutions of (1.4) with multiple delays and impulsive perturbations. Li, Liu and Yuan [21] investigated stability of positive equilibrium and the existence of Turing pattern. Additional results on competition models with saturation effect can be found in [14, 28].

Noting that in model (1.2), the authors did not consider the saturation effect. Building upon the system (1.2), Wu, Zhao, and Lin [37] also argued that the influence of the second population on the first population has saturation effect and cannot grow indefinitely. Consequently, they proposed the following two-population amensalism model with cover and saturation effect.

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = a_1 x - b_1 x^2 - \dfrac{c_1(1-k)xy}{1+d_1 y}, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = a_2 y - b_2 y^2, \end{cases} \tag{1.5}$$

where $\frac{y}{1+d_1 y}$ denotes the saturation effect term, and d_1 represents the half-saturation constant. The authors investigated the global stability of positive and boundary equilibria of the system, respectively.

It is well established that discrete equations are more appropriate for describing populations with short life cycles and non-overlapping generations. Discrete models can be more easily numerically simulated by computers, making the study of discrete ecosystems particularly important. Discrete-time models can exhibit even more complex dynamics than their continuous-time counterparts [6, 7, 15, 18, 19, 29, 30, 42, 46–48]. To our knowledge, the discrete system corresponding to system (1.5) has not been studied previously. Therefore, by applying the piecewise constant parameter method described in [17], we discretize system (1.5) to obtain a discrete amensalism model with cover and saturation effect as follows:

$$\begin{cases} x(n+1) = x(n)\exp\left(a_1 - b_1 x(n) - \dfrac{c_1(1-k)y(n)}{1+d_1 y(n)}\right), \\ y(n+1) = y(n)\exp(a_2 - b_2 y(n)), \end{cases} \tag{1.6}$$

where a_1, a_2, b_1, b_2, c_1, k, d_1 are positive constants. Our analysis focuses on system (1.6) within the biologically meaningful region $\Omega_1 = \{(x,y) : x \geq 0, y \geq 0\}$ of the $(x,y)-$ plane. A distinctive feature of this paper is the incorporation of a saturation effect term, which we examine in detail to elucidate its influence on the system's dynamics. The saturation effect term has been demonstrated to play a significant role in promoting an increase in the density of the first population. In contrast, the second population remains unaffected by this term, indicating a selective impact on the dynamics of the two populations within the system. Our model exhibits a richer array of dynamical behaviors compared to its continuous counterpart, including phenomena such as flip bifurcation and transcritical bifurcation.

The remainder of this paper is organized as follows. Section 2 examines the existence and local stability of all equilibria within the system. Section 3 investigates the global attractivity of both positive and boundary equilibria, respectively. Section 4 delves into bifurcation analysis, covering flip and transcritical bifurcations. Section 5 addresses the topic of chaos control. Numerical simulations are subsequently performed to substantiate our findings. Finally, we present a summary and outline future research directions based on the work discussed herein.

2. Existence and local stability of equilibria

To delve into the existence and local stability of equilibria within the system (1.6), we must first engage in the calculation of the ensuing system of equations.

$$\begin{cases} x = x\exp\left(a_1 - b_1x - \dfrac{c_1(1-k)y}{1+d_1y}\right), \\ y = y\exp(a_2 - b_2y). \end{cases}$$

Through a straightforward computation, we get that the system (1.4) always has equilibria $O(0,0), E_1(\frac{a_1}{b_1},0),$ and $E_2(0,\frac{a_2}{b_2})$. When $c_1 < \frac{a_1(d_1y^*+1)}{y^*(1-k)} := c_1^{**}$, system (1.6) has a unique positive equilibrium $E^*(x^*,y^*)$, where

$$x^* = \frac{a_1 + a_1d_1y^* - c_1y^*(1-k)}{(d_1y^*+1)b_1}, \quad y^* = \frac{a_2}{b_2}.$$

The Jacobi matrix of system (1.6) at the equilibrium $E(x,y)$ is

$$J(E) = \begin{pmatrix} (1-b_1x)M_1 & -\dfrac{c_1(1-k)x}{(d_1y+1)^2}M_1 \\ 0 & (1-b_2y)M_2 \end{pmatrix},$$

where $M_1 = \exp\left(a_1 - b_1x - \frac{c_1(1-k)y}{d_1y+1}\right)$ and $M_2 = \exp(a_2 - b_2y)$.

Now, we define the discrete equations to determine the local stability of the equilibria (see, [2]).

Lemma 2.1. *Consider the system $x(k+1) = f(k,x(k))$, $x \in \mathbb{R}^2$, assuming that $E(x,y)$ is one of its equilibrium, and the characteristic equation of $J(E)$ is $F(\lambda) = \lambda^2 + B\lambda + C = 0$, where B and C are constants. λ_1 and λ_2 are its two eigenvalues, then the stability of the system at the equilibrium $E(x,y)$ is concluded as follows.*

1. *If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $E(x,y)$ is a sink and is locally asymptotically stable;*
2. *If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then $E(x,y)$ is a source and is unstable;*
3. *If either ($|\lambda_1| > 1$ and $|\lambda_2| < 1$) or ($|\lambda_1| < 1$ and $|\lambda_2| > 1$), then $E(x,y)$ is a saddle and is unstable;*
4. *If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then $E(x,y)$ is non-hyperbolic.*

Theorem 2.1. *The system (1.6) is always unstable at the trivial equilibrium $O(0,0)$, that is a source.*

Proof. The Jacobi matrix of (1.6) at the trivial equilibrium $O(0,0)$ is

$$J(O) = \begin{pmatrix} \mathrm{e}^{a_1} & 0 \\ 0 & \mathrm{e}^{a_2} \end{pmatrix},$$

it is clear that the two eigenvalues of $J(O)$ are $\lambda_1 = \mathrm{e}^{a_1}$ and $\lambda_2 = \mathrm{e}^{a_2}$. Since $a_1, a_2 > 0$, there are always $|\lambda_1| > 1$, and $|\lambda_2| > 1$. Therefore, by Lemma 2.1, the equilibrium $O(0,0)$ is a source and is unstable. The proof of Theorem 2.1 is completed. □

Theorem 2.2. *The system (1.6) always has a boundary equilibrium $E_1(\frac{a_1}{b_1},0)$ and $E_1(\frac{a_1}{b_1},0)$ is*

1. *a source, if $a_1 > 2$;*

(2) a saddle, if $0 < a_1 < 2$*;*

(3) non-hyperbolic, if $a_1 = 2$*;*

In any case, $E_1(\frac{a_1}{b_1}, 0)$ *is always unstable.*

Proof. The Jacobi matrix of (1.6) at the boundary equilibrium $E_1(\frac{a_1}{b_1}, 0)$ is

$$J(E_1) = \begin{pmatrix} 1 - a_1 & -\dfrac{a_1 c_1 (1-k)}{b_1} \\ 0 & \mathrm{e}^{a_2} \end{pmatrix},$$

obviously, the two eigenvalues of $J(E_1)$ are $\lambda_1 = 1 - a_1$ and $\lambda_2 = \mathrm{e}^{a_2}$. Since $a_2 > 0$, there is $|\lambda_2| > 1$, then $E_1(\frac{a_1}{b_1}, 0)$ is always unstable. When $a_1 > 2$, there is $\lambda_1 < -1$, i.e. $|\lambda_1| > 1$, combined with $|\lambda_2| > 1$, so by Lemma 2.1, E_1 is a source. When $a_1 < 2$, there is $-1 < \lambda_1 < 1$, E_1 is a saddle. When $a_1 = 2$, $\lambda_1 = -1$, i.e. $|\lambda_1| = 1$, then E_1 is non-hyperbolic. The proof of Theorem 2.2 is completed. □

Theorem 2.3. *The system (1.6) always has a boundary equilibrium* $E_2(0, \frac{a_2}{b_2})$ *and* $E_2(0, \frac{a_2}{b_2})$ *is*

(1) a sink, if $c_1 > c_1^{**}$ *and* $a_2 < 2$*;*

(2) a source, if $c_1 < c_1^{**}$ *and* $a_2 > 2$ *hold;*

(3) a saddle and unstable, if one of the following conditions holds

  (i) $c_1 < c_1^{**}$ *and* $a_2 < 2$*,*

  (ii) $c_1 > c_1^{**}$ *and* $a_2 > 2$*;*

(4) non-hyperbolic, if either $c_1 = c_1^{**}$ *or* $a_2 = 2$*.*

Proof. The Jacobi matrix of system (1.6) at boundary equilibrium $E_2(0, \frac{a_2}{b_2})$ is

$$J(E_2) = \begin{pmatrix} \mathrm{e}^{a_1 - \frac{c_1(1-k)y^*}{d_1 y^* + 1}} & 0 \\ 0 & 1 - a_2 \end{pmatrix},$$

clearly, the two eigenvalues of $J(E_2)$ are $\lambda_1 = \mathrm{e}^{a_1 - \frac{c_1(1-k)y^*}{d_1 y^* + 1}}$, and $\lambda_2 = 1 - a_2 < 1$. By a direct calculation, we have

$$\mathrm{e}^{a_1 - \frac{c_1(1-k)y^*}{d_1 y^* + 1}} \begin{cases} > 1, & \text{if } 0 < c_1 < c_1^{**}, \\ = 1, & \text{if } c_1 = c_1^{**}, \\ \in (0,1), & \text{if } c_1 > c_1^{**}, \end{cases}$$

and

$$1 - a_2 \begin{cases} \in (-1,1), & \text{if } 0 < a_2 < 2, \\ = -1, & \text{if } a_2 = 2, \\ < -1, & \text{if } a_2 > 2. \end{cases}$$

Thus, when $c_1 > c_1^{**}$ and $a_2 < 2$, then $0 < \lambda_1 < 1$, $-1 < \lambda_2 < 1$, by Lemma 2.1, E_2 is a sink and is locally asymptotically stable. Similarly, it can be proved that, (2), (3), (4) hold. The proof of Theorem 2.3 is completed. □

Theorem 2.4. *When $0 < c_1 < c_1^{**}$, topological properties of the positive equilibrium $E^*(x^*, y^*)$ of (1.6) are shown in Table 1.*

Table 1. Topological properties of positive equilibrium $E^*(x^*, y^*)$.

Conditions			Eigenvalues		Properties
			λ_1	λ_2	
$0 < a_1 \leq 2$	$0 < a_2 < 2$	$0 < c_1 < c_1^{**}$	$\|\lambda_1\| < 1$	$\|\lambda_2\| < 1$	sink
	$a_2 = 2$			$\|\lambda_2\| = 1$	non-hyperbolic
	$a_2 > 2$			$\|\lambda_2\| > 1$	saddle
$a_1 > 2$	$0 < a_2 < 2$	$c_1 < c_1^*$	$\|\lambda_1\| > 1$	$\|\lambda_2\| < 1$	saddle
		$c_1 = c_1^*$	$\|\lambda_1\| = 1$		non-hyperbolic
		$c_1^* < c_1 < c_1^{**}$	$\|\lambda_1\| < 1$		sink
	$a_2 = 2$	$c_1 < c_1^*$	$\|\lambda_1\| > 1$	$\|\lambda_2\| = 1$	non-hyperbolic
		$c_1 = c_1^*$	$\|\lambda_1\| = 1$		non-hyperbolic
		$c_1^* < c_1 < c_1^{**}$	$\|\lambda_1\| < 1$		non-hyperbolic
	$a_2 > 2$	$c_1 < c_1^*$	$\|\lambda_1\| > 1$	$\|\lambda_2\| > 1$	source
		$c_1 = c_1^*$	$\|\lambda_1\| = 1$		non-hyperbolic
		$c_1^* < c_1 < c_1^{**}$	$\|\lambda_1\| < 1$		saddle

Proof. The Jacobi matrix of (1.6) at positive equilibrium $E^*(x^*, y^*)$ is

$$J(E^*) = \begin{pmatrix} 1 - b_1 x^* & -\dfrac{c_1(1-k)x^*}{(d_1 y^* + 1)^2} \\ 0 & 1 - a_2 \end{pmatrix},$$

clearly, the two eigenvalues of $J(E^*)$ are $\lambda_1 = 1 - b_1 x^* < 1$, and $\lambda_2 = 1 - a_2 < 1$. By a direct calculation, we have

$$\lambda_1 = 1 - b_1 x^* = 1 - a_1 + \frac{c_1(1-k)y^*}{d_1 y^* + 1}.$$

Let $\lambda_1 = -1$, we have $c_1 = \frac{(a_1-2)(d_1 y^*+1)}{(1-k)y^*} := c_1^*$. Further, when $a_1 > 2$ and $0 < c_1 < c^{**}$, there are

$$1 - b_1 x^* \begin{cases} < -1, & \text{if } 0 < c_1 < c_1^*, \\ = -1, & \text{if } c_1 = c_1^*, \\ \in (-1, 1), & \text{if } c_1^* < c_1 < c_1^{**}. \end{cases}$$

Since $\lambda_2 = 1 - a_2$ has been analysed in Theorem 2.3. Thus, the proof of Theorem 2.4 is completed. □

Remark 2.1. The system (1.6) has in common with the corresponding continuous system (1.5), the fact that equilibria $O(0,0)$ and $E_1(\frac{a_1}{b_1}, 0)$ are both unstable. In system (1.5) (see, by Theorem 2.1 of [37]), the boundary equilibrium $E_2(0, \frac{a_2}{b_2})$ is locally asymptotically stable if $k < 1 - \frac{a_1 a_2 d_1 + a_1 b_2}{a_2 c_1}$. But in discrete system (1.6), it follows from the Theorem 2.3 that if $c_1 > c_1^{**} \Longleftrightarrow k < 1 - \frac{a_1 a_2 d_1 + a_1 b_2}{a_2 c_1}$ and $a_2 < 2$, $E_2(0, \frac{a_2}{b_2})$ is locally stable. Thus, in contrast to the conclusion for the continuous system (1.5), the natural growth rate of the second population in the system (1.6) plays a vital role in the stability of the boundary equilibrium $E_2(0, \frac{a_2}{b_2})$. In system (1.5) (see, by Theorem 2.1 of [37]), it follows that positive equilibrium $E^*(x^*, y^*)$ is locally asymptotically stable if $k > 1 - \frac{a_1 a_2 d_1 + a_1 b_2}{a_2 c_1}$. However, the positive equilibrium $E^*(x^*, y^*)$ is the sink, and is locally asymptotically stable from Table 1, if (a) $0 < a_1 \leq 2$, $0 < a_2 < 2$, $0 < c_1 < c_1^{**}$ or (b) $a_1 > 2$, $0 < a_2 < 2$, $c_1^* < c_1 < c_1^{**} \Longleftrightarrow 1 - \frac{a_1 a_2 d_1 + a_1 b_2}{a_2 c_1} < k < \min\{1 - \frac{a_1 a_2 d_1 + a_1 b_2}{a_2 c_1} + \frac{2 a_2 d_1 + 2 b_2}{a_2 c_1}, 1\}$. Therefore, the natural growth rate of the second population and cover in the discrete system (1.6) play essential roles in the stability of the positive equilibrium $E^*(x^*, y^*)$, and by Theorem 2.4, the topological properties of the positive equilibrium $E^*(x^*, y^*)$ in the system (1.6) are more complex.

Remark 2.2. If the positive equilibrium $E^*(x^*, y^*)$ exists, there is

$$c_1 < c_1^{**} \Longleftrightarrow a_1(d_1 y^* + 1) - c_1 y^* (1 - k) > 0,$$

and following results can be obtained.

$$\begin{aligned} \frac{\partial(x^*)}{\partial k} &= \frac{c_1 a_2}{b_1(d_1 a_2 + b_2)} > 0, \\ \frac{\partial(x^*)}{\partial d_1} &= \frac{y^2 c_1 (1 - k)}{b_1 (d_1 y + 1)^2} > 0, \\ \frac{\partial(y^*)}{\partial k} &= 0, \\ \frac{\partial(y^*)}{\partial d_1} &= 0. \end{aligned} \tag{2.1}$$

Thus, x^* is strictly monotonically increasing for the parameters k and d_1, and as the cover increases, more and more of the first population can stay in the cover. Thus, the final equilibrium density of the population increases. The cover k and the half-saturation constant d_1 do not affect the second population y.

3. Global attraction

From Theorems 2.3 and 2.4, it follows that positive equilibrium $E^*(x^*, y^*)$ and the boundary equilibrium $E_2(0, \frac{a_2}{b_2})$ are sink and locally asymptotically stable under appropriate conditions. Next, we will provide sufficient conditions to ensure they are globally attractive.

Theorem 3.1. *Assume that*

$$0 < a_2 < 2,\ 0 < a_1 - \frac{c_1(1 - k) y^*}{1 + d_1 y^*} < \ln 2 + 1 \tag{3.1}$$

hold, then positive equilibrium $E^*(x^*, y^*)$ of system (1.6) is globally attractive, i.e.

$$\lim_{n \to +\infty} x(n) = x^*, \qquad \lim_{n \to +\infty} y(n) = y^*,$$

where $(x(n), y(n))$ is any positive solution of system (1.6).

Proof. When $a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} > 0$ holds, there exist 4 equilibria $O(0,0)$, $E_1(\frac{a_1}{b_1}, 0)$, $E_2(0, \frac{a_2}{b_2})$ and $E^*(x^*, y^*)$ of system (1.6), and by Theorem 2.1, O is a source, and when (3.1) holds, we have $0 < a_2 < 2$, and by Theorem 2.2, E_1 is a saddle. By Theorem 2.3, E_2 is a saddle, and all 3 equilibria are unstable. When (3.1) hold, there are $0 < a_2 < 2$, $0 < a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} < \ln 2 + 1 < 2$, the positive equilibrium E^* is a sink, which is locally asymptotically stable. Next, we will show that it is globally attractive.

Assume that $0 < a_2 < 2$ holds, it follows from Lemma 4 (see, [2]) that

$$\lim_{n\to+\infty} y(n) = \frac{a_2}{b_2} = y^*. \tag{3.2}$$

Thus, for any sufficiently small $\varepsilon > 0$, there exists a positive integer N_1 such that if $n \geq N_1$, there are

$$y^* - \varepsilon < y(n) < y^* + \varepsilon. \tag{3.3}$$

Now we consider the system

$$x_2(n+1) = x_2(n) \exp\left(a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} - b_1x_2(n)\right), \quad n \in \mathbb{N}, \tag{3.4}$$

the system (3.4) has a unique positive equilibrium $x^* = \frac{a_1(1+d_1y^*)-c_1(1-k)y^*}{b_1(1+d_1y^*)}$. $x_2(n)$ is any positive solution of system (3.4) if $0 < a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} < 2$. From Lemma 4 (see, [2]), it follows that $\lim_{n\to+\infty} x_2(n) = x^*$. Therefore, in order to prove that $\lim_{n\to+\infty} x(n) = x^*$, it is necessary to show that $\lim_{n\to+\infty}[x(n) - x_2(n)] = 0$. Let

$$x(n) = x_2(n)\exp[k(n)].$$

By the differential median theorem, the first equation of the system (1.6) reduces to

$$\begin{aligned}
& k(n+1) \\
&= \ln \frac{x(n+1)}{x_2(n+1)} \\
&= \ln \frac{x(n)\exp\left(a_1 - b_1x(n) - \frac{c_1(1-k)y(n)}{1+d_1y(n)}\right)}{x_2(n)\exp\left(a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} - b_1x_2(n)\right)} \\
&= \ln\frac{x(n)}{x_2(n)} - b_1x(n) - \frac{c_1(1-k)y(n)}{1+d_1y(n)} + b_1x_2(n) + \frac{c_1(1-k)y^*}{1+d_1y^*} \\
&= k(n) - b_1x_2(n)\exp[k(n)] - \frac{c_1(1-k)y(n)}{1+d_1y(n)} + b_1x_2(n) + \frac{c_1(1-k)y^*}{1+d_1y^*} \\
&= k(n) - b_1x_2(n)\{\exp[k(n)] - 1\} - \frac{c_1(1-k)y(n)}{1+d_1y(n)} + \frac{c_1(1-k)y^*}{1+d_1y^*} \\
&= k(n)\{1 - b_1x_2(n)\exp[\theta(n)k(n)]\} - \frac{c_1(1-k)y(n)}{1+d_1y(n)} + \frac{c_1(1-k)y^*}{1+d_1y^*},
\end{aligned} \tag{3.5}$$

where $\theta(n) \in [0,1]$, and $x_2(n)\exp[\theta(n)k(n)]$ between $x_2(n)$ and $x(n)$. In order to prove that $\lim_{n\to+\infty}[x(n) - x_2(n)] = 0$, it is sufficient to show that

$$\lim_{n\to+\infty} k(n) = 0.$$

From (3.3), if $n \geq N_1$, we have

$$x(n+1) \geq x(n)\exp\left(a_1 - \frac{c_1(1-k)(y^*+\varepsilon)}{1+d_1(y^*+\varepsilon)} - b_1x(n)\right),$$

$$x(n+1) \leq x(n)\exp\left(a_1 - \frac{c_1(1-k)(y^*-\varepsilon)}{1+d_1(y^*-\varepsilon)} - b_1x(n)\right),$$

by Lemma 1 and 2 (see, [41]), we have

$$\begin{aligned}\limsup_{n\to\infty} x(n) &\leq \frac{1}{b_1}\exp\left[a_1 - \frac{c_1(1-k)(y^*-\varepsilon)}{1+d_1(y^*-\varepsilon)} - 1\right] := U_1,\\ \liminf_{n\to\infty} x(n) &\geq \frac{1}{b_1}\left[a_1 - \frac{c_1(1-k)(y^*+\varepsilon)}{1+d_1(y^*+\varepsilon)}\right]\exp\left[a_1 - \frac{c_1(1-k)(y^*+\varepsilon)}{1+d_1(y^*+\varepsilon)} - b_1U_1\right]\\ &:= V_1.\end{aligned}$$

Moreover, from the system (3.4), Lemma 1 and Lemma 2 (see, [41]), it follows that there are

$$\begin{aligned}\limsup_{n\to\infty} x_2(n) &\leq \frac{1}{b_1}\exp\left[a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} - 1\right] \leq U_1,\\ \liminf_{n\to\infty} x_2(n) &\geq \frac{1}{b_1}\left[a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*}\right]\exp\left[a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} - b_1U_1\right] \geq V_1.\end{aligned}$$

Then for any sufficiently small $\varepsilon > 0$, there exists a positive integer $N_2 > N_1$, such that when $n \geq N_2$, there are

$$V_1 - \varepsilon \leq x_2(n),\ x(n) \leq U_1 + \varepsilon, \quad n \geq N_2. \tag{3.6}$$

Assume that

$$\lambda = \max\{|1 - b_1V_1|, |1 - b_1U_1|\} < 1.$$

Then, for any sufficiently small $\varepsilon > 0$, we can make the

$$\lambda_\varepsilon = \max\{|1 - b_1(V_1 - \varepsilon)|, |1 - b_1(U_1 + \varepsilon)|\} < 1. \tag{3.7}$$

Based on (3.3), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}|k(n+1)| &\leq \max\{|1 - b_1(V_1-\varepsilon)|, |1 - b_1(U_1+\varepsilon)|\}|k(n)| + \frac{c_1(1-k)\varepsilon}{1+d_1(y^*-\varepsilon)}\\ &\leq \lambda_\varepsilon|k(n)| + \frac{c_1(1-k)\varepsilon}{1+d_1(y^*-\varepsilon)}, \qquad n \geq N_2.\end{aligned} \tag{3.8}$$

From (3.8), there is

$$|k(n)| \leq \lambda_\varepsilon^{n-N_2}|k(N_2)| + \frac{(1-\lambda_\varepsilon^{n-N_2})[c_1(1-k)\varepsilon]}{(1-\lambda_\varepsilon)[1+d_1(y^*-\varepsilon)]}, \quad n \geq N_2.$$

Because $\lambda_\varepsilon < 1$ and ε is arbitrarily small, we have $\lim_{n\to\infty} k(n) = 0$. That is, we have $\lim_{n\to+\infty} k(n) = 0$ holds if $\lambda < 1$. Noticed

$$1 - b_1U_1 < 1 - b_1V_1 < 1,$$

thus, $\lambda < 1$ is equivalent to

$$1 - b_1U_1 > -1,$$

i.e.

$$\exp\left(a_1 - \frac{c_1(1-k)(y^* - \varepsilon)}{1 + d_1(y^* - \varepsilon)} - 1\right) < 2, \tag{3.9}$$

since ε is sufficiently small, and (3.1) holds, we have that (3.9) also holds, i.e., we have $\lim_{n\to+\infty}[x(n) - x_2(n)] = 0$.

The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that*

$$a_1 < \frac{a_2c_1(1-k)}{b_2 + a_2d_1},\ \ 0 < a_2 < 2 \tag{3.10}$$

hold, $(x(n), y(n))$ *is any positive solution of system (1.6), then there are*

$$\lim_{n\to+\infty} x(n) = 0, \qquad \lim_{n\to+\infty} y(n) = y^*,$$

i.e., boundary equilibrium $E_2(0, y^*)$ *is globally attractive.*

Proof. From the inequality (3.10), we have

$$\frac{a_1}{a_2} < \frac{c_1(1-k) - a_1d_1}{b_2}.$$

Take the positive constants γ and ζ, such that

$$\frac{a_1}{a_2} < \frac{\zeta}{\gamma} < \frac{c_1(1-k) - a_1d_1}{b_2},$$

thus

$$\gamma[c_1(1-k) - a_1d_1] - \zeta b_2 > 0, \tag{3.11}$$

and there exists a $\psi > 0$, making

$$\gamma a_1 - \zeta a_2 < -\psi < 0. \tag{3.12}$$

If $(x(n), y(n))$ is any positive solution of system (1.6), then for any $p \in \mathbb{N}$, there are

$$\begin{aligned}
\ln\frac{x(p+1)}{x(p)} &= a_1 - b_1x(p) - \frac{c_1(1-k)y(p)}{1 + d_1y(p)} \\
&= \frac{a_1 - b_1x(p) + a_1d_1y(p) - b_1d_1x(p)y(p) - c_1(1-k)y(p)}{1 + d_1y(p)} \\
&\le a_1 - b_1x(p) + a_1d_1y(p) - c_1(1-k)y(p), \\
\ln\frac{y(p+1)}{y(p)} &= a_2 - b_2y(p).
\end{aligned} \tag{3.13}$$

According to (3.11), (3.12) and (3.13), it is obtained that

$$\begin{aligned}
&\gamma\ln\frac{x(p+1)}{x(p)} - \zeta\ln\frac{y(p+1)}{y(p)} \\
&\le \gamma a_1 - \zeta a_2 - \gamma b_1x(p) - \{\gamma[c_1(1-k) - a_1d_1] - \zeta \mathrm{b}_2\}y(p) \\
&\le \gamma a_1 - \zeta a_2 < -\psi < 0.
\end{aligned} \tag{3.14}$$

Adding both sides of (3.14) from 0 to $n-1$ simultaneously, we get

$$\gamma \ln \frac{x(n)}{x(0)} - \zeta \ln \frac{y(n)}{y(0)} < -\psi n,$$

then

$$x(n) \leq \left[\left(\frac{y(n)}{y(0)} \right)^{\zeta} x(0)^{\gamma} \right]^{\frac{1}{\gamma}} \exp\left(-\frac{\psi}{\gamma} n \right). \tag{3.15}$$

By Theorem 1 (see, [4]), for any sufficiently small $\varepsilon > 0$, there exists a positive integer N_3 such that when $n > N_3$, there is

$$y(n) < M_2 + \varepsilon.$$

Further, this can be obtained from (3.15)

$$x(n) \leq \left[\left(\frac{(M_2+\varepsilon)}{y(0)} \right)^{\zeta} x(0)^{\gamma} \right]^{\frac{1}{\gamma}} \exp\left(-\frac{\psi}{\gamma} n \right). \tag{3.16}$$

According to inequality (3.16), we have that when $n \to +\infty$, $x(n) \to 0$. Moreover, when $0 < a_2 < 2$, by Lemma 4 (see, [2]), we have that

$$\lim_{n\to+\infty} y(n) = \frac{a_2}{b_2} = y^*.$$

The proof of Theorem 3.2 is completed. □

4. Bifurcation analysis

In this section, we apply central manifold theorem [23] and bifurcation theory [36] to discuss transcritical bifurcation of system (1.6) at boundary equilibrium $E_2(0, \frac{a_2}{b_2})$, and flip bifurcation of system (1.6) at boundary equilibrium $E_1(\frac{a_1}{b_1}, 0)$ and positive equilibrium $E^*(x^*, y^*)$.

4.1. Flip bifurcation at boundary equilibrium $E_1(\frac{a_1}{b_1}, 0)$

Theorem 4.1. *System (1.6) undergoes a flip bifurcation at boundary equilibrium* $E_1(\frac{a_1}{b_1}, 0)$ *when the parameter* a_1 *passes through* 2.

Proof. If $a_1 = 2$, then boundary equilibrium E_1 is non-hyperbolic, as the two eigenvalues of Jacobi matrix $J(E_1)$ are $\lambda_1 = -1$ and $\lambda_2 = e^{a_2} > 1$ by Theorem 2.2(3). Secondly, a simple calculation shows that $y = 0$ is the central manifold of the system (1.6) at E_1. Thus, the system (1.6) restricted to the central manifold can be expressed as the mapping

$$x \to f(x) = x\exp(a_1 - b_1 x),$$

its positive equilibrium is $x_1 = \frac{a_1}{b_1}$. By calculation we have $f'(x_1) = -1$ if $a_1 = 2$. From the bifurcation theory [36], it follows that system (1.6) undergoes a flip bifurcation at equilibrium $E_1(\frac{a_1}{b_1}, 0)$. □

4.2. Transcritical bifurcation at boundary equilibrium $E_2(0, \frac{a_2}{b_2})$

Theorem 4.2. *Assume that* $a_2 \neq 2$, *then system (1.6) undergoes a transcritical bifurcation at boundary equilibrium* $E_2(0, \frac{a_2}{b_2})$ *when the parameter* c_1 *passes through* c_1^{**}.

Proof. According to Theorem 2.4(4), the two eigenvalues of Jacobi matrix $J(E_2)$ are $\lambda_1 = 1$ and $|\lambda_2| = |1 - a_2| \neq 1$ when $c_1 = c_1^{**}$, $a_2 \neq 2$. We take c_1 as the bifurcation parameter and make $c_1 = c_1^{**} + \vartheta$, where $|\vartheta| \ll 1$ and is a very small perturbation parameter. Thus, we can represent system (1.6) as the following mapping form:

$$\begin{pmatrix} x \\ \\ y \end{pmatrix} \to \begin{pmatrix} x \exp\left(a_1 - b_1 x - \dfrac{(c_1^{**} + \vartheta)(1-k)y}{1 + d_1 y}\right) \\ y \exp(a_2 - b_2 y) \end{pmatrix}. \tag{4.1}$$

Next, translating the equilibrium $E_2(0, \frac{a_2}{b_2})$ to origin $(0,0)$, making the transformations $u = x - 0, v = y - \frac{a_2}{b_2}$ and Taylor expanding at origin, we have

$$\begin{pmatrix} u \\ \\ v \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 - a_2 \end{pmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v,\vartheta) \\ f_2(u,v,\vartheta) \end{pmatrix}, \tag{4.2}$$

and

$$\begin{aligned} f_1(u,v,\vartheta) = {} & -b_1 u^2 + \frac{(-1+k)c_1^{**}uv}{(d_1 y^* + 1)^2} + \frac{y^*(-1+k)u\vartheta}{d_1 y^* + 1} \\ & + \frac{b_1^2 u^3}{2} - \frac{b_1 c_1^{**}(-1+k)u^2 v}{(d_1 y^* + 1)^2} + \frac{b_1 y^*(-1+k)u^2\vartheta}{d_1 y^* + 1} \\ & + \frac{c_1^{**}(-1+k)(-2d_1^2 y^* + c_1^{**}k - c_1^{**} - 2d_1)uv^2}{2(d_1 y^* + 1)^4} \\ & + \frac{(-1+k)(c_1^{**}ky^* - c_1^{**}y^* + d_1 y^* + 1)uv\vartheta}{(d_1 y^* + 1)^3} \\ & + \frac{{y^*}^2(k^2 - 2k + 1)u\vartheta^2}{2(d_1 y^* + 1)^2} + O\big((|u| + |v| + |\vartheta|)^3\big), \\ f_2(u,v,\vartheta) = {} & \frac{b_2(a_2 - 2)v^2}{2} - \frac{b_2^2(a_2 - 3)v^3}{6} + O\big((|u| + |v| + |\vartheta|)^3\big). \end{aligned}$$

By the central manifold theorem, there exists a central manifold $W^c(0,0,0)$, of map (4.2) at equilibrium $(u,v) = (0,0)$ in a small neighbourhood of $\vartheta = 0$, which can be represented as

$$W^c(0,0,0) = \left\{(u,v,\vartheta) \in \mathbb{R}^3 : v = g(u,\vartheta), g(0,0) = 0, Dg(0,0) = 0\right\},$$

where u and ϑ are sufficiently small. We consider the following assumption

$$g(u,\vartheta) = g_1 u^2 + g_2 u\vartheta + g_3\vartheta^2 + O\big((|u| + |\vartheta|)^3\big), \tag{4.3}$$

and $g(u,\vartheta)$ has the following relation

$$g(u+f_1(u,g(u,\vartheta),\vartheta),\vartheta)=(1-a_2)g(u,\vartheta)+f_2(u,g(u,\vartheta),\vartheta). \tag{4.4}$$

Substituting (4.3) into (4.4) and comparing the coefficients u^2, $u\vartheta$, and the ϑ^2 terms of (4.4), we have

$$g_1=g_2=g_3=0.$$

Thus, system restricted to central manifold $W^c(0,0,0)$ can be expressed as follows:

$$F_1:\ u\to u-b_1u^2+\frac{y^*(-1+k)}{d_1y^*+1}u\vartheta+\frac{b_1^2}{2}u^3-\frac{b_1y^*(-1+k)}{d_1y^*+1}u^2\vartheta$$
$$+\frac{{y^*}^2(k^2-2k+1)}{2(d_1y^*+1)^2}u\vartheta^2+O\big((|u|+|\vartheta|)^3\big),$$

this is obtained by direct calculation

$$F_1(0,0)=0,\qquad \frac{\partial F_1}{\partial u}(0,0)=1,$$
$$\frac{\partial^2 F_1}{\partial u\partial\vartheta}(0,0)=\frac{y^*(-1+k)}{d_1y^*+1}\neq 0,\qquad \frac{\partial^2 F_1}{\partial u^2}(0,0)=-2b_1\neq 0.$$

Therefore, according to [36], the system (1.6) undergoes a transcritical bifurcation at equilibrium E_2 when $c_1=c_1^{**}$ and $a_2\neq 2$. □

4.3. Flip bifurcation at positive equilibrium $E^*(x^*,y^*)$

Theorem 4.3. *Assume that* $a_1>2$ *and* $a_2\neq 2$, *then system (1.6) undergoes a flip bifurcation at positive equilibrium* $E^*(x^*,y^*)$ *when the parameter* c_1 *passes through* c_1^*.

Proof. If $c_1=c_1^*, a_1>2$ and $a_2\neq 2$, then positive equilibrium E^* is non-hyperbolic, as the eigenvalues of Jacobi matrix $J(E^*)$ are $\lambda_1=-1$ and $|\lambda_2|=|1-a_2|\neq 1$ by Theorem 2.4. Next we show that the system flip bifurcation of (1.6) at positive equilibrium $E^*(x^*,y^*)$ when $c_1=c_1^*, a_1>2$ and $a_2\neq 2$.

We consider a perturbation of system (1.6), expressed in terms of a mapping as follows:

$$\begin{pmatrix}x\\ y\end{pmatrix}\to\begin{pmatrix}x\exp\left(a_1-b_1x-\dfrac{(c_1^*+\upsilon)(1-k)y}{1+d_1y}\right)\\ y\exp(a_2-b_2y)\end{pmatrix}, \tag{4.5}$$

where $c_1^*=\frac{(a_1-2)(a_2d_1+b_2)}{a_2(1-k)}$, and υ is a small perturbation parameter and $|\upsilon|\ll 1$. Next, by translating the positive equilibrium $E^*(x^*,y^*)$ to origin $(0,0)$ through the transformations $U=x-x^*, V=y-y^*$, and then expanding it by Taylor, the system (4.5) reduces to

$$\begin{pmatrix}U\\ V\end{pmatrix}\to\begin{pmatrix}-1 & p_{010}\\ 0 & 1-a_2\end{pmatrix}\begin{pmatrix}U\\ V\end{pmatrix}+\begin{pmatrix}f_3(U,V,\upsilon)\\ f_4(U,V,\upsilon)\end{pmatrix}, \tag{4.6}$$

where,

$$
\begin{aligned}
f_3(U,V,\upsilon) = {} & p_{110}UV + p_{101}U\upsilon + p_{020}V^2 + p_{011}V\upsilon + p_{300}U^3 + p_{201}U^2\upsilon \\
& + p_{120}UV^2 + p_{111}UV\upsilon + p_{030}V^3 + p_{021}V^2\upsilon + p_{012}V\upsilon^2 \\
& + O\big((|U| + |V| + |\upsilon|)^3\big), \\
f_4(U,V,\upsilon) = {} & q_{020}V^2 + q_{030}V^3 + O\big((|U| + |V| + |\upsilon|)^3\big),
\end{aligned}
$$

and

$$
\begin{aligned}
& p_{110} = \frac{b_2^2(a_1-2)}{a_2(d_1a_2+b_2)}, \quad p_{101} = -\frac{a_2(-1+k)}{d_1a_2+b_2}, \quad p_{011} = -\frac{b_2^2(-1+k)(a_1-4)}{(d_1a_2+b_2)^2b_1}, \\
& p_{020} = \frac{b_2^3(a_1-2)(a_1b_2+2d_1a_2-2b_2)}{(d_1a_2+b_2)^2a_2^2b_1}, \quad p_{111} = \frac{b_2^2(-1+k)(a_1-3)}{(d_1a_2+b_2)^2}, \\
& p_{201} = \frac{b_1a_2(-1+k)}{2(d_1a_2+b_2)}, \quad p_{120} = \frac{b_2^3(a_1-2)(a_1b_2+2d_1a_2-2b_2)}{2a_2^2(d_1a_2+b_2)^2}, \quad p_{300} = \frac{b_1^2}{6}, \\
& p_{021} = \frac{b_2^3(-1+k)(a_1^2b_2+2a_1a_2d_1-8a_1b_2-8d_1a_2+12b_2)}{2a_2(d_1a_2+b_2)^3b_1}, \; p_{010} = -\frac{2}{b_1}p_{110}, \\
& p_{030} = -\frac{b_2^4(a_1-2)(a_1^2b_2^2+6a_1a_2b_2d_1+6a_2^2d_1^2-4a_1b_2^2-12a_2b_2d_1+4b_2^2)}{3(d_1a_2+b_2)^3a_2^3b_1}, \\
& p_{012} = \frac{(-1+k)^2a_2b_2^2}{(d_1a_2+b_2)^3b_1}, \quad q_{020} = \frac{b_2(a_2-2)}{2}, \quad q_{030} = -\frac{b_2^2(a_2-3)}{6}.
\end{aligned}
$$

By means of a linear transformation

$$
\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} p_{010} & p_{010} \\ 0 & 2-a_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.
$$

Then, system (4.6) can be reduced to

$$
\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & 1-a_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} f_5(X,Y,\upsilon) \\ f_6(X,Y,\upsilon) \end{pmatrix}, \tag{4.7}
$$

where,

$$
\begin{aligned}
f_5(X,Y,\upsilon) = {} & \frac{1}{p_{010}}(p_{300}U^3 + p_{201}U^2\upsilon + p_{120}UV^2 + p_{111}UV\upsilon + p_{030}V^3 \\
& + p_{012}V\upsilon^2 + p_{110}UV + p_{101}U\upsilon + p_{020}V^2 + p_{011}\upsilon V + p_{021}V^2\upsilon) \\
& + \frac{q_{030}V^3 + q_{020}V^2}{-2+a_2} + O\big((|X| + |Y| + |\upsilon|)^3\big),
\end{aligned}
$$

$$f_6(X, Y, \upsilon) = -\frac{q_{030}V^3 + q_{020}V^2}{-2 + a_2} + O\big((|X| + |Y| + |\upsilon|)^3\big),$$

and $U = p_{010}(X + Y)$, $V = (2 - a_2)Y$.

According to the central manifold theorem, there exists a central manifold $W^c(0, 0, 0)$ of the map (4.7) in a small neighbourhood of $\upsilon = 0$ at equilibrium $(X, Y) = (0, 0)$, which can be represented as

$$W^c(0, 0, 0) = \big\{(X, Y, \upsilon) \in \mathbb{R}^3 : Y = g(X, \upsilon), g(0, 0) = 0, Dg(0, 0) = 0\big\},$$

where X and υ are sufficiently small. Suppose that the expression for $g(X, \upsilon)$ is

$$g(X, \upsilon) = g_1X^2 + g_2X\upsilon + g_3\upsilon^2, \tag{4.8}$$

which satisfies the following relation

$$g(-X + f_5(X, g(X, \upsilon), \upsilon), \upsilon) = (1 - a_2)g(X, \upsilon) + f_6(X, g(X, \upsilon), \upsilon). \tag{4.9}$$

Substituting (4.8) into (4.9) and comparing the coefficients X^2, $X\upsilon$, and υ^2 of (4.9), we have that

$$g_1 = g_2 = g_3 = 0.$$

Thus, the map restricted to the central manifold can be written as

$$F_2 : \; X \to -X + p_{101}X\upsilon + p_{010}p_{201}X^2\upsilon + p_{010}^2p_{300}X^3 + O\big((|X| + |\upsilon|)^3\big).$$

By a direct calculation, we have

$$\ell_1 = \left(\frac{\partial^2 F_2}{\partial X \partial \upsilon} + \frac{1}{2}\frac{\partial F_2}{\partial \upsilon}\frac{\partial^2 F_2}{\partial X^2}\right)(0, 0) = p_{101} \neq 0,$$

$$\ell_2 = \left[\frac{1}{6}\frac{\partial^3 F_2}{\partial X^3} + \left(\frac{1}{2}\frac{\partial^2 F_2}{\partial X^2}\right)^2\right](0, 0) = p_{010}^2p_{300} \neq 0.$$

Therefore, according to the bifurcation theory [36], the system (1.6) generates a flip bifurcation at positive equilibrium E^* when $c_1 = c_1^*, a_1 > 2$ and $a_2 \neq 2$. □

5. Chaos control

According to the previous bifurcation analysis, the system will undergo a flip bifurcation at positive equilibrium $E^*(x^*, y^*)$. As the bifurcation parameter c_1 increases, the system becomes chaotic. Thus, this subsection aims to control the chaotic behavior of the system (1.6) using hybrid control strategies [27]. Firstly, we write the control system as the following mapping

$$\begin{pmatrix} x \\ \\ y \end{pmatrix} \to \begin{pmatrix} \rho x \exp\left(a_1 - b_1x - \dfrac{c_1(1 - k)y}{1 + d_1y}\right) + (1 - \rho)x \\ \rho y \exp(a_2 - b_2y) + (1 - \rho)y \end{pmatrix}. \tag{5.1}$$

Where ρ is an external control parameter and $0 < \rho < 1$. A simple calculation shows that (5.1) and system (1.6) have the same positive equilibrium $E^*(x^*, y^*)$, which corresponds to the Jacobi matrix

$$J(E^*) = \begin{pmatrix} 1 - b_1\rho x^* & \dfrac{\rho x^* c_1(-1+k)}{(d_1 y^* + 1)^2} \\ 0 & 1 - \rho a_2 \end{pmatrix},$$

from eigenvalues of the upper triangular matrix on its diagonal, two eigenvalues of Jacobi matrix $J(E^*)$ are $\lambda_1 = 1 - b_1\rho x^* < 1$ and $\lambda_2 = 1 - \rho a_2 < 1$. According to Lemma 2.1(1), we can get the following result.

Theorem 5.1. *Assume that*

$$0 < \rho < \min\left\{\frac{2}{b_1 x^*}, \frac{2}{a_2}, 1\right\},$$

positive equilibrium $E^*(x^*, y^*)$ of system (5.1) is locally asymptotically stable.

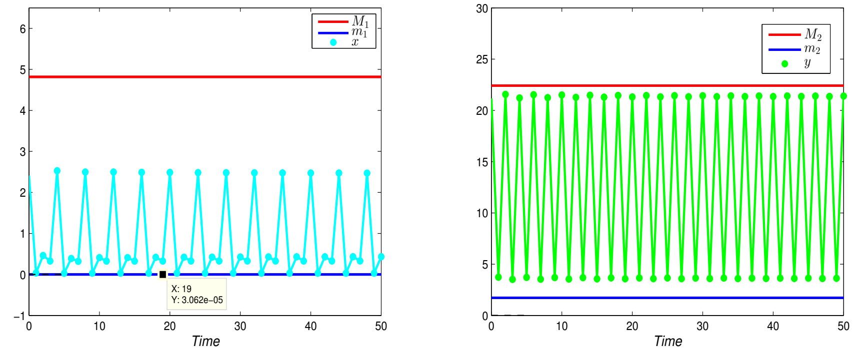
6. Numerical simulation

In this section, we present several examples and conduct numerical simulations to confirm above the theoretical analysis.

Example 6.1. This example demonstrates the permanence of system (1.6) and the global attraction of positive equilibrium E^*.

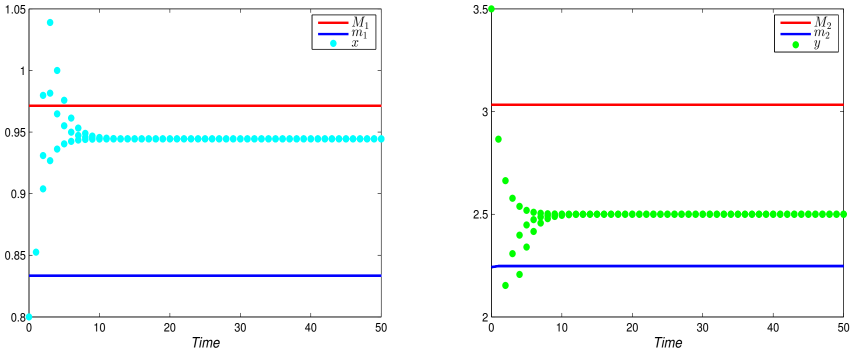
(1) Suppose that parameters $a_1 = 4, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 2.5, b_2 = 0.2, k = 0.5$ in system (1.6), then system (1.6) has a unique positive equilibrium $E^*(0.6, 12.5)$, as $c_1 = 1 < c_1^{**} = \frac{a_1(d_1y^*+1)}{y^*(1-k)} = 1.44$. And because $a_1 = 4 > 2, a_2 = 2.5 > 2, c_1 = 1 > c_1^* = \frac{(a_1-2)(d_1y^*+1)}{y^*(1-k)} = 0.72$, E^* is a saddle by Theorem 2.4, which is unstable. In addition, it is straightforward to compute that $M_2 = \frac{\exp(a_2-1)}{b_2} \approx 22.4, a_1 > \frac{c_1(1-k)M_2}{1+d_1M_2} \approx 3.45$, satisfying conditions of Theorem 3 (see, [4]). Thus, the system (1.6) is permanence (see, Figure 1(a)(b)). Taking the initial value $(x(0), y(0)) = (0.8, 3.5)$, numerical simulations show that the solution for this initial value cannot get closer to the positive equilibrium point as time increases, i.e., the permanence condition does not guarantee that positive equilibrium is globally stable (see, Figure 1).

(2) Suppose that parameters $a_1 = 1.5, b_1 = 1, c_1 = 1, d_1 = 0.5, a_2 = 0.5, b_2 = 0.2, k = 0.5$ and initial values $(x(0), y(0)) = (0.2, 1), (0.4, 1.6)$,and $(0.8, 3.5)$ in system (1.6). It can be calculated that $c_1 = 1 < c_1^{**} = \frac{a_1(d_1y^*+1)}{y^*(1-k)} = 2.7$, then there exists a unique positive equilibrium $E^*(0.94, 2.5)$ for system (1.6), and since $a_1 = 1.5 < 2, a_2 = 0.5 < 2$, by Theorem 2.4, E^* is a sink which is locally asymptotically stable. Moreover, it can be directly calculated that $M_2 = \frac{\exp(a_2-1)}{b_2} \approx 3.0327$, $a_1 = 1.5 > \frac{c_1(1-k)M_2}{1+d_1M_2} \approx 0.6$, by Theorem 2 and Theorem 3 (see, [4]), system (1.6) is permanence (Figure 2(a)(b)). Furthermore, because of $a_2 = 0.5 < 2, a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} \approx 0.94 < \ln 2 + 1$, by Theorem 3.1, there exists a unique globally attracting positive equilibrium $E^*(0.94, 2.5)$ of system (1.6) (see, Figure 2(a)(b)).



(a) (b)

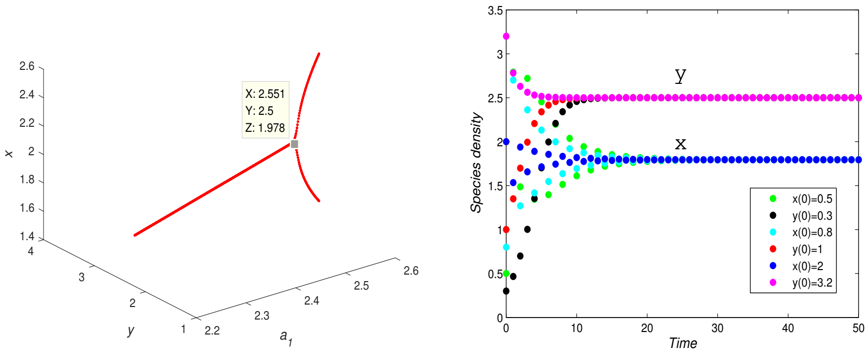
Figure 1. Permanence of system (1.6), where $a_1 = 4, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 2.5, b_2 = 0.2, k = 0.5$, and the initial value is $(x(0), y(0)) = (0.8, 3.5)$.



(a) (b)

Figure 2. Permanence of system (1.6), where $a_1 = 1.5, b_1 = 1, c_1 = 1, d_1 = 0.5, a_2 = 0.5, b_2 = 0.2, k = 0.5$, and the initial values are $(x(0), y(0)) = (0.2, 1), (0.4, 1.6)$ and $(0.8, 3.5)$.

(3) Suppose that parameters $b_1 = 1, c_1 = 1, d_1 = 0.5, a_2 = 0.5, b_2 = 0.2, k = 0.5$ in system (1.6), then positive equilibrium of system (1.6) is $E^*(a_1 - 0.55, 2.5)$. If $2.23 \leq a_1 < 2.55$, that can be calculated $\ln 2 + 1 \leq a_1 - 0.55 < 2$, by Theorem 2.4, then E^* is a sink which is locally asymptotically stable. The conditions of Theorem 3.1 are not satisfied; however, numerical simulations (see Figure 3(a)) show that there exists a unique stable positive equilibrium $E^*(a_1 - 0.55, 2.5)$ for the system when $2.23 \leq a_1 < 2.55$. It is worthwhile to take the initial values $(x(0), y(0))$ as $(0.5, 0.3), (0.8, 1)$ and $(2, 3.2)$ for $a_1 = 2.35$, and from the numerical simulation (see Figure 3(b)), we can see that there exists a unique globally attractive positive equilibrium $E^*(1.8, 2.5)$ of system (1.6).



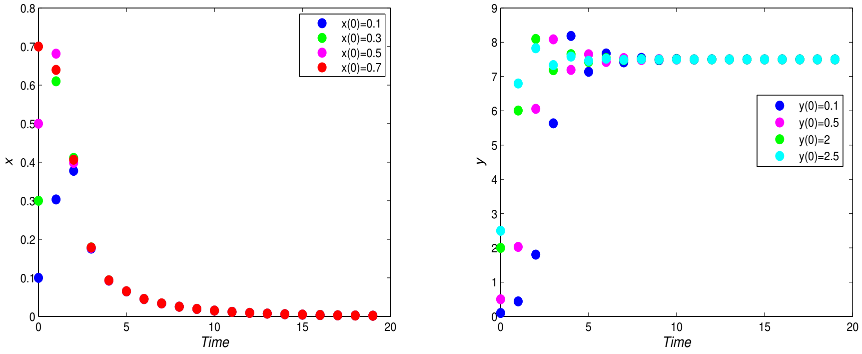
(a) (b)

Figure 3. (**a**) Plot of population density x and y versus natural growth rate a_1 where $b_1 = 1, c_1 = 1, d_1 = 0.5, a_2 = 0.5, b_2 = 0.2, k = 0.5$. (**b**) the global attractiveness of positive equilibrium E^*, where $a_1 = 2.35, b_1 = 1, c_1 = 1, d_1 = 0.5, a_2 = 0.5, b_2 = 0.2, k = 0.5$, and the initial values are $(x(0), y(0)) = (0.5, 0.3), (0.8, 1)$ and $(2, 3.2)$.

Example 6.2. When taking the parameters of system (1.6)

$$a_1 = 1.5, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 1.5, b_2 = 0.2, k = 0.6,$$

then, it is straightforward to calculate $a_1 = 1.5 < \frac{a_2 c_1 (1-k)}{b_2 + a_2 d_1} \approx 1.71$. By Theorem 3.2, boundary equilibrium $E_2(0, 7.5)$ is globally attractive (see, Figure (4)(a)(b)), i.e., the first population x tends to be extinct and the second population y tends to be y^*.



(a) (b)

Figure 4. Global attractiveness of boundary equilibrium E_2.

Example 6.3. When taking the parameters of system (1.6)

$$a_1 = 3.5, b_1 = 1, d_1 = 0.1, a_2 = 1.5, b_2 = 1, k = 0.3,$$

then, it can be calculated that $c_1^* \approx 1.64, c_1^{**} \approx 3.83$. Assume that the parameters $c_1 = c_1^{**}$, equilibrium $E_2(0, 1.5)$ is non-hyperbolic, as two eigenvalues of Jacobi matrix $J(E_2)$ of system (1.6) are $\lambda_1 = 1$ and $\lambda_2 = -0.5$. Bifurcation analysis of Theorem 4.2 shows that system (1.6) undergoes a transcritical bifurcation in a small neighbourhood of the equilibrium $E_2(0, 1.5)$.

Figure 5 is transcritical bifurcation diagram at equilibrium $E_2(0, 1.5)$ in the $c_1 - x$ plane. From Figure 5, it can be seen that $E^*(3.5 - 0.913c_1, 1.5)$ is saddle and unstable when $0 < c_1 < 1.64$ (indicated by cyan dotted line). The positive equilibrium $E^*(3.5 - 0.913c_1, 1.5)$ is a sink, which is locally asymptotically stable when $1.64 < c_1 < 3.83$ (indicated by the blue solid line). The positive equilibrium point $E^*(3.5 - 0.913c_1, 1.5)$ vanishes when $c_1 > 3.83$. When $0 < c_1 < 3.83$, the boundary equilibrium $E_2(0, 1.5)$ is a saddle and is unstable (indicated by the red dashed line). The boundary equilibrium $E_2(0, 1.5)$ is a sink and is locally asymptotically stable when $c_1 > 3.83$.

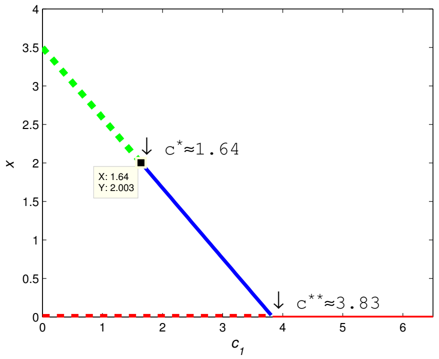


Figure 5. Transcritical bifurcation diagram for boundary equilibrium E_2.

Example 6.4. This example investigates the effect of the amensalism coefficient c_1 on the behavior of population dynamics.

(1) Suppose that the parameters $a_1 = 1, b_1 = 1, k = 0.5, d_1 = 0.5, a_2 = 1, b_2 = 1.5$ in system (1.6), it can be calculated that $c_1^* \approx -4$, $c_1^{**} \approx 4$. Assume that $0 < c_1 < 4$, then there exists a unique positive equilibrium $E^*(1-0.25c_1, 0.67)$ of system (1.6), and because $a_1 = 1 < 2, a_2 = 1 < 2$, by Theorem 2.4, $E^*(1 - 0.25c_1, 0.67)$ is a sink, which is stable (see Fig. 6(a)). Assume that $c_1 > 4$, positive equilibrium disappears and the first population x tends to become extinct (see, Figure 6(a)).

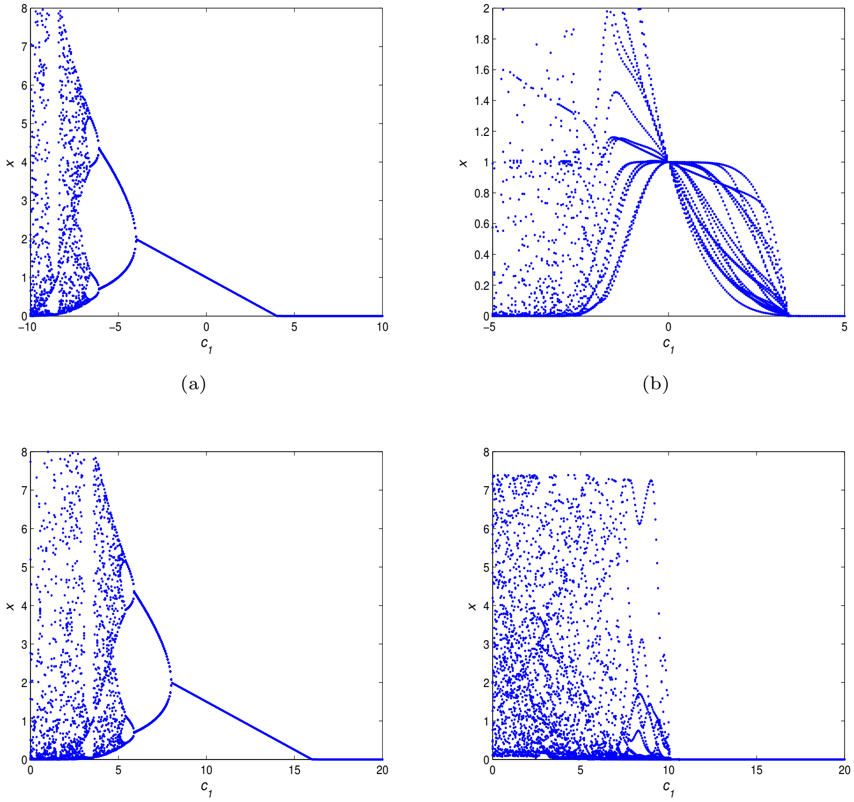
(2) Suppose that the parameters $a_1 = 1, b_1 = 1, k = 0.5, d_1 = 0.5, a_2 = 4, b_2 = 1.5$ in system (1.6), it can be computed that $c_1^* \approx -1.75$, $c_1^{**} \approx 1.75$, and system (1.6) has a unique positive equilibrium $E^*(1-0.57c_1, 2.66)$ when $0 < c_1 < 1.75$. Since $a_1 = 1 < 2, a_2 = 4 > 2$, by theorem 2.4, $E^*(1-0.57c_1, 2.66)$ is a saddle, which is unstable. The positive equilibrium disappears when $c_1 > 1.75$, i.e., the first population x tends to become extinct (see, Figure 6(b)).

(3) Suppose that the parameters $a_1 = 4, b_1 = 1, k = 0.5, d_1 = 0.5, a_2 = 1, b_2 = 1.5$ in system (1.6), if the parameter $c_1 = c_1^* \approx 8$ is taken, the two eigenvalues of $J(E^*)$ are $\lambda_1 = -1$ and $\lambda_2 = 0$, and then positive equilibrium E^* is nonhyperbolic, i.e., the positive equilibrium of system (1.6) E^* undergoes a flip bifurcation near the $c_1 = c_1^* \approx 8$ (Figure 6(c)). In addition, the 2-cycle track

bifurcating from the positive equilibrium is stable because of $ell_1 = 0.25 > 0$, $\ell_2 = 3.375 > 0$.

(4) Suppose that the parameters $a_1 = 3, b_1 = 1, k = 0.5, d_1 = 0.5, a_2 = 4, b_2 = 1.5$ in system (1.6), it can be computed that $c_1^* \approx 1.75$, $c_1^{**} \approx 5.25$, the system (1.6) has a unique positive equilibrium $E^*(3 - 0.57c_1, 2.67)$ if $0 < c_1 < 5.25$ (see, Figure 6(d)). At this point, the two eigenvalues of Jacobi matrix $J(E^*)$ are $\lambda_1 = -2 + 0.57c_1$ and $\lambda_2 = -3$. Since $|\lambda_2| > 1$, the system (1.6) is unstable.

Summarising the above four sets of parameters, it can be seen that as the amensalism coefficient c_1 increases, the first population of system (1.6) eventually goes to extinction in various forms. It can be interpreted as a commensalism species system if $c_1 < 0$.



(a) (b)

(c) (d)

Figure 6. Plot of changes in amensalism coefficient c_1 affecting population x.

Example 6.5. This example investigates the effect of the cover k and the half-saturation constant d_1 on the dynamical behavior of the system (1.6).

(1) When the parameters $a_1 = 4, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 1.5, b_2 = 0.2$, and $k > 0.067$ are taken, then there exists a unique positive equilibrium point $E^*(-0.286 + 4.286k, 7.5)$ for system (1.6). By observing Figure

7(a), it can be seen that the system (1.6) has a stable positive equilibrium $E^*(-0.286 + 4.286k, 7.5)$ when $0.067 < k < 0.53$. When k increases to 0.53, E^* begins to bifurcate into orbitals with a period of 2. As k increases, it successively bifurcates from a two-periodic orbit to a four-periodic orbit, and due to the continuous overturning of the system's periodic orbits, chaos eventually appears. This suggests that the k is within a certain range, which contributes to the stability of the first system population. Figure 7(c) is the Maximum Lyapunov exponents diagram, reflecting the chaotic set (consistent with Figure 7(a)).

(2) There exists a unique positive equilibrium $E^*(-0.778+2.778k, 12.5)$ for system (1.6) when the parameters $a_1 = 4, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 2.5, b_2 = 0.2$, and $k > 0.28$ are taken. Since $a_1 = 4 > 2, a_2 = 2.5 > 2$, by Theorem 2.4, one of the eigenvalues $|\lambda_2| > 1$ of $J(E^*)$, exists in the positive equilibrium E^* of system which is unstable. Subsequently, it is also clear from Figure 7(b) When $0.28 < k < 0.42$, the first population x generates a two-periodic trajectory; when $0.42 < k < 0.67$, the first population x emerges in a four-periodic trajectory; with the increase of k, the system keeps flipping towards chaos, and thus, when $k > 0.28$, the only positive equilibrium of system $E^*(-0.778 + 2.778k, 12.5)$ is always unstable. The comparison of two sets of parameter values in (1) and (2) shows that the natural growth rate of the second population contributes to the stability of the first population, i.e., it can delay the chaos of the first population.

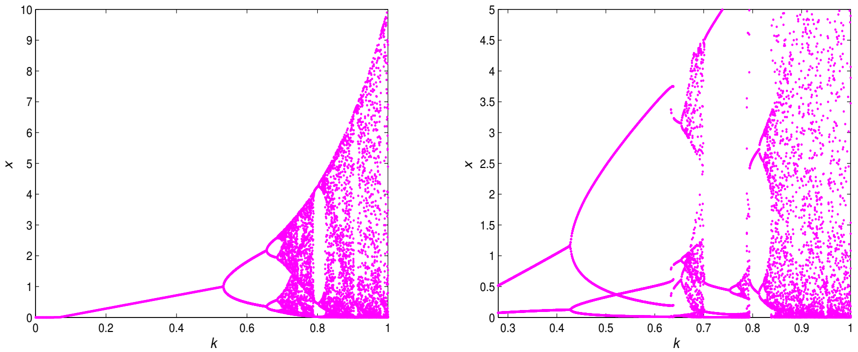
(3) When taking the parameters $a_1 = 1.5, b_1 = 0.5, a_2 = 1.5, b_2 = 1$, it can be computed as $x^* = \frac{-2+7.5d_1+5k}{2.5d_1+1}$, which is a strict function of k and half-saturation constant d_1. From Figure 8, x^* is a strictly monotonically increasing function of k and d_1, and thus both cover and half-saturation constant contribute to the increase in population density of the first population.

Example 6.6. Suppose that $a_1 = 4, b_1 = 2, c_1 = 1, d_1 = 0.1, a_2 = 1.5, b_2 = 0.2, k = 0.85$ in system (1.6), then we can see that system (1.6) produces chaotic behaviour by observing Figure 7(a). To control the chaos generated at this time, we obtain the following control system

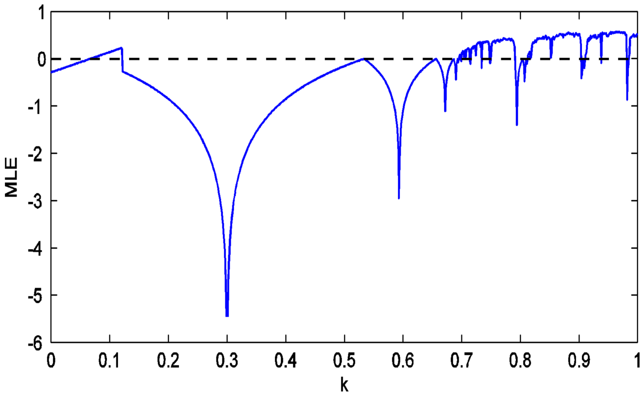
$$\begin{pmatrix} x \\ \\ y \end{pmatrix} \to \begin{pmatrix} \rho x \exp\left(4 - 2x - \dfrac{0.15y}{1 + 0.1y}\right) + (1-\rho)x \\ \rho y \exp(1.5 - 0.2y) + (1-\rho)y \end{pmatrix}. \tag{6.1}$$

The system has a positive equilibrium $E^*(1.91, 7.5)$. From Theorem 5.1, the positive equilibrium $E^*(1.91, 7.5)$ of system (6.1) is locally asymptotically stable if $0 < \rho \leq 0.5957$ holds, and positive equilibrium $E^*(1.91, 7.5)$ of system (6.1) is unstable if $0.5957 < \rho < 1$ holds. When $\rho = 0.59$, the system (6.1) is stable (see, Figure 9(a)); when $\rho = 0.6$, then the system (6.1) generates periodic solutions, and it is unstable (see, Figure 9(b)).

Remark 6.1. Numerical simulations (see Figure 3(a)) show that the system (1.6) is also globally stable when $0 < a_2 < 2, \ln 2 + 1 \leq a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} < 2$ holds, and we conjecture that the condition of Theorem 3.1 can be further relaxed as $0 < a_2 < 2, 0 < a_1 - \frac{c_1(1-k)y^*}{1+d_1y^*} < 2$, however, we are not able to give a rigorous proof at this moment, and this is to be followed up by further discussion.



(a) $a_2 = 1.5$ (b) $a_2 = 2.5$



(c)

Figure 7. Flip bifurcation and Maximal Lyapunov exponents maps.

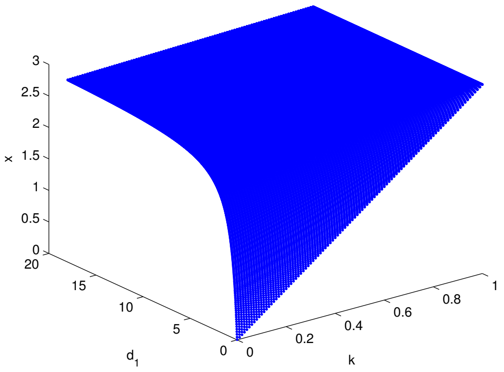
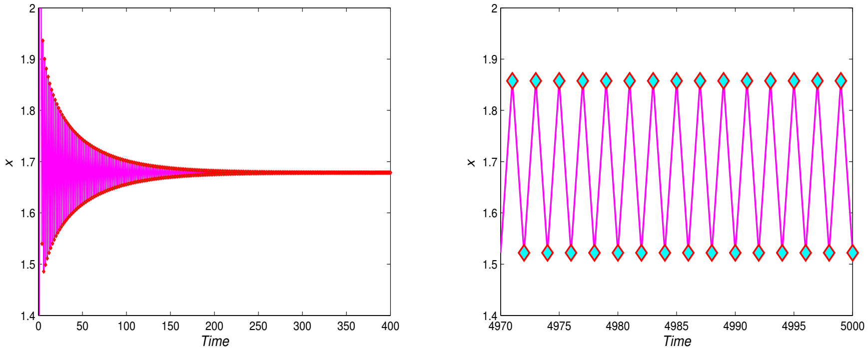


Figure 8. Plot of x^* versus parameters k and d_1.



(a) $\rho = 0.59$ (b) $\rho = 0.6$

Figure 9. Stability of the first population under different control parameters.

7. Conclusion

This article investigates global attractivity, bifurcation analysis, and chaos control in a discrete amensalism model incorporating cover and saturation effect. The method of piecewise constant arguments for differential equations is utilized to derive the discrete-time counterpart of the continuous one. The saturation effect is derived from the consideration of competition models with saturation in [11], while the constant cover is inspired by [32]. This presents an intriguing research problem. Consequently, we delve into the analysis of the discrete amensalism model (1.6).

Initially, we investigate the existence and local stability of the system's equilibria. Subsequently, through a detailed analysis of the difference equation, we derive a set of sufficient conditions that ensure the global attractivity of the system (1.6) at equilibria E^* and E_2. It is demonstrated that while discrete and continuous systems share the same number of equilibria, the topological structures in discrete systems are more complex (see,Table 1). Furthermore, we discover that the size of the cover and the natural growth rate of the second population significantly influence the system's stability, which differs from the corresponding continuous system.

Next, we discuss the flip bifurcations at the boundary equilibrium E_1 and the positive equilibrium E^*, as well as the transcritical bifurcation at the boundary equilibrium E_2, using the central manifold theorem and bifurcation theory. Subsequently, we propose a hybrid control strategy to delay or eliminate chaotic behavior induced by flip bifurcations.

Finally, the validity of our findings is confirmed through numerical simulations, which demonstrate that both the half-saturation constant and the size of the cover contribute to an increase in the density of the first population. These simulations (as shown in Fig. 3(a)) reveal that the system (1.6) exhibits globally stable when $0 < a_2 < 2$ and $\ln 2 + 1 \leq a_1 - \frac{c_1(1-k)y}{1+d_1 y} < 2$. We hypothesize that the conditions of Theorem 3.1 could potentially be relaxed to $0 < a_2 < 2$ and $0 < a_1 - \frac{c_1(1-k)y}{1+d_1 y} < 2$. However, at present, we are unable to provide rigorous proof for this relaxation, and this remains an area for future exploration. Difference equations are particularly suited for modeling populations with non-overlapping generations, where challenges such as mating and defense become significant when the population size is relatively small. Consequently, the Allee effect is incorporated into model (1.6) to account for these influences. This presents an intriguing research question that we aim to explore further in future work.

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