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THRESHOLD BEHAVIOR OF A STOCHASTIC SVI EPIDEMIC MODEL WITH JUMP NOISE AND SATURATED INCIDENCE RATE*

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Abstract This paper analyzes a stochastic SVI epidemic model with jump noise and saturated incidence rate. By applying Kunita's inequality, we derive the asymptotic pathwise estimation of the stochastic solution. We then present the basic reproduction number of the model, $\bar{R}_0^s$, which determines the extinction and persistence in the mean of the disease. Additionally, numerical simulations are conducted to verify the theoretical results. Consequently, both theoretical and numerical results indicate that the jump noise can hold off the spread of the disease.

Keywords Stochastic SVI model, jump noise, Kunita's inequality, threshold behavior, extinction.

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1. Introduction

The impact of epidemics on human society cannot be underestimated. As globalization advances, transportation has become increasingly developed. This has facilitated frequent migration across regions, accelerating the spread of epidemics on a global scale. Therefore, understanding and preventing the spread of infectious diseases are particularly important [8]. To study the spread of diseases like COVID-19, many researchers have developed various mathematical models [4, 11, 21].

To prevent epidemic outbreaks, vaccines are widely used in modern society, and epidemic models with vaccination have been studied extensively [4, 11]. Furthermore, many practical factors, such as media coverage, quarantine measures, and population density, may affect the incidence rate. Therefore, researchers often consider nonlinear incidence rates, such as the half-saturated incidence rate [9] and the saturated incidence rate [7], in their epidemic models.

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Liu et al. [15] considered the saturated incidence rate and analyzed the stationary distribution of an SVI epidemic model. They formulated the following model:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - (b+\theta)S(t) - \dfrac{\beta S(t)I(t)}{1+\eta S(t)}\right]\mathrm{d}t, \\ \mathrm{d}V(t) = \left[\theta S(t) - \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma_1+b)V(t)\right]\mathrm{d}t, \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)I(t)}{1+\eta S(t)} + \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma+b+\alpha)I(t)\right]\mathrm{d}t, \end{cases} \tag{1.1}$$

where S, V and I denote the susceptible, vaccinated, and infectious individuals, respectively. The saturating contact rates are assumed to be $\frac{\beta S}{1+\eta S}$ and $\frac{\beta V}{1+\eta V}$, where η is called the half-saturated constant in [15]. The saturated contact rates tend to $\frac{\beta}{\eta}$ when the number of susceptible and vaccinated individuals tend to infinite, respectively. It reflects the crowdedness effect and the behavioral changes of the individuals. The parameters in model (1.1) are all assumed to be positive, and their biological meanings are listed in the following Table 1:

Table 1. Biological meanings of parameters in model (1.1).

Notation	Biological meanings
Λ	constant recruitment rate
b	natural death rate
θ	fraction of susceptible individuals to vaccinated ones
β	average contact rate of an infected person
$\delta\beta$	average contact rate of a vaccinated person
η	half-saturated constant
γ	recovery rate of infectious individuals
γ_1	average rate at which a vaccinated individual acquires immunity
α	death rate caused by disease

Liu et al. [15] showed that the basic reproduction number for system (1.1) is

$$R_0 = \frac{\beta\left(\frac{\bar{S}}{1+\eta\bar{S}} + \frac{\delta\bar{V}}{1+\eta\bar{V}}\right)}{\gamma+b+\alpha}, \tag{1.2}$$

where $\bar{S} = \frac{\Lambda}{b+\theta}$, $\bar{V} = \frac{\Lambda\theta}{(b+\theta)(\gamma_1+b)}$. They proved that the disease will die out when $R_0 < 1$, and it has a disease-free equilibrium $E_1 = (\bar{S}, \bar{V}, 0)$, which is globally asymptotically stable in the invariant set $\Gamma = \left\{(S,V,I) \in \mathbb{R}^3_+ : S+V+I \leq \frac{\Lambda}{b}\right\}$. On the other hand, the disease will persist when $R_0 > 1$, and it has an unstable equilibrium E_1 and a locally asymptotically stable equilibrium $E_2 = (S^*, V^*, I^*)$, where $S^* > 0, V^* > 0, I^* > 0$.

Many researchers have established that infectious diseases spread randomly due to contacts between individuals and environmental fluctuations [10, 23, 24, 27]. Therefore, they often use stochastic differential equations with white noise to simulate infectious disease dynamics. Motivated by these works, Liu et al. [15] incorporated white noise perturbations into model (1.1) and established the following stochastic SVI epidemic model:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - (b+\theta)S(t) - \dfrac{\beta S(t)I(t)}{1+\eta S(t)}\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}V(t) = \left[\theta S(t) - \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma_1+b)V(t)\right]\mathrm{d}t + \sigma_2 V(t)\mathrm{d}B_2(t), \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)I(t)}{1+\eta S(t)} + \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma+b+\alpha)I(t)\right]\mathrm{d}t + \sigma_3 I(t)\mathrm{d}B_3(t), \end{cases} \tag{1.3}$$

where $B_i(t)$ $(i = 1, 2, 3)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. σ_i $(i = 1, 2, 3)$ represent the intensities of the white noise. By studying the dynamical behavior of the stochastic SVI model (1.3), Liu et al. [15] revealed the influence of white noise on epidemic models. They proved that the solution of model (1.3) is positive recurrent and has a unique ergodic stationary distribution when $R_0^* = \frac{\beta\Lambda(b+\theta)}{(b+\theta+\frac{\sigma_1^2}{2})(\eta\Lambda+b+\theta)} + \frac{\delta\theta\beta\Lambda(b+\theta)(\gamma_1+b)}{(b+\theta+\frac{\sigma_1^2}{2})(\gamma_1+b+\frac{\sigma_2^2}{2})(\eta\theta\Lambda+(b+\theta)(\gamma_1+b))} - \left(\gamma+b+\alpha+\frac{\sigma_3^2}{2}\right) > 0$.

However, white noise cannot account for discontinuous perturbations caused by large incidental environmental shocks. So many researchers have begun to describe large accidental fluctuations using Lévy noise [2, 3, 16, 20, 22, 28–30]. Privault et al. [20] investigated a stochastic SIR epidemic model driven by a multidimensional Lévy jump process with heavy-tailed increments and possible correlation between noise components. They first used Kunita's inequality to estimate the asymptotic pathwise behavior of the solution and demonstrated that the Lévy process can have a significant influence on the dynamical behavior of the epidemic system. Bao et al. [2] considered the competitive Lotka–Volterra population dynamics with jumps. They give an explicit solution for the one-dimensional model, and discuss the sample Lyapunov exponent and extinction of the n-dimensional model. Zhu et al. [30] studied a stochastic SIRS system with jump noise, saturated incidence rate, and vaccination strategies. They proved through both theoretical and numerical results that the jump noise can suppress disease outbreaks. Motivated by the research results from Liu et al. [15], Privault et al. [20], Bao et al. [2] and Zhu et al. [30], we incorporate jump noise perturbation into system (1.1) and get the stochastic SVI model with jump noise as follows:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - (b+\theta)S(t) - \dfrac{\beta S(t)I(t)}{1+\eta S(t)}\right]\mathrm{d}t + S\left(t^-\right)\mathrm{d}Z_1(t), \\ \mathrm{d}V(t) = \left[\theta S(t) - \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma_1+b)V(t)\right]\mathrm{d}t + V\left(t^-\right)\mathrm{d}Z_2(t), \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)I(t)}{1+\eta S(t)} + \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma+b+\alpha)I(t)\right]\mathrm{d}t + I\left(t^-\right)\mathrm{d}Z_3(t), \end{cases} \tag{1.4}$$

where the left limits of $S(t)$, $V(t)$ and $I(t)$ are denoted by $S(t^-)$, $V(t^-)$ and $I(t^-)$, respectively. $Z(t) = (Z_1(t), Z_2(t), Z_3(t))$, which models the random perturbation of the system, is a three-dimensional jump process satisfying the Lévy-Khintchine representation (Theorem 1.2.14, [1]):

$$\begin{aligned} &\mathbb{E}[e^{iu_1 Z_1(t)+iu_2 Z_2(t)+iu_3 Z_3(t)}] \\ &= \exp\left(-\frac{t}{2}(u, Au) + t\int_{\mathbb{R}\backslash 0}(e^{i(u,\eta(z))} - i(u,\eta(z)) - 1)\nu(\mathrm{d}z)\right), \end{aligned}$$

where $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and A is a 3×3 diagonal matrix with $\sqrt{\sigma_i}$ as its diagonal elements. The process $Z_i(t)$ can be characterized by

$$Z_i(t) = \sigma_i B_i(t) + \int_0^t \int_{\mathbb{Y}} \eta_i(z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad i = 1, 2, 3, \tag{1.5}$$

where $\widetilde{N}(\mathrm{d}t, \mathrm{d}z)$ is the compensated Poisson random measure defined by $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$. $N(\mathrm{d}t, \mathrm{d}z)$ is a Poisson counting measure with characteristic measure ν on a measurable subset $\mathbb{Y}$ of $[0, \infty)$ with $\nu(\mathbb{Y}) < \infty$.

By substituting (1.5) into model (1.4), we obtain the following SVI epidemic model with saturated incidence rate and jump noise:

$$\begin{cases} \mathrm{d}S(t) = \left[\Lambda - (b+\theta)S(t) - \dfrac{\beta S(t)I(t)}{1+\eta S(t)}\right]\mathrm{d}t + \sigma_1 S(t)\mathrm{d}B_1(t) \\ \qquad\qquad + \displaystyle\int_{\mathbb{Y}} \eta_1(z)S(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}V(t) = \left[\theta S(t) - \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma_1 + b)V(t)\right]\mathrm{d}t + \sigma_2 V(t)\mathrm{d}B_2(t) \\ \qquad\qquad + \displaystyle\int_{\mathbb{Y}} \eta_2(z)V(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)I(t)}{1+\eta S(t)} + \dfrac{\delta\beta V(t)I(t)}{1+\eta V(t)} - (\gamma + b + \alpha)I(t)\right]\mathrm{d}t + \sigma_3 I(t)\mathrm{d}B_3(t) \\ \qquad\qquad + \displaystyle\int_{\mathbb{Y}} \eta_3(z)I(t^-)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \end{cases} \tag{1.6}$$

where $\eta_i : \mathbb{Y} \to \mathbb{R}$ $(i = 1, 2, 3)$ represent the effects of jump and are suppose to be continuously differentiable and bounded. Moreover, for jump noise proposed in (1.6), we always assume that

Assumption 1.1. $\int_{\mathbb{Y}} |\eta_i(z)|^2 \nu(\mathrm{d}z) < \infty$ $(i = 1, 2, 3)$.

Assumption 1.2. *$\eta_i(z) > -1$,$\nu(\mathrm{d}z) - a.e.$, and* $\int_{\mathbb{Y}} |\eta_i(z) - \ln(1 + \eta_i(z))|\nu(\mathrm{d}z) < \infty$ $(i = 1, 2, 3)$;

Assumption 1.3. $\int_{\mathbb{Y}} [\ln(1 + \eta_i(z))]^2 \nu(dz) < \infty$ $(i = 1, 2, 3)$.

The study of transmission and extinction of epidemic is an important subject for the control of disease due to its theoretical and practical significance. Compared to the research of Liu et al. [15], we aim to investigate the sufficient conditions for extinction and persistence of the disease in (1.6), and derive the threshold behaviors for the stochastic model. Therefore, it is necessary to estimate the solution of stochastic system (1.6). Generally, for continuous martingales, Zhao and Jiang [25] used the Burkholder-Davis-Gundy (BDG) inequality to demonstrate the asymptotic pathwise estimation of the solution for stochastic SIS epidemic model (Lemma 2.1–2.2, [25]). In this paper, due to the introduction of jump noise, Kunita's inequality is utilized to estimate the asymptotic pathwise of the solution for stochastic system (1.6) instead of the BDG inequality.

In the rest of this paper, the existence of a unique global positive solution is proved in Section 2. The asymptotic pathwise estimation of the stochastic solution of system (1.6) is obtained in Section 3. Furthermore, by constructing appropriate Lyapunov functions, the critical value that can fully determine the extinction and persistence of the disease, i.e., threshold of the disease, is presented in Section 4. Finally, numerical simulations and some discussions are provided in Section 5.

2. Existence and uniqueness of the positive solution

To ensure an epidemic model is significant, it is necessary to verify that the system has a unique positive solution. In this section, we prove the existence and uniqueness of the positive solution of system (1.6) to prepare for studying the dynamic behavior of the stochastic epidemic system.

Theorem 2.1. *Assuming* Assumption 1.1 *and* Assumption 1.2 *hold, for any given initial value* $(S(0), V(0), I(0)) \in \mathbb{R}^3_+$ *a.s., the stochastic system* (1.6) *has a unique global solution* $(S(t), V(t), I(t)) \in \mathbb{R}^3_+$ *for* $t \geq 0$*, and the solution will remain in* $\mathbb{R}^3_+$ *with probability 1.*

Proof. According to Theorem 2.1 in [30], under Assumption 1.1, it is easy to prove that the coefficients of model (1.6) satisfy local Lipschitz conditions. Consequently, there exists a unique local solution $(S(t), V(t), I(t)) \in \mathbb{R}^3_+$ for any $t \in [0, \tau_e)$, where τ_e is the explosion time. Next, to show that the solution is global, we need to verify that $\tau_e = +\infty$ a.s.. In other words, $S(t), V(t)$ and $I(t)$ do not explode in any finite time. Let $k_0 > 0$ be sufficiently large such that $S(0), V(0)$ and $I(0)$ all lie within the interval $\left[\frac{1}{k_0}, k_0\right]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{S(t), V(t), I(t)\} \leqslant \frac{1}{k} \text{ or } \max\{S(t), V(t), I(t)\} \geqslant k\right\}. \tag{2.1}$$

According to (2.1), we set $\tau_\infty = \lim_{k\to\infty} \tau_k \leq \tau_e$ a.s.. So it is sufficient to prove that $\tau_\infty = +\infty$ a.s.. If not, then there exist constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P(\tau_\infty \leq T) \geq \epsilon.$$

Consequently, there is an integer $k_1 \geq k_0$ such that

$$P(\tau_k \leq T) \geq \epsilon, \quad \text{for} \quad \text{all} \quad k \geq k_1. \tag{2.2}$$

Define

$$\begin{aligned}
&V_1\big(S(t), V(t), I(t)\big) \\
=&S(t) + V(t) + I(t) - \ln S(t) - \ln V(t) - \ln I(t) + \frac{(1+\delta)\beta}{\gamma + b + a} I(t) \\
&+ \frac{(1+\delta)\beta}{\gamma + b + \alpha} V(t) + \frac{(1+\delta)\beta\theta}{(\gamma + b + \alpha)(b+\theta)} S(t) + \frac{(1+\delta)\beta}{\gamma + b + \alpha} S(t).
\end{aligned}$$

By Itô's formula with jump noise (Theorem 4.4.7, [1]), we have

$$\begin{aligned}
&\mathrm{d}V_1\big(S(t), V(t), I(t)\big) \\
=&LV_1 \mathrm{d}t + \sigma_1(S(t) - 1)\mathrm{d}B_1(t) + \sigma_2(V(t) - 1)\mathrm{d}B_2(t) + \sigma_3(I(t) - 1)\mathrm{d}B_3(t) \\
&+ \frac{(1+\delta)\beta}{\gamma + b + \alpha}\sigma_3 I(t)\mathrm{d}B_3(t) + \frac{(1+\delta)\beta}{\gamma + b + \alpha}\sigma_2 V(t)\mathrm{d}B_2(t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}\sigma_1 S(t)\mathrm{d}B_1(t)+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\sigma_1 S(t)\mathrm{d}B_1(t)\\
&+\int_{\mathbb{Y}}\big[(1+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}+\frac{(1+\delta)\beta}{\gamma+b+\alpha})\eta_1(z)S(t^-)\\
&+(1+\frac{(1+\delta)\beta}{\gamma+b+\alpha})\eta_2(z)V(t^-)+(1+\frac{(1+\delta)\beta}{\gamma+b+\alpha})\eta_3(z)I(t^-)\\
&-\ln\big(1+\eta_1(z)\big)-\ln(1+\eta_2(z))-\ln\left(1+\eta_3(z)\right)\big]\widetilde{N}(\mathrm{d}t,\mathrm{d}z),
\end{aligned}
$$

where LV_1 is the generating operator of the stochastic system (1.6):

$$
\begin{aligned}
LV_1:=&\Lambda-\frac{\Lambda}{S}-bS+b+\theta+\frac{\beta I}{1+\eta S}+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}\Lambda+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\Lambda\\
&-\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}(b+\theta)S-\frac{(1+\delta)\beta}{\gamma+b+\alpha}bS-\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}\frac{\beta SI}{1+\eta S}\\
&-\frac{(1+\delta)\beta}{\gamma+b+\alpha}(\gamma_1+b)V-\frac{\theta S}{V}+\frac{\delta\beta I}{1+\eta V}-(\gamma_1+b)V+\gamma_1+b-\frac{\beta S}{1+\eta S}\\
&-\frac{\delta\beta V}{1+\eta V}-\left(1+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\right)(\gamma+b+\alpha)I+\gamma+b+\alpha+\frac{1}{2}{\sigma_1}^2\\
&+\frac{1}{2}{\sigma_2}^2+\frac{1}{2}{\sigma_3}^2+\int_{\mathbb{Y}}[\eta_1(z)-\ln(1+\eta_1(z))+\eta_2(z)-\ln(1+\eta_2(z))\\
&+\eta_3(z)-\ln\ (1+\eta_3(z))]\nu(\mathrm{d}z).
\end{aligned}
$$

Since $\eta_i(z)-\ln(1+\eta_i(z))\geq 0$ for $\eta_i(z)>1$, under Assumption 1.2, we obtain

$$
\begin{aligned}
LV_1\leq&\Lambda+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}\Lambda+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\Lambda+3b+\theta+\gamma_1+\gamma+\alpha\\
&+\frac{1}{2}{\sigma_1}^2+\frac{1}{2}{\sigma_2}^2+\frac{1}{2}\sigma_3+(1+\delta)\beta I-\frac{(1+\delta)\beta}{\gamma+b+\alpha}(\gamma+b+\alpha)I\\
&+\int_{Y}[\eta_1(z)-\ln(1+\eta_1(z))+\eta_2(z)-\ln(1+\eta_2(z))+\eta_3(z)\\
&-\ln\left(1+\eta_3(z)\right)]\nu(\mathrm{d}z)\\
\leq&\Lambda+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}\Lambda+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\Lambda+3b+\theta+\gamma_1+\gamma+\alpha\\
&+\frac{1}{2}{\sigma_1}^2+\frac{1}{2}{\sigma_2}^2+\frac{1}{2}{\sigma_3}^2+\int_{Y}[\eta_1(z)-\ln(1+\eta_1(z))+\eta_2(z)\\
&-\ln(1+\eta_2(z))+\eta_3(z)-\ln\ (1+\eta_3(z))]\,\nu(\mathrm{d}z)\\
:=&K,
\end{aligned}
$$

where K is a positive constant independent of S,V,I and t. Therefore, we have

$$
\begin{aligned}
&\mathrm{d}V_1\big(S(t),V(t),I(t)\big)\\
\leq&Kdt+\sigma_1\left(S+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}S+\frac{(1+\delta)\beta}{\gamma+b+\alpha}S-1\right)\mathrm{d}B_1(t)\\
&+\sigma_2\left(V+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\sigma_2 V-1\right)\mathrm{d}B_2(t)+\sigma_3\left(I+\frac{(1+\delta)\beta}{\gamma+b+\alpha}I-1\right)\mathrm{d}B_3(t)\\
&+\int_{\mathbb{P}}[\left(1+\frac{(1+\delta)\beta\theta}{(\gamma+b+\alpha)(b+\theta)}+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\right)\eta_1(z)S(t^-)
\end{aligned}
$$

$$+\left(1+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\right)\eta_2(z)V(t^-)+\left(1+\frac{(1+\delta)\beta}{\gamma+b+\alpha}\right)\eta_3(z)I\left(t^-\right)$$
$$-\ln(1+\eta_1(z))-\ln(1+\eta_2(z))-\ln\left(1+\eta_3(z)\right)]\widetilde{N}(\mathrm{d}t,\mathrm{d}z). \tag{2.3}$$

Integrating both sides of (2.3) from 0 to $\tau_k \wedge T$ and then taking the expectation of it, we have

$$\mathbb{E}V_1\big(S(\tau_k\wedge T),V(\tau_k\wedge T),I(\tau_k\wedge T)\big)\le V_1\big(S(0),V(0),I(0)\big)+K\mathbb{E}(\tau_k\wedge T).$$

Therefore,

$$\mathbb{E}V_1\left(S(\tau_k\wedge T),V(\tau_k\wedge T),I(\tau_k\wedge T)\right)\le V_1\left(S(0),V(0),I(0)\right)+KT. \tag{2.4}$$

The remaining of the proof follows the lines of proof of Theorem 3.1 in [15]. Set $\Omega_k=\{\tau_k\le T\}$ for $k\ge k_1$ and by (2.2), we have $P(\Omega_k)\ge\epsilon$. Note that for every $\omega\in\Omega_k$, there exists $S(\tau_k,\omega)$, $V(\tau_k,\omega)$, or $I(\tau_k,\omega)$ that equals either k or $\frac{1}{k}$. Thus, $V_1\big(S(\tau_k\wedge T),V(\tau_k\wedge T),I(\tau_k\wedge T)\big)$ is no less than either

$$k-1-\ln k$$

or

$$\frac{1}{k}-1-\ln\frac{1}{k}=\frac{1}{k}-1+\ln k.$$

So we get

$$V_1(S(\tau_k,\omega),V(\tau_k,\omega),I(\tau_k,\omega))\ge(k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right).$$

By (2.4), we obtain

$$\begin{aligned}V_1\big(S(0),V(0),I(0)\big)+KT\ge&\mathbb{E}\big[1_{\Omega_k(\omega)}V_1\big(S(\tau_k,\omega),V(\tau_k,\omega),I(\tau_k,\omega)\big)\big]\\ \ge&\epsilon(k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right),\end{aligned}$$

where $1_{\Omega_k(\omega)}$ denotes the indicator function of Ω_k. Letting $k\to\infty$,

$$\infty>V_1(S(0),V(0),I(0))+KT=\infty,$$

which leads to the contradiction. Therefore, we must have $\tau_\infty=\infty$ a.s.. This completes the proof. $\square$

3. Asymptotic pathwise estimation of the stochastic solution

In this section, we present preliminary results for the asymptotic pathwise estimation of the solution to the stochastic system (1.6).

Similar to [20], we use the Kunita's inequality (Theorem 4.4.23 [1], Theorem 2.11 [17]) instead of the BDG inequality [18] to estimate the asymptotic pathwise for stochastic system (1.6). Kunita's inequality is stated as follows:

Lemma 3.1. **(Kunita's inequality)** *For any $p \geq 2$, there exists $C_p > 0$ such that*

$$\begin{aligned} E[\sup_{0<s\leq t}|Y(s)|^p] \leq & C_p\{E[(\int_0^t\int_{\mathbb{Y}}|H(s,z)|^2\mathrm{d}s\nu(\mathrm{d}z))^{p/2}] \\ & + E[\int_0^t\int_{\mathbb{Y}}|H(s,z)|^p\nu(\mathrm{d}z)]\mathrm{d}s\}, \end{aligned} \tag{3.1}$$

where $(s,z) \in \mathbb{R}_+ \times \mathbb{Y}$.

Theorem 3.1. *Suppose* Assumption 1.1 *and* Assumption 1.2 *hold. If $b - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$ for $p > 1$, then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \lim_{t\to\infty}\frac{V(t)}{t} = 0, \lim_{t\to\infty}\frac{I(t)}{t} = 0, a.s..$$

Proof. Let $X(t) = S(t) + V(t) + I(t)$. Define

$$V_2(X) = (1+X)^p,$$

where $p > 0$ is a constant to be determined later. It follows that

$$\begin{aligned} \mathrm{d}V_2(X(t)) = & LV_2\mathrm{d}t + p(1+X(t))^{p-1}[\sigma_1 S(t)\mathrm{d}B_1(t) + \sigma_2 V(t)\mathrm{d}B_2(t) + \sigma_3 I(t)\mathrm{d}B_3(t)] \\ & + \int_{\mathbb{Y}}[(1+X(t^-)+\eta_1(z)S(t^-)+\eta_2(z)V(t^-)+\eta_3(z)I(t^-))^p \\ & - (1+X(t^-))^p]\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} LV_2 := & p(1+X)^{p-1}[\Lambda - bS - (\gamma_1+b)V - (\gamma+b+\alpha)I] \\ & + \frac{p(p-1)}{2}(1+X)^{p-2}(\sigma_1^2S^2 + \sigma_2^2V^2 + \sigma_3^2I^2) \\ & + \int_{\mathbb{Y}}[(1+X+\eta_1(z)|S(t^-)+\eta_2(z)V(t^-)+\eta_3(z)I(t^-))^p - (1+X)^p \\ & - (\eta_1(z)S(t^-)+\eta_2(z)V(t^-)+\eta_3(z)I(t^-))p(1+X)^{p-1}]\nu(\mathrm{d}z) \\ = & p(1+X)^{p-1}[\Lambda - bS - (\gamma_1+b)V - (\gamma+b+\alpha)I] \\ & + \frac{p(p-1)}{2}(1+X)^{p-2}(\sigma_1^2S^2 + \sigma_2^2V^2 + \sigma_3^2I^2) \\ & + \int_{\mathbb{Y}}(1+X)^p[(\frac{1+X+\eta_1(z)S(t^-)+\eta_2(z)V(t^-)+\eta_3(z)I(t^-)}{1+X(t)})^p - 1]\nu(\mathrm{d}z) \\ \leq & p(1+X)^{p-1}[\Lambda - bS - (\gamma_1+b)V - (\gamma+b+\alpha)I] \\ & + \frac{p(p-1)}{2}(1+X)^{p-2}(\sigma_1^2S^2 + \sigma_2^2V^2 + \sigma_3^2I^2) \\ & + \int_{\mathbb{Y}}(1+X(t))^p[(1+\eta_1(z)\vee\eta_2(z)\vee\eta_3(z))^p - 1]\nu(\mathrm{d}z) \\ \leq & p(1+X)^{p-2}\{[\Lambda - bS - (\gamma_1+b)V - (\gamma+b+\alpha)I](1+X) \\ & + \frac{p-1}{2}\sigma^2X^2 + \frac{\lambda}{p}(1+X)^2\} \end{aligned}$$

$$\leq p(1+X)^{p-2}\left\{-\left[b-\frac{p-1}{2}\sigma^2-\frac{\lambda}{p}\right]X^2+\left(\Lambda-b+\frac{2\lambda}{p}\right)X+\Lambda+\frac{\lambda}{p}\right\},$$

where $\lambda=\int_{\mathbb{Y}}\left[(1+\eta_1(z)\vee\eta_2(z)\vee\eta_3(z))^p-1\right]\nu(\mathrm{d}z)$ and $\sigma^2=\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2$. Then we set $\rho=b-\frac{p-1}{2}\sigma^2-\frac{\lambda}{p}>0$. It follows that

$$LV_2\leq p(1+X)^{p-2}[-\rho X^2+\left(\Lambda-b+\frac{2\lambda}{p}\right)X+\Lambda+\frac{\lambda}{p}]. \tag{3.3}$$

According to (3.2), for any $k\in\mathbb{R}$ we obtain

$$\begin{aligned}de^{kt}V_2(X)=&L[e^{kt}V(X)]\mathrm{d}t+e^{kt}p(1+X)^{p-1}[\sigma_1S\mathrm{d}B_1(t)+\sigma_2V\mathrm{d}B_2(t)+\sigma_3I\mathrm{d}B_3(t)]\\&+e^{kt}\int_{\mathbb{Y}}[(1+X(t^-)+\eta_1(z)S(t^-)+\eta_2(z)V(t^-)+\eta_3(z)I(t^-))^p\\&-(1+X(t^-))^p]\widetilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned} \tag{3.4}$$

After integrating both sides of equation (3.4) from 0 to t, we have

$$\begin{aligned}&e^{kt}(1+X)^p\\=&(1+X(0))^p+\int_0^t[ke^{ks}(1+X(s))^p+e^{ks}LV_2(X(s))]\mathrm{d}s\\&+\int_0^t e^{ks}p(1+X(s))^{p-1}[\sigma_1S(s)\mathrm{d}B_1(s)+\sigma_2I(s)\mathrm{d}B_2(s)+\sigma_3R(s)\mathrm{d}B_3(s)]\\&+\int_0^t e^{ks}\int_{\mathbb{Y}}[(1+X(s^-)+\eta_1(z)S(s^-)+\eta_2(z)V(s^-)+\eta_3(z)I(s^-))^p\\&-(1+X(s^-))^p]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{3.5}$$

Then we take expectations of (3.5), which yields

$$e^{kt}E[(1+X(t))^p]=(1+X(0))^p+E[\int_0^t[ke^{ks}(1+X(s))^p+e^{ks}LV_2(X(s))]\mathrm{d}s]. \tag{3.6}$$

According to (3.3), we get that for any $k<\rho p$,

$$\begin{aligned}&ke^{ks}(1+X(s))^p+e^{ks}LV_2(X(s))\\&\leq ke^{ks}(1+X(s))^p+e^{ks}p(1+X)^{p-2}[-\rho X^2+\left(\Lambda-b+\frac{2\lambda}{p}\right)X+\Lambda+\frac{\lambda}{p}]\\&=e^{ks}p(1+X)^{p-2}[-(\rho-\frac{k}{p})X^2(t)+(\Lambda-b+\frac{2\lambda}{p}+\frac{2k}{p})X(t)+\Lambda+\frac{\lambda+k}{p}]\\&\leq e^{ks}pK_1,\end{aligned} \tag{3.7}$$

where $K_1=(1+X)^{p-2}[-(\rho-\frac{k}{p})X^2(t)+(\Lambda-b+\frac{2\lambda}{p}+\frac{2k}{p})X(t)+\Lambda+\frac{\lambda+k}{p}]$, and $K_1\in(0,\infty)$. Substituting (3.7) into (3.6), we obtain

$$\begin{aligned}e^{kt}E[(1+X(t))^p]&\leq(1+X(0))^p+pK_1\int_0^t e^{ks}\mathrm{d}s\\&=(1+X(0))^p+\frac{pK_1}{k}e^{kt}.\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} E[(1+X(t))^p] \le \frac{pK_1}{k}.$$

Here we set $M > 0$, then

$$E[(1+X(t))^p] \le M. \tag{3.8}$$

For $i = 1, 2, ...,$ $t \ge i\theta$ and sufficiently small $\theta > 0$, combining (3.2) and (3.3), we integrate (3.2) from $i\theta$ to t and obtain

$$\begin{aligned}
&(1+X(t))^p - (1+X(i\theta))^p \\
\le\, & p\int_{i\theta}^{t}(1+X(s))^{p-2}[-\rho X^2(s) + (\Lambda - b + \frac{2\lambda}{p})X(s) + \Lambda + \frac{\lambda}{p}]\mathrm{d}s \\
&+ p\int_{i\theta}^{t}(1+X(s))^{p-1}(\sigma_1 S(s)\mathrm{d}B_1(s) + \sigma_2 V(s)\mathrm{d}B_2(s) + \sigma_3 I(s)\mathrm{d}B_3(s)) \\
&+ \int_{i\theta}^{t}\int_{\mathbb{Y}}[(1+X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)V(s^-) + \eta_3(z)I(s^-))^p \\
&- (1+X(s^-))^p]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).
\end{aligned} \tag{3.9}$$

Taking the supremum of (3.9), we get

$$\begin{aligned}
&\sup_{i\theta\le t\le (i+1)\theta}(1+X(t))^p \\
\le\, & (1+X(i\theta))^p + \sup_{i\theta\le t\le (i+1)\theta} p\left|\int_{i\theta}^{t}(1+X(s))^{p-2}[-\rho X^2(s) + (\Lambda - b + \frac{2\lambda}{p})X(s)\right. \\
&\left. + \Lambda + \frac{\lambda}{p}]\mathrm{d}s\right| + \sup_{i\theta\le t\le (i+1)\theta} p\left|\int_{i\theta}^{t}(1+X(s))^{p-1}(\sigma_1 S(s)\,\mathrm{d}B_1(s) + \sigma_2 V(s)\mathrm{d}B_2(s)\right. \\
&\left. + \sigma_3 I(s)\mathrm{d}B_3(s))\right| + \sup_{i\theta\le t\le (i+1)\theta}\left|\int_{i\theta}^{t}\int_{\mathbb{Y}}[(1+X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)V(s^-)\right. \\
&\left. + \eta_3(z)I(s^-))^p - (1+X(s^-))^p]\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|.
\end{aligned}$$

After taking expectation of the both sides of above inequality and using (3.8), we have

$$\begin{aligned}
E[\sup_{i\theta\le t\le (i+1)\theta}(1+X(t))^p] &\le E[(1+X(i\theta))^p] + I_1 + I_2 + I_3 \\
&\le M + I_1 + I_2 + I_3,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
I_1 =\,& pE\{\sup_{i\theta\le t\le (i+1)\theta}\left|\int_{i\theta}^{t}(1+X(s))^{p-2}[-\rho X^2(s) + (\Lambda - b + \frac{2\lambda}{p})X(s) + \Lambda + \frac{\lambda}{p}]\mathrm{d}s\right|\}, \\
I_2 =\,& pE\{\sup_{i\theta\le t\le (i+1)\theta}\left|\int_{i\theta}^{t}(1+X(s))^{p-1}(\sigma_1 S(s)\,\mathrm{d}B_1(s) + \sigma_2 V(s)\mathrm{d}B_2(s)\right. \\
&\left. + \sigma_3 I(s)\mathrm{d}B_3(s))\right|\}, \\
I_3 =\,& E\{\sup_{i\theta\le t\le (i+1)\theta}\left|\int_{i\theta}^{t}\int_{\mathbb{Y}}[(1+X(s^-) + \eta_1(z)S(s^-) + \eta_2(z)V(s^-) + \eta_3(z)I(s^-))^p\right. \\
&\left. - (1+X(s^-))^p]\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|\}.
\end{aligned}$$

Using the same method as Zhu et al. (Theorem 3.1, [30]), we have

$$I_1 \leq c_1 \theta E\{\sup_{i\theta \leq t \leq (i+1)\theta} (1+X)^p\},$$

where $c_1 > 0$ is a constant. By using BDG inequality (Theorem 7.3, [18]), we obtain

$$I_2 \leq c_2 \theta^{\frac{1}{2}} p\sigma E\{\sup_{i\theta \leq t \leq (i+1)\theta} (1+X)^p\},$$

where $c_2 > 0$ is a constant. By the Kunita's inequality [1,17], it follows that

$$I_3 \leq 2\theta E[\sup_{i\theta \leq t \leq (i+1)\theta} (1+X)^p] \int_{\mathbb{Y}} \mid (1+\eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p - 1 \mid \nu(\mathrm{d}z).$$

Therefore, we have

$$\begin{aligned} E[\sup_{i\theta \leq t \leq (i+1)\theta} (1+X(t))^p] \leq & M + E[\sup_{i\theta \leq t \leq (i+1)\delta} (1+X(t))^p][c_1\theta + c_2\theta^{\frac{1}{2}} p\sigma \\ & + 2\theta \int_{\mathbb{Y}} \mid (1+\eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p - 1 \mid \nu(\mathrm{d}z)]. \end{aligned} \tag{3.11}$$

Thus there exist a small positive θ such that

$$c_1\theta + c_2\theta^{\frac{1}{2}} p\sigma + 2\theta \int_{\mathbb{Y}} \mid (1+\eta_1(z) \vee \eta_2(z) \vee \eta_3(z))^p - 1 \mid \nu(\mathrm{d}z) < \frac{1}{2}.$$

From (3.10) and (3.11), we get

$$E[\sup_{i\theta \leq t \leq (i+1)\theta} (1+X(t))^p] \leq 2M. \tag{3.12}$$

Using Chebyshev's inequality and (3.12), for arbitrary $\epsilon > 0$ and all $i > 1$, we get

$$\begin{aligned} P\{\sup_{i\theta \leq t \leq (i+1)\theta} (1+X(t))^p > (i\theta)^{1+\epsilon}\} \leq & \frac{E[\sup_{i\theta \leq t \leq (i+1)\theta} (1+X(t))^p]}{(i\theta)^{1+\epsilon}} \\ \leq & \frac{2M}{(i\theta)^{1+\epsilon}}. \end{aligned}$$

Applying Borel-Cantelli Lemma (Lemma 2.4, [18]), for all but finitely many i, we have

$$\sup_{i\theta \leq t \leq (i+1)\theta} (1+X(t))^p \leq (i\theta)^{1+\epsilon}.$$

For almost all $\omega \in \Omega$, there exists $k_0(\omega)$ such that for any $k \geq k_0$, we get

$$\frac{\ln(1+X(t))^p}{\ln t} \leq \frac{(1+\epsilon)\ln i\theta}{\ln i\theta} = 1+\epsilon, \epsilon > 0.$$

Then taking the limit superior of the inequality above, we obtain

$$\limsup_{t\to\infty} \frac{\ln(1+X(t))^p}{\ln t} \leq 1+\epsilon, a.s..$$

By the arbitrariness of ϵ, letting $\epsilon \to 0$, we have

$$\limsup_{t\to\infty} \frac{\ln(1+X(t))^p}{\ln t} \leq 1, a.s..$$

Then for $p > 1$, there exists

$$\limsup_{t\to\infty} \frac{\ln X(t)}{\ln t} \leq \limsup_{t\to\infty} \frac{\ln(1+X(t))}{\ln t} \leq \frac{1}{p}, a.s..$$

That is to say, for any $m \in (0, 1-\frac{1}{p})$, there exists a finite random time $\bar{T} = \bar{T}(\omega)$ such that

$$\ln X(t) \leq (\frac{1}{p} + m)\ln t, t \geq \bar{T},$$

which implies

$$\limsup_{t\to\infty} \frac{X(t)}{t} \leq \limsup_{t\to\infty} \frac{t^{\frac{1}{p}+m}}{t} = 0.$$

Combining with the positivity of the solution for system (1.6), we get $\lim\limits_{t\to\infty} \frac{X(t)}{t} = 0$, a.s.. Therefore,

$$\lim_{t\to\infty} \frac{X(t)}{t} = \lim_{t\to\infty} \frac{S(t)+I(t)+R(t)}{t} = 0,$$

which means that

$$\lim_{t\to\infty} \frac{S(t)}{t} = 0, \lim_{t\to\infty} \frac{I(t)}{t} = 0, \lim_{t\to\infty} \frac{R(t)}{t} = 0, a.s..$$

The proof is completed. □

Theorem 3.2. *Suppose* Assumption 1.1 *and* Assumption 1.2 *hold. If there exists $p > 2$ such that $b - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$, then*

$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{Y}} \eta_1(z) S(s^-) \widetilde{N}(\mathrm{d}s, \mathrm{d}z)}{t} = 0, \lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{Y}} \eta_2(z) V(s^-) \widetilde{N}(\mathrm{d}s, \mathrm{d}z)}{t} = 0,$$
$$\lim_{t\to\infty} \frac{\int_0^t \int_{\mathbb{Y}} \eta_3(z) I(s^-) \widetilde{N}(\mathrm{d}s, \mathrm{d}z)}{t} = 0, a.s..$$

Proof. For convenience, we denote

$$X_1(t) = \int_0^t \int_{\mathbb{R}} \eta_1(z) S(s^-) \widetilde{N}(\mathrm{d}s, \mathrm{d}z).$$

According to Kunita's inequality [1,17] and (3.8), there exists $C_p > 0$ for any $p \geq 2$ such that

$$\begin{aligned}
& E[\sup_{0\leq s\leq t} |X_1(s)|^p] \\
\leq & C_p E[(\int_0^t \int_{\mathbb{Y}} |\eta_1(z) S(s^-)|^2 \nu(\mathrm{d}z)\mathrm{d}s)^{\frac{p}{2}}] + C_p E[\int_0^t \int_{\mathbb{Y}} |\eta_1(z) S(s^-)|^p \nu(\mathrm{d}z)\mathrm{d}s] \\
= & C_p [\int_{\mathbb{Y}} \eta_1^2(z) \nu(\mathrm{d}z)]^{\frac{p}{2}} E[(\int_0^t |S(s)|^2 \mathrm{d}s)^{\frac{p}{2}}] + C_p (\int_{\mathbb{Y}} \eta_1^p(z) \nu(\mathrm{d}z)) E[\int_0^t |S(s)|^p \mathrm{d}s] \\
\leq & C_p t^{\frac{p}{2}} [\int_{\mathbb{Y}} \eta_1^2(z) \nu(\mathrm{d}z)]^{\frac{p}{2}} E[\sup_{0\leq s\leq t} |S(s)|^p] + C_p t M \int_{\mathbb{Y}} \eta_1^p(z) \nu(\mathrm{d}z).
\end{aligned}$$

By (3.12) and the above inequality, we get

$$\begin{aligned}&E[\sup_{i\theta\le t\le(i+1)\theta}|X_1(t)|^p]\\ \le&C_p2M((i+1)\theta)^{\frac{p}{2}}[\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z)]^{\frac{p}{2}}+C_pM(i+1)\theta\int_{\mathbb{Y}}\eta_1^p(z)\nu(\mathrm{d}z).\end{aligned}$$

We use Doob's martingale inequality, and suppose $\epsilon>0$ be arbitrary. Therefore,

$$\begin{aligned}&P\{\omega:\sup_{i\theta\le t\le(i+1)\theta}|X_1(t)|^p>(i\theta)^{1+\epsilon+\frac{p}{2}}\}\\ \le&\frac{E[\sup_{i\theta\le t\le(i+1)\theta}|X_1(t)|^p]}{(i\theta)^{1+\epsilon+\frac{p}{2}}}\\ \le&\frac{2MC_p((i+1)\theta)^{\frac{p}{2}}}{(i\theta)^{1+\epsilon+\frac{p}{2}}}[\int_{\mathbb{Y}}\eta_1^2(z)\nu(\mathrm{d}z)]^{\frac{p}{2}}+\frac{MC_p(i+1)\theta}{(i\theta)^{1+\epsilon+\frac{p}{2}}}\int_{\mathbb{Y}}\eta_1^p(z)\nu(\mathrm{d}z).\end{aligned}$$

According to Borel-Cantelli Lemma [18], for all but finite i, we have

$$\sup_{i\theta\le t\le(i+1)\theta}|X_1(t)|^p\le(i\theta)^{1+\epsilon+\frac{p}{2}},a.s..$$

Thus, there exists $i_0(\omega)>0$ such that for all $i>i_0$,

$$\frac{\ln|X_1(t)|^p}{\ln t}\le\frac{(1+\epsilon+\frac{p}{2})\ln(i\theta)}{\ln(i\theta)}=1+\epsilon+\frac{p}{2},\epsilon>0.$$

Taking limit superior of the inequality above, we obtain

$$\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t}\le\frac{1}{2}+\frac{1+\epsilon}{p}.$$

Letting $\epsilon\to0$, we have

$$\limsup_{t\to\infty}\frac{\ln|X_1(t)|}{\ln t}\le\frac{1}{2}+\frac{1}{p},p>2.$$

Similar to the argument of Theorem 3.1, we get

$$\limsup_{t\to\infty}\frac{|X_1(t)|}{t}\le\limsup_{t\to\infty}\frac{t^{\frac{1}{2}+\frac{1}{p}}}{t}=0.$$

Together with the positivity of the solution for system (1.6), it follows that

$$\lim_{t\to\infty}\frac{|X_1(t)|}{t}=0,a.s..$$

According to the same discussion, we can also prove $\lim_{t\to\infty}\frac{|\int_0^t\int_{\mathbb{Y}}\eta_2(z)V(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)|}{t}=0$ and $\lim_{t\to\infty}\frac{|\int_0^t\int_{\mathbb{Y}}\eta_3(z)I(s^-)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)|}{t}=0$, a.s..

Theorem 3.3. *Suppose* Assumption 1.1 *and* Assumption 1.2 *hold. If there exists* $p>1$ *such that* $b-\frac{p-1}{2}\sigma^2-\frac{1}{p}\lambda>0$, *then*

$$\lim_{t\to\infty}\frac{\int_0^tS(s)\mathrm{d}B_1(s)}{t}=0,\lim_{t\to\infty}\frac{\int_0^tV(s)\mathrm{d}B_2(s)}{t}=0,\lim_{t\to\infty}\frac{\int_0^tI(s)\mathrm{d}B_3(s)}{t}=0,a.s..$$

The proof is omitted here because it is similar to Lemma 2.2 in [26]. □

4. Threshold behavior of the disease

One of the main concerns in epidemiology is how to regulate disease dynamics to ensure that the disease becomes extinct or persists in the long term. In this section, we aim to provide the critical value for the extinction and persistence in the mean of the disease. Firstly, we quote the concepts of extinction and persistence in the mean of the disease [13].

For simplicity, we denote $\langle x(t)\rangle = \frac{1}{t}\int_0^t x(s)\mathrm{d}s$ and $\langle x(t)\rangle_* = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s$.

Definition 4.1. $x(t)$ is said to be extinct if $\lim_{t\to\infty} x(t) = 0, a.s.$.

Definition 4.2. $x(t)$ is said to be persistent in the mean if $\langle x(t)\rangle_* > 0, a.s.$.

Define

$$\bar{R}_0^s = \frac{1}{\gamma+b+\alpha}\Big\{\frac{\beta\bar{S}}{1+\eta\bar{S}} + \frac{\delta\beta\bar{V}}{1+\eta\bar{V}} - \frac{\sigma_3^2}{2} - \int_{\mathbb{Y}}\big[\eta_3(z) - \ln\big(1+\eta_3(z)\big)\big]\nu(\mathrm{d}z)\Big\}. \tag{4.1}$$

In order to study the sufficient conditions of extinction and persistence in the mean of the disease, we consider the following stochastic system:

$$\begin{cases} \mathrm{d}\bar{S}(t) = [\Lambda - (b+\theta)\bar{S}(t)]\mathrm{d}t + \sigma_1\bar{S}(t)\mathrm{d}B_1(t) + \displaystyle\int_{\mathbb{Y}}\eta_1(z)\bar{S}(t^-)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\ \mathrm{d}\bar{V}(t) = [\theta\bar{S}(t) - (\gamma_1+b)\bar{V}(t)]\mathrm{d}t + \sigma_2\bar{V}(t)\mathrm{d}B_2(t) + \displaystyle\int_{\mathbb{Y}}\eta_2(z)\bar{V}(t^-)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \end{cases}$$

with the same initial value $\bar{S}(0) = S(0) > 0$, $\bar{V}(0) = V(0) > 0$. Making use of the stochastic comparison theorem (Theorem 3.1, [19]), we have

$$S(t) \le \bar{S}(t), V(t) \le \bar{V}(t), a.s..$$

Moreover, we can get

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{S}(s)\mathrm{d}s &= \frac{\Lambda}{b+\theta}, \\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{V}(s)\mathrm{d}s &= \frac{\theta\Lambda}{(\gamma_1+b)(b+\theta)}. \end{aligned} \tag{4.2}$$

Theorem 4.1. *Suppose* Assumptions 1.1-1.3 *hold. Assume that there exists some* $p > 2$ *such that* $\mu - \frac{p-1}{2}\sigma^2 - \frac{1}{p}\lambda > 0$. *Let* $(S(t), V(t), I(t))$ *be a positive solution of system* (1.6) *with the initial value* $(S(0), V(0), I(0)) \in \mathbb{R}_+^3$. *We obtain*

(i) If $\bar{R}_0^s > 1$, *then*

$$\lim_{t\to\infty} I(t) = 0, a.s..$$

It indicates the disease will die out in a long term.

(ii) If $\bar{R}_0^s < 1$, *then*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)\mathrm{d}s \ge \frac{(\gamma+b+\alpha)(R_0^s-1)}{K_2} > 0, a.s..$$

Here $K_2 := (\gamma+b+\alpha)\left(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta(b+2\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\right) > 0$. *This implies the disease will persist in a long term.*

Proof. **(i).** By Itô's formula with jump noise [1], we have

$$\begin{aligned}
&\mathrm{d}\ln I(t)\\
=&\Big\{\frac{\beta S}{1+\eta S}+\frac{\delta\beta V}{1+\eta V}-(\gamma+b+\alpha)-\frac{\sigma_3^2}{2}-\int_{\mathbb{Y}}[\eta_3(z)-\ln\big(1+\eta_3(z)\big)]\nu(\mathrm{d}z)\Big\}\mathrm{d}t\\
&+\sigma_3\mathrm{d}B_3(t)+\int_{\mathbb{Y}}\ln(1+\eta_3(\mathrm{z}))\widetilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\ln I(t)-\ln I(0)}{t}=&\left\langle\frac{\beta S}{1+\eta S}\right\rangle+\left\langle\frac{\delta\beta V}{1+\eta V}\right\rangle-(\gamma+b+\alpha)-\frac{\sigma_3^2}{2}\\
&-\int_{\mathbb{Y}}[\eta_3(z)-\ln(1+\eta_3(z))]\,\nu(\mathrm{d}z)+\sigma_3\frac{B_3(t)}{t}\\
&+\frac{1}{t}\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_3(z))\widetilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}\tag{4.3}$$

Define

$$H_1(t)=\int_0^t\int_{\mathbb{Y}}\ln(1+\eta_3(z))\widetilde{N}(\mathrm{d}t,\mathrm{d}z).$$

By Assumption 1.3, we have

$$\int_0^t\frac{\mathrm{d}\langle H_1,H_1\rangle(s)}{(1+s)^2}\mathrm{d}s=\frac{t}{1+t}\int_{\mathbb{Y}}(\ln(1+\eta_3(z)))^2\nu(\mathrm{d}z)<+\infty.$$

According to the law of large numbers for local martingales (Theorem 1, [12]), we have

$$\lim_{t\to\infty}\frac{H_1(t)}{t}=0,a.s..$$

Moreover, by the law of large numbers (Theorem 3.4, [18]), we have

$$\lim_{t\to\infty}\frac{B_3(t)}{t}=0,a.s..$$

Taking the superior limit of (4.3) and using stochastic comparison theorem [19], combined with (4.2), we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\ln I(t)}{t}\\
=&\limsup_{t\to\infty}\left(\left\langle\frac{\beta S}{1+\eta S}\right\rangle+\left\langle\frac{\delta\beta V}{1+\eta V}\right\rangle\right)-(\gamma+b+\alpha)-\frac{\sigma_3^2}{2}\\
&-\int_{\mathbb{Y}}[\eta_3(z)-\ln(1+\eta_3(z))]\,\nu(\mathrm{d}z)\\
=&\limsup_{t\to\infty}\left[-\left(\frac{\beta}{\eta}\left\langle\frac{1}{1+\eta S}\right\rangle+\frac{\delta\beta}{\eta}\left\langle\frac{1}{1+\eta V}\right\rangle\right)\right]+\frac{\beta}{\eta}+\frac{\delta\beta}{\eta}-(\gamma+b+\alpha)-\frac{\sigma_3^2}{2}\\
&-\int_{\mathbb{Y}}[\eta_3(z)-\ln(1+\eta_3(z))]\,\nu(\mathrm{d}z)\\
\leq&-\frac{\beta}{\eta}\left\langle\frac{1}{1+\eta\bar{S}}\right\rangle-\frac{\delta\beta}{\eta}\left\langle\frac{1}{1+\eta\bar{V}}\right\rangle+\frac{\beta}{\eta}+\frac{\delta\beta}{\eta}-(\gamma+b+\alpha)-\frac{\sigma_3^2}{2}
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{Y}} \left[\eta_3(z) - \ln\left(1+\eta_3(z)\right)\right] \nu(\mathrm{d}z) \\
=& \frac{\beta\Lambda}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta\Lambda}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda} - (\gamma+b+\alpha) - \frac{\sigma_3^2}{2} \\
&- \int_{\mathbb{Y}} \left[\eta_3(z) - \ln\left(1+\eta_3(z)\right)\right] \nu(\mathrm{d}z) \\
=& (\gamma+b+\alpha)(\bar{R}_0^s - 1) < 0, a.s..
\end{aligned}
$$

Therefore, it indicates that

$$
\lim_{t\to\infty} I(t) = 0, a.s..
$$

Consequently, this means the disease will go to extinction exponentially in probability one.

(ii). Define a C^2-function V_3

$$
\begin{aligned}
V_3(S,V,I) =& -\ln I - \left(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\right)(S+I) \\
&- \frac{\delta\beta(b+\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}(V+I).
\end{aligned}
$$

Then

$$
\begin{aligned}
&LV_3 \\
=&(\gamma+b+\alpha) - \frac{\beta S}{1+\eta S} - \frac{\delta\beta V}{1+\eta V} + \frac{\sigma_3^2}{2} + \int_{\mathbb{Y}} \left[\eta_3(z) - \ln(1+\eta_3(z))\right] \nu(\mathrm{d}z) \\
&- \left(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\right) \\
&\times \left[\Lambda - (b+\theta)S + \frac{\delta\beta VI}{1+\eta V} - (\gamma+b+\alpha)I\right] \\
&- \frac{\delta\beta(b+\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\left[\theta S - (\gamma_1+b)V + \frac{\beta SI}{1+\eta S} - (\gamma+b+\alpha)I\right] \\
\le&(\gamma+b+\alpha) - \frac{\beta S}{1+\eta\bar{S}} - \frac{\delta\beta V}{1+\eta\bar{V}} + \frac{\sigma_3^2}{2} + \int_{\mathbb{Y}} \left[\eta_3(z) - \ln(1+\eta_3(z))\right] \nu(\mathrm{d}z) \\
&- \left(\frac{\beta\Lambda}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta\Lambda}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\right) + \frac{b+\theta}{b+\theta+\eta\Lambda}\beta S \\
&+ \left(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\right)(\gamma+b+\alpha)I \\
&+ \frac{(b+\theta)(\gamma_1+b)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\delta\beta V + \frac{\delta\beta(b+\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}(\gamma+b+\alpha)I \\
\le&(\gamma+b+\alpha) + \frac{{\sigma_3}^2}{2} + \int_{\mathbb{Y}} \left[\eta_3(z) - \ln(1+\eta_3(z))\right] v(\mathrm{d}z) \\
&- (\frac{\beta\Lambda}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta\Lambda}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}) \\
&+ (\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta(b+2\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda})(\gamma+b+\alpha)I \\
=& -(\gamma+b+\alpha)(R_0^s - 1) + K_2 I.
\end{aligned}
$$

Let $K_3 = \frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta(b+2\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}$. Consequently,

$$\begin{aligned}
&\mathrm{d}V_3(S,V,I)\\
=&LV_3\mathrm{d}t - \sigma_3\mathrm{d}B_3(t) - \int_{\mathbb{Y}} \ln\big(1+\eta_3(z)\big)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)\\
&-(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda})(\sigma_1 S\mathrm{d}B_1(t)\\
&+\int_{\mathbb{Y}} \eta_1(z)S(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)) - \frac{\delta\beta(b+\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}\big(\sigma_2 V\mathrm{d}B_2(t)\\
&+\int_{\mathbb{Y}} \eta_2(z)V(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)\big) - K_3(\sigma_3 I\mathrm{d}B_3(t) + \int_{\mathbb{Y}} \eta_3(z)I(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)).
\end{aligned} \tag{4.4}$$

Integrating both sides of (4.4), we obtain

$$\begin{aligned}
&\frac{V_3\left(S(t),V(t),I(t)\right) - V_3\left(S(0),V(0),I(0)\right)}{t}\\
\leq&-(\gamma+b+\alpha)(\bar{R}_0^s - 1) + K_2\langle I\rangle - \frac{\sigma_3 B_3(t)}{t} - \frac{1}{t}\int_0^t \int_{\mathbb{Y}} \ln\left(1+\eta_3(z)\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)\\
&-(\frac{\beta}{b+\theta+\eta\Lambda} + \frac{\delta\beta\theta}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda})(\frac{1}{t}\int_0^t \sigma_1 S\mathrm{d}B_1(t)\\
&+\frac{1}{t}\int_0^t \int_{\mathbb{Y}} \eta_1(z)S(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z))\\
&-\frac{\delta\beta(b+\theta)}{(\gamma_1+b)(b+\theta)+\eta\theta\Lambda}(\frac{1}{t}\int_0^t \sigma_2 V\mathrm{d}B_2(t) + \frac{1}{t}\int_0^t\int_{\mathbb{Y}} \eta_2(z)V(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z))\\
&-K_3(\frac{1}{t}\int_0^t \sigma_3 I\mathrm{d}B_3(t) + \frac{1}{t}\int_0^t\int_Y \eta_3(z)I(t^-)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)).
\end{aligned}$$

In view of Theorem 3.1 to 3.3, we obtain

$$\begin{aligned}
&\liminf_{t\to\infty} K_2\langle I(t)\rangle\\
\geq&(\gamma+b+\alpha)(\bar{R}_0^s - 1) + \liminf_{t\to\infty}\left(\frac{V_3(S(t),V(t),I(t)) - V_3(S(0),V(0),I(0))}{t}\right)\\
\geq&(\gamma+b+\alpha)(\bar{R}_0^s - 1) > 0, a.s..
\end{aligned}$$

Therefore, we provide the sufficient condition for the persistence in the mean of the disease. This completes the proof. □

5. Conclusions

In this paper, we study the stochastic SVI epidemic model with jump noise and saturated incidence rate. By using Kunita's inequality [1, 17], we complete the asymptotic pathwise estimation of system (1.6). Moreover, we prove that the disease will die out if $\bar{R}_0^s < 1$ and will persist in the mean if $\bar{R}_0^s > 1$ under certain conditions. Comparing (1.2) with (4.1), we obtain $\bar{R}_0^s = R_0 - \frac{1}{\gamma+b+\alpha}\{\frac{\sigma_3^2}{2} + \int_{\mathbb{Y}}\big[\eta_3(z) - \ln\big(1+\eta_3(z)\big)\big]\nu(\mathrm{d}z)\}$. It is easy to prove that $\bar{R}_0^s < R_0$, which indicates that jump noise can slow down the transmission of the disease.

Then we present some examples and use numerical simulations to verify the theoretical results obtained above. According to [27], the initial value of the system (1.6) is given by $(S(0), V(0), I(0)) = (0.3, 0.3, 0.2), \Lambda = 0.8, b = \frac{1}{24*365}, \theta = 0.5, \beta = 0.35, \eta = 1, \gamma_1 = \frac{1}{10*365}, \gamma = 0.1, \alpha = 0.05, \delta = 0.06$. According to [14], for computational convenience, we choose $\nu(\mathbb{Y}) = 1$.

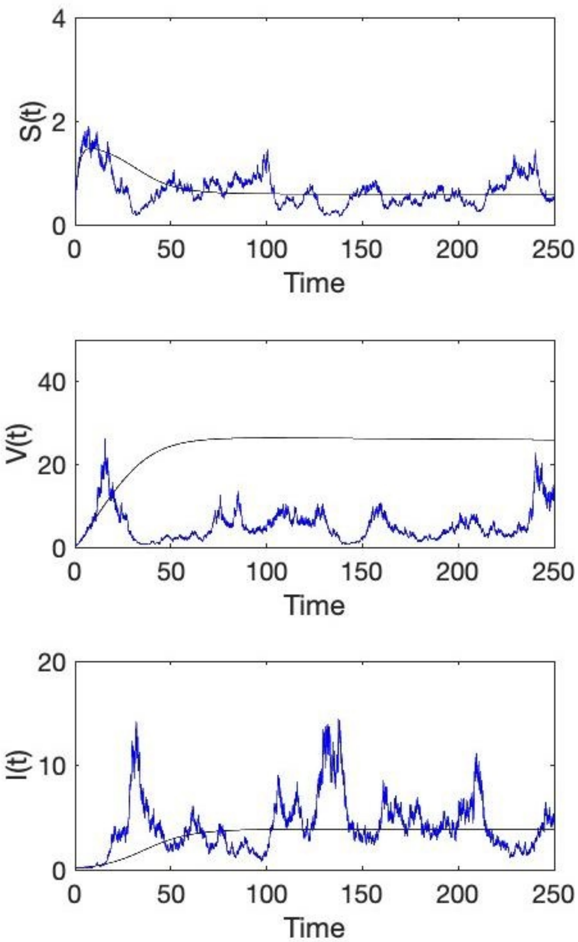

Figure 1. The blue lines show the paths of $S(t), V(t), I(t)$ with jump noise under the noise intensities $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.04, 0.16, 0.8), (\eta_1, \eta_2, \eta_3) = (0.1, 0.1, 0.1)$. The black ones are the paths of deterministic SVI model.

Example 5.1. We suppose $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.04, 0.16, 0.8), (\eta_1, \eta_2, \eta_3) = (0.1, 0.1, 0.1)$, which leads to $R_0 = 1.5746 > 1$, $\bar{R}_0^s = 1.5428 > 1$. From Theorem 4.1, it follows that the disease is expected to persist regardless of whether the system has jump noise or not. The simulation results are illustrated in Figure 1.

Example 5.2. We suppose $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.04, 0.16, 0.8), (\eta_1, \eta_2, \eta_3) = (0.1, 0.1, 0.8)$, which leads to $R_0 = 1.5746 > 1, \bar{R}_0^s = 0.9546 < 1$. In this case, the disease will be persist in deterministic model (1.1) but die out in stochastic system (1.6). The simulation results are provided in Figure 2. Moreover, it is clear that the jump phenomenon is shown in Figure 2.

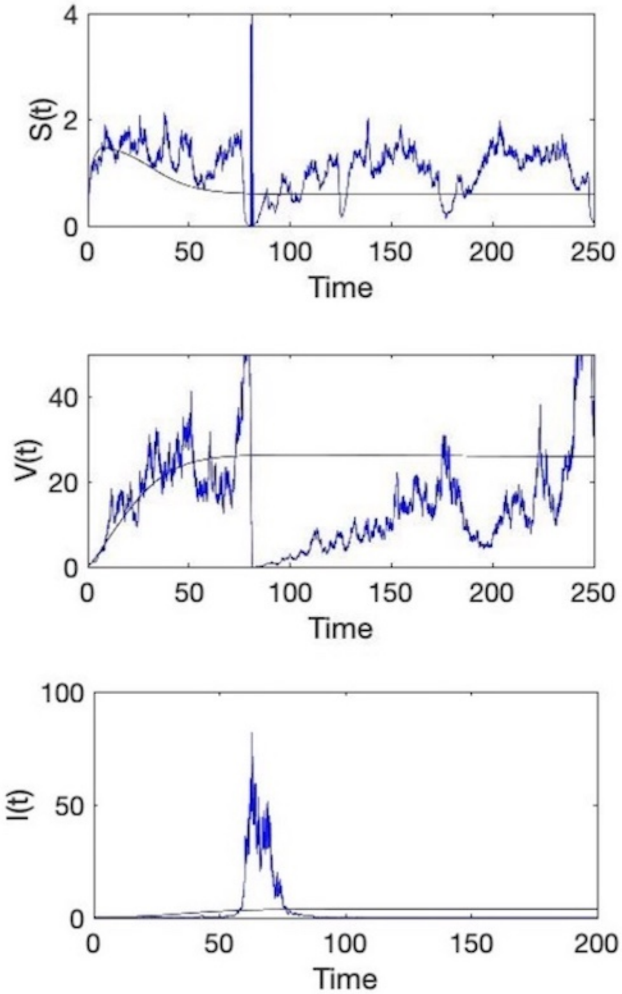

Figure 2. The blue lines show the paths of $S(t), V(t), I(t)$ with jump noise under the noise intensities $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.04, 0.16, 0.8)$, $(\eta_1, \eta_2, \eta_3) = (0.1, 0.1, 0.8)$. The black ones are the paths of deterministic SVI model.

The numerical simulation indicates that the disease will become extinct as the noises $\eta_i, i = 1, 2, 3$ become stronger. Jump noise curbs the spread of the disease.

Comparing with system (1.1), we derive a more general $\bar{R}_0^s$, which may enrich the research of threshold behavior in epidemic model and help us better understand the dynamics in jump noise sense. In addition, some interesting questions deserve further discussion. For example, we can consider the logistic growth [6, 10] and seasonality [5] in system (1.6). The research is carrying out.

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