Journal of Applied Analysis and Computation
Volume 15, Number 5, October 2025, 3045–3066
Website:http://www.jaac-online.com
DOI:10.11948/20240520

NEW TRAVELING WAVE SOLUTIONS OF THE (3+1)-DIMENSIONAL GENERALIZED BREAKING SOLITON EQUATION

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Abstract In this paper, we will study solutions of the (3+1)-dimensional generalized breaking soliton (gBS) equation which used to describe the interaction phenomena between Riemann wave and long wave via three space variables in nonlinear media. Firstly, we transform (3+1)-dimensional gBS equation to the bilinear form. Secondly, we apply the three-wave method to study bilinear form and then get many kinds of solutions for (3+1)-dimensional gBS equation, concluding periodic solitary wave solutions, bell solitary wave solutions, two-soliton solutions, breather lump wave solutions, et al. These solutions can describe interaction between waves and are presented by 3D and 2D graphs. Finally, we analyze the resolving thoughts of extended homoclinic test method and its correlation of three-wave method. Our results show the significance and efficiency of these methods.

Keywords Symbolic calculation, three-wave method, extended homoclinic test method, gBS equation.

MSC(2010) 34C25, 35C09, 35G20, 68W30.

1. Introduction

In recent years, many high-dimensional nonlinear partial differential equations which can better reflect complex and practical physical phenomena have been established in the fields of mechanics, control processes, ecological and economic systems, chemical recycling systems and epidemiology. Due to the fact that exact solutions can profoundly explain the physical model itself and predict the evolution process of the actual physical state, more mathematicians and physicists investigated exact solutions (including traveling waves and non-traveling waves) of high-dimensional nonlinear partial differential equations with constant coefficients and variable coefficients [2, 5, 6, 23, 24, 29]. Bilinear method, three-wave method and extended homoclinic test method have been widely used to solve partial differential equations (PDEs) [1, 7, 20, 25, 26]. Symbolic computations have been used in exploring lump solutions to nonlinear wave equations since 2015 [16, 17]. Moreover, Ma [18] used Darboux transformations with a general class of Darboux matrices to explore soliton solutions. Refs. [10,21,22] applied extended Jacobian elliptic function expansion approach, unified method, Sardar sub-equation method, etc. to study solutions of

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nonlinear partial differential equations. Inspired by the above literature, our article will explore traveling wave solutions of generalized breaking soliton equation that describes the phenomena of folded waves in nature.

The (2+1)-dimensional breaking soliton equation [26]

$$u_t + \alpha u_{xxx} + \beta u_{xxy} + \gamma u u_x + \lambda u u_y + \delta u_x \partial x^{-1} u_y = 0 \tag{1.1}$$

is used to describe the interaction phenomena for propagation of long wave along the x-axis and propagation of Riemann wave along the y-axis direction, where real constants α, β, γ and δ are nonzero hyper-parameters of the system, $u(x, y, t) : R \times R \times R \to R$ is the real function of variables x, y and t, representing the Riemann wave. Xu [27] discussed the Painlevé property and derived the bilinear form, N-soliton solutions, BT, Lax pair and infinite conservation laws to (1.1) as $\lambda = 2\delta$ and $\beta\gamma = 3\alpha\delta$. By introducing the nonzero seed solution, Hu [12] obtained the real non-static lumps, lump-soliton solutions and other relevant exact solutions of (1.1). Based on the resulting Hirota's bilinear equation and the extended homoclinic test theory, Ref. [28] constructed soliton solutions, homoclinic breather waves and rogue waves. Via the Hirota method, Wronskian technique, extended modified rational expansion method, Ref. [11, 13] derived bilinear forms, N-soliton solutions, parallel solitons and so on. of Eq. (1.1).

Based on the (2+1)-dimensional breaking soliton equation (1.1), considering the case of discontinuity in bottom depth, identifying the interaction phenomena between Riemann wave and long wave via three space variables in nonlinear media, the following (3+1)-dimensional generalized breaking soliton (gBS) equation

$$\partial x^{-1}(u_{xt} + u_{yt} + u_{zt}) + \alpha u_{xxx} + \beta u_{xxy} + \gamma u u_x + \lambda u u_y + \delta u_x \partial x^{-1} u_y = 0 \tag{1.2}$$

is derived. This equation can be used to model more general wave problems as these arbitrary constants might be related to more general physical conditions, which is of great importance in ocean engineering, fiber optics, mathematical physics, fluid dynamics, et al. $u(x, y, z, t) : R \times R \times R \times R \to R$ is the real function of variables x, y, z and t, real constants α, β, γ, λ and δ are hyper-parameters of the system, ∂x^{-1} presents integral operator of x. As $y = -z$, (1.2) can be reduced to (1.1). Refs. [8, 14, 15] have studied the lump-type solutions, rogue wave type solutions, double-periodic solutions, breather-wave, multi-wave and periodic lump-stripe interaction phenomena to (1.2). Ref. [3] introduced Bäcklund transformation, Wronskian solutions and interaction solutions to (1.2). Using bilinear neural network method, Refs. [9, 30] studied lump waves, lump-stripe solitons, rogue-type waves et al. for system (1.2). By using optimal system of Lie subalgebra, Ref. [19] studied the symmetry analysis, closed-form invariant solutions and dynamical wave structures of (1.2). Introducing the multi-dimensional Riemann theta function, Chen et al. [4] constructed one-periodic wave solutions, two-periodic wave solutions, and gave asymptotic properties of those solutions.

Letting λ=δ=3β, γ=6α, (1.2) can be transformed as follows

$$\partial x^{-1}(u_{xt} + u_{yt} + u_{zt}) + \alpha u_{xxx} + \beta u_{xxy} + 6\alpha u u_x + 3\beta u u_y + 3\beta u_x \partial x^{-1} u_y = 0. \tag{1.3}$$

In our paper, we investigate traveling wave solutions for (3+1)-dimensional gBS equation (1.3). Firstly, we use $u = 2(\ln f)_{xx}$ to get the bilinear form of (1.3). Then, applying two forms of three-wave method [7] and the extended homoclinic test method [16, 17], we obtain many exact solutions of (3+1)-dimensional generalized breaking soliton equation (1.3).

2. Exact solutions of the (3+1)-dimensional gBS equation

Via the transformation $u = 2(\ln f)_{xx}$, we will convert equation (1.3) into bilinear form

$$
\begin{aligned}
&(D_xD_t + D_yD_t + D_zD_t + \alpha D_x^4 + \beta D_x^3 D_y) f \cdot f \\
=&\alpha f f_{xxxx} - 4\alpha f_x f_{xxx} + 3\alpha f_{xx}^2 + \beta f f_{xxxy} - \beta f_{xxx} f_y - 3\beta f_x f_{xxy} \\
&+ 3\beta f_{xx} f_{xy} + f f_{xt} - f_x f_t + f f_{yt} - f_y f_t + f f_{zt} - f_z f_t \\
=&0.
\end{aligned} \tag{2.1}
$$

2.1. First kind of three-wave method

In this subsection, using the three-wave method, we solve the solutions of (2.1) and further obtain solutions of (1.3).

We set the first form of f with exp-function, trigonometric function and hyperbolic function

$$
\begin{cases} f_1 = a_2 \cos\xi_2 + a_3 \cosh\xi_3 + a_4 e^{\xi_4} + a_5 e^{-\xi_4}, \\ \xi_i = k_i x + l_i y + m_i z + c_i t + d_i, \quad i = 2, 3, 4, \end{cases} \tag{2.2}
$$

where $\xi_i = k_i x + l_i y + m_i z + c_i t + d_i$, k_i, l_i, m_i, c_i, d_i, a_i $(i = 2, 3, 4)$ and a_5 are some constants to be determined below. In summary, the following conclusion can be drawn.

Theorem 2.1. *Let f_1 be given by (2.2). If f_1 is the solution of bilinear equation (2.1), then combining $u = 2(\ln f_1)_{xx}$, we get solutions of (3+1)-dimensional gBS equation (1.3) with the following form*

$$
\begin{aligned}
u(x,y,z,t) =&2\frac{-a_2k_2^2\cos\xi_2 + a_3k_3^2\cosh\xi_3 + a_4k_4^2e^{\xi_4} + a_5k_4^2e^{-\xi_4}}{a_2\cos\xi_2 + a_3\cosh\xi_3 + a_4e^{\xi_4} + a_5e^{-\xi_4}} \\
&- 2\left(\frac{-a_2k_2\sin\xi_2 + a_3k_3\sinh\xi_3 + a_4k_4e^{\xi_4} - a_5k_4e^{-\xi_4}}{a_2\cos\xi_2 + a_3\cosh\xi_3 + a_4e^{\xi_4} + a_5e^{-\xi_4}}\right)^2,
\end{aligned} \tag{2.3}
$$

where $\xi_i = k_i x + l_i y + m_i z + c_i t + d_i$, k_i, l_i, m_i, c_i, d_i, a_i $(i = 2, 3, 4)$ and a_5 satisfy some corresponding relationships.

Next, taking (2.2) into (2.1) and combining it with linear independence, yields a system of determining equations about parameters k_i, l_i, m_i, c_i, a_i $(i = 2, 3, 4)$ and a_5 as follows

$$
\begin{aligned}
&a_2a_3(\alpha k_3^4 + \alpha k_2^4 - 6\alpha k_2^2k_3^2 + \beta k_2^3 l_2 + \beta k_3^3 l_3 - 3\beta k_2^2 k_3 l_3 - 3\beta k_2 l_2 k_3^2 \\
&+ c_3(k_3 + l_3 + m_3) - c_2(k_2 + l_2 + m_2)) = 0, \\
&a_2a_3(4\alpha k_2 k_3^3 - 4\alpha k_2^3 k_3 - \beta k_2^3 l_3 + \beta k_3^3 l_2 + 3\beta k_2 k_3^2 l_3 - 3\beta k_2^2 l_2 k_3 \\
&+ c_2(k_3 + l_3 + m_3) + c_3(k_2 + l_2 + m_2)) = 0, \\
&a_2a_4(\alpha k_2^4 + \alpha k_4^4 - 6\alpha k_2^2k_4^2 + \beta k_2^3 l_2 + \beta k_4^3 l_4 - 3\beta k_2^2 k_4 l_4 - 3\beta k_2 l_2 k_4^2 \\
&+ c_4(k_4 + l_4 + m_4) - c_2(k_2 + l_2 + m_2)) = 0, \\
&a_2a_4(4\alpha k_2 k_4^3 - 4\alpha k_2^3 k_4 - \beta k_2^3 l_4 + \beta k_4^3 l_2 + 3\beta k_2 k_4^2 l_4 - 3\beta k_2^2 l_2 k_4
\end{aligned}
$$

$$\begin{aligned}
&+c_2(k_4+l_4+m_4)+c_4(k_2+l_2+m_2))=0,\\
&a_2a_5(\alpha k_2^4+\alpha k_4^4-6\alpha k_2^2k_4^2+\beta k_2^3l_2+\beta k_4^3l_4-3\beta k_2^2k_4l_4-3\beta k_2l_2k_4^2\\
&+c_4(k_4+l_4+m_4)-c_2(k_2+l_2+m_2))=0,\\
&-a_2a_5(4\alpha k_2k_4^3-4\alpha k_2^3k_4-\beta k_2^3l_4+\beta k_4^3l_2+3\beta k_2k_4^2l_4-3\beta k_2^2l_2k_4\\
&+c_2(k_4+l_4+m_4)+c_4(k_2+l_2+m_2))=0,\\
&a_3a_4(\alpha k_3^4+\alpha k_4^4+6\alpha k_3^2k_4^2+\beta k_3^3l_3+\beta k_4^3l_4+3\beta k_3^2k_4l_4+3\beta k_3l_3k_4^2\\
&+c_4(k_4+l_4+m_4)+c_3(k_3+l_3+m_3))=0,\\
&-a_3a_4(4\alpha k_3k_4^3+4\alpha k_3^3k_4+\beta k_3^3l_4+\beta k_4^3l_3+3\beta k_3k_4^2l_4+3\beta k_3^2l_3k_4\\
&+c_3(k_4+l_4+m_4)+c_4(k_3+l_3+m_3))=0,\\
&a_3a_5(\alpha k_3^4+\alpha k_4^4+6\alpha k_3^2k_4^2+\beta k_3^3l_3+\beta k_4^3l_4+3\beta k_3^2k_4l_4+3\beta k_3l_3k_4^2\\
&+c_4(k_4+l_4+m_4)+c_3(k_3+l_3+m_3))=0,\\
&a_3a_5(4\alpha k_3k_4^3+4\alpha k_3^3k_4+\beta k_3^3l_4+\beta k_4^3l_3+3\beta k_3k_4^2l_4+3\beta k_3^2l_3k_4\\
&+c_3(k_4+l_4+m_4)+c_4(k_3+l_3+m_3))=0,\\
&a_4a_5(16\alpha k_4^4+16\beta k_4^3l_4+4c_4(k_4+l_4+m_4))+a_2^2(4\alpha k_2^4+4\beta k_2^3l_2\\
&-c_2(k_2+l_2+m_2))+a_3^2(4\alpha k_3^4+4\beta k_3^3l_3+c_3(k_3+l_3+m_3))=0.
\end{aligned}\tag{2.4}$$

Then, solving equation (2.4) and combining (2.3), we obtain the following solutions of (3+1)-dimensional gBS equation (1.3).

- **Case 1**

$$l_2=-\frac{\alpha}{\beta}k_2,\quad m_2=\frac{\alpha-\beta}{\beta}k_2,$$

$$l_3=-\frac{\alpha}{\beta}k_3,\quad m_3=\frac{\alpha-\beta}{\beta}k_3,$$

$$l_4=-\frac{\alpha}{\beta}k_4,\quad m_4=\frac{\alpha-\beta}{\beta}k_4,$$

where a_i, k_i, c_i, d_i $(i=2,3,4)$ and a_5 are free constants.

Combining conditions of Case 1 with (2.3), yields the solution of (3+1) -dimensional gBS equation (1.3)

$$\begin{aligned}
u_1(x,y,z,t)=&\frac{2(-a_2k_2^2\cos\xi_2+a_3k_3^2\cosh\xi_3+a_4k_4^2\mathrm{e}^{\xi_4}+a_5k_4^2\mathrm{e}^{-\xi_4})}{a_2\cos\xi_2+a_3\cosh\xi_3+a_4\mathrm{e}^{\xi_4}+a_5\mathrm{e}^{-\xi_4}}\\
&-2\Big(\frac{-a_2k_2\sin\xi_2+a_3k_3\sinh\xi_3+a_4k_4\mathrm{e}^{\xi_4}-a_5k_4\mathrm{e}^{-\xi_4}}{a_2\cos\xi_2+a_3\cosh\xi_3+a_4\mathrm{e}^{\xi_4}+a_5\mathrm{e}^{-\xi_4}}\Big)^2,
\end{aligned}\tag{2.5}$$

where $\xi_i=k_ix-\frac{\alpha}{\beta}k_iy+\frac{\alpha-\beta}{\beta}k_iz+c_it+d_i$, a_i, k_i, c_i, d_i $(i=2,3,4)$ and a_5 are free constants.

Furthermore, through detailed analysis, we can obtain the following solutions.

(1.1) If $a_2=0$ in Case 1, one can obtain the solution of (1.3)

$$\begin{aligned}
u_2(x,y,z,t)=&\frac{2a_3\big(k_5(a_4\mathrm{e}^{\xi_4}+a_5\mathrm{e}^{-\xi_4})\cosh\xi_3-2k_3k_4(a_4\mathrm{e}^{\xi_4}-a_5\mathrm{e}^{-\xi_4})\sinh\xi_3\big)}{(a_3\cosh\xi_3+a_4\mathrm{e}^{\xi_4}+a_5\mathrm{e}^{-\xi_4})^2}\\
&+\frac{8a_4a_5k_4^2+2a_3^2k_3^2}{(a_3\cosh\xi_3+a_4\mathrm{e}^{\xi_4}+a_5\mathrm{e}^{-\xi_4})^2}.\quad (k_5=k_3^2+k_4^2)
\end{aligned}\tag{2.6}$$

Especially, if $a_4a_5 > 0$, solution u_2 can be rewritten as two-soliton solution

$$u_3(x,y,z,t) = \frac{\pm 4a_3\sqrt{a_4a_5}\big(k_5\cosh\widetilde{\xi}_4\cosh\xi_3 - 2k_3k_4\sinh\widetilde{\xi}_4\sinh\xi_3\big)}{(a_3\cosh\xi_3 \pm 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2} + \frac{8a_4a_5k_4^2 + 2a_3^2k_3^2}{(a_3\cosh\xi_3 \pm 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \tag{2.7}$$

where $\widetilde{\xi}_4 = \xi_4 + \theta_1$, $\theta_1 = \ln\sqrt{\frac{a_4}{a_5}}$ and $k_5 = k_3^2 + k_4^2$. When $a_4 > 0$, the sign of u_3 takes the positive sign. Otherwise, it takes the negative sign. Moreover, as $a_4 > 0$, $a_5 > 0$ with $a_3 > 0$, or $a_4 < 0$, $a_5 < 0$ with $a_3 < 0$, solution u_3 does not have singularities. Otherwise, solution u_3 have singularities.

If $a_4a_5 < 0$, solution u_2 can be simplified as

$$u_4(x,y,z,t) = \frac{\pm 4a_3\sqrt{-a_4a_5}\big(k_5\sinh\xi_5\cosh\xi_3 - 2k_3k_4\cosh\xi_5\sinh\xi_3\big)}{(a_3\cosh\xi_3 \pm 2\sqrt{-a_4a_5}\sinh\xi_5)^2} + \frac{8a_4a_5k_4^2 + 2a_3^2k_3^2}{(a_3\cosh\xi_3 \pm 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \tag{2.8}$$

where $\xi_5 = \xi_4 + \theta_2$, $\theta_2 = \ln\sqrt{-\frac{a_4}{a_5}}$ and $k_5 = k_3^2 + k_4^2$. The sign of u_4 takes the positive sign as $a_4 > 0$. Otherwise, it takes the negative sign.

Remark 2.1. If we take $k_3 = \pm k_4$, the solution u_3 becomes hyperbolic function solutions

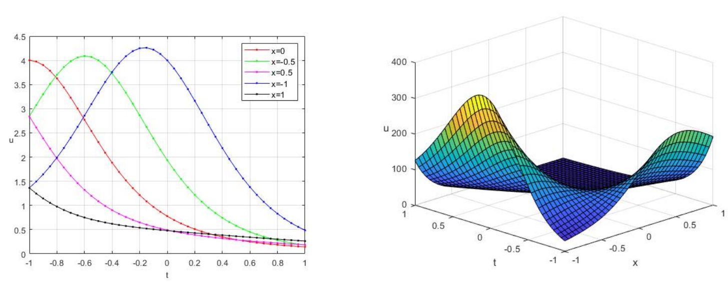
$$u_5(x,y,z,t) = \frac{8a_3\sqrt{a_4a_5}k_3^2\cosh(\xi_3 \mp \widetilde{\xi}_4) + (8a_4a_5 + 2a_3^2)k_3^2}{(a_3\cosh\xi_3 + 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \quad a_4 > 0, \tag{2.9}$$

$$u_6(x,y,z,t) = \frac{-8a_3\sqrt{a_4a_5}k_3^2\cosh(\xi_3 \mp \widetilde{\xi}_4) + (8a_4a_5 + 2a_3^2)k_3^2}{(a_3\cosh\xi_3 - 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \quad a_4 < 0. \tag{2.10}$$

Additionally, as $k_3 = \pm k_4$, solution u_4 reduces to

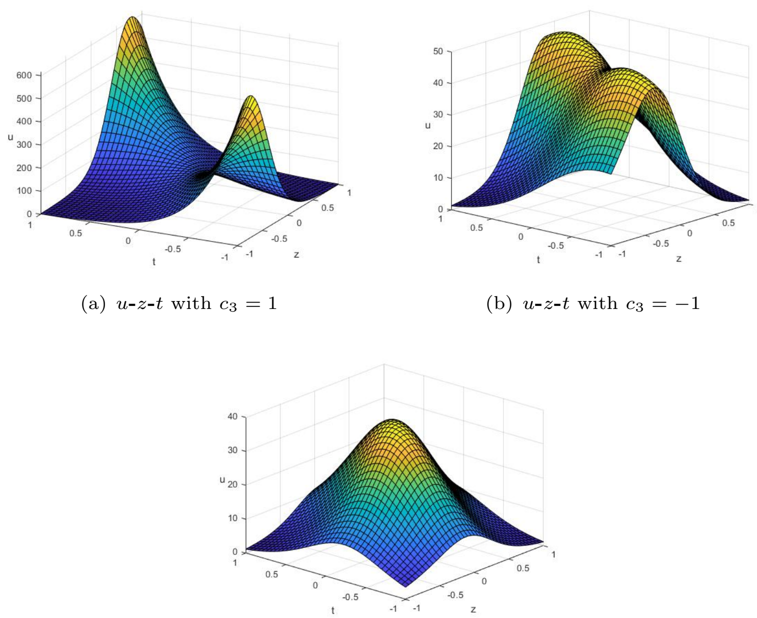
$$u_7(x,y,z,t) = \frac{8a_3\sqrt{-a_4a_5}k_3^2\sinh(\xi_5 \mp \xi_3) + (8a_4a_5 + 2a_3^2)k_3^2}{(a_3\cosh\xi_3 + 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \quad a_4 > 0, \tag{2.11}$$

$$u_8(x,y,z,t) = \frac{-8a_3\sqrt{-a_4a_5}k_3^2\sinh(\xi_5 \mp \xi_3) + (8a_4a_5 + 2a_3^2)k_3^2}{(a_3\cosh\xi_3 - 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \quad a_4 < 0. \tag{2.12}$$



(a) u-t (b) u-x-t

Figure 1. u_2 with $a_3 = a_4 = a_5 = 1$, $y = z = 0$, $k_3 = 2$, $k_4 = 1$, $c_3 = d_3 = 1$, $c_4 = d_4 = 2$.



(a) u-z-t with $c_3 = 1$ (b) u-z-t with $c_3 = -1$

(c) u-z-t with $c_3 = -2$

Figure 2. hyperbolic function solutions u_5 as $a_3 = a_4 = a_5 = 1$, $m_3 = m_4 = -2$, $c_4 = 1$, $d_3 = d_4 = 0$, $x = y = 0$.

Solutions $u_2 - u_8$ can be seen as combinations of two types of hyperbolic function solutions which reflect the interaction between waves. In Figure 2, we fix one hyperbolic function and change another hyperbolic function to better reflect the interaction between two solitary waves. Figure 3 reflects characteristics of singular waves.

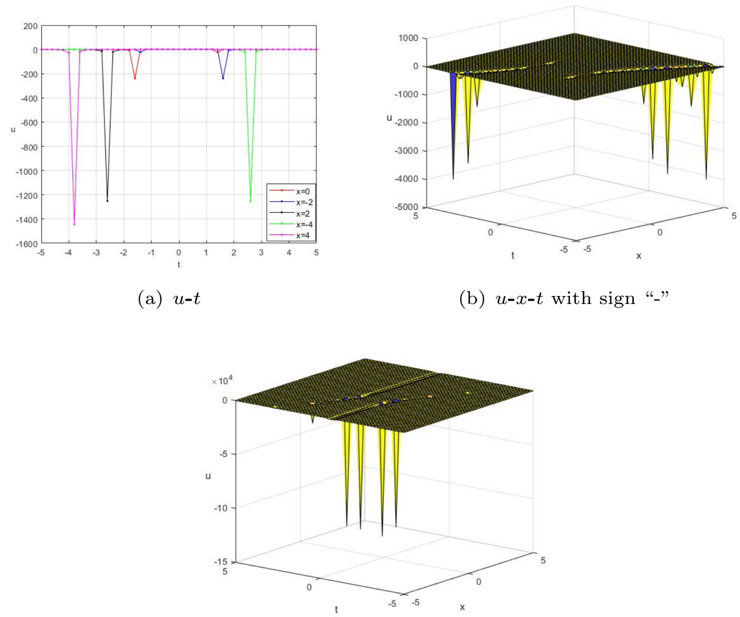
(1.2) If $a_3 = 0$, combining Case 1 and (2.5), yields the solution of (1.3)

$$u_9(x,y,z,t) = \frac{2a_2(k_4^2 - k_2^2)(a_4 e^{\xi_4} + a_5 e^{-\xi_4})\cos\xi_2 + 4a_2k_2k_4\sin\xi_2(a_4e^{\xi_4} - a_5e^{-\xi_4})}{(a_2\cos\xi_2 + a_4e^{\xi_4} + a_5e^{-\xi_4})^2} + \frac{8a_4a_5k_4^2 - 2a_2^2k_2^2}{(a_2\cos\xi_2 + a_4e^{\xi_4} + a_5e^{-\xi_4})^2}. \tag{2.13}$$

Moreover, as $a_4a_5 > 0$, one can get the following solution from (2.13)

$$u_{10}(x,y,z,t) = \frac{\pm 4a_2\sqrt{a_4a_5}\big((k_4^2 - k_2^2)\cosh\widetilde{\xi}_4\cos\xi_2 + 2k_2k_4\sinh\widetilde{\xi}_4\sin\xi_2\big)}{(a_2\cos\xi_2 \pm 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2} + \frac{8a_4a_5k_4^2 - 2a_2^2k_2^2}{(a_2\cos\xi_2 \pm 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \tag{2.14}$$

where $\widetilde{\xi}_4 = \xi_4 + \theta_1$, $\theta_1 = \ln\sqrt{\frac{a_4}{a_5}}$. As $a_4 > 0$, the sign of u_{10} takes positive. Otherwise, it takes the negative sign. Moreover, as $a_4 > 0$, $a_5 > 0$, $a_2 + 2\sqrt{a_4a_5} > 0$ with $2\sqrt{a_4a_5} - a_2 > 0$ or $a_4 < 0$, $a_5 < 0$, $a_2 - 2\sqrt{a_4a_5} < 0$ with $-a_2 - 2\sqrt{a_4a_5} < 0$, solution u_{10} does not have singularities. Otherwise, solution u_{10} have singularities.



(a) u-t

(b) u-x-t with sign "-"

(c) u-x-t with sign "+"

Figure 3. u_6 with $a_3 = a_4 = a_5 = 1$, $y = z = 0$, $k_3 = k_4 = 1$, $c_3 = 2$, $c_4 = 1$, $d_3 = d_4 = 1$.

In addition, if $a_4a_5 < 0$, from (2.13), one obtains the solution

$$u_{11}(x,y,z,t) = \frac{\pm 4a_2\sqrt{-a_4a_5}\big((k_4^2-k_2^2)\sinh\xi_5\cos\xi_2 + 2k_2k_4\cosh\xi_5\sin\xi_2\big)}{(a_2\cos\xi_2 \pm 2\sqrt{-a_4a_5}\sinh\xi_5)^2} + \frac{8a_4a_5k_4^2 - 2a_2^2k_2^2}{(a_2\cos\xi_2 \pm 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \tag{2.15}$$

where $\xi_5 = \xi_4 + \theta_2$, $\theta_2 = \ln\sqrt{-\frac{a_4}{a_5}}$. As $a_4 > 0$, the sign of u_{11} takes positive. Otherwise, it takes the negative sign.

Remark 2.2. Especially, for $k_2 = \pm k_4$, solution u_{10} can be rewritten as breather lump wave solutions

$$u_{12}(x,y,z,t) = \frac{\pm 8a_2\sqrt{a_4a_5}k_2^2\sinh\widetilde{\xi}_4\sin\xi_2 + (8a_4a_5 - 2a_2^2)k_2^2}{(a_2\cos\xi_2 + 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \quad a_4 > 0, \tag{2.16}$$

$$u_{13}(x,y,z,t) = \frac{\mp 8a_2\sqrt{a_4a_5}k_2^2\sinh\widetilde{\xi}_4\sin\xi_2 + (8a_4a_5 - 2a_2^2)k_2^2}{(a_2\cos\xi_2 - 2\sqrt{a_4a_5}\cosh\widetilde{\xi}_4)^2}, \quad a_4 < 0. \tag{2.17}$$

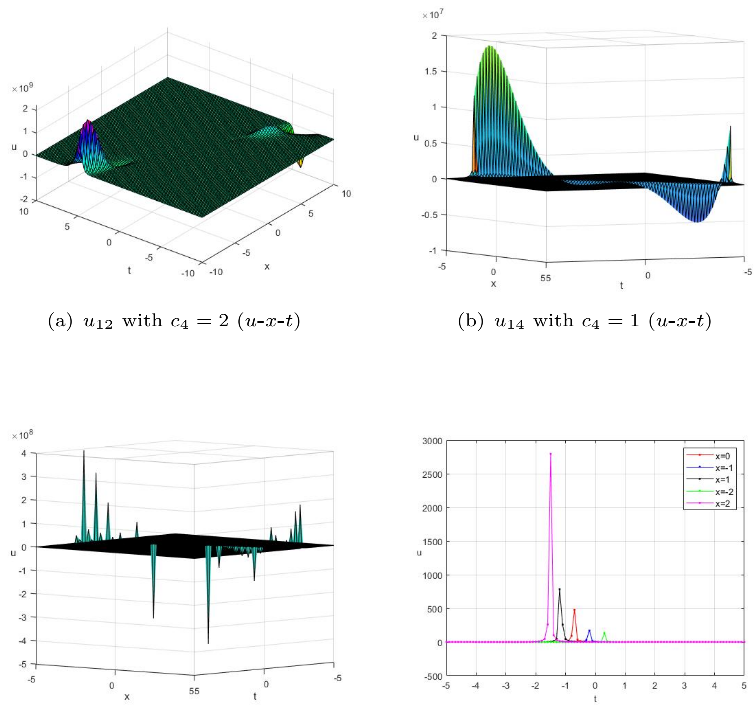
If $k_2 = \pm k_4$, the solution u_{11} reduces to breather lump wave solutions

$$u_{14}(x,y,z,t) = \frac{\pm 8a_2\sqrt{-a_4a_5}k_2^2\cosh\xi_5\sin\xi_2 + (8a_4a_5 - 2a_2^2)k_2^2}{(a_2\cos\xi_2 + 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \quad a_4 > 0, \tag{2.18}$$

$$u_{15}(x,y,z,t) = \frac{\mp 8a_2\sqrt{-a_4a_5}k_2^2\cosh\xi_5\sin\xi_2 + (8a_4a_5 - 2a_2^2)k_2^2}{(a_2\cos\xi_2 - 2\sqrt{-a_4a_5}\sinh\xi_5)^2}, \quad a_4 < 0, \tag{2.19}$$

where the signs of u_{14} and u_{15} take the symbol above as $k_2 = k_4$. Otherwise, they take the symbol below.

Solutions $u_{12} - u_{15}$ reflcet the interaction of periodic solutions and hyperbolic function solutions. The solutions all have the characteristics of periodic solutions and hyperbolic function solutions, reflecting by Figure 4. Moreover, we fix the periodic solution and change the type of hyperbolic function solution to show the different interactions between two solutions in Figure 4 (b) and (c).



(a) u_{12} with $c_4 = 2$ (u-x-t)

(b) u_{14} with $c_4 = 1$ (u-x-t)

(c) u_{14} with $c_4 = 2$ (u-x-t)

(d) u-t

Figure 4. $a_2 = a_4 = a_5 = k_2 = k_4 = c_2 = d_2 = d_4 = 1$, $x = y = 0$.

(1.3) In Case 1, if $a_4 = 0$, one obtains the solution of (1.3)

$$u_{16}(x,y,z,t) = \frac{4a_2a_3k_2k_3 \sin\xi_2 \sinh\xi_3 + 2a_2a_3(k_3^2 - k_2^2)\cos\xi_2 \cosh\xi_3}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_5 \mathrm{e}^{-\xi_4})^2} + \frac{(A_1 + B_1)e^{-\xi_4} + 2a_3^2k_3^2 - 2a_2^2k_2^2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_5\mathrm{e}^{-\xi_4})^2}, \tag{2.20}$$

where

$$A_1 = 2a_3a_5(k_3^2 + k_4^2)\cosh\xi_3 + 4a_3a_5k_3k_4\sinh\xi_3,$$
$$B_1 = 2a_2a_5(k_4^2 - k_2^2)\cos\xi_2 - 4a_2a_5k_2k_4\sin\xi_2.$$

Especially, if $k_2^2 = k_3^2 = k_4^2$ and $\xi_3 = \pm\xi_4$, yields the solution

$$u_{17}(x,y,z,t) = \frac{2k_2^2\widetilde{a} + 4a_2k_2k_3\sin\xi_2(a_3\sinh\xi_3 \mp a_5e^{\mp\xi_3})}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_5\mathrm{e}^{\mp\xi_3})^2}, \tag{2.21}$$

where $\widetilde{a} = a_3^2 - a_2^2 + 2a_3a_5$.

Furthermore, one gets the solutions from (2.21) as follows

$$u_{18}(x,y,z,t) = \frac{2k_2^2\widetilde{a} + 4a_2k_2k_3\sqrt{a_3(a_3+2a_5)}\sin\xi_2\sinh(\xi_3\pm\theta_3)}{(a_2\cos\xi_2 + \sqrt{a_3(a_3+2a_5)}\cosh(\xi_3\pm\theta_3))^2}, \tag{2.22}$$

where $a_3 > 0$, $a_3 + 2a_5 > 0$, $\theta_3 = \ln\sqrt{\frac{a_3}{a_3+2a_5}}$ and $\widetilde{a} = a_3^2 - a_2^2 + 2a_3a_5$.

$$u_{19}(x,y,z,t) = \frac{2k_2^2\widetilde{a} - 4a_2k_2k_3\sqrt{a_3(a_3+2a_5)}\sin\xi_2\sinh(\xi_3\pm\theta_3)}{(a_2\cos\xi_2 - \sqrt{a_3(a_3+2a_5)}\cosh(\xi_3\pm\theta_3))^2}, \tag{2.23}$$

where $a_3 < 0$, $a_3 + 2a_5 < 0$, $\theta_3 = \ln\sqrt{\frac{a_3}{a_3+2a_5}}$ and $\widetilde{a} = a_3^2 - a_2^2 + 2a_3a_5$. The signs of u_{18} and u_{19} take positive as $\xi_3 = \xi_4$, and take negative sign as $\xi_3 = -\xi_4$. Moreover, as $a_2 + \sqrt{a_3(a_3+2a_5)} > 0$ and $\sqrt{a_3(a_3+2a_5)} - a_2 > 0$, solution u_{18} does not have singularities. As $-a_2 - \sqrt{a_3(a_3+2a_5)} < 0$ and $-\sqrt{a_3(a_3+2a_5)} + a_2 < 0$, solution u_{19} does not possess singularities. Otherwise, they all have singularities. The subsequent partial results have similar properties, so we will not list one by one.

$$u_{20}(x,y,z,t) = \frac{2k_2^2\widetilde{a} \pm 4a_2k_2k_3\sqrt{-a_3(a_3+2a_5)}\sin\xi_2\cosh(\xi_3\pm\theta_4)}{(a_2\cos\xi_2 \pm \sqrt{-a_3(a_3+2a_5)}\sinh(\xi_3\pm\theta_4))^2}, \tag{2.24}$$

where $a_3 > 0$, $a_3 + 2a_5 < 0$, $\theta_4 = \ln\sqrt{-\frac{a_3}{a_3+2a_5}}$ and $\widetilde{a} = a_3^2 - a_2^2 + 2a_3a_5$.

$$u_{21}(x,y,z,t) = \frac{2k_2^2\widetilde{a} \mp 4a_2k_2k_3\sqrt{-a_3(a_3+2a_5)}\sin\xi_2\cosh(\xi_3\pm\theta_4)}{(a_2\cos\xi_2 \mp \sqrt{-a_3(a_3+2a_5)}\sinh(\xi_3\pm\theta_4))^2}, \tag{2.25}$$

where $a_3 < 0$, $a_3 + 2a_5 > 0$, $\theta_4 = \ln\sqrt{-\frac{a_3}{a_3+2a_5}}$ and $\widetilde{a} = a_3^2 - a_2^2 + 2a_3a_5$. The signs of u_{20} and u_{21} take the symbol above as $\xi_3 = \xi_4$. Otherwise, they take the symbol below as $\xi_3 = -\xi_4$.

(1.4) If $a_5 = 0$ in Case 1, it yields solution of (1.3)

$$\begin{aligned} u_{22}(x,y,z,t) = &\frac{4a_2a_3k_2k_3\sin\xi_2\sinh\xi_3 + 2a_2a_3(k_3^2 - k_2^2)\cos\xi_2\cosh\xi_3}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_4\mathrm{e}^{\xi_4})^2} \\ &+\frac{(A_2+B_2)e^{\xi_4} + 2a_3^2k_3^2 - 2a_2^2k_2^2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_4\mathrm{e}^{\xi_4})^2}, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} A_2 &= 2a_3a_4(k_3^2 + k_4^2)\cosh\xi_3 - 4a_3a_4k_3k_4\sinh\xi_3, \\ B_2 &= 2a_2a_4(k_4^2 - k_2^2)\cos\xi_2 + 4a_2a_4k_2k_4\sin\xi_2. \end{aligned}$$

And then, if $k_2^2 = k_3^2 = k_4^2$ and $\xi_3 = \pm\xi_4$, one gets the solution

$$u_{23}(x,y,z,t) = \frac{2k_2^2\overline{a} + 4a_2k_2k_3\sin\xi_2(a_3\sinh\xi_3 \pm a_4e^{\pm\xi_3})}{(a_2\cos\xi_2 + a_3\cosh\xi_3 + a_4\mathrm{e}^{\pm\xi_3})^2}, \tag{2.27}$$

where $\overline{a} = a_3^2 - a_2^2 + 2a_3a_4$.

Furthermore, solution u_{23} can be rewritten as the following periodic-solitary wave solutions

$$u_{24}(x,y,z,t) = \frac{2k_2^2\overline{a} + 4a_2k_2k_3\sqrt{a_3(a_3+2a_4)}\sin\xi_2\sinh(\xi_3 \pm \theta_5)}{(a_2\cos\xi_2 + \sqrt{a_3(a_3+2a_4)}\cosh(\xi_3 \pm \theta_5))^2}, \tag{2.28}$$

where $a_3 > 0$, $a_3 + 2a_4 > 0$, $\theta_5 = \ln\sqrt{\frac{a_3+2a_4}{a_3}}$ and $\overline{a} = a_3^2 - a_2^2 + 2a_3a_4$.

$$u_{25}(x,y,z,t) = \frac{2k_2^2\overline{a} - 4a_2k_2k_3\sqrt{a_3(a_3+2a_4)}\sin\xi_2\sinh(\xi_3 \pm \theta_5)}{(a_2\cos\xi_2 - \sqrt{a_3(a_3+2a_4)}\cosh(\xi_3 \pm \theta_5))^2}, \tag{2.29}$$

where $a_3 < 0$, $a_3 + 2a_4 < 0$, $\theta_5 = \ln\sqrt{\frac{a_3+2a_4}{a_3}}$ and $\overline{a} = a_3^2 - a_2^2 + 2a_3a_4$.

$$u_{26}(x,y,z,t) = \frac{2k_2^2\overline{a} \mp 4a_2k_2k_3\sqrt{-a_3(a_3+2a_4)}\sin\xi_2\cosh(\xi_3 \mp \theta_6)}{(a_2\cos\xi_2 \mp \sqrt{-a_3(a_3+2a_4)}\sinh(\xi_3 \mp \theta_6))^2}, \tag{2.30}$$

where $a_3 > 0$, $a_3 + 2a_4 < 0$, $\theta_6 = \ln\sqrt{-\frac{a_3}{a_3+2a_4}}$ and $\overline{a} = a_3^2 - a_2^2 + 2a_3a_4$.

$$u_{27}(x,y,z,t) = \frac{2k_2^2\overline{a} \mp 4a_2k_2k_3\sqrt{-a_3(a_3+2a_4)}\sin\xi_2\cosh(\xi_3 \pm \theta_6)}{(a_2\cos\xi_2 \mp \sqrt{-a_3(a_3+2a_4)}\sinh(\xi_3 \pm \theta_6))^2}, \tag{2.31}$$

where $a_3 < 0$, $a_3 + 2a_4 > 0$, $\theta_6 = \ln\sqrt{-\frac{a_3}{a_3+2a_4}}$ and $\overline{a} = a_3^2 - a_2^2 + 2a_3a_4$.

(1.5) Combining $a_2 = a_3 = 0$ with Case 1, (1.3) has solution

$$u_{28}(x,y,z,t) = \frac{8a_4a_5k_4^2}{(a_4\mathrm{e}^{\xi_4} + a_5\mathrm{e}^{-\xi_4})^2}. \tag{2.32}$$

Then, in the condition of $a_4a_5 > 0$, one can obtain bell solitary wave solution

$$u_{29}(x,y,z,t) = 2k_4^2\mathrm{sech}^2(\xi_4 + \theta_1). \tag{2.33}$$

Similarly, as $a_4a_5 < 0$, it yields singular travelling wave solution

$$u_{30}(x,y,z,t) = 2k_4^2\mathrm{csch}^2(\xi_4 + \theta_2). \tag{2.34}$$

(1.6) In Case 1, if $a_2 = a_4 = 0$, yields solution of (1.3)

$$u_{31}(x,y,z,t) = \frac{2a_3^2k_3^2 + 2a_3a_5\big((k_3^2 + k_4^2)\cosh\xi_3 + 2k_3k_4\sinh\xi_3\big)e^{-\xi_4}}{(a_3\cosh\xi_3 + a_5e^{-\xi_4})^2}. \tag{2.35}$$

Moreover, as $a_3(2a_5 + a_3) > 0$, $\xi_3 = \pm\xi_4$, (2.35) becomes

$$u_{32}(x,y,z,t) = \frac{2a_3^2k_3^2 + 4a_3a_5k_3^2}{a_3(2a_5 + a_3)}\mathrm{sech}^2(\xi_3 \pm \theta_3). \tag{2.36}$$

As $a_3(2a_5 + a_3) < 0$, $\xi_3 = \pm\xi_4$, solution u_{31} can reduce to

$$u_{33}(x,y,z,t) = \frac{2a_3^2k_3^2 + 4a_3a_5k_3^2}{a_3(2a_5 + a_3)}\mathrm{csch}^2(\xi_3 \pm \theta_4). \tag{2.37}$$

(1.7) Combining $a_2 = a_5 = 0$ with Case 1, yields solution of (1.3)

$$u_{34}(x,y,z,t) = \frac{2a_3^2k_3^2 + 2a_3a_4\big((k_3^2+k_4^2)\cosh\xi_3 - 2k_3k_4\sinh\xi_3\big)e^{\xi_4}}{(a_3\cosh\xi_3 + a_4e^{\xi_4})^2}. \tag{2.38}$$

Moreover, as $a_3(2a_4+a_3) > 0$, $\xi_3 = \pm\xi_4$, solution u_{34} can reduce to

$$u_{35}(x,y,z,t) = \frac{2a_3^2k_3^2 + 4a_3a_4k_3^2}{a_3(2a_4+a_3)}\operatorname{sech}^2(\xi_3 \pm \theta_5). \tag{2.39}$$

As $a_3(2a_4+a_3) < 0$, $\xi_3 = \pm\xi_4$, (2.38) can be rewritten as

$$u_{36}(x,y,z,t) = \frac{2a_3^2k_3^2 + 4a_3a_5k_3^2}{a_3(2a_4+a_3)}\operatorname{csch}^2(\xi_3 \pm \theta_6). \tag{2.40}$$

(1.8) If $a_3 = a_4 = 0$ in Case 1, one gets solution of (1.3)

$$\begin{aligned} u_{37}(x,y,z,t) &= \frac{-2a_2^2k_2^2 + 2a_2a_5\big((k_4^2-k_2^2)\cos\xi_2 - 2k_2k_4\sin\xi_2\big)e^{-\xi_4}}{(a_2\cos\xi_2 + a_5e^{-\xi_4})^2} \\ &= \frac{-2a_2^2k_2^2 + 2a_2a_5(k_2^2+k_4^2)\cos(\xi_2+\theta_7)e^{-\xi_4}}{(a_2\cos\xi_2 + a_5e^{-\xi_4})^2}, \end{aligned} \tag{2.41}$$

where θ_7 satisfies $\tan\theta_7 = \frac{2k_2k_4}{k_4^2-k_2^2}$.

(1.9) Similar to case (1.8), if $a_3 = a_5 = 0$ in Case 1, one obtains

$$\begin{aligned} u_{38}(x,y,z,t) &= \frac{-2a_2^2k_2^2 + 2a_2a_5\big((k_4^2-k_2^2)\cos\xi_2 + 2k_2k_4\sin\xi_2\big)e^{-\xi_4}}{(a_2\cos\xi_2 + a_4e^{\xi_4})^2} \\ &= \frac{-2a_2^2k_2^2 + 2a_2a_4(k_2^2+k_4^2)\cos(\xi_2-\theta_7)e^{\xi_4}}{(a_2\cos\xi_2 + a_4e^{\xi_4})^2}. \end{aligned} \tag{2.42}$$

(1.10) When Case 1 with $a_4 = a_5 = 0$, yields

$$\begin{aligned} u_{39}(x,y,z,t) &= \frac{2a_2a_3\big((k_3^2-k_2^2)\cos\xi_2\cosh\xi_3 + 4k_2k_3\sin\xi_2\sinh\xi_3\big)}{(a_2\cos\xi_2 + a_3\cosh\xi_3)^2} \\ &\quad + \frac{-2a_2^2k_2^2 + 2a_3^2k_3^2}{(a_2\cos\xi_2 + a_3\cosh\xi_3)^2}. \end{aligned} \tag{2.43}$$

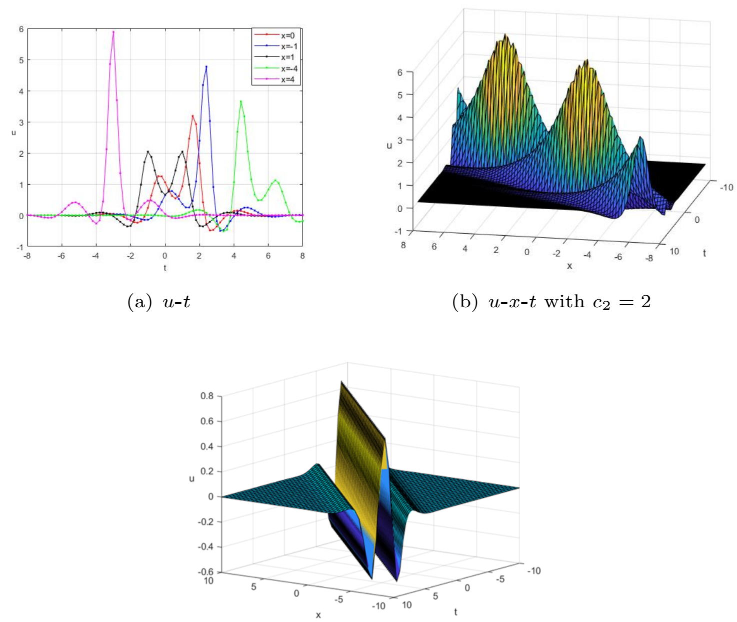
Especially, if $k_2 = \pm k_3$, one gets the solution from (2.43)

$$u_{40}(x,y,z,t) = \frac{2k_2^2(a_3^2-a_2^2) \pm 4a_2a_3k_2^2\sin\xi_2\sinh\xi_3}{(a_2\cos\xi_2 + a_3\cosh\xi_3)^2}. \tag{2.44}$$

Solution u_{40} can be seen as the superposition of periodic solution and hyperbolic function solution. Figure 5 shows the properties of periodicity and hyperbolic function. Moreover, (b) and (c) in Figure 5 reflect the impact of parameter perturbations on the behavior of solutions.

Remark 2.3. In solutions $u_1 - u_{40}$,

$$\xi_2 = k_2x - \frac{\alpha}{\beta}k_2y + \frac{\alpha-\beta}{\beta}k_2z + c_2t + d_2,$$



(a) u-t (b) u-x-t with $c_2 = 2$

(c) u-x-t with $c_2 = 1$

Figure 5. u_{40} as $a_2 = 1$, $a_3 = 2$, $k_2 = k_3 = c_3 = 1$, $d_2 = d_3 = -1$, $y = z = 0$.

$$\xi_3 = k_3 x - \frac{\alpha}{\beta} k_3 y + \frac{\alpha - \beta}{\beta} k_3 z + c_3 t + d_3,$$
$$\xi_4 = k_4 x - \frac{\alpha}{\beta} k_4 y + \frac{\alpha - \beta}{\beta} k_4 z + c_4 t + d_4,$$

a_i, k_i, c_i, d_i $(i = 2, 3, 4)$ and a_5 are free constants.

- **Case 2**

$$a_3 = a_4 = a_5 = 0, \quad l_2 = \frac{(k_2 + m_2)c_2 - 4\alpha k_2^4}{4\beta k_2^2 - c_2},$$

or

$$a_3 = a_4 = a_5 = 0, \quad m_2 = \frac{\alpha}{\beta} k_2 - k_2, \quad c_2 = 4\beta k_2^3,$$

or

$$a_3 = a_4 = a_5 = 0, \quad l_2 = -\frac{\alpha}{\beta} k_2, \quad m_2 = \frac{\alpha - \beta}{\beta} k_2.$$

Combining the conditions of Case 2 with (2.3), it gets the periodic solution of (3+1)-dimensional gBS equation (1.3)

$$u_{41}(x, y, z, t) = -2k_2^2(1 + \tan \xi_2^2), \tag{2.45}$$

where

$$\xi_2 = k_2 x + \frac{(k_2 + m_2)c_2 - 4\alpha k_2^4}{4\beta k_2^2 - c_2} y + m_2 z + c_2 t + d_2,$$

k_2, m_2, c_2 and d_2 are free constants, or

$$\xi_2 = k_2x + l_2y + (\frac{\alpha}{\beta}k_2 - k_2)z + 4\beta k_2^3 t + d_2,$$

k_2, l_2 and d_2 are free constants, or

$$\xi_2 = k_2x - \frac{\alpha}{\beta}k_2y + \frac{\alpha-\beta}{\beta}k_2z + c_2t + d_2,$$

k_2, c_2 and d_2 are free constants.

- **Case 3**

$$a_2 = 0, \quad a_5 = \frac{a_3^2}{4a_4}, \quad k_3 = k_4, \quad c_3 = c_4 = -\frac{8\alpha k_4^4 + 4\beta k_4^3 l_4 + 4\beta k_4^3 l_3}{2k_4 + l_3 + l_4 + m_3 + m_4}.$$

From condition of Case 3 and (2.3), yields the following two-soliton solution

$$\begin{aligned} u_{42}(x,y,z,t) &= \frac{2a_3(k_3^2+k_4^2)(a_4 e^{\xi_4} + \frac{a_3^2}{4a_4}e^{-\xi_4})\cosh\xi_3 + 2a_3^2k_4^2 + 2a_3^2k_3^2}{(a_3\cosh\xi_3 + a_4e^{\xi_4} + \frac{a_3^2}{4a_4}e^{-\xi_4})^2} \\ &\quad - \frac{4a_3k_3k_4\sinh\xi_3(a_4e^{\xi_4} - \frac{a_3^2}{4a_4}e^{-\xi_4})}{(a_3\cosh\xi_3 + a_4e^{\xi_4} + \frac{a_3^2}{4a_4}e^{-\xi_4})^2} \\ &= \pm 4k_3^2 \frac{\cosh\xi_3\cosh(\xi_4+\theta_8) - \sinh\xi_3\sinh(\xi_4+\theta_8) \pm 1}{(\cosh\xi_3 \pm \cosh(\xi_4+\theta_8))^2} \\ &= \pm 4k_3^2 \frac{\cosh(\xi_4+\theta_8-\xi_3) \pm 1}{(\cosh\xi_3 \pm \cosh(\xi_4+\theta_8))^2}, \end{aligned} \tag{2.46}$$

where $\theta_8 = \ln|\frac{2a_4}{a_3}|$,

$$\xi_3 = k_4x + l_3y + m_3z - \frac{8\alpha k_4^2 + 4\beta k_4^3 l_4 + 4\beta k_4^3 l_3}{2k_4 + l_3 + l_4 + m_3 + m_4}t + d_3,$$
$$\xi_4 = k_4x + l_4y + m_4z - \frac{8\alpha k_4^2 + 4\beta k_4^3 l_4 + 4\beta k_4^3 l_3}{2k_4 + l_3 + l_4 + m_3 + m_4}t + d_4,$$

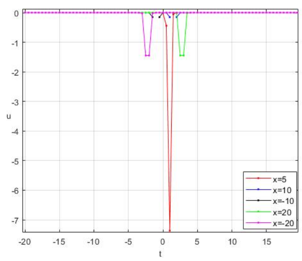
a_3, a_4, k_2, k_4, c_2, l_i and m_i $(i = 2,3,4)$ are free constants. As $a_3a_4 > 0$, the sign of (2.46) takes positive. Otherwise, it takes negative sign.

Solution u_{42} can be seen as the superposition of two hyperbolic function solutions. When taking the positive and negative signs respectively, solution u_{42} shows different properties in Figure 6. When taking the negative sign, the solution u_{42} reflects singularity properties in Figure 6(b).

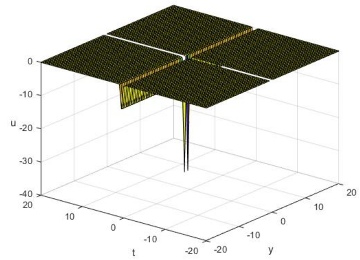
- **Case 4**

$$a_3 = 0, \quad a_2^2 = 4a_4a_5, \quad k_2 = c_2 = 0, \quad m_2 = -\frac{\beta k_4^2 l_2 + c_4 l_2}{c_4},$$
$$l_4 = -\frac{\alpha}{\beta}k_4, \quad m_4 = \frac{\alpha-\beta}{\beta}k_4.$$

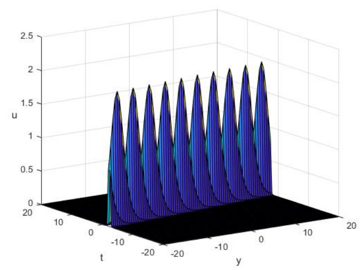
Following condition of Case 4 and (2.3), one gets the breather lump wave solution of Eq. (1.3)



(a) u_{42} with sign “-” (u-t)



(b) u_{42} with sign “-” (u-y-t)



(c) u_{42} with sign “+” (u-y-t)

Figure 6. $a_4 = k_3 = k_4 = -l_3 = d_3 = d_4 = 1$, $c_3 = c_4 = -4$, $a_3 = l_4 = 2$, $x = z = 0$.

$$\begin{aligned} u_{43}(x,y,z,t) =& \frac{\pm 4a_2\sqrt{a_4a_5}\big(\widetilde{k_5}\cosh(\xi_4+\theta_1)\cos\xi_2 + 2k_2k_4\sinh(\xi_4+\theta_1)\sin\xi_2\big)}{(a_2\cos\xi_2 \pm 2\sqrt{a_4a_5}\cosh(\xi_4+\theta_1))^2} \\ &+ \frac{2a_2^2\widetilde{k_5}}{(a_2\cos\xi_2 \pm 2\sqrt{a_4a_5}\cosh(\xi_4+\theta_1))^2} \\ =& \frac{\pm 2\big(\widetilde{k_5}\cosh(\xi_4+\theta_1)\cos\xi_2 + 2k_2k_4\sinh(\xi_4+\theta_1)\sin\xi_2\big)}{(\cos\xi_2 \pm \cosh(\xi_4+\theta_1))^2} \\ &+ \frac{2\widetilde{k_5}}{(\cos\xi_2 \pm \cosh(\xi_4+\theta_1))^2}, \end{aligned} \tag{2.47}$$

where $\theta_1 = \ln\sqrt{\frac{a_4}{a_5}}$, $\widetilde{k_5} = k_4^2 - k_2^2$,

$$\begin{aligned} \xi_2 &= l_2 y - \frac{\beta k_4^2 l_2 + c_4 l_2}{c_4} t + d_2, \\ \xi_4 &= k_4 x - \frac{\alpha}{\beta} k_4 y + \frac{\alpha-\beta}{\beta} k_4 z + c_4 t + d_4, \end{aligned}$$

a_4, a_5, k_4, l_2, c_4, d_2 and d_4 are free constants. As $a_2 > 0$, the sign of u_{43} takes positive. Otherwise, it takes the negative sign.

Especially, if $k_2^2 = k_4^2$, solution u_{43} can reduce to

$$u_{44}(x,y,z,t) = \frac{\pm 4k_2k_4\sinh(\xi_4+\theta_1)\sin\xi_2}{(\cos\xi_2 \pm \cosh(\xi_4+\theta_1))^2}. \tag{2.48}$$

• **Case 5**

$$a_4=0,\quad k_2=k_3=0,\quad l_2=-m_2,\quad l_3=-m_3,$$
$$c_2=\frac{\beta k_4^3 m_2}{k_4+l_4+m_4},\quad c_3=\frac{\beta k_4^3 m_3}{k_4+l_4+m_4},\quad c_4=\frac{-\alpha k_4^4-\beta k_4^3 l_4}{k_4+l_4+m_4}.$$

Combining Case 5 with (2.3), one obtains the solution

$$u_{45}(x,y,z,t)=\frac{2a_5k_4^2(a_3\cosh\xi_3+a_2\cos\xi_2)\mathrm{e}^{-\xi_4}}{(a_2\cos\xi_2+a_3\cosh\xi_3+a_5\mathrm{e}^{-\xi_4})^2}, \tag{2.49}$$

where

$$\xi_2=-m_2y+m_2z+\frac{\beta k_4^3 m_2}{k_4+l_4+m_4}t+d_2,$$
$$\xi_3=-m_3y+m_3z+\frac{\beta k_4^3 m_3}{k_4+l_4+m_4}t+d_3,$$
$$\xi_4=k_4x+l_4y+m_4z-\frac{\alpha k_4^4+\beta k_4^3 l_4}{k_4+l_4+m_4}t+d_4,$$

a_2, a_3, a_5, k_4, l_4, m_i and $d_i (i=2,3,4)$ are free constants.

• **Case 6**

$$a_5=0,\quad k_2=k_3=0,\quad l_2=-m_2,\quad l_3=-m_3,$$
$$c_2=\frac{\beta k_4^3 m_2}{k_4+l_4+m_4},\quad c_3=\frac{\beta k_4^3 m_3}{k_4+l_4+m_4},\quad c_4=\frac{-\alpha k_4^4-\beta k_4^3 l_4}{k_4+l_4+m_4}.$$

Similar to Case 5, from (2.3), we obtain the solution

$$u_{46}(x,y,z,t)=\frac{2a_4k_4^2(a_3\cosh\xi_3+a_2\cos\xi_2)\mathrm{e}^{\xi_4}}{(a_2\cos\xi_2+a_3\cosh\xi_3+a_4\mathrm{e}^{\xi_4})^2}, \tag{2.50}$$

where ξ_2, ξ_3 and ξ_4 are similar to Case 5, k_4, l_4, a_i, m_i and $d_i (i=2,3,4)$ are free constants.

• **Case 7**

$$k_2=k_3=0,\quad c_2=c_3=0,\quad m_4=\frac{\alpha-\beta}{\beta}k_4,$$
$$l_2=-\frac{c_4m_2}{\beta k_4^3+c_4},\quad l_3=-\frac{c_4m_3}{\beta k_4^3+c_4},\quad l_4=-\frac{\alpha}{\beta}k_4.$$

From condition of Case 7 and (2.3), as $a_4a_5>0$, we yield the following solution

$$u_{47}(x,y,z,t)=\frac{8a_4a_5k_4^2\pm 2k_4^2\sqrt{a_4a_5}\cosh(\xi_4+\theta_1)(a_2\cos\xi_2+a_3\cosh\xi_3)}{(a_2\cos\xi_2+a_3\cosh\xi_3\pm\sqrt{a_4a_5}\cosh(\xi_4+\theta_1))^2}, \tag{2.51}$$

where

$$\xi_2=-\frac{c_4m_2}{\beta k_4^3+c_4}y+m_2z+d_2,$$
$$\xi_3=-\frac{c_4m_3}{\beta k_4^3+c_4}y+m_3z+d_3,$$

$$\xi_4 = k_4 x - \frac{\alpha}{\beta} k_4 y + \frac{\alpha - \beta}{\beta} k_4 z + c_4 t + d_4,$$

k_4, c_4, m_2, m_3, a_5, a_i and $d_i (i = 2, 3, 4)$ are free constants.

In addition, as $a_4 a_5 < 0$, we can get the solution

$$u_{48}(x, y, z, t) = \frac{8a_4 a_5 k_4^2 \pm 2k_4^2 \sqrt{-a_4 a_5} \sinh(\xi_4 + \theta_2)(a_2 \cos \xi_2 + a_3 \cosh \xi_3)}{\big(a_2 \cos \xi_2 + a_3 \cosh \xi_3 \pm \sqrt{-a_4 a_5} \sinh(\xi_4 + \theta_1)\big)^2}. \quad (2.52)$$

As $a_4 > 0$, the sign of u_{47} and u_{48} takes positive. Otherwise, it takes the negative sign.

- **Case 8**

$$k_2 = k_4 = 0, \quad c_2 = c_4 = 0,$$
$$l_2 = -\frac{c_3 m_2}{\beta k_3^3 + c_3}, \quad l_4 = -\frac{c_3 m_4}{\beta k_3^3 + c_3}, \quad l_3 = -\frac{\alpha}{\beta} k_3, \quad m_3 = \frac{\alpha - \beta}{\beta} k_3.$$

From condition of Case 8 and (2.3), as $a_4 a_5 > 0$, one gets the following multi-wave solution

$$u_{49}(x, y, z, t) = \frac{2a_3 k_3^2 + 2a_3 k_3^2 (a_2 \cos \xi_2 \pm \sqrt{a_4 a_5} \cosh(\xi_4 + \theta_1)) \cosh \xi_3}{(a_2 \cos \xi_2 + a_3 \cosh \xi_3 \pm \sqrt{a_4 a_5} \cosh(\xi_4 + \theta_1))^2}, \quad (2.53)$$

where

$$\xi_2 = -\frac{c_3 m_2}{\beta k_3^3 + c_3} y + m_2 z + d_2,$$
$$\xi_3 = k_3 x - \frac{\alpha}{\beta} k_3 y + \frac{\alpha - \beta}{\beta} k_3 z + c_3 t + d_3,$$
$$\xi_4 = -\frac{c_3 m_4}{\beta k_3^3 + c_3} y + m_4 z + d_4,$$

k_3, c_3, m_2, m_4, a_5, a_i and $d_i (i = 2, 3, 4)$ are free constants.

In addition, as $a_4 a_5 < 0$, we can get the multi-wave solution

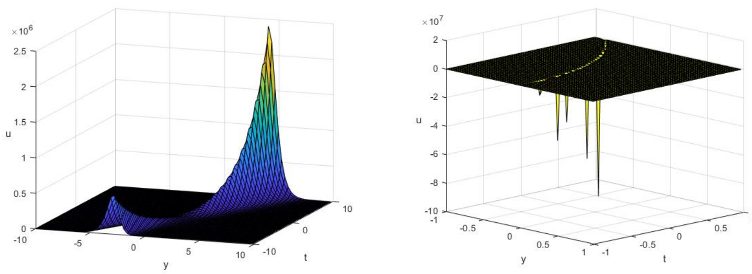
$$u_{50}(x, y, z, t) = \frac{2a_3 k_3^2 + 2a_3 k_3^2 (a_2 \cos \xi_2 \pm \sqrt{-a_4 a_5} \sinh(\xi_4 + \theta_1)) \cosh \xi_3}{(a_2 \cos \xi_2 + a_3 \cosh \xi_3 \pm \sqrt{-a_4 a_5} \sinh(\xi_4 + \theta_1))^2}. \quad (2.54)$$

As $a_4 > 0$, the sign of u_{49} and u_{50} takes positive. Otherwise, it takes the negative sign.

Solutions u_{49} and u_{50} can be seen as the interaction of three waves. When the positive and negative signs are taken in the solution u_{49}, the solution exhibits different properties. When taking the positive sign, the solution u_{49} shows periodicity and hyperbolic function properties in Figure 7(a). Otherwise, u_{49} has periodicity and singularity properties in Figure 7(b).

- **Case 9**

$$k_3 = k_4 = 0, \quad c_2 = 0, \quad l_2 = -\frac{\alpha}{\beta} k_2,$$
$$l_3 = -m_3 = \frac{c_3 (m_2 - \frac{\alpha - \beta}{\beta} k_2)}{\beta k_2^3},$$



(a) u_{49} with sign "+" (u-y-t)   (b) u_{49} with sign "-" (u-y-t)

Figure 7. u_{49} as $a_2 = a_3 = a_4 = 1$, $k_3 = 1$, $c_3 = 1$, $l_3 = -1$, $l_2 = l_4 = -\frac{1}{2}$, $x = z = 0$.

$$l_4 = -m_4 = \frac{c_4(m_2 - \frac{\alpha-\beta}{\beta}k_2)}{\beta k_2^3}.$$

From condition of Case 9 and (2.3), one obtains multi-wave solution

$$u_{51}(x,y,z,t) = \frac{2a_2k_2^2 - 2a_2k_2^2(a_3\cosh\xi_3 \pm \sqrt{a_4a_5}\cosh(\xi_4+\theta_1))\cos\xi_2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 \pm \sqrt{a_4a_5}\cosh(\xi_4+\theta_1))^2}, \quad (2.55)$$

where $a_4a_5 > 0$,

$$\xi_2 = k_2x - \frac{\alpha}{\beta}k_2y + m_2z + d_2,$$
$$\xi_3 = c_3\Big(\frac{m_2 - \frac{\alpha-\beta}{\beta}k_2}{\beta k_2^3}y - \frac{m_2 - \frac{\alpha-\beta}{\beta}k_2}{\beta k_2^3}z + t\Big) + d_3,$$
$$\xi_4 = c_4\Big(\frac{m_2 - \frac{\alpha-\beta}{\beta}k_2}{\beta k_2^3}y - \frac{m_2 - \frac{\alpha-\beta}{\beta}k_2}{\beta k_2^3}z + t\Big) + d_4,$$

k_2, m_2, c_3, c_4, a_5, a_i and $d_i (i = 2,3,4)$ are free constants.

In addition, as $a_4a_5 < 0$, we can get the solution

$$u_{52}(x,y,z,t) = \frac{2a_2k_2^2 - 2a_2k_2^2(a_3\cosh\xi_3 \pm \sqrt{-a_4a_5}\sinh(\xi_4+\theta_1))\cos\xi_2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 \pm \sqrt{-a_4a_5}\sinh(\xi_4+\theta_1))^2}. \quad (2.56)$$

As $a_4 > 0$, the sign of u_{51} and u_{52} take positive. Otherwise, it takes the negative sign.

- **Case 10**

$$k_2 = l_2 = m_2 = c_2 = 0, \quad k_3 = 0, \quad c_4 = 0,$$
$$l_3 = -m_3 = -\frac{c_3(m_4 - \frac{\alpha-\beta}{\beta}k_4)}{\beta k_4^3}, \quad l_4 = -\frac{\alpha}{\beta}k_4.$$

From condition of Case 10 and (2.3), one gets the two-soliton solution

$$u_{53}(x,y,z,t) = \frac{\pm 2k_4^2\sqrt{a_4a_5}\cosh(\xi_4+\theta_1)(a_3\cosh\xi_3 + a_2\cos\xi_2) + 8a_4a_5k_4^2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 \pm \sqrt{a_4a_5}\cosh(\xi_4+\theta_1))^2}, \quad (2.57)$$

where $a_4a_5 > 0$,

$$\xi_2 = d_2,$$
$$\xi_3 = c_3\Big(-\frac{c_3(m_4 - \frac{\alpha-\beta}{\beta}k_4)}{\beta k_4^3}y + \frac{c_3(m_4 - \frac{\alpha-\beta}{\beta}k_4)}{\beta k_4^3}z + t\Big) + d_3,$$
$$\xi_4 = k_4x - \frac{\alpha}{\beta}k_4y + m_4z + d_4,$$

k_2, m_2, c_3, c_4, a_5, a_i and $d_i (i = 2, 3, 4)$ are free constants.

In addition, as $a_4a_5 < 0$, we can get the solution

$$u_{54}(x,y,z,t) = \frac{\pm 2k_4^2\sqrt{-a_4a_5}\sinh(\xi_4+\theta_1)(a_3\cosh\xi_3 + a_2\cos\xi_2) + 8a_4a_5k_4^2}{(a_2\cos\xi_2 + a_3\cosh\xi_3 \pm \sqrt{-a_4a_5}\sinh(\xi_4+\theta_1))^2}. \quad (2.58)$$

As $a_4 > 0$, the sign of u_{53} and u_{54} take positive. Otherwise, it takes the negative sign.

Remark 2.4. Moreover, if $\alpha = \beta$ in Remark 2.3, we can get solutions $u_1 - u_{40}$ for the (2+1)-dimensional generalized breaking soliton (gBS) equation

$$\partial x^{-1}(u_{xt} + u_{yt}) + \alpha u_{xxx} + \alpha u_{xxy} + 6\alpha uu_x + 3\alpha uu_y + 3\alpha u_x\partial x^{-1}u_y = 0. \quad (2.59)$$

If $\alpha = \beta$ in the third condition of Case 2, Case 4, Case 7, Case 8, Case 9, Case 10 with $m_2 = m_3 = m_4 = 0$, yields the periodic solutions, breather lump wave solutions, hyperbolic function solitary solutions u_{41}, u_{43}, u_{44} and $u_{47} - u_{54}$ of Eq. (2.59).

If $m_2 = 0$ in the first condition of Case 2, $k_2 = -\frac{\beta}{\alpha}$ in the second condition of Case 2, $m_3 = m_4 = 0$ in Case 3, $m_2 = m_3 = 0$ in Case 5 and Case 6, yield the periodic solution u_{41}, two-soliton solution u_{42}, bell soliton solutions u_{45} and u_{46} of (2+1)-dimensional generalized breaking soliton (gBS) equation

$$\partial x^{-1}(u_{xt} + u_{yt}) + \alpha u_{xxx} + \beta u_{xxy} + 6\alpha uu_x + 3\beta uu_y + 3\beta u_x\partial x^{-1}u_y = 0. \quad (2.60)$$

In (2.59) and (2.60), $u(x,y,t) : R \times R \times R \to R$ is the real function of variables x, y and t, real constants α and β are hyper-parameters of the system, ∂x^{-1} presents integral operator of x.

2.2. Application of other methods

In this section, we will analyze other three methods for obtaining exact solutions.

Firstly, for the three-wave method, we can set the second form of f with exp-function, trigonometric function and hyperbolic function

$$\begin{cases} f_2 = a_2\sin\xi_2 + a_3\sinh\xi_3 + a_4e^{\xi_4} + a_5e^{-\xi_4}, \\ \xi_i = k_ix + l_iy + m_iz + c_it + d_i, \quad i = 2,3,4, \end{cases} \quad (2.61)$$

where a_i, k_i, l_i, m_i, c_i, d_i and a_5 are some constants to be determined below. Taking (2.61) into (2.1), we let the coefficients of $\sin\xi_2\sinh\xi_3$, $e^{\xi_4}\sin\xi_2$, $e^{-\xi_4}\sin\xi_2$, $e^{\xi_4}\sinh\xi_2$, $e^{-\xi_4}\sinh\xi_2$, $\cos\xi_2\cosh\xi_3$, $e^{\xi_4}\cos\xi_2$, $e^{-\xi_4}\cos\xi_2$, $e^{\xi_4}\cosh\xi_2$, $e^{-\xi_4}\cosh\xi_2$ and the constant term are zero, and then yield a system of determining equations about parameters a_i, k_i, l_i, m_i, c_i, d_i and a_5 which is the same as (2.4). Similarly, the following conclusion can be drawn.

Theorem 2.2. *Let f_2 be given by (2.61). If f_2 is the solution of bilinear equation (2.1), then combining $u = 2(\ln f_2)_{xx}$, we get the following form of solutions for (3+1)-dimensional gBS equation (1.3)*

$$\begin{aligned} u(x,y,z,t) =&2\frac{-a_2k_2^2\sin\xi_2 + a_3k_3^2\sinh\xi_3 + a_4k_4^2\mathrm{e}^{\xi_4} + a_5k_4^2\mathrm{e}^{-\xi_4}}{a_2\sin\xi_2 + a_3\sinh\xi_3 + a_4\mathrm{e}^{\xi_4} + a_5\mathrm{e}^{-\xi_4}}\\ &-2(\frac{a_2k_2\cos\xi_2 + a_3k_3\sinh\xi_3 + a_4k_4\mathrm{e}^{\xi_4} - a_5k_4\mathrm{e}^{-\xi_4}}{a_2\sin\xi_2 + a_3\sinh\xi_3 + a_4\mathrm{e}^{\xi_4} + a_5\mathrm{e}^{-\xi_4}})^2, \end{aligned} \tag{2.62}$$

where $\xi_i = k_ix + l_iy + m_iz + c_it + d_i$, k_i, l_i, m_i, c_i, d_i, a_i $(i = 2, 3, 4)$ and a_5 are some constants.

Similar to Cases 1-10, we can obtain 54 kinds of exact solutions.

Secondly, applying the extended homoclinic test method, we set an auxiliary function of the following form

$$\begin{cases} f_3 = 1 + b_1\mathrm{e}^{\xi_2}\cos\xi_1 + b_2\mathrm{e}^{2\xi_2} + b_3\mathrm{e}^{\xi_2}\cosh\xi_1,\\ \xi_i = k_ix + l_iy + m_iz + c_it + d_i, \quad i = 1, 2, \end{cases} \tag{2.63}$$

where $\xi_i = k_ix + l_iy + m_iz + c_it + d_i$, b_i, k_i, l_i, m_i, c_i, d_i $(i = 1, 2)$ and b_3 are some constants to be determined below. We can see the forms of $f_1e^{\xi_4}$ and f_3 are similar. Then, taking (2.63) into (2.1), we get the system of coefficients

$$\begin{cases} b_1b_3(-4\alpha k_1^4 - 4\beta k_1^3l_1) = 0,\\ b_1b_3(c_1(k_1 + l_1 + m_1)) = 0,\\ b_1(\alpha(k_1^4 + k_2^4 - 6k_1^2k_2^2) + \beta(k_1^3l_1 + k_2^3l_2 - 3k_1^2k_2l_2 - 3k_1l_1k_2^2)\\ +c_2(k_2 + l_2 + m_2) - c_1(k_1 + l_1 + m_1)) = 0,\\ b_1(-\alpha(4k_1k_2^3 - 4k_1^3k_2) - \beta(k_2^3l_1 - k_1^3l_2 - 3k_1^2l_1k_2 + 3k_1k_2^2l_2)\\ -c_2(k_1 + l_1 + m_1) - c_1(k_2 + l_2 + m_2)) = 0,\\ b_3(\alpha(k_1^4 + k_2^4 + 6k_1^2k_2^2) + \beta(k_1^3l_1 + k_2^3l_2 + 3k_1^2k_2l_2 + 3k_1l_1k_2^2)\\ +c_2(k_2 + l_2 + m_2) + c_1(k_1 + l_1 + m_1)) = 0,\\ b_1(\alpha(4k_1k_2^3 - 4k_1^3k_2) + \beta(k_2^3l_1 + k_1^3l_2 + 3k_1^2l_1k_2 + 3k_1k_2^2l_2)\\ +c_2(k_1 + l_1 + m_1) + c_1(k_2 + l_2 + m_2)) = 0,\\ b_2\big(16\alpha k_2^4 + 16\beta k_2^3l_2 + 4c_2(k_2 + l_2 + m_2)\big) + b_1^2\big(4\alpha k_1^4 + 4\beta k_1^3l_1\\ -c_1(k_1 + l_1 + m_1)\big) + b_3^2\big(4\alpha k_1^4 + 4\beta k_1^3l_1 + c_1(k_1 + l_1 + m_1)\big) = 0. \end{cases} \tag{2.64}$$

With the solution $u(x,y,z,t) = 2\frac{f_{3xx}f_3 - f_{3x}^2}{f_3^2}$ and the relation between f_1 and f_3, we get the solution same as $u(x,y,z,t) = 2\frac{f_{1xx}f_1 - f_{1x}^2}{f_1^2}$, which just make some changes to the coefficients.

Finally, if assuming (2.63) as

$$\begin{cases} f_4 = 1 + b_1\mathrm{e}^{\xi_2}\sin\xi_1 + b_2\mathrm{e}^{2\xi_2} + b_3\mathrm{e}^{\xi_2}\sinh\xi_1,\\ \xi_i = k_ix + l_iy + m_iz + c_it + d_i, \quad i = 1, 2 \end{cases} \tag{2.65}$$

and carrying (2.65) into Eq. (2.1), we yield a system of determining equations about parameters k_i, l_i, m_i, c_i, b_i $(i = 1, 2)$ and b_3 same as (2.64).

3. Conclusion

In this paper, we derive a series of new traveling wave solutions of (3+1)-generalized breaking soliton equation by two types of three-wave methods and two types of the extended homoclinic test methods. These solutions contain bell solitary solutions, singular solitary solutions, periodic-solitary solutions and many interaction solutions between periodic waves and hyperbolic solutions. Our work contains the breather lump wave solutions [8] and multi-wave soluions [15]. If α, β, m_2, m_3 and m_4 take appropriate values, we can obtain many solutions of (2+1)-dimensional generalized breaking soliton (gBS) equation (2.59) and (2.60). Our results greatly enrich and expand the existing results. From our research process, we find that three wave method and extended homoclinic test method are two convenient, feasible, and efficient methods for solving exact solutions of nonlinear partial differential equations.

Moreover, we investigate several wave patterns for the free parameter values and also show the interaction of two waves propagation with various 2D and 3D graphs. Figure 3, Figure 4 and Figure 6 show the interaction of periodic waves and hyperbolic waves. Figure 1, Figure 2 and Figure 5 describe the interaction between two hyperbolic waves. The obtained results are very helpful in the study of interaction phenomena in mathematical physics, fluid dynamics, engineering and many other various areas of scientific fields. In the future, we will apply three-wave method to study exact solutions for four-component nonlinear Schrödinger integrable models and novel nonlocal nonlinear Schrödinger equations. Moreover, we will attempt to explore traveling wave solutions of (3+1)-dimensional gBS equation by other methods.

Acknowledgements

The authors would like to express their thanks to the anonymous referee for their valuable remarks and helpful suggestions on the earlier version of the paper.

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Received November 2024; Accepted April 2025; Available online April 2025.