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A RIEMANN-HILBERT METHOD TO FINITE GAP SOLUTIONS OF THE EXTENDED MODIFIED KORTEWEG-DE VRIES EQUATION

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Abstract In this paper we use finite gap integration method and Riemann-Hilbert method to obtain finite gap solutions of the extended modified Korteweg-de Vries equation. Based on the techniques developed in long-time asymptotics, we show that the Baker-Akhiezer function of the extended modified Korteweg-de Vries equation can be described by solvable matrix Riemann-Hilbert problems with σ_2- symmetry condition on $\mathbb{C}$. The procedure avoids solving Dubrovin's equations and Jacobi inverse problem.

Keywords Finite gap solutions, Riemann-Hilbert method, Baker-Akhiezer function, long-time asymptotics.

MSC(2010) 35Q15, 37K20.

1. Introduction

It is well-known that the modified Korteweg-de Vries (mKdV) equation is a fundamentally completely integrable model in solitary wave theory, with the canonical form

$$u_t + 6\sigma u^2 u_x + u_{xxx} = 0, \qquad \sigma = \pm 1, \qquad u = u(x,t), \tag{1.1}$$

which has many important applications in different physical backgrounds such as anharmonic lattices, Alfvén waves, ion acoustic solitons and fluid mechanics [26, 30, 41, 47]. Additionally, the more extensive form of KdV equation has been studied in detail, such as nonlocal mKdV equation [43], matrix mKdV equation [54] and extended mKdV equation. In this paper, we investigate extended mKdV (emKdV) equation

$$u_t + \alpha(6u^2u_x + u_{xxx}) + \beta(30u^4u_x + 10u_x^3 + 40uu_xu_{xx} + 10u^2u_{xxx} + u_{xxxxx}) = 0, \tag{1.2}$$

where $\alpha > 0$, $\beta > 0$ represent the third-order and fifth-order dispersion coefficients respectively. It is noted that equation (1.2) has some applications in describing the nonlinear internal waves stratified by density and current in a fluid [24, 48]. Moreover, the Painlevé integrability and multi-soliton solutions for equation (1.2) were studied by the simplified Hirota's direct method in [52]. The infinitely many conservation laws, periodic and rational solution have been derived by the N-fold Darboux transformation in [51]. The explicit leading-order asymptotics of the solution for (1.2) has been obtained by nonlinear steepest descent method associated matrix Riemann-Hilbert (RH) problem in [40]. Rogue waves on the general periodic traveling wave background have been found in [42]. N-soliton solutions of the nonlocal emKdV equation have been derived by the $\bar{\partial}$-dressing methods in [39]. Recently, the multi-soliton solutions of (1.2)

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under the condition with arbitrary order zeros based on RH problem have been explored in [55], and the solutions of (1.2) with zero/nonzero boundary conditions at infinity by the inverse scattering transform (IST) with the RH problem have been considered in [53]. However, as far as we know, it has not been reported to study finite gap solutions of the emKdV equation with RH problem.

The theory of finite gap solutions of nonlinear partial differential equations has been developed in integrable systems using algebraic geometry since the 1970s [15, 19, 27, 37]. It has been developed into one of the most effective tools for studying quasi-periodic problems in integrable systems. Finite gap integration method involves the spectral theory of differential operators, compactified Riemann surfaces, Riemann theta functions, inverse problems and other theories. Finite gap solutions of soliton equations are an important class of exact solution, which reveal inherent structural mechanism of solutions and describe the quasi-periodic behavior of nonlinear phenomena. Based on the hyperelliptic curve theory, several systematic methods were developed to obtain quasi-periodic solutions of soliton equations, such as the finite gap integration method, IST method for periodic problem, nonlinearization approach and others [5, 7, 18, 20, 22]. The Baker-Akhiezer (BA) function theory in finite gap solutions brought very interesting and important results in the spectral theory of almost periodic operators and theory of completely integrable nonlinear equations such as Korteweg-de Vries (KdV) equation, nonlinear Schrödinger equation, sine-Gordon equation, Kadomtsev-Petviashvili equation [2, 16, 17, 21, 28, 29, 33, 35, 36, 44, 46].

The RH problem is a modern version of IST method, which can be used to investigate the soliton solutions and long-time behaviors of integrable systems. In 1993, a completely rigorous method for analyzing the long-time asymptotics of solutions of integrable systems was introduced by Deift and Zhou, which is called the nonlinear steepest descent method [8, 14]. This method has been applied in a variety of problems and is very successful for asymptotic analysis the oscillatory RH problems. Through a series of deformations, the original RH problem can be reduced to canonical "model problems", which are characterized by piece-wise constant jumps across certain contours. Up to now, various integrable equations have been studied, such as the KdV equation, modified nonlinear Schrödinger equation, sine-Gordon equation and others [1, 6, 23, 25, 31, 32, 45, 50]. The elements of the finite gap integration method have been largely used in numerous studies of asymptotic behavior of nonlinear equations, such as the study of the large time asymptotics for the KdV equation (with decaying initial conditions) in the so-called "collisionless shock wave" region [11], the small dispersive limit for the KdV equation [12, 13], the asymptotics for the orthogonal matrices [9, 10], the large time asymptotic results for the KdV equation with the periodic background [38], semi-classical limits papers [3,4].

The main goal of the paper is to give the BA function of the emKdV equation. One more goal is applications in asymptotic analysis. It is convenient to describe the BA function through the corresponding RH problem. Compared to the classical method which transform the solution problem into an inverse problem on the Riemann surface, the method in this paper does not require the use of Dubrovin's equations and only needs to solve a relatively simple algebraic equation. The emKdV equation is equivalent to the compatibility condition $\Psi_{xt} = \Psi_{tx}$ of

$$\begin{aligned} \Psi_x(x,t,k) &= X(x,t,k)\Psi(x,t,k), \\ \Psi_t(x,t,k) &= T(x,t,k)\Psi(x,t,k), \end{aligned}$$

where

$$X = ik\sigma_3 + U, \quad T = (-16i\beta k^5 + 4i\alpha k^3)\sigma_3 + V,$$

with $k \in \mathbb{C}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $U = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}$, $V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ and

$$\begin{aligned} A &= 8i\beta k^3 u^2 - i(6\beta u^4 + 2\alpha u^2 + 4\beta u u_{xx} - 2\beta u_x^2)k, \\ B &= -16\beta k^4 u + 8i\beta k^3 u_x + (8\beta u^3 + 4\alpha u + 4\beta u_{xx})k^2 - i(12\beta u^2 u_x + 2\beta u_{xxx} \\ &\quad + 2\alpha u_x)k - 6\beta u^5 - 2\alpha u^3 - 10\beta u^2 u_{xx} - 10\beta u u_x^2 - \beta u_{xxxx} - \alpha u_{xx}, \\ C &= -B + 16i\beta k^3 u_x - i(24\beta u^2 u_x + 4\alpha u_x + 4\beta u_{xxx})k. \end{aligned}$$

Starting from the Lax pair, we use a combination of the finite gap method introduced by Krichever and the RH method developed by Kotlyarov and Shepelsky [34], which allows us to redefine the classical BA function in terms of RH problem, while simultaneously retaining its relations with finite gap solutions.

The paper is organized as follows. In section 2, we present the specific hyperelliptic Riemann surface and introduce some basic concepts related to it, including divisors, holomorphic differentials, homology basis, Riemann matrix, Riemann theta function, Riemann constants and Abel map. we also give the description of BA function of the emKdV equation in terms of the RH problem. In section 3, we present the asymptotic analysis of solutions of the spectral problem with respect to the spectral parameter k, which allows to recover solutions of the emKdV equation from its BA function. In section 4, we provide the exact construction of the BA function by the phase functions and a matrix-valued function M. Then we derive Riemann theta function representation of M by solving the associated RH problem. In section 5, we present finite gap solutions of the emKdV equation.

2. Preliminaries

In this section, we briefly review some basic concepts and results that exist in the theory of hyperelliptic Riemann surface. In addition, we provide the definition for the BA function of the emKdV equation.

Definition 2.1. Let $\{E_j, \hat{E}_j\}_{j=0}^n$ be $2n+2$ distinct points in $\mathbb{C}$. Then a Riemann surface $\mathcal{X}$ of genus n is defined by the following algebraic curve

$$\mathcal{X}: \ \mathcal{F}(k,y) = y^2 - P(k) = 0, \ P(k) = \prod_{j=0}^{n} (k - E_j)(k - \hat{E}_j),$$

with cuts along the arcs $\Gamma_j = (E_j, \hat{E}_j)$ connecting branch points E_j with $\hat{E}_j$, $j = 0, 1, 2, \ldots, n$. The Riemann surface $\mathcal{X}$ can be viewed as a double covering of the complex k-plane: Two sheets of k-plane are glued along the cuts Γ_j. The upper and lower sheets of $\mathcal{X}$ are denoted by $\mathcal{X}_+$ and $\mathcal{X}_-$ respectively. They are fixed by the relations

$$\sqrt{P(k)} = \pm k^{n+1}(1 + O(k^{-1})), \quad k = \pi(\mathcal{P}) \to \infty, \quad \mathcal{P} \in \mathcal{X}_\pm,$$

where $k = \pi(\mathcal{P})$ is the projection of $\mathcal{P} = (k, y(k)) \in \mathcal{X}$ on the Riemann sphere $\mathbb{CP}^1$. Thus each point on the k-plane has two preimages $\mathcal{P}_\pm$ except for the branch points. Denote the preimage of $k = \infty$ on $\mathcal{X}_\pm$ by $\infty^\pm$ respectively. In standard theory, $\mathcal{X}$ is compactified by two points $\infty^\pm$ at infinity. $y(\cdot)$ is the meromorphic function on $\mathcal{X}$, which has $2n+2$ zeros at E_j and $\hat{E}_j$, $j = 0, 1, \ldots, n$, and two poles at ∞^+ and ∞^-, each of multiplicity $n+1$.

Define the corresponding positive divisor of degree n on $\mathcal{X}$ by

$$\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n}:\ \mathcal{X}\to\mathbb{N}_0,\quad \mathcal{P}\mapsto\mathcal{D}_{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n}=\begin{cases} m, & \text{if } \mathcal{P} \text{ occurs m times} \\ & \text{in } \{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n\}, \\ 0, & \text{if } \mathcal{P}\ \notin\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n\}.\end{cases}$$

On the Riemann surface $\mathcal{X}$, n linearly independent holomorphic differentials $\eta_j(k)$ are defined by

$$\eta_j(k)=\frac{k^{j-1}}{\sqrt{P(k)}}dk,\quad j=1,\ldots,n.$$

There are $2n$ topologically distinct closed cycles which are split into two classes on $\mathcal{X}$, denoted a_j cycles and b_j cycles ($j=1,2,\ldots n$). Each of these cycles has a specified direction in which it is to be traversed. The rules for constructing a and b cycles are as follows:

(1) a_j cycles do not cross any other a_j cycles; b_j cycles do not cross any other b_j cycles.

(2) The cycle a_k intersects b_k only once and intersects no other b cycles.

This is symbolized as

$$a_j\circ b_l=\delta_{jl},\quad a_j\circ a_l=0,\quad b_j\circ b_l=0,\quad j,l=1,\cdots,n.$$

Using the a and b cycles we can construct the **A**-period and **B**-period $n\times n$ matrices,

$$A_{lj}=\int_{a_j}\eta_l(k),\quad B_{lj}=\int_{b_j}\eta_l(k),\quad j,l=1,\cdots,n.$$

The normalized Abelian differentials of the first kind are defined by

$$\omega_j(k)=\sum_{l=1}^{n}c_{jl}\eta_l(k),\qquad j=1,2,\cdots,n,$$

and the coefficients c_{jl} are uniquely determined by the normalization relationships

$$\int_{a_l}\omega_j=\delta_{jl},\quad j,l=1,2,\cdots,n.$$

Associated with ω_j, the b-periods of ω_j are defined by

$$B_{jl}=\int_{b_l}\omega_j,\quad j,l=1,2,\cdots,n.$$

Given the Riemann surface $\mathcal{X}$, the homology basis $\{a_j,b_j\}_{j=1}^n$, and the matrix $\mathbf{B}$ of b-periods of the differentials of the first kind $\{\omega_j\}_{j=1}^n$, the Riemann theta function associated with $\mathcal{X}$ and the homology basis is defined as

$$\theta(\mathbf{s})=\sum_{\mathbf{m}\in\mathbb{Z}^n}exp\big(2\pi i(\mathbf{m},\mathbf{s})+\pi i(\mathbf{m},\mathbf{mB})\big),\quad \mathbf{s}\in\mathbb{C}^n,$$

which has properties

$$\theta(s_1,\ldots,s_{j-1},-s_j,s_{j+1},\ldots,s_n)=\theta(\mathbf{s}),\ \ \mathbf{s}=(s_1,\ldots,s_n)\in\mathbb{C}^n,$$

$$\theta(\mathbf{s}+\mathbf{e_j})=\theta(\mathbf{s}),\quad \theta(\mathbf{s}\pm\mathbf{B_j})=\theta(\mathbf{s})exp\big(\mp 2\pi i s_j-\pi i B_{jj}\big),$$

where $(\cdot,\cdot)$ denotes the scalar product of two vectors in $\mathbb{C}^n$, with $\mathbf{B_j}=\mathbf{B}\mathbf{e_j}$, $\mathbf{e_j}=(0,\cdots,1,\cdots,0)$ being the j-th basis vector in $\mathbb{C}^n$.

The Jacobi variety of $\mathcal{X}$ is defined by

$$Jac(\mathcal{X})=\mathbb{C}^n/L_n,\quad L_n=\{\mathbf{k}\in\mathbb{C}^n|\mathbf{k}=\mathbf{p}+\mathbf{q}\mathbf{B},\ \mathbf{p},\ \mathbf{q}\in\mathbb{Z}^n\},$$

and the Abelian map is defined by

$$\begin{aligned}&\mathbf{A}_{\mathcal{P}_0}:\mathcal{X}\to Jac\{\mathcal{X}\},\\ &\mathcal{P}\mapsto\mathbf{A}_{\mathcal{P}_0}(\mathcal{P})=\big(A_{\mathcal{P}_0,1}(\mathcal{P}),\cdots,A_{\mathcal{P}_0,n}(\mathcal{P})\big)=\bigg(\int_{\mathcal{P}_0}^{\mathcal{P}}\omega_1,\ldots,\int_{\mathcal{P}_0}^{\mathcal{P}}\omega_n\bigg),\end{aligned}$$

where $\mathcal{P}_0\in\mathcal{X}$ is a fixed base point and the same path is chosen from $\mathcal{P}_0$ to $\mathcal{P}$ for all $j=1,\ldots,n$. The Abelian map is also defined by the summation of integral divisors $\mathcal{D}=\mathcal{P}_1+\mathcal{P}_2+\cdots+\mathcal{P}_n$, where the form is

$$\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})=\mathbf{A}_{\mathcal{P}_0}(\mathcal{P}_1)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{P}_2)+\cdots+\mathbf{A}_{\mathcal{P}_0}(\mathcal{P}_n).$$

Moreover, let $\mathcal{P}\in\{\mathcal{P}_1,\mathcal{P}_2,\cdots,\mathcal{P}_n\}$ and assume $\mathcal{D}$ to be non-special. Then

$$\theta(-\mathbf{A}_{\mathcal{P}_0}(\mathcal{P})+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0})=0,$$

where the vector of Riemann constants $\mathbf{K}_{\mathcal{P}_0}=(K_{\mathcal{P}_0,1},\ldots,K_{\mathcal{P}_0,n})$ is denoted by

$$K_{\mathcal{P}_0,j}=\frac{1}{2}-\frac{B_{jj}}{2}+\sum_{\substack{l=1\\ l\neq j}}^{n}\int_{a_l}\Big(\int_{\mathcal{P}_0}^{\mathcal{P}}\omega_j\Big)\omega_l,\quad j=1,\ldots,n.$$

In the hyperelliptic case,

$$K_{\mathcal{P}_0,j}=\frac{1}{2}\sum_{l=1}^{n}B_{lj}-\frac{j}{2},\quad j=1,\ldots,n.$$

The classical vector-valued BA function $\psi(P,x,t)=\binom{\psi_1}{\psi_2}$ is characterized by the following conditions:

- $\psi(\mathcal{P})$ is a meromorphic function on $\mathcal{X}\backslash(\{\infty^-\}\cup\{\infty^+\})$ with pole divisor

$$\mathcal{D}=\sum_{j=1}^{n}\mathcal{P}_j,$$

  which is non-special in the sense that the following conditions hold:

$$\pi(\mathcal{P}_j)\neq E_j,\hat{E}_j,\quad \pi(\mathcal{P}_j)\neq\pi(\mathcal{P}_l),\quad j\neq l.$$

- $\psi(\mathcal{P})$ has the following asymptotics:

$$\begin{aligned}&\psi(x,t,\mathcal{P})=\left[\begin{pmatrix}1\\0\end{pmatrix}+O(k^{-1})\right]e^{i\left(kx+(-16\beta k^5+4\alpha k^3)t\right)\sigma_3},\quad k=\pi(\mathcal{P}),\quad \mathcal{P}\to\infty^-,\\ &\psi(x,t,\mathcal{P})=ck\left[\begin{pmatrix}0\\1\end{pmatrix}+O(k^{-1})\right]e^{-i\left(kx+(-16\beta k^5+4\alpha k^3)t\right)\sigma_3},\quad k=\pi(\mathcal{P}),\quad \mathcal{P}\to\infty^+,\end{aligned}$$

  where c is a complex number.

The BA function, redefined in term of RH problem, is given in Definition 2.2.

Definition 2.2 (**Baker-Akhiezer function**)**.** Let $\Gamma_j = (E_j, \hat{E}_j)$, $j = 0, 1, \cdots, n$, be a set of oriented arcs which connect E_j with $\hat{E}_j$. They are symmetric with respect to 0. A 2×2 matrix $\Psi(x, t, k)$ is called the Baker-Akhiezer function of emKdV equation if it satisfies

(1) $\Psi(x, t, k)$ is analytic in $\mathbb{C} \backslash \Gamma$, $\Gamma = (\bigcup_{j=0}^{n} \Gamma_j) \cup (\bigcup_{j=1}^{n} \hat{\Gamma}_j)$, $\Gamma_j = (E_j, \hat{E}_j)$, $\hat{\Gamma}_j = (\hat{E}_{j-1}, E_j)$;

(2) $\Psi(x, t, k)$ has inverse fourth root singularities at E_j and $\hat{E}_j$, $j = 0, \cdots, n$;

(3) $\Psi_x \Psi^{-1}$ and $\Psi_t \Psi^{-1}$ all are entire functions. Meanwhile, Ψ satisfies

$$\Psi_-(x, t, k) = \Psi_+(x, t, k) J(k), \quad k \in \Gamma, \tag{2.1}$$

where $J(k)$ is a piece-wise constant matrix on Γ and

$$\text{Case A: } J(k) = \begin{pmatrix} 0 & ie^{i\phi_j} \\ ie^{-i\phi_j} & 0 \end{pmatrix}, \quad k \in \big(E_j, \hat{E}_j\big), \quad j = 0, 1, \cdots, n, \tag{2.2}$$

$$\text{Case B: } J(k) = \begin{cases} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in \Gamma_j = \big(E_j, \hat{E}_j\big), \quad j = 0, 1, \cdots, n, \\ \begin{pmatrix} e^{i\hat{\phi}_j} & 0 \\ 0 & e^{-i\hat{\phi}_j} \end{pmatrix}, & k \in \hat{\Gamma}_j = \big(\hat{E}_{j-1}, E_j\big), \quad j = 1, \cdots, n, \end{cases} \tag{2.3}$$

with ϕ_j and $\hat{\phi}_j$ are real constants, with $\phi_0 = 0$;

(4) Ψ satisfies the following symmetry condition

$$\Psi(x, t, k) = \sigma_2 \Psi(x, t, -k) \sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \tag{2.4}$$

(5) Ψ has essential singularity at infinity point,

$$\Psi(x, t, k) = \big(I + O(k^{-1})\big) e^{i\big(kx + (-16\beta k^5 + 4\alpha k^3)t\big)\sigma_3}, \quad k \to \infty.$$

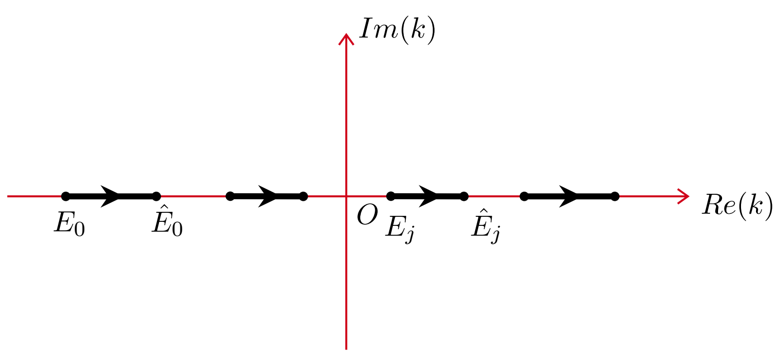

Figure 1. The oriented contour in the k-plane.

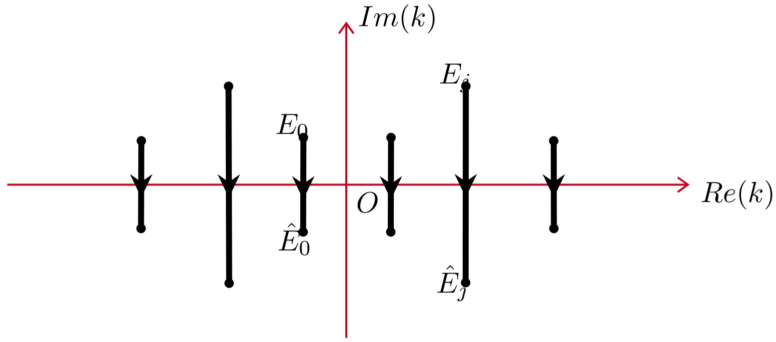

Figure 2. The oriented contour in the k-plane.

Remark 2.3. Once we get the existence result for the RH problem, the condition (2.4) becomes redundant and can be removed from Definition 2.2.

Remark 2.4. Since $\Psi_x\Psi^{-1}$ and $\Psi_t\Psi^{-1}$ are entire functions in (3), $J(k)$ is either diagonal or anti-diagonal, which guarantee the uniqueness of the RH problem. For the details one may refer to section 4.

Moreover, the uniqueness also indicates that the jump matrix $J(k)$ in Definition 2.2 should satisfy the symmetry condition

$$J(k)^{-1} = \sigma_2 J(-k)\sigma_2. \tag{2.5}$$

The relation (2.5) can be derived as follows. From (2.1), we have

$$\Psi_-(x,t,k) = \Psi_+(x,t,k)J(k),\ \ k \in \Gamma.$$

Converting k to $-k$, it follows

$$\Psi_+(x,t,-k) = \Psi_-(x,t,-k)J(-k),\ \ k \in \Gamma.$$

The action of σ_2 on both sides then yields

$$\sigma_2\Psi_+(x,t,-k)\sigma_2 = \{\sigma_2\Psi_-(x,t,-k)\sigma_2\}\{\sigma_2 J(-k)\sigma_2\},\ \ k \in \Gamma.$$

Taking into account (2.4), we finally arrive at (2.5).

3. Asymptotic analysis

In this section, we firstly discuss the asymptotic properties of a general matrix function ψ at infinity. Then we establish the relation between the solutions of the emKdV equation and ψ by introducing appropriate constraint conditions. Thus, once the BA function Ψ defined in section 2 satisfies all the conditions of ψ, finite gap solution of the emKdV equation can be directly derived from the asymptotic conditions of associated RH problems.

Next we consider the most general analytic properties of ψ, which can be summarized in the following two propositions.

Theorem 3.1. *Let $\psi(k)$ be an arbitrary 2×2 matrix function, which is holomorphic in a neighborhood of infinity on Riemann sphere, smoothly depends on x and t, and has the following expansion at infinity*

$$\psi(x,t,k) = \Big(I + \frac{D_1}{k} + \frac{D_2}{k^2} + \frac{D_3}{k^3} + \frac{D_4}{k^4} + \frac{D_5}{k^5} + O(\frac{1}{k^6})\Big)e^{i\left(kx+(-16\beta k^5+4\alpha k^3)t\right)\sigma_3}C(k), \tag{3.1}$$

where $C(k)$ is a 2×2 invertible matrix, independent of x and t. Then the function described above satisfies the following asymptotic estimations:

$$\psi_x\psi^{-1} = \widetilde{X} + O(k^{-1}), \tag{3.2}$$

$$\psi_t\psi^{-1} = \widetilde{T}(k) + O(k^{-1}), \quad k\to\infty, \tag{3.3}$$

where

$$\widetilde{X}(k) = ik\sigma_3 - i[\sigma_3, D_1], \tag{3.4}$$

$$\begin{aligned}
\widetilde{T}(k) =& -16i\beta k^5\sigma_3 + 16i\beta k^4[\sigma_3, D_1] + \Big(4i\alpha\sigma_3 + 16i\beta[\sigma_3, D_2] - 16i\beta[\sigma_3, D_1]D_1\Big)k^3\\
&+ \Big(-4i\alpha[\sigma_3, D_1] + 16i\beta[\sigma_3, D_3] - 16i\beta[\sigma_3, D_2]D_1 + 16i\beta[\sigma_3, D_1]D_1^2\\
&- 16i\beta[\sigma_3, D_1]D_2\Big)k^2 + \Big(-4i\alpha[\sigma_3, D_2] + 4i\alpha[\sigma_3, D_1]D_1 + 16i\beta[\sigma_3, D_4]\\
&- 16i\beta[\sigma_3, D_3]D_1 + 16i\beta[\sigma_3, D_2]D_1^2 - 16i\beta[\sigma_3, D_2]D_2 - 16i\beta[\sigma_3, D_1]D_1^3\\
&+ 16i\beta[\sigma_3, D_1]D_2D_1 + 16i\beta[\sigma_3, D_1]D_1D_2 - 16i\beta k[\sigma_3, D_1]D_3\Big)k\\
&- 4i\alpha[\sigma_3, D_3] + 4i\alpha[\sigma_3, D_2]D_1 - 4i\alpha[\sigma_3, D_1]D_1^2 + 4i\alpha[\sigma_3, D_1]D_2 + 16i\beta[\sigma_3, D_5]\\
&- 16i\beta[\sigma_3, D_4]D_1 + 16i\beta[\sigma_3, D_3]D_1^2 - 16i\beta[\sigma_3, D_3]D_2 - 16i\beta[\sigma_3, D_2]D_1^3\\
&+ 16i\beta[\sigma_3, D_2]D_2D_1 + 16i\beta[\sigma_3, D_2]D_1D_2 - 16i\beta[\sigma_3, D_2]D_3\\
&+ 16i\beta[\sigma_3, D_1]D_1^4 - 16i\beta[\sigma_3, D_1]D_2D_1^2 - 16i\beta[\sigma_3, D_1]D_1D_2D_1\\
&+ 16i\beta[\sigma_3, D_1]D_3D_1 - 16i\beta[\sigma_3, D_1]D_1^2D_2 + 16i\beta[\sigma_3, D_1]D_2^2\\
&+ 16i\beta[\sigma_3, D_1]D_1D_3 - 16i\beta[\sigma_3, D_1]D_4.
\end{aligned} \tag{3.5}$$

Proof. Obviously, due to $\psi\psi^{-1} = I$, we easily obtain

$$\psi^{-1}(x,t,k) = e^{-i(kx+(-16\beta k^5+4\alpha k^3)t)\sigma_3}\left(I + \frac{E_1}{k} + \frac{E_2}{k^2} + \frac{E_3}{k^3} + \frac{E_4}{k^4} + \frac{E_5}{k^5} + O(\frac{1}{k^6})\right), \tag{3.6}$$

where $k\to\infty$, with

$$\begin{aligned}
E_1 =& - D_1, \quad E_2 = -D_2 + D_1^2,\\
E_3 =& - D_3 + \big(D_1D_2 + D_2D_1\big) - D_1^3,\\
E_4 =& - D_4 + \big(D_2^2 + D_1D_3 + D_3D_1\big) - \big(D_1D_2D_1 + D_2D_1^2 + D_1^2D_2) + D_1^4,\\
E_5 =& - D_5 + \big(D_1D_4 + D_4D_1 + D_2D_3 + D_3D_2\big) - \big(D_1D_3D_1 + D_3D_1^2 + D_1^2D_3\\
&+ D_2^2D_1 + D_1D_2^2 + D_2D_1D_2\big) + \big(D_1^3D_2 + D_1^2D_2D_1 + D_1D_2D_1^2 + D_2D_1^3\big) - D_1^5.
\end{aligned}$$

Taking the x,t-derivatives of ψ and using (3.6), one may finally obtain (3.2), (3.3). □

In Theorem 3.1, we do not require ψ to be eigenfunctions of the emKdV spectral problem and hence its relation with the emKdV equation does not exist. In order to connect ψ with solutions of the emKdV equation, we should impose some constraint conditions on ψ. We have considered the following constraint condition:

$$\psi(x,t,k) = \sigma_2\psi(x,t,-k)\sigma_2. \tag{3.7}$$

The properties of ψ can be simplified under the constraints (3.7). We observe $\widetilde{X}$ and $\widetilde{T}$ satisfy

$$\widetilde{X}(k) = \sigma_2\widetilde{X}(-k)\sigma_2, \quad \widetilde{T}(k) = \sigma_2\widetilde{T}(-k)\sigma_2. \tag{3.8}$$

Then based on above relations, we can obtain the following result.

Theorem 3.1. *Assume ψ satisfies the conditions of Theorem 3.1 and the constraint conditions (3.7). Besides, we presume $\psi_x\psi^{-1}$, $\psi_t\psi^{-1}$ are both entire functions. Then if we define*

$$u(x,t) = -2i \lim_{k\to\infty} k\{\psi e^{-i(kx+(-16\beta k^5+4\alpha k^3)t)\sigma_3} - I\}_{12}, \tag{3.9}$$

or equivalently

$$U = -i[\sigma_3, D_1], \tag{3.10}$$

it may be shown that $u(x,t)$ is an exact solution to the emKdV equation.

Proof. For notational convenience we denote D_i by

$$D_i = \begin{pmatrix} a_i\ b_i \\ c_i\ d_i \end{pmatrix}, \quad i = 1, 2, 3, 4.$$

From (3.7), we derive

$$a_i = (-1)^i d_i, \quad b_i = (-1)^{i-1} c_i. \tag{3.11}$$

To prove this theorem one has to show that $X = \widetilde{X}$, $T = \widetilde{T}$. Set $u = -2ib_1$. Firstly, we check

$$[\sigma_3, D_2] - [\sigma_3, D_1]D_1 = \frac{1}{2}(u^2\sigma_3 + u_x\sigma_1), \quad \sigma_1 = \begin{pmatrix} 0\ 1 \\ 1\ 0 \end{pmatrix}.$$

Since $\psi_x\psi^{-1}$ is an entire function, the coefficient of the term $O(k^{-1})$ in (3.2) is zero, that is,

$$i([\sigma_3, D_2] - [\sigma_3, D_1]D_1) = D_{1,x}. \tag{3.12}$$

The main and the anti-diagonal elements have the following relations,

$$i([\sigma_3, D_2] - [\sigma_3, D_1]D_1)_{OD} = (D_{1,x})_{OD} = \frac{1}{2} i u_x \sigma_1,$$
$$i([\sigma_3, D_2] - [\sigma_3, D_1]D_1)_D = (-i[\sigma_3, D_1]D_1)_D = -2ib_1c_1\sigma_3 = \frac{1}{2} i u^2 \sigma_3.$$

Next, we need to prove

$$[\sigma_3, D_3] - [\sigma_3, D_2]D_1 + [\sigma_3, D_1]D_1^2 - [\sigma_3, D_1]D_2 = \frac{1}{4}(2u^3 + u_{xx})\sigma_2. \tag{3.13}$$

As previously, using the conditions (3.7) and (3.12), one obtains

$$([\sigma_3, D_3] - [\sigma_3, D_2]D_1 + [\sigma_3, D_1]D_1^2 - [\sigma_3, D_1]D_2)_D = 0.$$

Since

$$i([\sigma_3, D_3] - [\sigma_3, D_2]D_1 + [\sigma_3, D_1]D_1^2 - [\sigma_3, D_1]D_2) = D_{2,x} - D_{1,x}D_1, \tag{3.14}$$

it follows that the compatibility of (3.13) with (3.14) requires validation of

$$D_{2,x} - D_{1,x}D_1 = \frac{i}{4}(2u^3 + u_{xx})\sigma_2. \tag{3.15}$$

Obviously,

$$(D_{2,x} - D_{1,x}D_1)_{OD} = \frac{i}{4}(2u^3 + u_{xx})\sigma_2.$$

Thirdly, it remains to prove that

$$\begin{aligned}&[\sigma_3, D_4] - [\sigma_3, D_3]D_1 + [\sigma_3, D_2]D_1^2 - [\sigma_3, D_1]D_1^3 + [\sigma_3, D_1]D_2D_1 - [\sigma_3, D_2]D_2\\&+ [\sigma_3, D_1]D_1D_2 - [\sigma_3, D_1]D_3\\={}& -\frac{1}{8}(3u^4 + 2uu_{xx} - u_x^2)\sigma_3 - \frac{1}{8}(6u^2u_x + u_{xxx})\sigma_1.\end{aligned} \tag{3.16}$$

Similarly, since

$$\begin{aligned}&i([\sigma_3, D_4] - [\sigma_3, D_3]D_1 + [\sigma_3, D_2]D_1^2 - [\sigma_3, D_1]D_1^3 + [\sigma_3, D_1]D_2D_1 - [\sigma_3, D_2]D_2\\&+ [\sigma_3, D_1]D_1D_2 - [\sigma_3, D_1]D_3)\\={}& D_{3,x} - D_{2,x}D_1 + D_{1,x}D_1^2 - D_{1,x}D_2,\end{aligned} \tag{3.17}$$

the comparison of (3.16) and (3.17) reveals that the following relation need to be established:

$$D_{3,x} - D_{2,x}D_1 + D_{1,x}D_1^2 - D_{1,x}D_2 = -\frac{i}{8}(3u^4 + 2uu_{xx} - u_x^2)\sigma_3 - \frac{i}{8}(6u^2u_x + u_{xxx})\sigma_1.$$

Obviously, using the condition (3.7), (3.12) and (3.14), one obtains

$$\begin{aligned}&([\sigma_3, D_4] - [\sigma_3, D_3]D_1 + [\sigma_3, D_2]D_1^2 - [\sigma_3, D_1]D_1^3 + [\sigma_3, D_1]D_2D_1 - [\sigma_3, D_2]D_2\\&+ [\sigma_3, D_1]D_1D_2 - [\sigma_3, D_1]D_3)_D\\={}& -\frac{1}{8}(3u^4 + 2uu_{xx} - u_x^2)\sigma_3,\end{aligned}$$

and

$$(D_{3,x} - D_{2,x}D_1 + D_{1,x}D_1^2 - D_{1,x}D_2)_{OD} = -\frac{i}{8}(6u^2u_x + u_{xxx})\sigma_1.$$

To complete the proof, it remains to check

$$\begin{aligned}&[\sigma_3, D_5] - [\sigma_3, D_4]D_1 + [\sigma_3, D_3]D_1^2 - [\sigma_3, D_2]D_1^3 + [\sigma_3, D_1]D_1^4 - [\sigma_3, D_1]D_2D_1^2\\&+ [\sigma_3, D_2]D_2D_1 - [\sigma_3, D_1]D_1D_2D_1 + [\sigma_3, D_1]D_3D_1 - [\sigma_3, D_3]D_2 + [\sigma_3, D_2]D_1D_2\\&- [\sigma_3, D_1]D_1^2D_2 + [\sigma_3, D_1]D_2^2 - [\sigma_3, D_2]D_3 + [\sigma_3, D_1]D_1D_3 - [\sigma_3, D_1]D_4\\={}& -\frac{1}{16}(6u^5 + 10u^2u_{xx} + 10uu_x^2 + u_{xxxx})\sigma_2,\end{aligned} \tag{3.18}$$

and this can be done by Theorem 3.2. □

Theorem 3.2. *Under the conditions of Theorem 3.1, the coefficient of k^0 in $\widetilde{T}$ is a anti-diagonal matrix and can be represented by u:*

$$\begin{aligned}&[\sigma_3, D_5] - [\sigma_3, D_4]D_1 + [\sigma_3, D_3]D_1^2 - [\sigma_3, D_2]D_1^3 + [\sigma_3, D_1]D_1^4 - [\sigma_3, D_1]D_2D_1^2\\&+ [\sigma_3, D_2]D_2D_1 - [\sigma_3, D_1]D_1D_2D_1 + [\sigma_3, D_1]D_3D_1 - [\sigma_3, D_3]D_2 + [\sigma_3, D_2]D_1D_2\\&- [\sigma_3, D_1]D_1^2D_2 + [\sigma_3, D_1]D_2^2 - [\sigma_3, D_2]D_3 + [\sigma_3, D_1]D_1D_3 - [\sigma_3, D_1]D_4\\={}& -\frac{1}{16}(6u^5 + 10u^2u_{xx} + 10uu_x^2 + u_{xxxx})\sigma_2.\end{aligned} \tag{3.19}$$

Proof. From the constraint (3.8), we can find that the left hand of (3.19) is a anti-diagonal matrix. First the matrix form of (3.12),

$$\begin{pmatrix} 0 & 2ib_2 \\ -2ic_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2ib_1 \\ -2ic_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_{1x} & b_{1x} \\ c_{1x} & d_{1x} \end{pmatrix},$$

directly implies that

$$a_{1x} = -2ib_1c_1, \quad d_{1x} = 2ib_1c_1, \tag{3.20}$$
$$2ib_2 - b_{1x} - 2ib_1d_1 = 0, \quad -2ic_2 - c_{1x} + 2ia_1c_1 = 0. \tag{3.21}$$

Next from (3.14)

$$\begin{aligned} &\begin{pmatrix} 0 & 2ib_3 \\ -2ic_3 & 0 \end{pmatrix} - \left\{ \begin{pmatrix} 0 & 2ib_2 \\ -2ic_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2ib_1 \\ -2ic_1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \right\} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \\ &- \begin{pmatrix} 0 & 2ib_1 \\ -2ic_1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ = &\begin{pmatrix} a_{2x} & b_{2x} \\ c_{2x} & d_{2x} \end{pmatrix} - \begin{pmatrix} a_{1x} & b_{1x} \\ c_{2x} & d_{1x} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \end{aligned}$$

and (3.11), (3.20), (3.21), we obtain

$$a_{2x} = d_{2x} = b_1b_{1x} - 2ia_1b_1^2, \tag{3.22}$$
$$b_3 = c_3 = -\frac{1}{4}b_{1xx} + \frac{i}{2}a_1b_{1x} + b_1^3 + a_2b_1. \tag{3.23}$$

Similarly, from (3.17) and (3.11), (3.20), (3.21), (3.22), (3.23), we obtain

$$a_{3x} = -d_{3x} = a_1b_1b_{1x} - 2ib_1^2a_2 - 2ib_1^4 + \frac{i}{2}b_1b_{1xx}, \tag{3.24}$$
$$b_4 = -c_4 = \frac{i}{8}b_{1xxx} - \frac{5i}{2}b_1^2b_{1x} + \frac{1}{4}a_1b_{1xx} - a_1b_1^3 - b_1a_3 - \frac{i}{2}a_2b_{1x}. \tag{3.25}$$

Based on

$$\begin{aligned} &D_{4,x} - D_{3,x}D_1 + D_{2,x}D_1^2 - D_{1,x}D_1^3 + D_{1,x}D_2D_1 - D_{2,x}D_2 + D_{1,x}D_1D_2 - D_{1,x}D_3 \\ =\ &i\big([\sigma_3, D_5] + (-[\sigma_3, D_4] + [\sigma_3, D_3]D_1 - [\sigma_3, D_2]D_1^2 + [\sigma_3, D_1]D_1^3 - [\sigma_3, D_1]D_2D_1 \\ &+ [\sigma_3, D_2]D_2 - [\sigma_3, D_1]D_1D_2 + [\sigma_3, D_1]D_3)D_1 + (-[\sigma_3, D_3] + [\sigma_3, D_2]D_1 \\ &- [\sigma_3, D_1]D_1^2 + [\sigma_3, D_1]D_2)D_2 + (-[\sigma_3, D_2] + [\sigma_3, D_1]D_1)D_3 - [\sigma_3, D_1]D_4\big), \end{aligned} \tag{3.26}$$

it remains to establish the identity

$$\begin{aligned} &\big(D_{4,x} - D_{3,x}D_1 + D_{2,x}D_1^2 - D_{1,x}D_1^3 + D_{1,x}D_2D_1 - D_{2,x}D_2 + D_{1,x}D_1D_2 - D_{1,x}D_3\big) \\ = &-\frac{i}{16}(6u^5 + 10u^2u_{xx} + 10uu_x^2 + u_{xxxx})\sigma_2. \end{aligned}$$

Using (3.9), (3.11), (3.20), (3.21), (3.22), (3.23), (3.24) and (3.25), we obtain

$$\begin{aligned}
&\big([\sigma_3, D_5] - [\sigma_3, D_4]D_1 + [\sigma_3, D_3]D_1^2 - [\sigma_3, D_2]D_1^3 + [\sigma_3, D_1]D_1^4 - [\sigma_3, D_1]D_2D_1^2 \\
&+ [\sigma_3, D_2]D_2D_1 - [\sigma_3, D_1]D_1D_2D_1 + [\sigma_3, D_1]D_3D_1 - [\sigma_3, D_3]D_2 + [\sigma_3, D_2]D_1D_2 \\
&- [\sigma_3, D_1]D_1^2D_2 + [\sigma_3, D_1]D_2^2 - [\sigma_3, D_2]D_3 + [\sigma_3, D_1]D_1D_3 - [\sigma_3, D_1]D_4\big)_D = 0, \\
&\quad \big(D_{4,x} - D_{3,x}D_1 + D_{2,x}D_1^2 - D_{1,x}D_1^3 + D_{1,x}D_2D_1 - D_{2,x}D_2 + D_{1,x}D_1D_2 - D_{1,x}D_3\big)_{OD} \\
&= -\frac{i}{16}(6u^5 + 10u^2u_{xx} + 10uu_x^2 + u_{xxxx})\sigma_2,
\end{aligned}$$

which completes the proof. □

Corollary 3.3. *Assume $\psi(x,t,k)$ satisfies the conditions of Theorem 3.1 and the constraint conditions (3.7). In addition, suppose $\psi(x,t,k)$ is an exact solution to the following linear problem*

$$\begin{aligned}
\psi_x(x,t,k) &= \widetilde{X}(x,t,k)\psi(x,t,k), \\
\psi_t(x,t,k) &= \widetilde{T}(x,t,k)\psi(x,t,k),
\end{aligned} \tag{3.27}$$

where $\widetilde{X}$ and $\widetilde{T}$ are given by (3.4) and (3.5). Then if we define u by (3.9), it may be shown that $u(x,t)$ is an exact solution to the emKdV equation (1.2).

Proof. It suffices to prove that for all $k \in \mathbb{C}$, $\widetilde{X}$, $\widetilde{T}$ satisfy the zero curvature equation

$$\widetilde{X}_t(k) - \widetilde{T}_x(k) + [\widetilde{X}(k), \widetilde{T}(k)] = 0, \quad k \in \mathbb{C}. \tag{3.28}$$

Cross differentiation of (3.27) gives

$$\Big\{\widetilde{X}_t(k) - \widetilde{T}_x(k) + [\widetilde{X}(k), \widetilde{T}(k)]\Big\}\psi(k) = 0, \tag{3.29}$$

for all $k \in \mathbb{C}$. From the asymptotic condition (3.1) we know that ψ is invertible in a neighborhood of infinity. Thus, multiplying ψ^{-1} to both side of (3.29), one concludes that the zero-curvature equation (3.28) holds in the neighborhood of infinity. Further, noticing that $\widetilde{X}_t(k) - \widetilde{T}_x(k) + [\widetilde{X}(k), \widetilde{T}(k)]$ is an entire function in the k-plane, we conclude that (3.28) holds for all $k \in \mathbb{C}$. □

4. Explicit construction of the Baker-Akhiezer function

The important aspect that we discuss in section 3 is that it allows us directly to connect the BA functions defined in the k-plane to finite gap solutions of the emKdV equation. Note that the original RH problem has piece-wise constant jumps across the contour Γ, we can explicitly solve it based on standard method [34].

The strategy is as follow. We first seek the representation of the BA function Ψ in terms of $f(k)$, $g(k)$, $h(k)$ and a new matrix-valued function M. Then we introduce three phase functions $f(k)$, $g(k)$, $h(k)$ by using Cauchy integrals. Further, we construct Riemann theta representation of M by solving associated RH problem. Finally we obtain finite gap solutions of the emKdV equation by investigating the asymptotic expansion of M near infinity.

4.1. Representation of Ψ with explicitly separated exponential factors

The proof of existence for the BA function proceeds by an explicit construction of Ψ, which is obtained by solving original RH problem. To transform the initial RH problem to a form allowing an explicit solution, let us seek $\Psi(x,t,k)$ in the following theorem:

Theorem 4.1. *Let Ψ be the BA function of emKdV defined in Definition 2.2. Then Ψ is unique and has the following representation*

$$\Psi(x,t,k) = e^{-i\left(f_0 x + (g_0+h_0)t\right)\sigma_3} M(x,t,k) e^{i\left(f(k)x+(g(k)+h(k))t\right)\sigma_3}, \tag{4.1}$$

where constants f_0, g_0, h_0, scalar functions $f(k)$, $g(k)$, $h(k)$ and matrix $M(x,t,k)$ are to be determined. Due to the definition of Ψ we obtain the RH problem above. Indeed, all above statements will be true if $f(k)$, $g(k)$ and $h(k)$ possess properties:

- *$f(k)$, $g(k)$, $h(k)$ are analytic in $\mathbb{C}\backslash\Gamma$, $\Gamma = (\bigcup_{j=0}^{n}\Gamma_j)\cup(\bigcup_{j=0}^{n}\hat{\Gamma}_j)$, $\Gamma_j = (E_j, \hat{E}_j)$, $\hat{\Gamma}_j = (\hat{E}_{j-1}, E_j)$, and*

$$f(k) = k + f_0 + O(\frac{1}{k}), \tag{4.2}$$

$$g(k) = -\,16\beta k^5 + g_0 + O(\frac{1}{k}), \tag{4.3}$$

$$h(k) = 4\alpha k^3 + h_0 + O(\frac{1}{k}), \tag{4.4}$$

  where f_0, g_0, h_0 are zero according to (2.4);

- *$f(k)$, $g(k)$ and $h(k)$ have to satisfy the symmetry conditions*

$$-f(-k) = f(k), \quad -g(-k) = g(k), \quad -h(-k) = h(k);$$

- *$f(k)$, $g(k)$ and $h(k)$ satisfy the jump conditions*

$$f_+(k) + f_-(k) = \ C_j^f, \quad g_+(k) + g_-(k) = \ C_j^g, \quad h_+(k) + h_-(k) = \ C_j^h, \tag{4.5}$$

  $k \in \Gamma_j$, $j = 0, \ldots, n$, where C_j^f, C_j^g, C_j^h are real constants;

$$f_+(k) - f_-(k) = \ \hat{C}_j^f, \quad g_+(k) - g_-(k) = \ \hat{C}_j^g, \quad h_+(k) - h_-(k) = \ \hat{C}_j^h, \tag{4.6}$$

  $k \in \hat{\Gamma}_j$, $j = 1, \ldots, n$, where $\hat{C}_j^f$, $\hat{C}_j^g$, $\hat{C}_j^h$ all are real constants.

Above $f(k)$, $g(k)$, $h(k)$, and $M(x,t,k)$ are to be determined.

The jump matrix of $M(x,t,k)$ in the Case A can be derived by using (2.2) and (4.5). Indeed, for $k \in \Gamma_j$, we have

$$\begin{aligned} J_A(k) &= e^{i(f_+(k)x+(g_+(k)+h_+(k))t)\sigma_3} J(k) e^{-i(f_-(k)x+(g_-(k)+h_-(k))t)\sigma_3} \\ &= \begin{pmatrix} 0 & ie^{\triangle_j} \\ ie^{-\triangle_j} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ie^{i\left(C_j^f x + \left(C_j^g + C_j^h\right)t + \phi_j\right)} \\ ie^{-i\left(C_j^f x + \left(C_j^g + C_j^h\right)t + \phi_j\right)\sigma_3} & 0 \end{pmatrix}, \end{aligned}$$

where $\triangle_j = i\left((f_+ + f_-)x + (g_+ + g_- + h_+ + h_-)t + \phi_j\right)$. The jump matrix of $M(x,t,k)$ in the Case B can be derived similarly.

4.2. Construct $f(k)$, $g(k)$ and $h(k)$ in Case A

In this section we address the construction of phase functions on the k-plane. The phase functions $f(k)$, $g(k)$, $h(k)$ are three scalar functions defined by Cauchy integral

$$f(k)=\begin{cases}\omega(k)\big(1+\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_0}\dfrac{C_0^f}{\omega_+(\xi)(\xi-k)}d\xi\big), & n=0,\\ \dfrac{\omega(k)}{2\pi i}\displaystyle\sum_{j=0}^{n}\int_{\Gamma_j}\dfrac{C_j^f}{\omega_+(\xi)(\xi-k)}d\xi, & n\geq 1,\end{cases} \tag{4.7}$$

$$g(k)=\begin{cases}\omega(k)\big(-16\beta+\dfrac{1}{2\pi i}\displaystyle\sum_{j=0}^{4}\int_{\Gamma_j}\dfrac{C_j^g}{\omega_+(\xi)(\xi-k)}d\xi\big), & n=4,\\ \dfrac{\omega(k)}{2\pi i}\displaystyle\sum_{j=0}^{n}\int_{\Gamma_j}\dfrac{C_j^g}{\omega_+(\xi)(\xi-k)}d\xi, & n\geq 5,\end{cases} \tag{4.8}$$

$$h(k)=\begin{cases}\omega(k)\big(4\alpha+\dfrac{1}{2\pi i}\displaystyle\sum_{j=0}^{2}\int_{\Gamma_j}\dfrac{C_j^h}{\omega_+(\xi)(\xi-k)}d\xi\big), & n=2,\\ \dfrac{\omega(k)}{2\pi i}\displaystyle\sum_{j=0}^{n}\int_{\Gamma_j}\dfrac{C_j^h}{\omega_+(\xi)(\xi-k)}d\xi, & n\geq 3,\end{cases} \tag{4.9}$$

where C_j^f, C_j^g, C_j^h, $j=0,1,\ldots,n$, are undetermined constants, and $\omega(k)$ is defined by

$$\omega(k)=\sqrt{\prod_{j=0}^{n}(k-E_j)(k-\hat{E}_j)}.$$

Notice that $\omega(k)$ is analytic outside $k\in\Gamma_j$, $j=0,\ldots,n$, and

$$\omega(k)=\sum_{j=0}^{\infty}c_jk^{n+1-j}+O(k^n),\quad k\to\infty,$$

where $c_0=1$ and

$$c_j=\sum_{\substack{j_0,\cdots,j_{2n+1}=0\\ j_0+\cdots+j_{2n+1}=j}}^{j}\frac{(2j_0)!\cdots(2j_{2n+1})!E_0^{j_0}\hat{E}_0^{j_1}\cdots E_n^{j_{2n}}\hat{E}_n^{j_{2n+1}}}{2^{2j}(j_0!)^2\cdots(j_{2n+1}!)^2(2j_0-1)\cdots(2j_{2n+1}-1)},\quad j\geqslant 1.$$

Moreover, it is not difficult to verify that $\frac{f(k)}{\omega(k)}$, $\frac{g(k)}{\omega(k)}$ and $\frac{h(k)}{\omega(k)}$ are analytic in $\mathbb{C}\backslash\overline{\Gamma}$ and have jumps along each contour Γ_j:

$$\frac{f_+(k)}{\omega_+(k)}-\frac{f_-(k)}{\omega_-(k)}=\frac{C_j^f}{\omega_+(k)},\quad \frac{g_+(k)}{\omega_+(k)}-\frac{g_-(k)}{\omega_-(k)}=\frac{C_j^g}{\omega_+(k)},\quad \frac{h_+(k)}{\omega_+(k)}-\frac{h_-(k)}{\omega_-(k)}=\frac{C_j^h}{\omega_+(k)}. \tag{4.10}$$

Their asymptotic behaviors as $k\to\infty$ are given by

$$f(k)=\begin{cases}k-\dfrac{1}{2}(E_0+\hat{E}_0)-\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_0}\dfrac{C_0^f}{\omega_+(\xi)}d\xi+O(k^{-1}), & n=0,\\ c_0d_0k^n+(c_0d_1+c_1d_0)k^{n-1}+\cdots+(c_0d_{n-1}+\cdots+c_{n-1}d_0)k & \\ +c_0d_n+\cdots+c_nd_0+O(k^{-1}), & n\geq 1,\end{cases} \tag{4.11}$$

$$g(k)=\begin{cases}-16\beta k^5+(-16\beta c_1+e_0)k^4+(-16\beta c_2+c_1e_0+e_1)k^3+(-16\beta c_3\\ +c_2e_0+c_1e_1+e_2)k^2+(-16\beta c_4+c_3e_0+c_2e_1+c_1e_2+e_3)k\\ +(-16\beta c_5+c_4e_0+c_3e_1+c_2e_2+c_1e_3+e_4), & n=4,\\ c_0e_0k^n+(c_0e_1+c_1e_0)k^{n-1}+\cdots+(c_0e_{n-2}+\cdots+c_{n-2}e_0)k^2\\ +(c_0e_{n-1}+\cdots+c_{n-1}e_0)k+c_0e_n+\cdots+c_ne_0+O(k^{-1}), & n\geq 5,\end{cases} \tag{4.12}$$

$$h(k)=\begin{cases}4\alpha k^3+(4\alpha c_1+l_0)k^2+(4\alpha c_2+c_1l_0+l_1)k\\ 4\alpha c_3+c_2l_0+c_1l_1+l_2+O(k^{-1}), & n=2,\\ c_0l_0k^n+(c_0l_1+c_1l_0)k^{n-1}+\cdots+(c_0l_{n-1}+\cdots+c_{n-1}l_0)k\\ +c_0l_n+\cdots+c_nl_0+O(k^{-1}), & n\geq 2,\end{cases} \tag{4.13}$$

where $p=0,\ldots,n$, with

$$d_p=-\frac{1}{2\pi i}\sum_{j=0}^{n}\int_{\Gamma_j}\frac{C_j^f\xi^p}{\omega_+(\xi)}d\xi,\quad e_p=-\frac{1}{2\pi i}\sum_{j=0}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^p}{\omega_+(\xi)}d\xi,\quad l_p=-\frac{1}{2\pi i}\sum_{j=0}^{n}\int_{\Gamma_j}\frac{C_j^h\xi^p}{\omega_+(\xi)}d\xi.$$

To obtain the desired asymptotic properties, we have to choose appropriate C_j^f, C_j^g, C_j^h and this can be achieved by the following lemma.

Lemma 4.1. *Let $C_0^f=C_0^g=C_0^h=0$ and C_j^f, C_j^g, C_j^h, $j=1,\ldots,n$, be unique solutions of the following algebraic system*

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^f\xi^k}{\omega_+(\xi)}d\xi=-2\pi i\delta_{k,n-1},\quad k=0,\ldots,n-1, \tag{4.14}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^k}{\omega_+(\xi)}d\xi=32\pi i\beta\delta_{k,n-5},\quad k=0,1,\ldots,n-5, \tag{4.15}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^{n-4}}{\omega_+(\xi)}d\xi=-32\pi i\beta c_1, \tag{4.16}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^{n-3}}{\omega_+(\xi)}d\xi=-32\pi i\beta c_2+32\pi i\beta c_1^2, \tag{4.17}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^{n-2}}{\omega_+(\xi)}d\xi=-32\pi i\beta c_3+64\pi i\beta c_1c_2-32\pi i\beta c_1^3, \tag{4.18}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^g\xi^{n-1}}{\omega_+(\xi)}d\xi=-32\pi i\beta c_4+64\pi i\beta c_1c_3+32\pi i\beta c_2^2-96\pi i\beta c_2c_1^2+32\pi i\beta c_1^4, \tag{4.19}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^h\xi^k}{\omega_+(\xi)}d\xi=-8\pi i\alpha\delta_{k,n-3},\quad k=0,1,\ldots,n-3, \tag{4.20}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^h\xi^{n-2}}{\omega_+(\xi)}d\xi=8\pi i\alpha c_1, \tag{4.21}$$

$$\sum_{j=1}^{n}\int_{\Gamma_j}\frac{C_j^h\xi^{n-1}}{\omega_+(\xi)}d\xi=-8\pi i\alpha c_1^2+8\pi i\alpha c_2. \tag{4.22}$$

Then $f(k)$, $g(k)$, $h(k)$ defined by (4.7), (4.8) and (4.9) have the following properties:

- *$f(k)$, $g(k)$ and $h(k)$ are analytic in $\mathbb{C}\backslash\overline{\Gamma}$;*
- *$f(k)$, $g(k)$ and $h(k)$ have the expansions (4.2), (4.3) and (4.4);*
- *$f(k)$, $g(k)$ and $h(k)$, satisfy the symmetry conditions*

$$-f(-k) = f(k), \quad -g(-k) = g(k), \quad -h(-k) = h(k);$$

- *$f(k)$, $g(k)$ and $h(k)$, $k \in \Gamma_j$, $j = 0, \ldots, n$, satisfy the jump conditions*

$$f_+(k) + f_-(k) = C_j^f,\ g_+(k) + g_-(k) = C_j^g,\ h_+(k) + h_-(k) = C_j^h.$$

Proof. The system (4.14)-(4.22) is equivalent to the following conditions

$$\begin{aligned}
&d_0 = d_1 = \cdots = d_{n-2} = 0, \quad d_{n-1} = 1,\\
&e_0 = e_1 = \cdots = e_{n-6} = 0, \quad e_{n-5} = -16\beta, \quad e_{n-4} - 16\beta c_1 = 0,\\
&e_{n-3} + 16\beta c_1^2 - 16\beta c_2 = 0, \quad e_{n-2} - 16\beta c_1^3 + 32\beta c_1 c_2 - 16\beta c_3 = 0,\\
&e_{n-1} + 16\beta c_1^4 - 48\beta c_1^2 c_2 + 32\beta c_1 c_3 + 16\beta c_2^2 - 16\beta c_4 = 0,\\
&l_0 = l_0 = \cdots = l_{n-4} = 0, \quad l_{n-3} = 4\alpha, \quad l_{n-2} + 4\alpha c_1 = 0, \quad l_{n-1} - 4\alpha c_1^2 + 4\alpha c_2 = 0,
\end{aligned}$$

from which we can reduce (4.11), (4.12) and (4.13) to (4.2), (4.3) and (4.4), respectively. The uniqueness of solutions of the system (4.14)-(4.22) is well-known (see [49]). □

4.3. Explicit construction of M in Case A

In this section we seek for explicit representation of $M(x, t, k)$ by solving associated RH problem. In Case A, $M(x, t, k)$ is a solution of the following RH problem:

- $M(k)$ is analytic in $\mathbb{C}\backslash\Gamma_A$, where $\Gamma_A = \cup_{j=0}^n (E_j, \hat{E}_j)$;
- $M_-(k) = M_+(k)J_A(k),\ k \in \Gamma_A$, where $J(k)$ is

$$J_A(k) = \begin{cases} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in (E_0, \hat{E}_0), \\ \begin{pmatrix} 0 & ie^{i(C_j^f x + (C_j^g + C_j^h)t + \phi_j)} \\ ie^{-i(C_j^f x + (C_j^g + C_j^h)t + \phi_j)} & 0 \end{pmatrix}, & k \in (E_j, \hat{E}_j); \end{cases} \tag{4.23}$$

- $M(x, t, -k) = \sigma_2 M(x, t, k)\sigma_2$;
- $M(k) = I + O(k^{-1}),\ k \to \infty$;
- $M(k)$ is bounded on Γ_A except for branch points $E_j, \hat{E}_j$, where it has at most inverse fourth root singularities.

Firstly, we define

$$\varkappa(k) = \prod_{j=0}^n \sqrt[4]{\frac{k - E_j}{k - \hat{E}_j}},\ k \in \mathbb{C}\backslash\Gamma_A,$$

where the cuts are alone $\big(E_j, \hat{E}_j\big)$, $j = 0, 1, \cdots, n$, and the branch is fixed by the condition $\varkappa(\infty) = 1$. Then it is evident that

$$\varkappa_-(k) = i\varkappa_+(k),\ k \in \Gamma_A. \tag{4.24}$$

Moreover,

$$\varkappa(k) = (k - \hat{E}_j)^{-1/4} + O(1), \quad k \to \hat{E}_j,$$
$$\varkappa(k) = 1 + \sum_{j=0}^{n} \frac{\hat{E}_j - E_j}{4k} + O(k^{-2}), \quad k \to \infty.$$

The equation

$$\varkappa^4(k) - 1 = 0, \text{ or } \big(\varkappa(k) + \varkappa^{-1}(k)\big)\big(\varkappa(k) - \varkappa^{-1}(k)\big) = 0$$

has exact n zeros $k_1, \dots, k_n$. We assume

$$\varkappa(k_j) + \varkappa^{-1}(k_j) = 0, \quad j = 1, \dots, m,$$

for some fixed $m \in \mathbb{N}$, and

$$\varkappa(k_l) - \varkappa^{-1}(k_l) = 0, \quad l = m+1, \dots, n.$$

Associated with k_j, we introduce a non-special divisor

$$\mathcal{D} = \mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_n = \mathcal{D}_1 + \mathcal{D}_2,$$

where $\mathcal{D}_1 = \mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_m$, $\mathcal{D}_2 = \mathcal{P}_{m+1} + \mathcal{P}_{m+2} + \cdots + \mathcal{P}_n$, and $\mathcal{D}_1 \in \mathcal{X}_-$, $\mathcal{D}_2 \in \mathcal{X}_+$. According to Riemann vanishing theorem, we have

- $\theta(\mathbf{A}_{\mathcal{P}_0}(\mathcal{P}) + \mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) + \mathbf{K}_{\mathcal{P}_0})$ has m zeros $\mathcal{P}'_1, \mathcal{P}'_2, \cdots, \mathcal{P}'_m$ on $\mathcal{X}_+$ and $n-m$ zeros $\mathcal{P}'_{m+1}, \mathcal{P}'_{m+2}, \cdots, \mathcal{P}'_n$ on $\mathcal{X}_-$.
- $\theta(\mathbf{A}_{\mathcal{P}_0}(\mathcal{P}) - \mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) - \mathbf{K}_{\mathcal{P}_0})$ has m zeros $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_m$ on $\mathcal{X}_-$ and $n-m$ zeros $\mathcal{P}_{m+1}, \mathcal{P}_{m+2}, \cdots, \mathcal{P}_n$ on $\mathcal{X}_+$.

We choose a_l cycles as ovals on the upper sheet of $\mathcal{X}$ around the interval $\big(E_l, \hat{E}_l\big)$, $l = 1, 2, \cdots, n$. b_l cycles start from $\big(E_0, \hat{E}_0\big)$ and go to intervals $\big(E_l, \hat{E}_l\big)$ on the upper sheet, and then return to the starting point on the lower sheet.

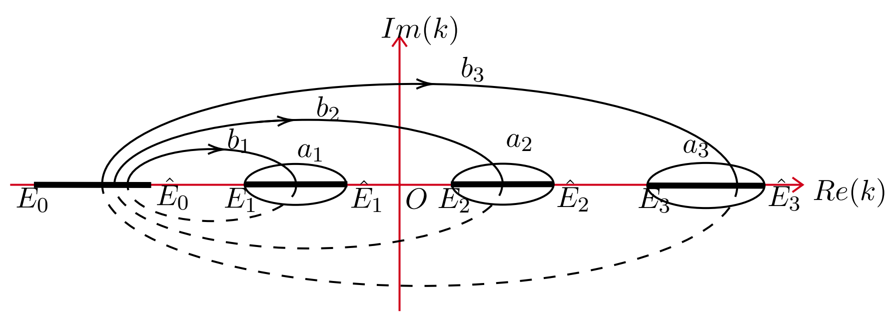

Figure 3. Case A; cuts on real axis for defocusing emKdV.

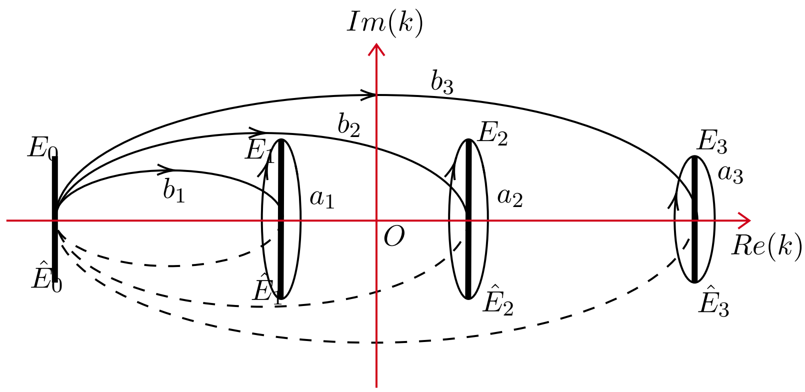

Figure 4. Case A; vertical cuts for focusing emKdV.

Using the normalized holomorphic differentials

$$\int_{a_j} \omega_j(\mathcal{P}) = 2\int_{E_l}^{\hat{E}_l} \omega_j(k) = \delta_{jl}, \quad j, l = 1, 2, \ldots, n,$$

we can define the b-period matrix as

$$\mathbf{B} = (B_{jl})_{n\times n}, \quad B_{jl} = \int_{b_l} \omega_j(\mathcal{P}) = 2\sum_{m=1}^{l} \int_{\hat{E}_{m-1}}^{E_m} \omega_j(k), \quad j, l = 1, 2, \cdots, n.$$

The Abelian mapping $\mathbf{A}_{\mathcal{P}_0}:\ \mathcal{X} \to Jac\{\mathcal{X}\}$ is defined as follows:

$$A_{\mathcal{P}_0,j}(\mathcal{P}) = \int_{\mathcal{P}_0}^{\mathcal{P}} \omega_j(\mathcal{Q}), \quad j = 1, 2, \ldots, n,$$

where the point $\mathcal{P}_0$ is fixed by condition $\pi(\mathcal{P}_0) = \hat{E}_0$ and $\mathcal{Q}$ is the integration variable. The Abelian integrals $\mathbf{A}_{\mathcal{P}_0}(k)$ on the upper sheet of $\mathcal{X}$ have the properties:

$$\mathbf{A}_{\mathcal{P}_0,-}(k) - \mathbf{A}_{\mathcal{P}_0,+}(k) = 0, \qquad k \in \left(\hat{E}_{j-1}, E_j\right), \quad j = 1, 2, \cdots, n, \tag{4.25}$$

$$\mathbf{A}_{\mathcal{P}_0,-}(k) + \mathbf{A}_{\mathcal{P}_0,+}(k) = \mathbf{B}\mathbf{e_j}, \qquad k \in \left(E_j, \hat{E}_j\right), \quad j = 0, 1, \cdots, n, \tag{4.26}$$

where $\mathbf{e_j} = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the j-th basis vector in $\mathbb{C}^n$. Define $(s = 1, 2)$

$$F_s(x,t,k) = \frac{\theta(\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{c} + \mathbf{d}_s)}{\theta(\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{d}_s)}, \quad H_s(x,t,k) = \frac{\theta(-\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{c} + \mathbf{d}_s)}{\theta(-\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{d}_s)}, \quad k \in \mathbb{C}\backslash\Gamma_A, \tag{4.27}$$

where

$$\mathbf{c} = (c_0, c_1, \ldots, c_j, \ldots, c_n), \quad j = 0, 1, \ldots, n,$$

and $\mathbf{d}_1 = \mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) + \mathbf{K}$, $\mathbf{d}_2 = -\mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) - \mathbf{K}$. Then from (4.25) and (4.26) we can obtain

$$\begin{aligned}
&F_{s_-}(k) = F_{s_+}(k), && H_{s_-}(k) = H_{s_+}(k), && k \in \left(\hat{E}_{j-1}, E_j\right),\ j = 1, 2, \cdots, n,\\
&F_{s_-}(k) = e^{-2\pi i c_j} H_{s_+}(k), && H_{s_-}(k) = e^{2\pi i c_j} F_{s_+}(k), && k \in \left(E_j, \hat{E}_j\right),\ j = 0, 1, \cdots, n.
\end{aligned}$$

Associated with (4.27) we define a 2×2 matrix-valued function

$$N(x,t,k) := \begin{pmatrix} F_1(x,t,k) \; H_1(x,t,k) \\ F_2(x,t,k) \; H_2(x,t,k) \end{pmatrix}, \quad k \in \mathbb{C}\backslash\Gamma. \tag{4.28}$$

The jump condition of $N(x,t,k)$ is as follows:

$$N_-(x,t,k) = N_+(x,t,k) J_N(x,t,k), \quad k \in \Gamma,$$

where

$$J_N(x,t,k) = \begin{cases} \begin{pmatrix} 1 \, 0 \\ 0 \, 1 \end{pmatrix}, & k \in \big(\hat{E}_{j-1}, E_j\big), \quad j = 1, \cdots, n. \\ \begin{pmatrix} 0 & e^{2\pi i c_j} \\ e^{-2\pi i c_j} & 0 \end{pmatrix}, & k \in \big(E_j, \hat{E}_j\big), \quad j = 0, 1, \cdots, n. \end{cases}$$

Next, taking into account (4.24) and (4.28), we introduce

$$\begin{aligned} M(x,t,k) :=& \frac{1}{2} \begin{pmatrix} \dfrac{1}{F_1(\infty)} & 0 \\ 0 & \dfrac{1}{H_2(\infty)} \end{pmatrix} \\ & \times \begin{pmatrix} \big(\varkappa(k) + \varkappa^{-1}(k)\big) F_1(k) \; \big(\varkappa(k) - \varkappa^{-1}(k)\big) H_1(k) \\ \big(\varkappa(k) - \varkappa^{-1}(k)\big) F_2(k) \; \big(\varkappa(k) + \varkappa^{-1}(k)\big) H_2(k) \end{pmatrix}. \end{aligned} \tag{4.29}$$

M is analysis in $\mathbb{C}\backslash\Gamma$, and it has the jumps

$$M_-(k) = M_+(k) \begin{pmatrix} 0 & i e^{2\pi i c_j} \\ i e^{-2\pi i c_j} & 0 \end{pmatrix}, \quad k \in \big(E_j, \hat{E}_j\big), \quad j = 0, 1, \cdots, n,$$

where the jump matrix of $M(x,t,k)$ is defined by (4.23) if we derive $c_j := \frac{x C_j^f + t(C_j^g + C_j^h) + \phi_j}{2\pi}$, $j = 1, \cdots, n$, $(c_0 = 0)$. Thus, the function $M(x,t,k)$ is the solution of the RH problem.

4.4. Construct $f(k)$, $g(k)$ and $h(k)$ in Case B

In this section we consider Case B. The phase functions $f(k)$, $g(k)$, $h(k)$ are defined by Cauchy integral

$$f(k) = \frac{\omega(k)}{2\pi i} \sum_{j=1}^{n} \int_{\hat{\Gamma}_j} \frac{\hat{C}_j^f}{\omega(\xi)(\xi - k)} d\xi, \quad n \geq 1, \tag{4.30}$$

$$g(k) = \begin{cases} \omega(k)\big(- 16\beta + \dfrac{1}{2\pi i} \displaystyle\sum_{j=1}^{4} \int_{\hat{\Gamma}_j} \frac{\hat{C}_j^g}{\omega(\xi)(\xi - k)} d\xi\big), & n = 4, \\ \dfrac{\omega(k)}{2\pi i} \displaystyle\sum_{j=1}^{n} \int_{\hat{\Gamma}_j} \frac{\hat{C}_j^g}{\omega(\xi)(\xi - k)} d\xi, & n \geq 5, \end{cases} \tag{4.31}$$

$$h(k)=\begin{cases}\omega(k)\big(4\alpha+\dfrac{1}{2\pi i}\displaystyle\sum_{j=1}^{2}\int_{\hat{\Gamma}_j}\frac{\hat{C}_j^h}{\omega(\xi)(\xi-k)}d\xi\big), & n=2,\\ \dfrac{\omega(k)}{2\pi i}\displaystyle\sum_{j=1}^{n}\int_{\hat{\Gamma}_j}\frac{\hat{C}_j^h}{\omega(\xi)(\xi-k)}d\xi, & n\geq 3,\end{cases}\tag{4.32}$$

where $\hat{C}_j^f$, $\hat{C}_j^g$, $\hat{C}_j^h$, $j=1,\ldots,n$ are undetermined constants. Their asymptotic behaviors as $k\to\infty$ are given by

$$\begin{aligned}f(k)=\ & c_0\hat{d}_0k^n+(c_0\hat{d}_1+c_1\hat{d}_0)k^{n-1}+\cdots+(c_0\hat{d}_{n-1}+\cdots+c_{n-1}\hat{d}_0)k\\ &+c_0\hat{d}_n+\cdots+c_n\hat{d}_0+O(k^{-1}),\quad n\geq 1,\end{aligned}\tag{4.33}$$

$$g(k)=\begin{cases}-16\beta k^5+(-16\beta c_1+\hat{e}_0)k^4+(-16\beta c_2+c_1\hat{e}_0+\hat{e}_1)k^3+(-16\beta c_3\\ +c_2\hat{e}_0+c_1\hat{e}_1+\hat{e}_2)k^2+(-16\beta c_4+c_3\hat{e}_0+c_2\hat{e}_1+c_1\hat{e}_2+\hat{e}_3)k\\ +(-16\beta c_5+c_4\hat{e}_0+c_3\hat{e}_1+c_2\hat{e}_2+c_1\hat{e}_3+\hat{e}_4), & n=4,\\ c_0\hat{e}_0k^n+(c_0\hat{e}_1+c_1\hat{e}_0)k^{n-1}+\cdots+(c_0\hat{e}_{n-2}+\cdots+c_{n-2}\hat{e}_0)k^2\\ +(c_0\hat{e}_{n-1}+\cdots+c_{n-1}\hat{e}_0)k+c_0\hat{e}_n+\cdots+c_n\hat{e}_0+O(k^{-1}), & n\geq 5,\end{cases}\tag{4.34}$$

$$h(k)=\begin{cases}4\alpha k^3+(4\alpha c_1+\hat{l}_0)k^2+(4\alpha c_2+c_1\hat{l}_0+\hat{l}_1)k\\ 4\alpha c_3+c_2\hat{l}_0+c_1\hat{l}_1+\hat{l}_2+O(k^{-1}), & n=2,\\ c_0\hat{l}_0k^n+(c_0\hat{l}_1+c_1\hat{l}_0)k^{n-1}+\cdots+(c_0\hat{l}_{n-1}+\cdots+c_{n-1}\hat{l}_0)k\\ +c_0\hat{l}_n+\cdots+c_n\hat{l}_0+O(k^{-1}), & n\geq 2,\end{cases}\tag{4.35}$$

where $p=0,\ldots,n$, with

$$\hat{d}_p=-\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\hat{\Gamma}_j}\frac{\hat{C}_j^f\xi^p}{\omega(\xi)}d\xi,\quad \hat{e}_p=-\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\hat{\Gamma}_j}\frac{\hat{C}_j^g\xi^p}{\omega(\xi)}d\xi,\quad \hat{l}_p=-\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\hat{\Gamma}_j}\frac{\hat{C}_j^h\xi^p}{\omega(\xi)}d\xi.$$

To obtain the desired asymptotic properties, we have to choose appropriate $\hat{C}_j^f$, $\hat{C}_j^g$, $\hat{C}_j^h$, which can be achieved by the Lemma 4.1 similar with Case A.

4.5. Explicit construction of M in Case B

In Case B, M needs to meet the following RH problem:

- $M(k)$ is analytic in $\mathbb{C}\backslash\Gamma$, where $\Gamma=\bigcup_{j=1}^{n}\big(\Gamma_j\cup\hat{\Gamma}_j\big)$;
- $M_-(k)=M_+(k)J_B(k)$, $k\in\Gamma$, where

$$J_B(k)=\begin{cases}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, & k\in\Gamma_j,\\ \begin{pmatrix}e^{(i\hat{C}_j^fx+i(\hat{C}_j^g+\hat{C}_j^h)t+i\hat{\phi}_j)} & 0\\ 0 & e^{-(i\hat{C}_j^fx+i(\hat{C}_j^g+\hat{C}_j^h)t+i\hat{\phi}_j)}\end{pmatrix}, & k\in\hat{\Gamma}_j,\end{cases}\tag{4.36}$$

  with $\Gamma_j=\big(E_j,\hat{E}_j\big),\ j=0,1,\cdots,n$, $\hat{\Gamma}_j=\big(\hat{E}_{j-1},E_j\big),\ j=1,\cdots,n$;
- $M(x,t,-k)=\sigma_2M(x,t,k)\sigma_2$;

- $M(k) = I + O(k^{-1})$, $k \to \infty$;
- M is bounded on Γ except for the branch points $E_j, \hat{E}_j$, where it has at most fourth root singularities.

In case B, we choose another homological basis. a_l cycles start from $\hat{E}_{l-1}$ and go to E_l on the upper sheet, and then them return to the starting points on the lower sheet. b_l cycles are ovals on the upper sheet enclosing intervals $\left(E_0, \hat{E}_{l-1}\right)$, $l = 1, 2, \cdots, n$.

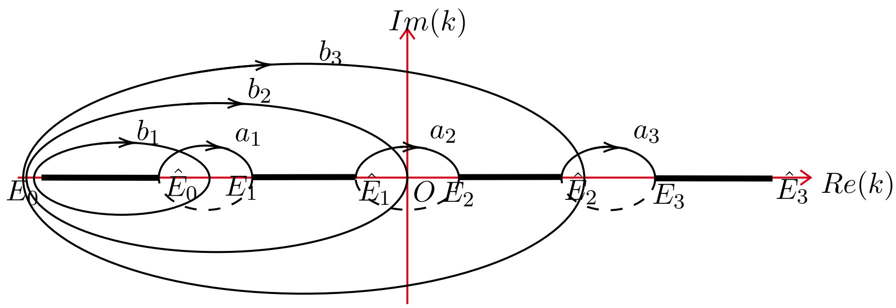

Figure 5. Case B; cut on real axis for defocusing emKdV.

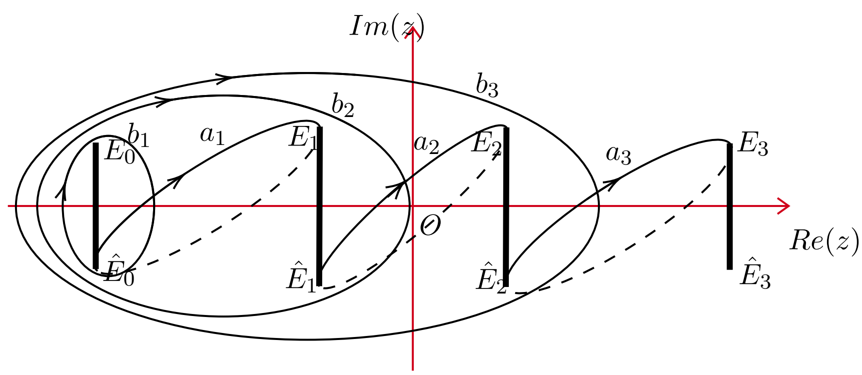

Figure 6. Case B; vertical cuts for focusing emKdV.

We normalize Abelian differentials $\omega_j(k)$ by

$$\int_{a_l} \omega_j(\mathcal{P}) = 2\int_{\hat{E}_{l-1}}^{E_l} \omega_j(k) = \delta_{jl},\ j, l = 1, 2, \cdots, n.$$

Then b-period matrix is defined as

$$B_{jl} = \int_{b_l} \omega_j(\mathcal{P}) = 2\sum_{k=0}^{l-1} \int_{E_k}^{\hat{E}_k} \omega_j(k).$$

The Abelian mapping $\mathbf{A}_{\mathcal{P}_0}:\ \mathcal{X} \to Jac\{\mathcal{X}\}$ is defined as follows:

$$A_{\mathcal{P}_0,j}(\mathcal{P}) = \int_{\mathcal{P}_0}^{\mathcal{P}} \omega_j(\mathcal{Q}), \quad j = 1, 2, \ldots, n,$$

where the point $\mathcal{P}_0$ is fixed by condition $\pi(\mathcal{P}_0) = E_0$ and $\mathcal{Q}$ is the integration variable. The Abelian maps $\mathbf{A}_{\mathcal{P}_0}(k)$ on the upper sheet of $\mathcal{X}$ have the following properties

$$\mathbf{A}_{\mathcal{P}_0,-}(k) + \mathbf{A}_{\mathcal{P}_0,+}(k) = 0, \qquad k \in \Gamma_j = \big(E_j, \hat{E}_j\big), \quad j = 0, 1, \cdots, n, \tag{4.37}$$

$$\mathbf{A}_{\mathcal{P}_0,-}(k) - \mathbf{A}_{\mathcal{P}_0,+}(k) = -\mathbf{B}\mathbf{e_j}, \qquad k \in \hat{\Gamma}_j = \big(\hat{E}_{j-1}, E_j\big), \quad j = 1, \cdots, n, \tag{4.38}$$

where $\mathbf{e_j} = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the j-th basis vector in $\mathbb{C}^n$. Define $(s = 1, 2)$

$$F_s(x,t,k) = \frac{\theta(\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{c} + \mathbf{d}_s)}{\theta(\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{d}_s)}, \quad H_s(x,t,k) = \frac{\theta(-\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{c} + \mathbf{d}_s)}{\theta(-\mathbf{A}_{\mathcal{P}_0}(k) + \mathbf{d}_s)}, \quad k \in \mathbb{C}\backslash\Gamma,$$

where

$$\mathbf{c} = (c_0, c_1, \ldots, c_j, \ldots, c_n), \quad j = 0, 1, \ldots, n,$$

and $\mathbf{d}_1 = \mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) + \mathbf{K}_{\mathcal{P}_0}$, $\mathbf{d}_2 = -\mathbf{A}_{\mathcal{P}_0}(\mathcal{D}) - \mathbf{K}_{\mathcal{P}_0}$. From (4.37) and (4.38) we can get

$$\begin{aligned} &F_{s-}(k) = H_{s+}(k), && H_{s-}(k) = F_{s+}(k), && k \in \Gamma_j, \quad j = 0, 1, \cdots, n, \\ &F_{s-}(k) = e^{2\pi i c_j} F_{s+}(k), && H_{s-}(k) = e^{-2\pi i c_j} H_{s+}(k), && k \in \hat{\Gamma}_j, \quad j = 1, \cdots, n. \end{aligned}$$

Associated with (4.27) we define a 2×2 matrix-valued function

$$N(x,t,k) := \begin{pmatrix} F_1(x,t,k) & H_1(x,t,k) \\ F_2(x,t,k) & H_2(x,t,k) \end{pmatrix}, \quad k \in \mathbb{C}\backslash\Gamma. \tag{4.39}$$

The jump condition of $N(x,t,k)$ is as follows:

$$N_-(x,t,k) = N_+(x,t,k) J_N(x,t,k), \quad k \in \Gamma,$$

where

$$J_N(x,t,k) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \in \big(E_j, \hat{E}_j\big),\ j = 0, 1, \cdots, n, \\ \begin{pmatrix} e^{2\pi i c_j} & 0 \\ 0 & e^{-2\pi i c_j} \end{pmatrix}, & k \in \big(\hat{E}_{j-1}, E_j\big),\ j = 1, 2, \cdots, n. \end{cases}$$

Set $c_j := \frac{x\hat{C}_j^f + t(\hat{C}_j^g + \hat{C}_j^h) + \hat{\phi}_j}{2\pi}$, $j = 1, \cdots, n$, $c_0 = 0$. The jump condition of M is

$$M_-(k) = \begin{cases} M_+(k) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & k \in \Gamma_j, \\ M_+(k) \begin{pmatrix} e^{(i\hat{C}_j^f x + i(\hat{C}_j^g + \hat{C}_j^h)t + i\hat{\phi}_j)} & 0 \\ 0 & e^{-(i\hat{C}_j^f x + i(\hat{C}_j^g + \hat{C}_j^h)t + i\hat{\phi}_j)} \end{pmatrix}, & k \in \hat{\Gamma}_j, \end{cases}$$

where $\Gamma_j = \big(E_j, \hat{E}_j\big)$, $j = 0, 1, \cdots, n$, $\hat{\Gamma}_j = \big(\hat{E}_{j-1}, E_j\big)$, $j = 1, \cdots, n$. Thus, the function $M(x,t,k)$ is the solution of the RH problem.

5. Finite gap solutions of the emKdV equation

We analyze Case A; the solutions for Case B are of the same form as those in Case A. Let us assume that $\Psi(x,t,k)$ satisfies all the conditions of $\psi(x,t,k)$ in Theorem 3.1. Then going back to the original RH problem and taking into account (3.9) and (4.1), we obtain

$$\begin{aligned} u(x,t) = & -2ik\{\psi e^{-i(kx+(-16\beta k^5+4\alpha k^3)t)\sigma_3} - I\}_{12} \\ = & -2ik\{e^{-if_0x\sigma_3-i(g_0+h_0)t\sigma_3} M(x,t,k) e^{if_0x\sigma_3+i(g_0+h_0)t\sigma_3} - I\}_{12}. \end{aligned} \tag{5.1}$$

Expanding $M(x,t,k)$ at $k=\infty$,

$$M(x,t,k) = I + \frac{M_1(x,t)}{k} + O(\frac{1}{k^2}), \quad k \to \infty, \tag{5.2}$$

we obtain

$$\begin{aligned} \big(M_1\big)_{12}(x,t) = & \frac{1}{4}\sum_{j=0}^{n}\big(\hat{E}_j - E_j\big) \\ & \times \frac{\theta\big(-\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}+\mathbf{c}(x,t)\big)\theta\big(\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}\big)}{\theta\big(-\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}\big)\theta\big(\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}+\mathbf{c}(x,t)\big)}. \end{aligned} \tag{5.3}$$

A combination of (5.1) and (5.3) finally gives rise to the following theta function representation of $u(x,t)$.

Theorem 5.1. *Assume also $u(x,t)$ is an n-gap solution of the emKdV equation (1.2) associated with the BA function $\Psi(x,t,k)$. Moreover, let $\mathcal{D}$ be a non-special divisor. Then $u(x,t)$ has the following Riemann theta function representation*

$$u(x,t) = -2iC_0\frac{\theta\big(-\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}+\mathbf{c}(x,t)\big)\theta\big(\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}\big)}{\theta\big(-\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}\big)\theta\big(\mathbf{A}_{\mathcal{P}_0}(\infty)+\mathbf{A}_{\mathcal{P}_0}(\mathcal{D})+\mathbf{K}_{\mathcal{P}_0}+\mathbf{c}(x,t)\big)}, \tag{5.4}$$

where $C_0 = \frac{1}{4}\sum_{j=0}^{n}\big(\hat{E}_j - E_j\big)$ and $\mathbf{c}(x,t)$ is a $(n+1)$ dimensional vector defined by (4.3).

Proof. Due to the structure of the jump (2.2) (x,t-independent), it is not difficult to verify $\Psi_x(x,t,k)\Psi^{-1}(x,t,k)$ and $\Psi_t(x,t,k)\Psi^{-1}(x,t,k)$ are both entire functions. Indeed, for $k \in \Gamma$, we have

$$\begin{aligned} (\Psi_-(x,t,k))_x = & (\Psi_+(x,t,k)J(x,t,k))_x \\ = & (\Psi_+(x,t,k))_x J(x,t,k) + \Psi_+(x,t,k)J_x(x,t,k) \\ = & (\Psi_+(x,t,k))_x J(x,t,k), \quad k \in \Gamma, \end{aligned}$$

and therefore

$$\begin{aligned} (\Psi_+(x,t,k))_x\Psi_+^{-1}(x,t,k) = & (\Psi_-(x,t,k))_x J^{-1}(x,t,k)\Psi_+^{-1}(x,t,k) \\ = & (\Psi_-(x,t,k))_x\Psi_-^{-1}(x,t,k). \end{aligned} \tag{5.5}$$

The relation (5.5) implies that $\Psi_x(x,t,k)\Psi^{-1}(x,t,k)$ is continuous at $k \in \Gamma$. Then we can conclude $\Psi_x(x,t,k)\Psi^{-1}(x,t,k)$ is analytic at any $k \in \mathbb{C}$. Similarly we can prove the analytic property of $\Psi_t(x,t,k)\Psi^{-1}(x,t,k)$. Thus, the function $\Psi(x,t,k)$ satisfies all the conditions of Theorem 3.1. Using (5.1) and (5.3), we finally obtain the formula (5.4). □

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Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] K. Andreiev, I. Egorova, T. L. Lange and G. Teschl, *Rarefaction waves of the Korteweg-de Vries equation via nonlinear steepest descent*, J. Differential Equations, 2016, 261, 5371–5410.

[2] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations*, Springer, 1994.

[3] R. J. Buckingham and P. D. Miller, *The sine-Gordon equation in the semiclassical limit: Dynamics of fluxon condensates*, Mem. Am. Math. Soc., 2013, 225, 1059.

[4] R. J. Buckingham and P. D. Miller, *Large-degree asymptotics of rational Painlevé-II functions: Noncritical behavior*, Nonlinearity, 2014, 27, 2489.

[5] C. W. Cao, Y. T. Wu and X. G. Geng, *Relation between the Kadometsev-Petviashvili equation and the confocal involutive system*, J. Math. Phys., 1999, 40, 3948–3970.

[6] P. J. Cheng, S. Venakides and X. Zhou, *Long-time asymptotics for the pure radiation solution of the sine-Gordon equation*, Commun. Part. Diff. Eq., 1999, 24, 1195–1262.

[7] P. Deift, A. Its, A. Kapaev and X. Zhou, *On the algebro-geometric integration of the Schlesinger equations*, Commun. Math. Phys., 1999, 203, 613–633.

[8] P. Deift, A. Its and X. Zhou, *Long-time asymptotics for integrable nonlinear wave equations*, Springer, 1993, 181–204.

[9] P. Deift, A. Its and X. Zhou, *A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics*, Ann. Math., 1997, 146, 149–235.

[10] P. Deift, T. Kriecherbauer, K. T. R. McLaughlin, S. Venakides and X. Zhou, *Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory*, Commun. Pure Appl. Math., 1999, 52, 1335–1425.

[11] P. Deift, S. Venakides and X. Zhou, *The collisionless shock region for the long-time behavior of solutions of the KdV equation*, Commun. Pure Appl. Math., 1994, 47, 199–206.

[12] P. Deift, S. Venakides and X. Zhou, *New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems*, Int. Math. Res. Not., 1997, 6, 285–299.

[13] P. Deift, S. Venakides and X. Zhou, *An extension of the steepest descent method for Riemann-Hilbert problems: The small dispersion limit of the Korteweg-de Vries (KdV) equation*, Proc. Natl. Acad. Sci. USA, 1998, 95, 450–454.

[14] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems, asymptotics for the MKdV equation*, Ann. Math., 1993, 137, 295–368.

[15] B. A. Dubrovin, *Inverse problem for periodic finite-zoned potentials in the theory of scattering*, Funct. Anal. Appl., 1975, 9, 61–62.

[16] B. A. Dubrovin, *Theta functions and non-linear equations*, Russ. Math. Surv., 1981, 36, 11.

[17] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russ. Math. Surv., 1976, 31, 59.

[18] X. Geng and C. Cao, *Decomposition of the (2 + 1)-dimensional Gardner equation and its quasi-periodic solutions*, Nonlinearity, 2001, 14, 1433.

[19] X. Geng and X. Zeng, *Algebro-geometric quasi-periodic solutions to the Satsuma–Hirota hierarchy*, Physica D, 2023, 448, 133738.

[20] J. S. Geronimo, F. Gesztesy and H. Holden, *Algebro-geometric solutions of the Baxter-Szegö difference equation*, Commun. Math. Phys., 2005, 258, 149–177.

[21] F. Gesztesy and H. Holden, *Soliton Equations and their Algebro-Geometric Solutions: Volume 1: (1+1)-Dimensional Continuous Models*, Cambridge University Press, 2003.

[22] F. Gesztesy and R. Ratneseelan, *An alternative approach to algebro-geometric solutions of the AKNS hierarchy*, Rev. Math. Phys., 1998, 10, 345–391.

[23] P. Giavedoni, *Long-time asymptotic analysis of the Korteweg-de Vries equation via the dbar steepest descent method: The soliton region*, Nonlinearity, 2017, 30, 1165.

[24] R. Grimshaw, E. Pelinovsky and O. Poloukhina, *Higher-order Korteweg-de Vries models for internal solitary waves in a stratified shear flow with a free surface*, Nonlin. Processes in Geophys., 2002, 9, 221–235.

[25] K. Grunert and G. Teschl, *Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent*, Math. Phys. Anal. Geom., 2009, 12, 287–324.

[26] M. A. Helal, *Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics*, Chaos Solitons Fractals, 2002, 13(9), 1917.

[27] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg–de Vries equation*, Theor. Math. Phys., 1975, 23, 343–355.

[28] A. R. Its and V. B. Matveev, *Hill operators with finitely many gaps*, Funct. Anal. Appl., 1975, 9, 65–66.

[29] A. R. Its and V. B. Matveev, *Algebrogeometric integration of the MNS equation, finite-gap solutions and their degeneration*, J. Math. Sci., 1983, 23, 2412–2420.

[30] A. H. Khater, O. H. El-Kalaawy and D. K. Callebaut, *Bäcklund transformations and exact solutions for Alfvén solitons in a relativistic electron–positron plasma*, Phys. Scr., 1998, 58, 545.

[31] A. V. Kitaev and A. H. Vartanian, *Leading-order temporal asymptotics of the modified nonlinear Schrödinger equation: Solitonless sector*, Inverse Problems, 1997, 13, 1311–1339.

[32] A. V. Kitaev and A. H. Vartanian, *Asymptotics of solutions to the modified nonlinear Schrödinger equation: Solitons on a nonvanishing continuous background*, SIAM J. Math. Anal., 1999, 30, 787–832.

[33] V. P. Kotlyarov, *A Matrix Baker-Akhiezer function associated with the Maxwell-Bloch equations and their finite-gap solutions*, SIGMA, 2018, 14, 082.

[34] V. Kotlyarov and D. Shepelsky, *Planar unimodular Baker-Akhiezer function for the nonlinear Schrödinger equation*, Ann. Math. Sci. Appl., 2017, 2, 343–384.

[35] I. M. Krichever, *Algebraic-geometric construction of the Zakharov–Shabat equations and their periodic solutions*, Dokl. Akad. Nauk SSSR, 1976, 227, 291–294.

[36] I. M. Krichever, *Integration of nonlinear equations by the methods of algebraic geometry*, Funct. Anal. Appl., 1977, 11, 12–26.

[37] P. D. Lax, *Periodic solutions of the KdV equation*, Commun. Pure Appl. Math., 1975, 28, 141–188.

[38] A. Mikikits-Leitner and G. Teschl, *Long-time asymptotics of perturbed finite-gap Korteweg-de Vries solutions*, J. Anal. Math., 2012, 116, 163–218.

[39] Z. Q. Li and S. F. Tian, *A hierarchy of nonlocal nonlinear evolution equations and $\bar{\partial}$-dressing method*, Appl. Math. Lett., 2021, 120, 107254.

[40] N. Liu, B. Guo, D. Wang and Y. Wang, *Long-time asymptotic behavior for an extended modified Korteweg-de Vries equation*, Commun. Math. Sci., 2019, 17(7), 1877–1913.

[41] K. E. Lonngren, *Ion acoustic soliton experiments in a plasma*, Opt. Quant. Electron., 1998, 30, 615–630.

[42] Y. Lou, Y. Zhang and R. Ye, *Rogue waves on the general periodic traveling wave background for an extended modified Korteweg–de Vries equation*, Math. Meth. Appl. Sci., 2021, 44, 13711–13722.

[43] J. Luo and E. Fan, *$\bar{\partial}$-dressing method for the nonlocal mKdV equation*, J. Geom. Phys., 2022, 177, 104550.

[44] V. B. Matveev, *30 years of finite-gap integration theory*, Phil. Trans. R. Soc. A, 2008, 366, 837–875.

[45] A. Boutet de Monvel, J. Lenells and D. Shepelsky, *The focusing NLS equation with step-like oscillating background: Scenarios of long-time asymptotics*, Comm. Math. Phys., 2021, 383, 893–952.

[46] S. P. Novikov, *The periodic problem for the Korteweg–de Vries equation*, Funct. Anal. Appl., 1974, 8, 236–246.

[47] H. Ono, *Soliton fission in anharmonic lattices with reflectionless inhomogeneity*, J. Phys. Soc. Jpn., 1992, 61, 4336–4343.

[48] E. N. Pelinovskii, O. E. Polukhina and K. Lamb, *Nonlinear internal waves in the ocean stratified in density and current*, Oceanology, 2000, 40, 757–766.

[49] T. Trogdon and B. Deconinck, *A Riemann–Hilbert problem for the finite-genus solutions of the KdV equation and its numerical solution*, Physica D, 2013, 251, 1–18.

[50] K. Wang, X. Geng and M. Chen, *Riemann–Hilbert approach and long-time asymptotics of the positive flow short-pulse equation*, Physica D, 2022, 439, 133383.

[51] X. Wang, J. Zhang and L. Wang, *Conservation laws, periodic and rational solutions for an extended modified Korteweg–de Vries equation*, Nonlinear Dyn., 2018, 92, 1507–1516.

[52] A. M. Wazwaz and G. Xu, *An extended modified KdV equation and its Painlevé integrability*, Nonlinear Dyn., 2016, 86, 1455–1460.

[53] Y. Xiao, Q. Zhu and X. Wu, *Riemann-Hilbert problem and multiple poles solution for an extended modified Korteweg-de Vries equation with zero/nonzero boundary conditions*, Nonlinear Dyn., 2022, 110, 1723–1746.

[54] J. Yang, S. Tian and Z. Li, *Inverse scattering problem for the matrix modified Korteweg–de Vries equation with finite density type initial data*, Physica D, 2023, 455, 133911.

[55] J. Yang, S. Tian and Z. Li, *Riemann-Hilbert method and multi-soliton solutions of an extended modified Korteweg-de Vries equation with N distinct arbitrary-order poles*, J. Math. Anal. Appl., 2022, 511, 126103.

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