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INVARIANT SOLITONS AND TRAVELLING-WAVE SOLUTIONS TO A HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION IN AN OPTICAL FIBER WITH AN IMPROVED TANH-FUNCTION ALGORITHM

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Abstract Optical fiber connection is essential to modern communication. A high-order nonlinear Schrödinger equation (NLSE) with additional dispersion of high-order and nonlinear components is studied in an inhomogeneous optical fiber. We provide numerous new analytic solutions via the improved modified extended tanh-function algorithm, including rational solution, singular periodic solution, Jacobi elliptic solutions (JESs), (bright, singular, dark) soliton, Weierstrass elliptic doubly periodic type solutions, and exponential solution. By employing the previously outlined method, they demonstrate their uniqueness for the given challenge. The results are presented in a clear and concise manner for various values of the necessary free parameters. Wolfram Mathematica's contour plot and 2D and 3D visualisations are used to show this process. The outcomes show how accurate, knowledgeable, and effective the computational procedures were. They may be used for increasingly complicated phenomena by integrating them with representational calculations. This finding constitutes a major advancement in our comprehension of the intricate and capricious behavior of this mathematical model.

Keywords Higher-order NLSE, inhomogeneous optical fiber, solitons, powerful analytic method.

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1. Introduction

A vital component of many scientific investigations and studies, non-linear partial differential equations (NLPDEs) are found in the fields of engineering physics, chemistry, biology, climate, and earth sciences, among others [2,26,38,43]. Intricate and nonlinear behaviors can also be mathematically explained using NLPDEs. In quantum physics, the NLSE is one type of equation that describes the behavior of a quantum state over time. NLSEs have gained importance in many fields of applied mathematics and physics throughout the past several decades, including fluid dynamics, quantum mechanics, molecular biology, hydrodynamics, elastic media, physics of plasmas, and non-linear optics [3,6]. Compared to the linear PDEs, NPDEs often have some simpler analytical solutions. While the analytical solutions may exist for some basic NPDEs, closed-form solutions are often absent from the real-world settings involving certain complex nonlinearities [15–19]. Precise solutions for nonnegative NLPDEs are crucial for obtaining correct qualitative comprehension and scientific interpretation of various physical processes [1, 29, 40]. Analytic results from NLPDEs give pictorial and mathematical support for the mechanical concept behind various complicated nonlinear phenomena, incorporating spatial localization transfer techniques, peaking regimes, and the presence or absence of stable states under certain circumstances [4,28,50]. Modern communication has benefited greatly from optical fiber connections [9, 42]. Because of their inherent value and possible uses, NLPDEs have garnered a lot of attention [37, 46, 47]. Applications for the high-order nonlinear Schrödinger equation, which includes nonlinear components and high-order dispersion, may be found in the Heisenberg spin chain [12], ocean waves [8], and fiber optics [49]. Pulse shaping, a remarkable technique based on nonlinear processes in optical fibers, may be used to modify the temporal and spectral content of light signals [24]. A variety of optical waveforms may be produced using this method, such as parabolic, triangular, rectangular profiles, and ultra-short compressed pulses [14,23,36]. In particular, since the non-linear dynamical behavior of pulses propagating in fibres with normal group-velocity dispersion (GVD) are usually sensitive to the initial conditions of the pulse, By adjusting the starting pulse's temporal intensity and/or phase profile, standard laser pulses may be nonlinearly shaped into a variety of specialised waveforms [27]. Changes in the sign of the GVD cause discrete changes in the phase profile, spectrum, and temporal structure of the pulse, exposing a variety of aspects of its nonlinear growth. The recent studies on nonlinear wave dynamics and soliton theory have provided significant insights into complex physical systems. Gao investigated the (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system, considering two-layer liquid interactions and lattice effects, which are crucial in fluid mechanics and condensed matter physics [20]. Wu et al. examined dark-soliton asymptotics in a repulsive nonlinear system under baroclinic flow conditions, highlighting the role of solitons in atmospheric and oceanic dynamics [45]. Gao and Tian explored similarity reductions in a (2+1)-dimensional modified Kadomtsev-Petviashvili system, contributing to the understanding of electromagnetic wave propagation in thin films [21]. Feng et al. analyzed the bilinear form and soliton solutions of a (3+1)-dimensional Korteweg-de Vries equation with time-dependent coefficients, furthering the mathematical framework for wave modeling in fluids [13]. Additionally, Shan et al. extended these investigations by deriving $N-$soliton solutions for a generalized Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation, which governs shallow water wave behavior under varying temporal conditions [41]. These studies collectively advance the theoretical and applied aspects of nonlinear wave phenomena across diverse physical contexts. Recent studies have advanced soliton theory across various nonlinear systems. Lan explored multi-soliton and breather-like solutions in optical fibers [31], $N-$soliton solutions in cylindrical Kadomtsev–Petviashvili equations [32], and soliton asymptotics in Bose–Einstein condensates [33]. Zhao analyzed the integrability of a (2+1)-dimensional Date–Jimbo–Kashiwara–Miwa equation [48]. Lan also investigated semirational rogue waves in higher-order Schrödinger equations [34] and bound-state solitons in three-wave resonant interactions [35]. These works contribute significantly to nonlinear wave modeling in optics, quantum fluids, and plasma physics.

Unlike previous studies, this work presents an innovative framework by integrating higher-order perturbative effects with advanced analytical techniques to obtain new soliton solutions. This novel approach not only extends existing models but also addresses unresolved complexities, offering fresh insights into nonlinear wave dynamics in optical fibers.

2. Related works

The study of high-order Nonlinear Schrödinger Equations (NLSEs) is essential in optical fiber research due to their ability to accurately model complex wave dynamics, including soliton interactions, self-phase modulation, and higher-order dispersion effects. In modern optical communication systems, where data transmission rates are continually increasing, traditional NLSE models often fall short in capturing critical nonlinear effects pertinent to ultra-short pulse propagation and high-power regimes. These conventional models typically overlook higher-order dispersion, self-steepening, and Raman scattering, leading to discrepancies between theoretical predictions and experimental observations.

Previous studies have primarily focused on the standard NLSE, which is limited in its capacity to describe pulse evolution under extreme conditions. While some extensions have incorporated third-order dispersion and self-steepening, they often lack a unified framework to handle complex nonlinear interactions in highly dispersive and birefringent fibers. Recent research has addressed these limitations by introducing higher-order nonlinear terms and perturbative effects, thereby refining the mathematical framework to enhance predictive accuracy for real-world fiber-optic systems. For instance, Murad et al. investigated optical soliton solutions of the time-fractional higher-order NLSE with Kudryashov's nonlinear refractive index, providing insights into ultra-short pulse propagation in optical fibers [39]. Similarly, Chen et al. explored periodic soliton interactions using a high-order NLSE, offering valuable understanding of soliton transmission characteristics in optical fibers [11]. These advancements contribute to the development of next-generation high-speed optical networks, facilitating more efficient and reliable data transmission.

In this work, we investigate the upcoming high-order NLSE in an inhomogeneous optical fibre, which can be read as [22, 44]:

$$\begin{aligned} &i\mathcal{T}_x + \frac{1}{2}\mathcal{T}_{tt} + \mathcal{T}|\mathcal{T}|^2 - i\eta\left(6|\mathcal{T}|^2\mathcal{T}_t + \mathcal{T}_{ttt}\right) \\ &+ \gamma\Big(\mathcal{T}_{tttt} + 6\mathcal{T}^*\mathcal{T}_t^2 + 4\mathcal{T}\left|\mathcal{T}_t\right|^2 + 6\mathcal{T}|\mathcal{T}|^4 + 2\mathcal{T}^2\mathcal{T}_{tt}^* + 8\mathcal{T}_{tt}|\mathcal{T}|^2\Big) = 0, \end{aligned} \tag{2.1}$$

where the envelope of the waves is represented by $\mathcal{T}(t,x)$, the propagation variable is x, and time in the moving frame is t. The complex conjugate is indicated by the superscript "$*$", and the partial derivatives are represented by the subscripts. In addition, η, γ are two real-valued constants which are represent the coefficients of third-order, fourth-order linear dispersion, respectively. $\mathcal{T}|\mathcal{T}|^4$ and $\mathcal{T}|\mathcal{T}|^2$ respectively elucidate the quintic and cubic nonlinear effects of self-phase modulation (SPM) on optical wave propagation. The additional nonlinear components are taken into account, including self-frequency shift, self-steepening, and Kerr effects [10]. Higher-order dispersion and nonlinear effects are crucial in the transmission of ultrashort pulses, including femtosecond pulses, due to the incredibly short pulse width. The shape and stability of the pulse are greatly influenced by the nonlinear effects of the optical signal, such as SPM and cross-phase modulation, especially in high-power or high-intensity fiber optic systems. We are aware of various published solutions for both rogue periodic waves and Jacobian elliptic functions for Eq. (2.1) in [44].

By analyzing the proposed model in (2.1) and applying the improved modified extended tanh-function algorithm (IMETFA), we hope to present novel optical modulated envelope soliton solutions in this work. We shall acquire various wave patterns that are modified, such as dark, bright, singular soliton, JESs, rational, singular periodic, exponential, and Weierstrass elliptic double periodic type functions. The effectiveness and potency of the used procedure are demonstrated by these extracted solutions.

Despite significant advancements in the study of high-order NLSEs within optical fibers, existing research has predominantly concentrated on specific cases or limited formulations of these equations. For instance, Kuang and Tian utilized an improved Riemann–Hilbert method to derive higher-order soliton solutions for the derivative NLSE [30]. While these studies have contributed valuable insights, they often address simplified versions of the governing equations or employ solution techniques that do not fully encapsulate the range of nonlinear effects pertinent to ultra-short pulse propagation and high-power regimes. In contrast, our work introduces a novel approach by incorporating higher-order perturbative terms and advanced analytical methods to derive new soliton solutions, thereby filling a crucial gap and making a unique contribution to the field [39].

3. Materials and methods

3.1. Motivation and advantages of the method

The IMETFA proves highly effective in obtaining precise exact solutions and enhancing the understanding of soliton dynamics compared to other recent methods. Its key advantages include efficiently generating accurate solutions, accommodating a broad range of nonlinear equations, and providing hyperbolic function-based solutions, which are crucial in physical applications. Additionally, it extends the traditional tanh method, offering improved flexibility and precision in solution derivation. However, its limitations involve dependence on selecting appropriate solution forms, potential complexity in handling higher-order nonlinear terms, and challenges in solving PDEs with non-hyperbolic structures. Moreover, achieving reliable results requires careful coefficient selection.

3.2. Mathematical preliminaries

A useful tool for PDE solutions is the IMETFA. In addition to handling complicated boundary conditions, it provides precise and effective solutions for both linear and nonlinear equations. The answers it offers are also simple to comprehend and apply. Understanding the underlying phenomena is aided by the fact that it offers a clear physical explanation of the solutions. In this portion, we highlight the important components of the IMETFA that this study will employ. Consider the succeeding NLPDE [5, 7, 25]:

$$\mathcal{R}\left(\mho, \mho_t, \mho_x, \mho_{xx}, \mho_{tt}, \mho_{xt}, \mho_{xxx}, \ldots\right) = 0, \tag{3.1}$$

where $\mathcal{R}$ denotes a polynomial function with its argument $\mho(x,t)$ accompanied by its respective partial derivatives with respect to the two independent variables.

Step 1. Here, our goal is to change Eq. (3.1), an NLPDE, into a non-linear ordinary differential equation (NODE). To do this, we use the following transformation:

$$\mho(x,t) = \mathcal{G}(\zeta)e^{i(\kappa x - ct)}, \qquad \zeta = \mathbf{a}x - \omega t, \tag{3.2}$$

where $\mathcal{G}(\zeta)$ denotes the amplitude component of the solution, and κ, c, $\mathbf{a}$, and ω are defined as real-valued constants that will be calculated later in the progress of the work.
Next, we combine Eq. (3.2) with Eq. (3.1), allowing us to build the necessary NODE as follows:

$$\mathcal{S}(\mathcal{G},\ \mathcal{G}',\ \mathcal{G}'',\ \mathcal{G}''',\ \ldots) = 0. \tag{3.3}$$

Step 2. According to the used algorithm, the general form of the solution for Eq. (3.3) is as follows:

$$\mathcal{G}(\zeta) = \sum_{j=0}^{\mathbb{M}} \mathbb{A}_j \mathcal{H}^j(\zeta) + \sum_{j=1}^{\mathbb{M}} \mathbb{B}_j \mathcal{H}^{-j}(\zeta), \tag{3.4}$$

here the parameters $\mathbb{A}_j$ and $\mathbb{B}_j$ $(j = 1, 2, ..., \mathbb{M})$ stand for constants in the solution equation that will be computed. This provides the necessary condition that neither $\mathbb{A}_{\mathbb{M}}$ nor $\mathbb{B}_{\mathbb{M}}$ can be zero at the same time.

Step 3. In order to assess the positive integer $\mathbb{M}$, the balancing principle (BP) is employed to Eq. (3.3). And the function $\mathcal{H}(\zeta)$ also satisfies the following constrain:

$$(\mathcal{H}'(\zeta))^2 = \left(\frac{d\mathcal{H}}{d\zeta}\right)^2 = \tau_0 + \tau_1\mathcal{H}(\zeta) + \tau_2\mathcal{H}^2(\zeta) + \tau_3\mathcal{H}^3(\zeta) + \tau_4\mathcal{H}^4(\zeta), \tag{3.5}$$

while τ_l $(0 \le l \le 4)$ represent constant values that shall assist in identifying potential solution scenarios.
Eq. (3.5) has the following general solutions with different possible values of $\tau_0, \tau_1, \tau_2, \tau_3$ and τ_4:

Family 1. When $\tau_0 = \tau_1 = \tau_3 = 0$, the following solutions are raised:

$$\mathcal{H}(\zeta) = \sqrt{-\frac{\tau_2}{\tau_4}}\operatorname{sech}\left[\zeta\sqrt{\tau_2}\right], \qquad \tau_2 > 0,\ \tau_4 < 0,$$

$$\mathcal{H}(\zeta) = \sqrt{-\frac{\tau_2}{\tau_4}}\sec\left[\zeta\sqrt{-\tau_2}\right], \qquad \tau_2 < 0,\ \tau_4 > 0.$$

These solutions are localized and describe soliton-like structures. The sech function corresponds to bright solitons, while the sec function represents singular periodic structures.

Family 2. When $\tau_1 = \tau_3 = 0$, the following solutions are raised:

$$\mathcal{H}(\zeta) = \sqrt{-\frac{\mu^2\tau_2}{(2\mu^2-1)\,\tau_4}}\ \mathrm{cn}\left[\zeta\sqrt{\frac{\tau_2}{2\mu^2-1}}\ \right],\ \tau_2>0,\ \tau_4<0,\ \tau_0=\frac{\mu^2\left(1-\mu^2\right)\tau_2^2}{4\left(2\mu^2-1\right)^2\tau_4},$$

$$\mathcal{H}(\zeta) = \sqrt{-\frac{\mu^2}{(2-\mu^2)\,\tau_4}}\ \mathrm{dn}\left[\zeta\sqrt{\frac{\tau_2}{2-\mu^2}}\ \right],\qquad \tau_2>0,\ \tau_4<0,\ \tau_0=\frac{\left(1-\mu^2\right)\tau_2^2}{\left(2-\mu^2\right)^2\tau_4},$$

$$\mathcal{H}(\zeta) = \sqrt{-\frac{\mu^2\tau_2}{(\mu^2+1)\,\tau_4}}\ \mathrm{sn}\left[\zeta\sqrt{\frac{-\tau_2}{\mu^2+1}}\ \right],\qquad \tau_2<0,\ \tau_4>0,\ \tau_0=\frac{\mu^2\tau_2^2}{\left(\mu^2+1\right)^2\tau_4},$$

$$\mathcal{H}(\zeta) = \epsilon\sqrt{-\frac{\tau_2}{2\tau_4}}\tanh\left(\sqrt{-\frac{\tau_2}{2}}\ \zeta\right),\qquad \tau_2<0,\tau_4>0,\tau_0=\frac{\tau_2^2}{4\tau_4},$$

$$\mathcal{H}(\zeta) = \epsilon\sqrt{\frac{\tau_2}{2\tau_4}}\tan\left(\sqrt{\frac{\tau_2}{2}}\ \zeta\right),\qquad \tau_2>0,\tau_4>0,\tau_0=\frac{\tau_2^2}{4\tau_4},$$

where μ is the modulus of the Jacobi elliptic functions, $0\le\mu\le1$, and $\epsilon=\pm1$. These solutions describe periodic and quasi-periodic wave structures, in addition to the dark solitons.

Family 3. When $\tau_2 = \tau_4 = 0,\ \tau_0 \neq 0,\ \tau_1 \neq 0,\ \tau_3 > 0$,the following solution is raised:

A Weierstrass elliptic doubly periodic type solution is obtained:

$$\mathcal{H}(\zeta) = \wp\left[\frac{\sqrt{\tau_3}}{2}\zeta,\ \mathcal{A}_2,\ \mathcal{A}_3\right],$$

where $\mathcal{A}_2 = -\dfrac{4\tau_1}{\tau_3}$ and $\mathcal{A}_3 = -\dfrac{4\tau_0}{\tau_3}$ are called Weierstrass elliptic function invariants.

Family 4. When $\tau_3 = \tau_4 = 0$, the following solution is raised:

$$\mathcal{H}(\zeta) = -\frac{\tau_1}{2\tau_2} + \exp\left(\epsilon\sqrt{\tau_2}\ \zeta\right),\qquad \tau_2>0,\tau_0=\frac{\tau_1^2}{4\tau_2},$$

$$\mathcal{H}(\zeta) = -\frac{\tau_1}{2\tau_2} + \frac{\epsilon\tau_1}{2\tau_2}\sin\left(\sqrt{-\tau_2}\ \zeta\right),\qquad \tau_0=0,\tau_2<0,$$

$$\mathcal{H}(\zeta) = -\frac{\tau_1}{2\tau_2} + \frac{\epsilon\tau_1}{2\tau_2}\sinh\left(2\sqrt{\tau_2}\ \zeta\right),\qquad \tau_0=0,\tau_2>0,$$

$$\mathcal{H}(\zeta) = \epsilon\sqrt{-\frac{\tau_0}{\tau_2}}\sin\left(\sqrt{-\tau_2}\ \zeta\right),\qquad \tau_1=0,\tau_0>0,\tau_2<0,$$

$$\mathcal{H}(\zeta) = \epsilon\sqrt{\frac{\tau_0}{\tau_2}}\sinh\left(\sqrt{\tau_2}\ \zeta\right),\qquad \tau_1=0,\tau_0>0,\tau_2>0.$$

Family 5. When $\tau_0 = \tau_1 = 0$, and $\tau_4 > 0$, the following solution is raised:

$$\mathcal{H}(\zeta) = -\frac{\tau_2\sec^2\left(\frac{\sqrt{-\tau_2}}{2}\ \zeta\right)}{2\varepsilon\sqrt{-\tau_2\tau_4}\tan\left(\frac{\sqrt{-\tau_2}}{2}\ \zeta\right)+\tau_3},\qquad \tau_2<0,$$

$$\mathcal{H}(\zeta) = \frac{\tau_2 \mathrm{sech}^2\left(\frac{\sqrt{\tau_2}}{2}\ \zeta\right)}{2\varepsilon\sqrt{\tau_2\tau_4}\tanh\left(\frac{\sqrt{\tau_2}}{2}\ \zeta\right) - \tau_3}, \qquad \tau_2 > 0, \tau_3 \neq 2\varepsilon\sqrt{\tau_2\tau_4},$$

$$\mathcal{H}(\zeta) = \frac{1}{2}\varepsilon\sqrt{\frac{\tau_2}{\tau_4}}\left(1 + \tanh\left(\frac{\sqrt{\tau_2}}{2}\ \zeta\right)\right), \qquad \tau_2 > 0, \tau_3 = 2\varepsilon\sqrt{\tau_2\tau_4}.$$

Step 4. Rendering Eq. (3.3) with the solution that seems to be provided in Eqs. (3.4) and (3.5) will generate a polynomial in $\mathcal{H}(\zeta)$. Mathematical software like Mathematica program may be utilized to solve an algebraic system of nonlinear equations that arises when the coefficients of $\mathcal{H}^k(\zeta)$, $(k = 0, \pm 1, \pm 2, ...)$, are equalised to zero. For the travelling wave in Eq. (3.1), there are thus several analytic solutions that we can obtain. The following flow chart summarizes all the aforementioned steps.

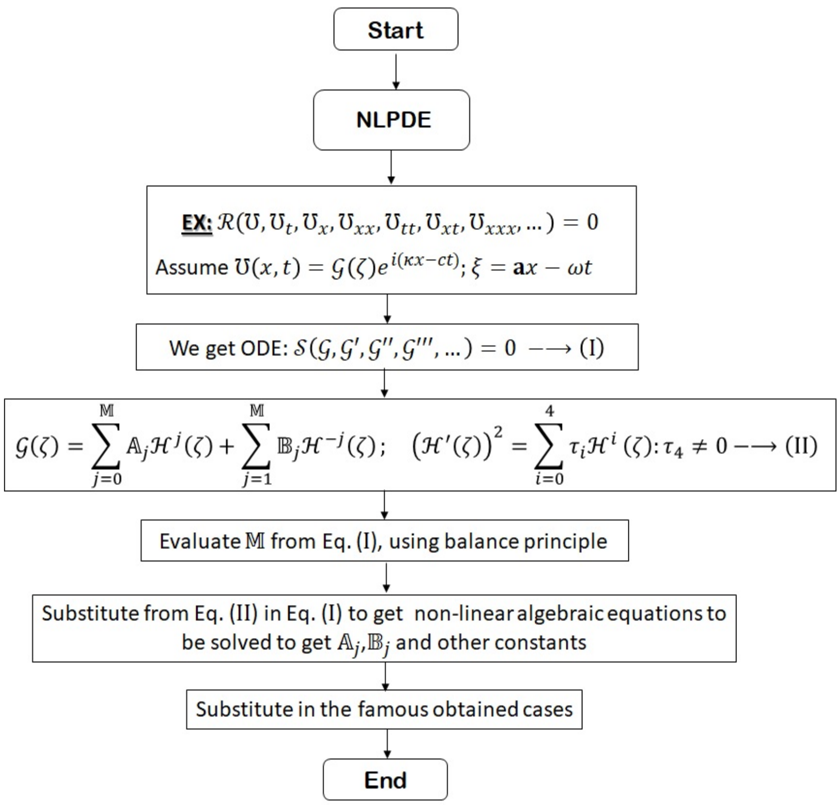

Figure 1. A flow chart of the IMETFA steps.

4. Results

In this part, the IMETFA is used to create all possible solutions for Eq. (2.1). To achieve this, we use the wave transformation shown below:

$$\mathcal{T}(t, x) = \phi(\zeta)e^{i\beta x}, \qquad \zeta = t - \varrho x, \tag{4.1}$$

where $\phi(\zeta)$ denotes the amplitude of the solution and $\beta,\ \varrho$ are constant parameters. When Eq. (4.1) is substituted into Eq. (2.1) and the real and imaginary sections are separated, the results are

$$-\beta\phi+\gamma\left(\phi^{(4)}+10\phi\left(\phi\phi''+(\phi')^2\right)+6\phi^5\right)+\phi^3+\frac{\phi''}{2}=0, \tag{4.2}$$

$$-\eta\phi^{(3)}-6\eta\phi^2\phi'-\varrho\phi'=0. \tag{4.3}$$

After integrating the imaginary part in Eq. (4.3) with respect to (ζ) once, while considering the integration's constant to be zero, yields:

$$\phi''=\frac{-2\eta\phi^3-\varrho\phi}{\eta}. \tag{4.4}$$

Using what is obtained in (4.4), Eq. (4.2) can be reduced to be as follows:

$$-(2\beta\eta+\varrho)\phi-28\gamma\eta\phi^5+2\gamma\eta\phi^{(4)}+20\gamma\eta\phi\,(\phi')^2-20\gamma\varrho\phi^3=0. \tag{4.5}$$

Therefore, using the BP described in Section 3.2 between $\phi^{(4)}$ and ϕ^5, we may establish the analytic solution form for Eq. (4.5), as follows:

$$\phi(\zeta)=\mathbb{A}_0+\mathbb{A}_1\mathcal{H}(\zeta)+\frac{\mathbb{B}_1}{\mathcal{H}(\zeta)}. \tag{4.6}$$

If the solution form in Eq. (4.6) is replaced with the limitation in Eq. (3.5), then there exists a polynomial in $\mathcal{H}(\zeta)$ by Eq. (4.5). When all terms with the same powers are added together and eventually equal to zero, an algebraic system of nonlinear equations is produced. Solving these equations with the Wolfram Mathematica program allows us to get the following results. The conditions are such that $\mathbb{A}_1$ and $\mathbb{B}_1$ cannot both be zero simultaneously.

Theorem 4.1. *Let $\tau_0=\tau_1=\tau_3=0$, then Eq.* (2.1) *has bright soliton, singular periodic, and rational solutions.*

Proof. Since $\tau_0=\tau_1=\tau_3=0$, then the solutions of the system of nonlinear algebraic equations are:

(1.a) $\mathbb{A}_0=\mathbb{B}_1=0,\ \mathbb{A}_1=\pm\sqrt{-\tau_4},\ \gamma=\dfrac{\eta(2\beta\eta+\varrho)}{2\varrho^2},\ \tau_2=-\dfrac{\varrho}{\eta}.$

(1.b) $\mathbb{A}_0=\mathbb{B}_1=0,\ \mathbb{A}_1=\sqrt{\dfrac{3\tau_4}{7}}\pm 2,\ \gamma=\dfrac{169\eta(2\beta\eta+\varrho)}{72\varrho^2},\ \tau_2=\dfrac{6\varrho}{13\eta}.$

By considering set (1.a), the solutions of Eq. (2.1) will be:

(1.a,1) If $\tau_2>0,\ \tau_4<0$, so:

$$\mathcal{T}_{1.a,1}(t,x)=\pm\sqrt{\tau_2}\ \mathrm{sech}\left[(t-\varrho x)\sqrt{\tau_2}\right]e^{i\beta x}, \tag{4.7}$$

that denotes a bright soliton solution. It is indicating its localized and non-dispersive nature. The soliton maintains its shape due to the balance between dispersion and nonlinearity, with its amplitude proportional to $\sqrt{\tau_2}$ and its propagation influenced by the parameters ϱ and β. The conditions $\tau_2>0$ and $\tau_4<0$ ensure the existence of this stable, localized wave structure.

By considering set (1.b), Eq. (2.1) will have the following solutions:

(1.b,1) If $\tau_2 < 0,\ \tau_4 > 0$, so:

$$\mathcal{T}_{1.b,1}(t,x) = \pm 2\sqrt{-\frac{3}{7}\tau_2}\sec\left[(t-\varrho x)\sqrt{-\tau_2}\right]e^{i\beta x}, \tag{4.8}$$

that describes a singular periodic solution. It indicates its singularity at specific points where the function diverges. The solution oscillates with periodic singularities due to the secant term, and its amplitude depends on τ_2. The conditions $\tau_2 < 0$ and $\tau_4 < 0$ ensure the existence of this solution, with its propagation influenced by the parameters ϱ and β.

(1.b,2) If $\tau_2 = 0,\ \tau_4 > 0$, so:

$$\mathcal{T}_{1.b,2}(t,x) = \frac{\mp 2\sqrt{\frac{3}{7}}}{t-\varrho x}e^{i\beta x}, \tag{4.9}$$

which denotes a rational solution. It is characterized by a fraction where the denominator introduces singularities, indicating localized structures that decay algebraically rather than exponentially, with propagation influenced by parameters ϱ and β.

□

Theorem 4.2. *Let $\tau_1 = \tau_3 = 0$, then Eq.* (2.1) *has Jacobi elliptic function, singular soliton, and singular periodic solutions.*

Proof. Since $\tau_1 = \tau_3 = 0$, then the solutions of the system of nonlinear algebraic equations are:

(2.a) $\mathbb{A}_0 = \mathbb{B}_1 = 0, \mathbb{A}_1 = \pm\sqrt{-\tau_4},\ \gamma = \dfrac{\eta(2\beta\eta+\varrho)}{2\left(2\eta^2\tau_0\tau_4+\varrho^2\right)},\ \tau_2 = -\dfrac{\varrho}{\eta}.$

(2.b) $\mathbb{A}_0 = \mathbb{B}_1 = 0, \mathbb{A}_1 = \pm 2\sqrt{\dfrac{3\tau_4}{7}},\ \gamma = \dfrac{1183\eta(2\beta\eta+\varrho)}{68952\eta^2\tau_0\tau_4+504\varrho^2},\ \tau_2 = \dfrac{6\varrho}{13\eta}.$

(2.c) $\mathbb{A}_0 = \mathbb{A}_1 = 0,\ \mathbb{B}_1 = \pm\sqrt{-\tau_0}, \gamma = \dfrac{\eta(2\beta\eta+\varrho)}{2\left(2\eta^2\tau_0\tau_4+\varrho^2\right)},\ \tau_2 = -\dfrac{\varrho}{\eta}.$

(2.d) $\mathbb{A}_0 = \mathbb{A}_1 = 0,\ \mathbb{B}_1 = \pm 2\sqrt{\dfrac{3\tau_0}{7}},\ \gamma = \dfrac{1183\eta(2\beta\eta+\varrho)}{68952\eta^2\tau_0\tau_4+504\varrho^2}, \tau_2 = \dfrac{6\varrho}{13\eta}.$

Considering the solutions' set (2.a), the solutions of Eq. (2.1) will be:

(2.a,1) If $\tau_2 > 0,\ \tau_4 < 0,\ \tau_0 = \frac{\aleph^2(1-\aleph^2)\tau_2^2}{(2\aleph^2-1)^2\tau_4}$, and $0 < \aleph \leq 1$, we get a Jacobi elliptic function (JEF) solution provided that $\aleph \neq \dfrac{1}{\sqrt{2}}$:

$$\mathcal{T}_{2.a,1}(t,x) = \aleph\sqrt{\frac{\tau_2}{2\aleph^2-1}}\operatorname{cn}(t-\varrho x)e^{i\beta x}. \tag{4.10}$$

This solution describes periodic wave structures that transition between solitonic and sinusoidal behaviors, depending on the modulus parameter. By setting $\aleph = 1$, we obtain a bright soliton solution:

$$\mathcal{T}_{2.a,2}(t,x) = \sqrt{\tau_2}\ \operatorname{sech}[t-\varrho x]e^{i\beta x}. \tag{4.11}$$

(2.a,2) If $\tau_2 > 0,\ \tau_4 < 0,\ \tau_0 = \frac{(1-\aleph^2)\tau_2^2}{(2-\aleph^2)^2\tau_4}$, and $0 < \aleph \leq 1$, a JEF solution is retrieved as:

$$\mathcal{T}_{2.a,3}(t,x) = \pm\aleph\sqrt{\frac{1}{2-\aleph^2}}\mathrm{dn}(t-\varrho x)e^{i\beta x}. \tag{4.12}$$

The "dn" function is known for maintaining a bell-shaped profile, making it useful for modeling soliton-like structures in nonlinear wave equations. By setting $\aleph = 1$, we obtain a bright soliton solution in the form:

$$\mathcal{T}_{2.a,4}(t,x) = \mathrm{sech}[t-\varrho x]e^{i\beta x}. \tag{4.13}$$

Through (2.b), the solutions are obtained as follows:

(2.b,1) If $\tau_2 < 0,\ \tau_4 > 0$, and $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, then:

$$\mathcal{T}_{2.b,1}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{14}}\tanh\left[(t-\varrho x)\sqrt{-\frac{\tau_2}{2}}\right]e^{i\beta x}, \tag{4.14}$$

and it is considered a dark soliton solution. The "tanh" function ensures a localized structure with a smooth transition between asymptotic states, making it suitable for modeling solitary waves in nonlinear systems.

(2.b,2) If $\tau_2 > 0,\ \tau_4 > 0$, and $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, then:

$$\mathcal{T}_{2.b,2}(t,x) = \pm 2\sqrt{\frac{3\tau_2}{14}}\tan\left[(t-\varrho x)\sqrt{\frac{\tau_2}{2}}\right]e^{i\beta x}, \tag{4.15}$$

that is a singular periodic solution.

(2.b,3) If $\tau_2 < 0,\ \tau_4 > 0,\ \tau_0 = \frac{\aleph^2\tau_2^2}{(1+\aleph^2)^2\tau_4}$, and $0 < \aleph \leq 1$, a JEF solution is raised as:

$$\mathcal{T}_{2.b,3}(t,x) = \pm 2\aleph\sqrt{-\frac{3\tau_2}{7\left(\aleph^2+1\right)}}\mathrm{sn}(t-\varrho x)e^{i\beta x}. \tag{4.16}$$

The presence of "sn" suggests that the wave exhibits oscillatory behavior, rather than a strictly localized soliton form. Especially, when setting $\aleph = 1$, we find a dark soliton solution:

$$\mathcal{T}_{2.b,4}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{14}}\tanh[t-\varrho x]e^{i\beta x}. \tag{4.17}$$

From (2.c), the solutions are obtained as follows:

(2.c,1) If $\tau_2 > 0,\ \tau_4 < 0,\ \tau_0 = \frac{\aleph^2(1-\aleph^2)\tau_2^2}{(2\aleph^2-1)^2\tau_4}$, and $0 < \aleph \leq 1$, a JEF solution is raised provided that $\aleph \neq \dfrac{1}{\sqrt{2}}$:

$$\mathcal{T}_{2.c,1}(t,x) = \pm\sqrt{\frac{\tau_2}{2\aleph^2-1}}\mathrm{cn}(t-\varrho x)e^{i\beta x}. \tag{4.18}$$

It describes a periodic and bounded wave structure. Unlike hyperbolic solitons, this type of solution is periodic and oscillatory in nature, meaning it does not decay to zero at infinity but instead repeats over a finite interval. By setting $\aleph = 1$, we obtain a bright soliton solution:

$$\mathcal{T}_{2.c,2}(t,x) = \pm\sqrt{\tau_2}\,\mathrm{sech}[t-\varrho x]e^{i\beta x}. \tag{4.19}$$

(2.c,2) If $\tau_2 > 0,\ \tau_4 < 0,\ \tau_0 = \frac{(1-\aleph^2)\tau_2^2}{(2-\aleph^2)^2\tau_4}$, and $0 < \aleph < 1$, a JEF solution is constructed as:

$$\mathcal{T}_{2.c,2}(t,x) = \pm\frac{\tau_2}{\aleph}\sqrt{\frac{1-\aleph^2}{2-\aleph^2}}\mathrm{nd}(t-\varrho x)e^{i\beta x}, \tag{4.20}$$

It exhibits periodic behavior and transitions between rational and solitonic waveforms depending on the modulus parameter.

Through (2.d), the solutions are obtained as follows:

(2.d,1) If $\tau_2 < 0,\ \tau_4 > 0$, and $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, then:

$$\mathcal{T}_{2.d,1}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{14}}\coth\left[(t-\varrho x)\sqrt{-\frac{\tau_2}{2}}\right]e^{i\beta x}, \tag{4.21}$$

and it is thought to be a singular soliton solution. This solution exhibits abrupt changes in amplitude with singularities, making it relevant for wave-breaking phenomena and discontinuous wave structures in nonlinear systems.

(2.d,2) If $\tau_2 > 0,\ \tau_4 > 0$, and $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, then:

$$\mathcal{T}_{2.d,2}(t,x) = \pm 2\sqrt{\frac{3\tau_2}{14}}\cot\left[(t-\varrho x)\sqrt{\frac{\tau_2}{2}}\right]e^{i\beta x}, \tag{4.22}$$

that represents a singular periodic solution.

(2.d,3) If $\tau_2 < 0,\ \tau_4 > 0,\ \tau_0 = \frac{\aleph^2\tau_2^2}{(1+\aleph^2)^2\tau_4}$, and $0 \le \aleph \le 1$, a JEF solution is established as:

$$\mathcal{T}_{2.d,3}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{7\left(\aleph^2+1\right)}}\mathrm{ns}(t-\varrho x)e^{i\beta x}. \tag{4.23}$$

It characterizes periodic wave structures with singularities, influenced by the parameter ϱ. Special cases, when $\aleph = 1$, we find a singular soliton solution:

$$\mathcal{T}_{2.d,4}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{14}}\coth[t-\varrho x]e^{i\beta x}. \tag{4.24}$$

In addition, setting $\aleph = 0$ will produce a singular periodic solution as:

$$\mathcal{T}_{2.d,5}(t,x) = \pm 2\sqrt{-\frac{3\tau_2}{7}}\csc[t-\varrho x]e^{i\beta x}. \tag{4.25}$$

□

Theorem 4.3. *Let* $\tau_2 = \tau_4 = 0,\ \tau_0 \neq 0,\ \tau_1 \neq 0$ *and* $\tau_3 > 0$*, then Eq.* (2.1) *has Weierstrass elliptic doubly periodic solutions.*

Proof. Since $\tau_2 = \tau_4 = 0,\ \tau_0 \neq 0,\ \tau_1 \neq 0$ and $\tau_3 > 0$, then, the solutions of the system of nonlinear algebraic equations are:

(3.a) $\mathbb{A}_0 = \pm\frac{\tau_1}{4\sqrt{-\tau_0}},\ \mathbb{A}_1 = 0,\ \mathbb{B}_1 = \pm\sqrt{-\tau_0},\ \beta = \frac{23\gamma\tau_1^4 - 24\tau_0\tau_1^2}{128\tau_0^2},\ \varrho = \frac{3\eta\tau_1^2}{8\tau_0},$ $\tau_3 = -\frac{\tau_1^3}{8\tau_0^2}.$

(3.b) $\mathbb{A}_0 = \pm\frac{\tau_1}{2}\sqrt{\frac{3}{7\tau_0}}$, $\mathbb{A}_1 = 0$, $\mathbb{B}_1 = \pm 2\sqrt{\frac{3\tau_0}{7}}$, $\beta = \frac{159\gamma\tau_1^4 + 91\tau_0\tau_1^2}{224\tau_0^2}$, $\varrho = -\frac{13\eta\tau_1^2}{16\tau_0}$, $\tau_3 = -\frac{\tau_1^3}{8\tau_0^2}$.

Using sets (3.a) and (3.b), one shall obtain Weierstrass elliptic doubly periodic type solutions respectively as:

$$\mathcal{T}_{3.a}(t,x) = \pm\left(\frac{\sqrt{-\tau_0}}{\wp\left(\frac{1}{2}(t-\varrho x)\sqrt{\tau_3}; -\frac{4\tau_1}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)} + \frac{\tau_1}{4\sqrt{-\tau_0}}\right) e^{i\beta x}, \quad \tau_0 < 0, \tag{4.26}$$

$$\mathcal{T}_{3.b}(t,x) = \pm\sqrt{\frac{3}{7}}\left(\frac{2\sqrt{\tau_0}}{\wp\left(\frac{1}{2}(t-\varrho x)\sqrt{\tau_3}; -\frac{4\tau_1}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)} + \frac{\tau_1}{2}\sqrt{\frac{1}{\tau_0}}\right) e^{i\beta x}, \quad \tau_0 > 0. \tag{4.27}$$

These solutions characterize complex periodic structures. The presence of the Weierstrass function indicates a solution that combines elliptic periodicity in both space and time. □

Theorem 4.4. *Let* $\tau_3 = \tau_4 = 0$*, then Eq.* (2.1) *has singular soliton, singular periodic, and exponential solutions.*

Proof. Since $\tau_3 = \tau_4 = 0$, then the solutions of the system of nonlinear algebraic equations are:

(4.a) $\mathbb{A}_0 = \mathbb{A}_1 = \tau_1 = 0$, $\mathbb{B}_1 = \pm 2\sqrt{\frac{3\tau_0}{7}}$, $\gamma = \frac{\beta}{\tau_2^2} + \frac{13}{12\tau_2}$, $\eta = \frac{6\varrho}{13\tau_2}$.

(4.b) $\mathbb{A}_0 = \pm\sqrt{\frac{3\tau_2}{7}}$, $\mathbb{A}_1 = 0$, $\mathbb{B}_1 = \pm 2\sqrt{\frac{3\tau_0}{7}}$, $\gamma = \frac{7(24\beta - 13\tau_2)}{348\tau_2^2}$, $\eta = -\frac{12\varrho}{13\tau_2}$, $\tau_1 = \pm 2\sqrt{\tau_0\tau_2}$.

From the set (4.a), we can construct the upcoming solutions:

(4.a,1) If $\tau_0 > 0$ and $\tau_2 < 0$, the following singular periodic solution shall be retrieved:

$$\mathcal{T}_{4.a,1}(t,x) = \pm 2\sqrt{-\frac{-3\tau_2}{7}}\csc\left[(t-\varrho x)\sqrt{-\tau_2}\right] e^{i\beta x}. \tag{4.28}$$

(4.a,2) If $\tau_0 > 0$ and $\tau_2 > 0$, the following singular soliton solution will be found:

$$\mathcal{T}_{4.a,2}(t,x) = \pm 2\sqrt{\frac{3\tau_2}{7}}\operatorname{csch}\left[(t-\varrho x)\sqrt{\tau_2}\right] e^{i\beta x}. \tag{4.29}$$

From the set (4.b), we may build the next exponential solution as follows:

$$\mathcal{T}_{4.b}(t,x) = \pm\sqrt{\frac{3}{7}}\left(\frac{2\sqrt{\tau_0}}{e^{\sqrt{\tau_2}(t-\varrho x)} - \frac{\tau_1}{2\tau_2}} + \sqrt{\tau_2}\right) e^{i\beta x}, \tag{4.30}$$

provided that $\tau_2 > 0$, $\tau_0 > 0$, and $e^{\sqrt{\tau_2}(t-\varrho x)} - \frac{\tau_1}{2\tau_2} \neq 0$. This solution represents a hyperbolic-type wave function incorporating an exponential term in the denominator, which suggests a localized or soliton-like behavior. □

Theorem 4.5. *If $\tau_0 = \tau_1 = 0$ and $\tau_4 > 0$, then we obtain:*

$$\mathbb{A}_0 = \pm\sqrt{\frac{3\tau_2}{7}},\ \mathbb{A}_1 = \pm\tau_3\sqrt{\frac{3}{7\tau_2}},\ \mathbb{B}_1 = 0,\ \gamma = \frac{7\,(24\beta - 13\tau_2)}{348\tau_2^2},\ \eta = -\frac{12\varrho}{13\tau_2},\ \tau_4 = \frac{\tau_3^2}{4\tau_2},$$

which gives the dark soliton solution as follows:

$$\mathcal{T}_5(t,x) = \pm\sqrt{\frac{3\tau_2}{7}}\left(\tanh\left[\frac{1}{2}(t-\varrho x)\sqrt{\tau_2}\right] + 2\right)e^{i\beta x}, \tag{4.31}$$

such that $\tau_2 > 0$.

5. Discussion

For Eq. (2.1), numerous categories of solutions were extracted when changing the parameter values in the model being examined. As so, some amazing results that have never been recorded or achieved have been obtained using this algorithm. Highlighting the physical characteristics of the retrieved solutions are drawings of numerous specific solutions obtained by the two-dimensional, three-dimensional, and contour plot simulations. In Figure 2, Eq. (4.7) is showed with its bright soliton solution of parameters $\varrho = -0.9,\ \eta = 0.5,\ \beta = -0.8,$ while $-8 \le x \le 8$. It illustrates a bright soliton solution, representing a localized wave that maintains its shape while traveling, which is crucial in optical fibers and fluid dynamics. The singular periodic solution of Eq. (4.8) is indicated in Figure 3 by setting $\varrho = -0.8,\ \eta = 0.5,\ \beta = 0.8,$ and $-15 \le x \le 15$. This figure displays a singular periodic solution, indicating wave patterns with repeating singularities, often linked to instability or resonance phenomena in nonlinear systems. A rational wave solution that was introduced in Eq. (4.9), is clarified in Figure 4 through parameters $\varrho = 0.8,\ \beta = 0.7,$ with $-15 \le x \le 15$. It showcases a rational wave solution, characterized by localized structures with algebraic decay, which are relevant in fluid turbulence and plasma waves. A dark soliton solution that was introduced in Eq. (4.14), is clarified in Figure 5 through parameters $\varrho = 0.9,\ \beta = 0.7,\ \eta = -0.5,$ with $-15 \le x \le 15$. This plot presents a dark soliton solution, depicting a dip in the wave profile commonly observed in Bose-Einstein condensates and nonlinear optics. Additionally, the graphs of the singular soliton solution represent Eq. (4.21) are sketched in Figure 6 as we choose the parameters as $\varrho = -0.9,\ \eta = 0.5,\ \beta = 0.6,$ and $-15 \le x \le 15$. It illustrates a singular soliton solution, highlighting wave structures with singularities that arise in certain physical settings, such as shallow water waves and nonlinear lattices. These diverse waveforms emphasize the richness and applicability of the obtained solutions in various physical contexts.

6. Conclusion

In today's world of communication, optical fiber connections are crucial. Our work aimed at identifying the precise traveling solutions and investigating their physical properties through mathematical analysis. The high-order NLSE, or Eq. (2.1), in an inhomogeneous optical fibre with additional higher-order dispersion and nonlinear components, is the subject of this investigation. We have effectively transformed

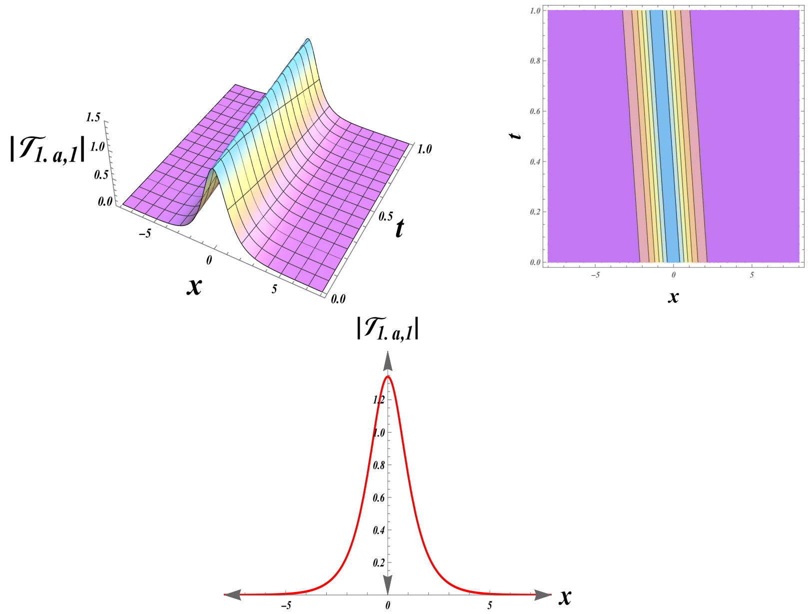

Figure 2. Bright soliton solution simulations of Eq. (4.7).

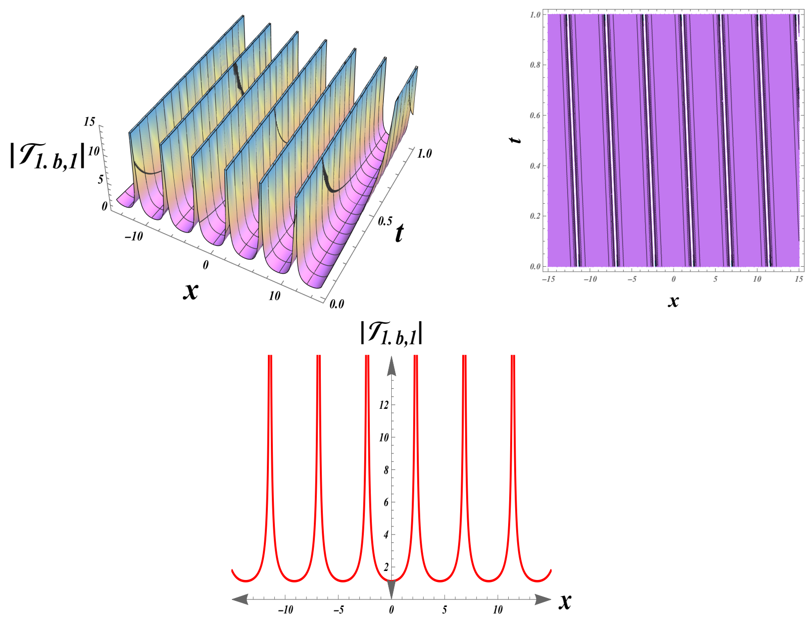

Figure 3. Singular periodic solution simulations of Eq. (4.8).

this problem, which we have dubbed the IMETFA—into an NLODE by using our study to illuminate its basic technique. The employed scheme leads to the discovery of a broad range of unique solutions, including exponential, rational, singular periodic, (bright, dark, singular) solitons, JEFs, and Weierstrass elliptic double periodic solutions. Mathematical physics can benefit from these solutions in multiple

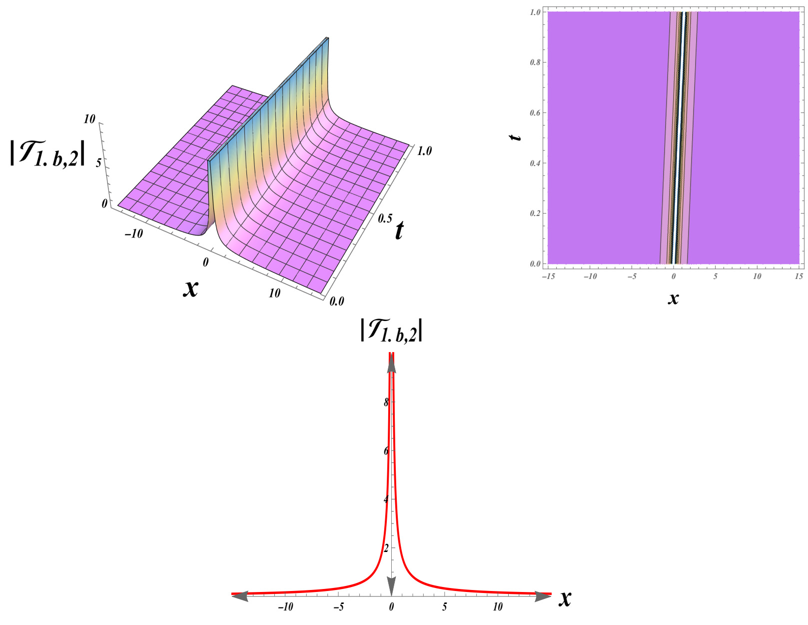

Figure 4. Rational solution simulations of Eq. (4.9).

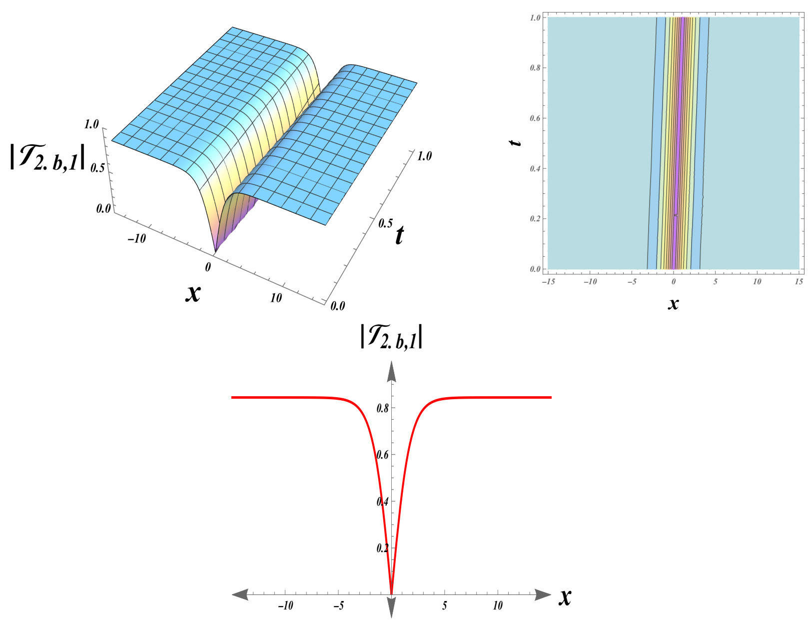

Figure 5. Dark soliton solution simulations of Eq. (4.14).

ways, including data carrier performance optimization and prediction in nonlinear optics. This finding represents a significant advancement in our comprehension of the complex and frequently erratic behaviour of the NLSEs. The book invites readers on an engrossing voyage into the world of non-linear waves, optical fibres, and dynamic systems, promising further revelations and insights.

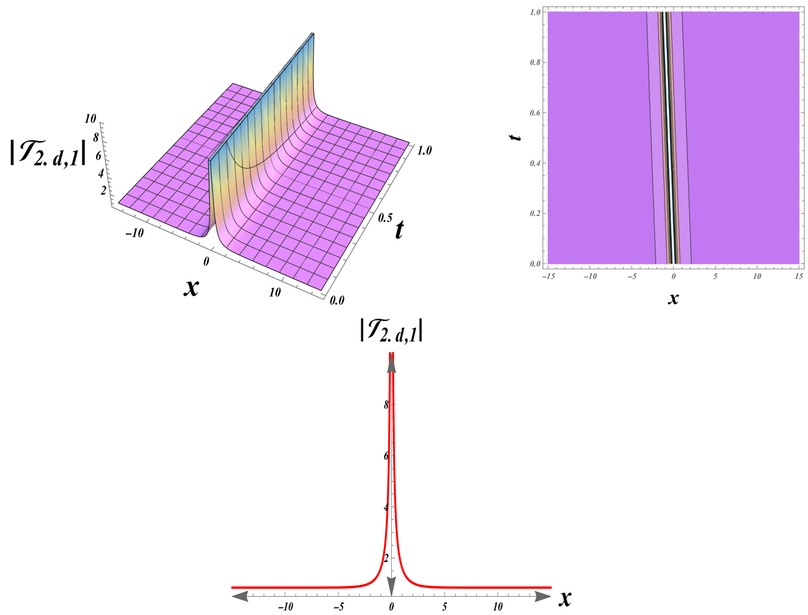

Figure 6. Singular soliton solution simulations of Eq. (4.21).

7. Future work

Future research could focus on analyzing the stability and long-term behavior of the obtained solitary wave solutions in stochastic systems. Examining the impact of random fluctuations and parametric variations on wave dynamics may uncover new phenomena. Integrating analytical and numerical stochastic methods could provide deeper insights into the complex interplay between noise and nonlinear wave structures.

Declarations

Ethics approval and consent to participate. Not applicable.

Consent for publication. Not applicable.

Availability of data and materials Not applicable.

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Authors' contributions. The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the manuscript.

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References

[1] K. K. Ahmed, H. M. Ahmed, N. M. Badra and W. B. Rabie, *Optical solitons retrieval for an extension of novel dual-mode of a dispersive non-linear*

Schrödinger equation, Optik, 2024, 307, 171835.

[2] K. K. Ahmed, N. M. Badra, H. M. Ahmed and W. B. Rabie, *Soliton solutions and other solutions for Kundu–Eckhaus equation with quintic nonlinearity and Raman effect using the improved modified extended tanh-function method*, Mathematics, 2022, 10(22), 4203.

[3] K. K. Ahmed, N. M. Badra, H. M. Ahmed and W. B. Rabie, *Soliton solutions of generalized Kundu-Eckhaus equation with an extra-dispersion via improved modified extended tanh-function technique*, Opt. Quantum Electron., 2023, 55(4), 299.

[4] K. K. Ahmed, N. M. Badra, H. M. Ahmed, W. B. Rabie, M. Mirzazadeh, M. Eslami and M. S. Hashemi, *Investigation of solitons in magneto-optic waveguides with Kudryashov's law nonlinear refractive index for coupled system of generalized nonlinear Schrödinger's equations using modified extended mapping method*, Nonlinear Anal-Model, 2024, 1–19.

[5] K. K. Ahmed, H. H. Hussein, H. M. Ahmed, W. B. Rabie and W. Alexan, *Analysis of the dynamical behaviors for the generalized Bogoyavlvensky–Konopelchenko equation and its analytical solutions occurring in mathematical physics*, Ain Shams Eng. J., 2024, 15(11), 103000.

[6] N. H. Ali, S. A. Mohammed and J. Manafian, *Study on the simplified MCH equation and the combined KdV–mKdV equations with solitary wave solutions*, Partial Differ. Equ. Appl. Math., 2024, 9, 100599.

[7] C. L. Bai and H. Zhao, *Generalized extended tanh-function method and its application*, Chaos, Solitons Fract., 2006, 27(4), 1026–1035.

[8] I. Bendahmane, H. Triki, A. Biswas, A. S. Alshomrani, Q. Zhou, S. P. Moshokoa and M. Belic, *Bright, dark and W-shaped solitons with extended nonlinear Schrödinger's equation for odd and even higher-order terms*, Superlattice. Microst., 2018, 114, 53–61.

[9] S. Boscolo and C. Finot, *Artificial neural networks for nonlinear pulse shaping in optical fibers*, Opt. Laser Techol., 2020, 131, 106439.

[10] B. Cao and H. Zhang, *Conversions and interactions of the nonlinear waves in a generalized higher-order nonlinear Schrödinger equation*, Optik, 2018, 158, 112–117.

[11] J. Chen, Z. Luan, Q. Zhou, A. K. Alzahrani, A. Biswas and W. Liu, *Periodic soliton interactions for higher-order nonlinear Schrödinger equation in optical fibers*, Nonlinear Dyn., 2020, 100, 2817–2821.

[12] S. Chettouh, H. Triki, A. El-Akrmi, Q. Zhou, S. P. Moshokoa, M. Z. Ullah, A. Biswas and M. Belic, *Dipole solitons in an extended nonlinear Schrödinger's equation with higher-order even and odd terms*, Optik, 2017, 145, 644–649.

[13] C. H. Feng, B. Tian and X. T. Gao, *Bilinear form, N solitons, breathers and periodic waves for a (3+1)-dimensional Korteweg-de Vries equation with the time-dependent coefficients in a fluid*, Qual. Theory Dyn. Syst., 2024, 23(1), 291.

[14] C. Finot and S. Boscolo, *Nonlinear shaping of light in optical fibers*, Adv. Nonlin. Photon., 2023, 381–412.

[15] X. Y. Gao, *In an ocean or a river: Bilinear auto-Bäcklund transformations and similarity reductions on an extended time-dependent (3+1)-dimensional shallow water wave equation*, China Ocean Eng., 2025, 39, 160–165.

[16] X. Y. Gao, *Hetero-Bäcklund transformation, bilinear forms and multi-solitons for a (2+1)-dimensional generalized modified dispersive water-wave system for the shallow water*, Chin. J. Phys., 2024, 92, 1233–1239.

[17] X. Y. Gao, *Symbolic computation on a (2+1)-dimensional generalized nonlinear evolution system in fluid dynamics, plasma physics, nonlinear optics and quantum mechanics*, Qual. Theory Dyn. Syst., 2024, 23(5), 202.

[18] X. Y. Gao, *In plasma physics and fluid dynamics: Symbolic computation on a (2+1)-dimensional variable-coefficient Sawada-Kotera system*, Appl. Math. Lett., 2025, 159, 109262.

[19] X. Y. Gao, *Oceanic shallow-water investigations on a generalized Whitham–Broer–Kaup–Boussinesq–Kupershmidt system*, Phys. Fluids, 2023, 35(12), 127106.

[20] X. Y. Gao, *Two-layer-liquid and lattice considerations through a (3+1)-dimensional generalized Yu-Toda-Sasa-Fukuyama system*, Appl. Math. Lett., 2024, 152, 109018.

[21] X. T. Gao and B. Tian, *Similarity reductions on a (2+1)-dimensional variable-coefficient modified Kadomtsev-Petviashvili system describing certain electromagnetic waves in a thin film*, Int. J. Theor. Phys., 2024, 63(4), 99.

[22] W. Y. Guan and B. Q. Li, *New observation on the breather for a generalized nonlinear Schrödinger system with two higher-order dispersion operators in inhomogeneous optical fiber*, Optik, 2019, 181, 853–861.

[23] N. Gupta, A. K. Alex, R. Partap and R. Johari, *Self action effects of q-Gaussian laser beam in preformed parabolic plasma channels: Effect of nonlinear absorption*, J. Opt., 2024, 1–14.

[24] K. Hammani, B. Kibler, J. Fatome, S. Boscolo, G. Genty, J. M. Dudley, G. Millot and C. Finot, *Nonlinear spectral shaping and optical rogue events in fiber-based systems*, Opt. Fiber. Technol., 2012, 18, 248–256.

[25] H. H. Hussein, H. M. Ahmed, W. B. Rabie, K. K. Ahmed, M. S. Hashemi and M. Bayram, *Multiple soliton solutions and other travelling wave solutions to new structured (2+1)-dimensional integro-partial differential equation using efficient technique*, Phys. Scr., 2024, 99(10), 105270.

[26] Y. Jiang, B. Tian, W.-J. Liu, K. Sun, M. Li and P. Wang, *Soliton interactions and complexes for coupled nonlinear Schrödinger equations*, Phys. Rev. E, 2012, 85(3), 036605.

[27] E. Kengne, *Engineering of chirp localized waves in optical media with positive group velocity dispersion*, Chaos Solitons Fract., 2024, 187, 115354.

[28] A. S. Khalifa, H. M. Ahmed, N. M. Badra and W. B. Rabie, *Exploring solitons in optical twin-core couplers with Kerr law of nonlinear refractive index using the modified extended direct algebraic method*, Opt. Quantum Electron., 2024, 56(6), 1060.

[29] A. S. Khalifa, N. M. Badra, H. M. Ahmed and W. B. Rabie, *Retrieval of optical solitons in fiber Bragg gratings for high-order coupled system with arbitrary refractive index*, Optik, 2023, 287, 171116.

[30] Y. Kuang and L. Tian, *Higher-order soliton solutions for the derivative nonlinear Schrödinger equation via improved Riemann–Hilbert method*, J. Nonlinear Math. Phys., 2024, 31, 58.

[31] Z. Z. Lan, *Multi-soliton solutions, breather-like and bound-state solitons for complex modified Korteweg–de Vries equation in optical fibers*, Chin. Phys. B, 2024, 33(6), 060201.

[32] Z. Z. Lan, *N−soliton solutions, Bäcklund transformation and Lax Pair for a generalized variable-coefficient cylindrical Kadomtsev–Petviashvili equation*, Appl. Math. Lett., 2024, 158, 109239.

[33] Z. Z. Lan, *Multiple soliton asymptotics in a spin-1 Bose–Einstein condensate*, Chin. Phys. Lett., 2024, 41(9), 090501.

[34] Z. Z. Lan, *Semirational rogue waves of the three coupled higher-order nonlinear Schrödinger equations*, Appl. Math. Lett., 2024, 147, 108845.

[35] Z. Z. Lan, *Bound-state solitons in three-wave resonant interactions*, Nonlinear Dyn., 2024, 112(22), 20173–20181.

[36] J. Li, C. Wang and P. Wang, *Narrow-bandwidth picosecond dissipative soliton resonance in an all-anomalous-dispersion Er/Yb co-doped fiber laser*, Opt. Fiber. Technol., 2024, 82, 103637.

[37] F. Y. Liu, Y. T. Gao, X. Yu and C. C. Ding, *Wronslian, Gramian, Pfaffian and periodic-wave solutions for a (3+1)-dimensional generalized nonlinear evolution equation arising in the shallow water waves*, Nonlinear Dynam., 2022, 108, 1599–1616.

[38] S. S. Mahmood and M. A. S. Murad, *Soliton solutions to time-fractional nonlinear Schrödinger equation with cubic-quintic-septimal in weakly nonlocal media*, Phys. Lett. A., 2024, 130183.

[39] M. A. S. Murad, H. F. Ismael, T. A. Sulaiman and H. Bulut, *Analysis of optical solutions of higher-order nonlinear Schrödinger equation by the new Kudryashov and Bernoulli's equation approaches*, Opt. Quantum Electron., 2024, 56(1), 76.

[40] W. B. Rabie, K. K. Ahmed, N. M. Badra, H. M. Ahmed, M. Mirzazadeh and M. Eslami, *New solitons and other exact wave solutions for coupled system of perturbed highly dispersive CGLE in birefringent fibers with polynomial nonlinearity law*, Opt. Quantum Electron., 2024, 56(5), 1–22.

[41] H. W. Shan, B. Tian, C. D. Cheng, X. T. Gao, Y. Q. Chen and H. D. Liu, *N−soliton and other analytic solutions for a (3+1)-dimensional Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation with the time-dependent coefficients for the shallow water waves*, Qual. Theory Dyn. Syst., 2024, 23(Suppl 1), 267.

[42] E. Tego, F. Matera and D. D. Buono, *Experimental investigation on transmission control protocol throughput behavior in optical fiber access networks*, Fiber Integr. Opt., 2016, 35, 72–85.

[43] H. Triki, A. Biswas, D. Milovi'c and M. Beli'c, *Chirped optical solitons in birefringent fibers with parabolic law nonlinearity and four-wave mixing*, Acta Phys. Pol. A, 2016, 130(3), 718–726.

[44] C. C. Wei, B. Tian, D. Y. Yang and S. H. Liu, *Jacobian-elliptic-function and rogue-periodic-wave solutions of a high-order nonlinear Schrödinger equation in an inhomogeneous optical fiber*, Chin. J. Phys., 2023, 81, 354–361.

[45] X. H. Wu, Y. T. Gao and X. Yu, *Dark-soliton asymptotics for a repulsive nonlinear system in a baroclinic flow*, Phys. Fluids, 2024, 36(5), 056615.

[46] X. H. Wu, Y. T. Gao, X. Yu, C. C. Ding and L. Q. Li, *Modified generalized Darboux transformation and solitons for a Lakshmanan-Porsezian-Daniel equation*, Chaos Solitons Fract., 2022, 162, 112399.

[47] H. M. Yin, Q. Pan and K. W. Chow, *The Fermi–Pasta–Ulam–Tsingou recurrence for discrete systems: Cascading mechanism and machine learning for the Ablowitz–Ladik equation*, Commun. Nonlinear Sci. Numer. Simul., 2022, 114, 106664.

[48] X. H. Zhao, *Multi-solitons and integrability for a (2+1)-dimensional variable coefficients Date–Jimbo–Kashiwara–Miwa equation*, Appl. Math. Lett., 2024, 149, 108895.

[49] X. H. Zhao and S. X. Li, *Dark soliton solutions for a variable coefficient higher-order Schrödinger equation in the dispersion decreasing fibers*, Appl. Math. Lett., 2022, 132, 108159.

[50] Q. Zhou, Q. Zhu and A. Biswas, *Optical solitons in birefringent fibers with parabolic law nonlinearity*, Opt. Appl., 2014, 44(3), 399–409.

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