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COMPARISON OF RIEMANN SOLUTIONS TO THE EULER EQUATIONS FOR MODIFIED AND GENERALIZED CHAPLYGIN GAS DYNAMICS*

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Abstract In this article, utilizing the method of characteristic analysis, we construct the Riemann solutions of the Euler equations for the modified and the generalized Chaplygin gas, which contains rarefaction waves, shock waves, contact discontinuities and $\delta-$ shock waves. For the $\delta-$ shock waves, we propose generalized Rankine-Hugoniot relation and entropy conditon. Moreover, by studing the limiting behaviour, we find that the Riemann solutions of modified Chaplygin gas is the same as generalized Chaplygin gas including the $\delta-$shock waves. Moreover, we give some numerical simulations to verify the theoretical analysis.

Keywords Characteristic analysis, $\delta-$shock waves, entropy conditon, generalized Rankine-Hugoniot conditions.

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1. Introduction

The one-dimensional Euler equations read

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ \left(\frac{1}{2}\rho u^2 + H\right)_t + \left(\left(\frac{1}{2}\rho u^2 + H + p\right)u\right)_x = 0, \end{cases} \tag{1.1}$$

where u, ρ, H and p denote the velocity, the density, the internal energy and the pressure, respectively. In this paper, we consider the Riemann problem of (1.1) for

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Chaplygin gas with initial data

$$(\rho, u, H)(x,t)|_{t=0} = \begin{cases} (\rho_l, u_l, H_l), & x < 0, \\ (\rho_r, u_r, H_r), & x > 0, \end{cases} \tag{1.2}$$

where $\rho_{l,r}, u_{l,r}, H_{l,r}$ are all given constants. Chaplygin [4] and Tsien [27] introduce the Chaplygin gas as a mathematical approximation for determining the lifting force on an airplane wing. The reader is referred to [1, 2, 8, 22] for a more detailed exploration of the physical implications of Chaplygin gas.

The state equations for Chaplygin gas can be expressed as

$$p = A\rho - \frac{Bf(s) + C}{\rho^\alpha}, \tag{1.3}$$

where s represents the specific entropy with $f(s) > 0$, $f'(s) > 0$, $A \geq 0$, $B \geq 0$, $C \geq 0$ and $0 < \alpha \leq 1$. For a different gas state, the Riemann solutions display many fascinating properties, including the accumulation of mass. Next, we discuss several cases of the Riemann problem of the compressible Euler equations for the Chaplygin gas with different gas states.

For the Riemann problem of the compressible Euler equations for the Chaplygin gas with a state equation $p = -\frac{1}{\rho}$, Pang obtained a kind of Riemann solution which involves a delta shock wave [17]. Wang et al. constructively obtained a global existence of generalized solutions and obtained the stability of the generalized solutions by making use of the perturbation of the initial data [29]. Qu et al. proved that the Riemann solutions are stable under the local small perturbations of the Riemann initial data even when the initial perturbed density depending on the parameter but with no mass concentration limit [20]. Cheng et al. studied the Riemann problem for relative Chaplygin Euler equations [6].

Consider another case involving the Chaplygin gas with a state equation $p = -\frac{A}{\rho^\alpha}$, Pang obtained a kind of solution in which a delta shock wave is formed, featuring Dirac delta distributions in the variables of density and internal energy [16]. Wang constructed the Riemann solutions, analyzed the interaction of the delta-shock wave with the elementary waves and proved the stability of the Riemann solutions [28]. Ding et al. [7] investigated the emergence of delta waves and vacuum states in the limit of vanishing pressure within the Riemann solutions to the equations of non-isentropic generalized Chaplygin gas. Sheng et al. [23] proved the limit solutions tend to the two kinds of Riemann solutions to the transport equations in zero-pressure flow, which include a Delta wave formed by a weighted δ-measure and a vacuum state.

For the Riemann problem of the compressible Euler equations for the Chaplygin gas with state equation $p = A\rho - \frac{B}{\rho}$, Yang et al. investigated the phenomena of concentration and cavitation, as well as the emergence of δ-shocks and vacuum states in Riemann solutions, as the pressure approaches zero with a double parameter [30].

For the Riemann problem of the compressible Euler equations for the Chaplygin gas with state equation $p = -\frac{f(s)}{\rho}$, Zhu et al. constructed the Riemann solutions where the delta shock wave with Dirac delta function only in the density appeared in solutions [32].

For more research on the Riemann problems of conservation laws for the Chaplygin gas, we refer the readers to [3, 9, 12, 15, 19, 21, 26] and the references cited therein.

Recently, the relationship between different solutions of the compressible Euler equations for the Chaplygin gas with different state equations has attracted intensive attention. Song et al. [25] and Jiang et al. [10] examined the asymptotic behavior of Riemann solutions for the non-isentropic Euler equations involving modified Chaplygin gases with state equations $p = A\rho - \frac{1}{\rho}$ and $p = A\rho - \frac{f(s)}{\rho}$, respectively. Through their analysis of the limiting behavior, they discovered that the Riemann solutions for the modified Chaplygin gases converge to those of the pure Chaplygin gas, including the presence of δ-shock waves. In [11], Jiang et al. made several comparisons of Riemann solutions of compressible Euler equations in the modified Chaplygin gas state $p = A\rho - \frac{1}{\rho}$ and classical Chaplygin gas state $p = -\frac{1}{\rho}$. Chaturvediet et al. studied the solution of the Riemann problem for modified Chaplygin gas equations in the presence of constant external force, which influenced the Riemann solution for the modified Chaplygin gas equation [5]. In [13], Kumar et al. solved the generalized Riemann problem through a characteristic shock(s). For several classes of non-constant and smooth initial data, the solution to the generalized Riemann problem is presented. In [24], Singh et al. obtained the solution to the generalized Riemann problem (GRP) for the one-dimensional Euler's equation for Chaplygin gas equations with Coulomb-like friction term by using the differential constraint method. For more research on the limiting behavior of Riemann problems to the Euler equations of compressible fluid flow for the modified Chaplygin gas, we refer the readers to [14, 18, 31] and the references cited therein.

In this paper, we continute to study the limiting behavior of Riemann solutions to the Euler equations for modified Chaplygin gas, characterized by the state equation

$$p = A\rho - \frac{1}{\rho^{\alpha}}. \tag{1.4}$$

Letting $A \to 0$, we get the corresponding state equation

$$p = -\frac{1}{\rho^{\alpha}}. \tag{1.5}$$

In the following, we investigate the Riemann problem for the system given by (1.1), (1.2) and (1.4), as well as the Riemann problem for the system given by (1.1), (1.2) and (1.5). The central theorem of this paper is stated below:

Theorem 1.1. *(i)Suppose that the initial data satisfies* $u_r + \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$*. Then the solution of the Riemann problem of (1.1), (1.2) and (1.4) converges to the delta shock wave solution of the Riemann problem of (1.1), (1.2) and (1.5) as* A *tends to zero.*
(ii)Suppose that the initial data satisfies $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$*. Then the solution of the Riemann problem of (1.1), (1.2) and (1.4) converges to the corresponding Riemann solutions of the problem of (1.1), (1.2) and (1.5) as* A *tends to zero.*

The organization of this paper is as follows. In Section 2, we present some preliminary knowledge for the Riemann problem of (1.1), (1.2) and (1.5). In Section 3, we construct the Riemann solutions to the Riemann problem of (1.1), (1.2) and (1.4) by characteristic analysis. In Section 4, we give the proof of the main theorem case by case. In Section 5, we give the numerical simulation to verify the theory proposed in Section 4.

2. Riemann problem of (1.1), (1.2) and (1.5)

2.1. Solutions involving the classical waves

The Riemann solution for the problem defined by (1.1), (1.2) and (1.5) has been extensively examined in [16]. Here we show the relevant results.

Suppose that these solutions satisfy the thermodynamical constraint

$$Tds - de = pd\frac{1}{\rho}. \tag{2.1}$$

Then, the physically meaningful region as specified by (2.1) is defined as follows:

$$\Omega := \{(\rho, u, H) : \rho > 0, u \in R, H \geq \frac{1}{(\alpha+1)\rho^{\alpha}}\}.$$

For systems (1.1) associated with (1.5), there exist three eigenvalues

$$\lambda_1 = u - \sqrt{\frac{\alpha}{\rho^{\alpha+1}}}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{\frac{\alpha}{\rho^{\alpha+1}}}, \tag{2.2}$$

with the corresponding right eigenvectors

$$\vec{r}_1 = (1, -\sqrt{\frac{\alpha}{\rho^{\alpha+3}}}, \frac{H}{\rho} - \frac{1}{\rho^{\alpha+1}})^T, \quad \vec{r}_2 = (0,0,1)^T, \quad \vec{r}_3 = (1, \sqrt{\frac{\alpha}{\rho^{\alpha+3}}}, \frac{H}{\rho} - \frac{1}{\rho^{\alpha+1}})^T. \tag{2.3}$$

By (2.2) and (2.3), we deduce that

$$\nabla\lambda_2 \cdot \vec{r}_2 = 0 \;\; and \;\; \nabla\lambda_i \cdot \vec{r}_i \neq 0 \;\; for \;\; i = 1, 3,$$

which imply that λ_2 is linearly degenerate, while λ_1 and λ_3 are genuinely nonlinear. Consequently, the elementary waves that can arise from the Riemann problem for this system comprise a set of all rarefaction waves, a contact discontinuity, and shock waves.

The rarefaction wave curve R in the space (ρ, u, H), passing through (ρ_l, u_l, H_l), can be expressed as:

$$R_1(\rho_l, u_l, H_l) : \begin{cases} \xi_1 = u_l - \sqrt{\dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u - \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho^{-\frac{\alpha+1}{2}} = u_l - \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_l^{-\frac{\alpha+1}{2}}, \\ \dfrac{H}{\rho} - \dfrac{1}{(\alpha+1)\rho^{\alpha+1}} = \dfrac{H_l}{\rho_l} - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \qquad \rho < \rho_l, \tag{2.4}$$

and

$$R_3(\rho_l, u_l, H_l) : \begin{cases} \xi_3 = u_l + \sqrt{\dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u + \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho^{-\frac{\alpha+1}{2}} = u_l + \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_l^{-\frac{\alpha+1}{2}}, \\ \dfrac{H}{\rho} - \dfrac{1}{(\alpha+1)\rho^{\alpha+1}} = \dfrac{H_l}{\rho_l} - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \qquad \rho > \rho_l. \tag{2.5}$$

Moreover, the shock wave curve can be defined as

$$S_1(\rho_l,u_l,H_l):\begin{cases}\sigma_1=u_l-\sqrt{\dfrac{[p]\rho}{[\rho]\rho_l}},\\ u-u_l=-\sqrt{\Big(\dfrac{1}{\rho_l^\alpha}-\dfrac{1}{\rho^\alpha}\Big)\Big(\dfrac{1}{\rho_l}-\dfrac{1}{\rho}\Big)}, & \rho>\rho_l,\\ \dfrac{H}{\rho}-\dfrac{\frac{1}{\rho^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho}=\dfrac{H_l}{\rho_l}-\dfrac{\frac{1}{\rho^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho_l},\end{cases}\tag{2.6}$$

and

$$S_3(\rho_l,u_l,H_l):\begin{cases}\sigma_3=u_l+\sqrt{\dfrac{[p]\rho}{[\rho]\rho_l}},\\ u-u_l=-\sqrt{\Big(\dfrac{1}{\rho_l^\alpha}-\dfrac{1}{\rho^\alpha}\Big)\Big(\dfrac{1}{\rho_l}-\dfrac{1}{\rho}\Big)}, & \rho<\rho_l.\\ \dfrac{H}{\rho}-\dfrac{\frac{1}{\rho^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho}=\dfrac{H_l}{\rho_l}-\dfrac{\frac{1}{\rho^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho_l},\end{cases}\tag{2.7}$$

The contact discontinuity curve is given by

$$J(\rho_l,u_l,H_l):\begin{cases}u=u_l,\rho=\rho_l,\\ H\neq H_l.\end{cases}\tag{2.8}$$

By direct calculation, we have

$$\begin{aligned}&\frac{du}{d\rho}|_{R_1}=-\sqrt{\frac{\alpha}{\rho^{\alpha+3}}}<0,\quad \frac{d^2u}{d\rho^2}|_{R_1}=\frac{\alpha+3}{2}\sqrt{\alpha}\rho^{-\frac{\alpha+5}{2}}>0,\\ &\frac{du}{d\rho}|_{R_3}=\sqrt{\frac{\alpha}{\rho^{\alpha+3}}}>0,\quad \frac{d^2u}{d\rho^2}|_{R_3}=-\frac{\alpha+3}{2}\sqrt{\alpha}\rho^{-\frac{\alpha+5}{2}}<0,\\ &\frac{du}{d\rho}|_{S_1}=-\frac{\alpha\rho^{-(\alpha+1)}\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)+\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\rho^{-2}}{2\sqrt{\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)}}<0,\end{aligned}$$

and

$$\begin{aligned}&\frac{d^2u}{d\rho^2}|_{S_1}\\ =&-\frac{1}{2}\Big\{-\frac{1}{2}\Big[\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)\Big]^{-\frac{3}{2}}\Big[\alpha\rho^{-(\alpha+1)}\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)+\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\rho^{-2}\Big]^2\\ &+\Big[\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)\Big]^{-\frac{1}{2}}\\ &\times\Big[\alpha(-\alpha-1)\rho^{-\alpha-2}\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)+2\alpha\rho^{-\alpha-3}-2\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\rho^{-3}\Big]\Big\}\\ =&-\frac{1}{2}\Big[\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)\Big]^{-\frac{1}{2}}\Big\{-\frac{1}{2}\alpha^2\rho^{-2\alpha-2}\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)^{-1}\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)\\ &-\frac{1}{2}\rho^{-4}\Big(\frac{1}{\rho_l^\alpha}-\frac{1}{\rho^\alpha}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)^{-1}+(\alpha^2+2\alpha+2)\rho^{-\alpha-3}\end{aligned}$$

$$
\begin{aligned}
&-\alpha(\alpha+1)\rho^{-\alpha-2}\rho_l^{-1}-2\rho_l^{-\alpha}\rho^{-3}\Big\}\\
>&-\frac{1}{2}\Big[\Big(\frac{1}{\rho_l^{\alpha}}-\frac{1}{\rho^{\alpha}}\Big)\Big(\frac{1}{\rho_l}-\frac{1}{\rho}\Big)\Big]^{-\frac{1}{2}}\Big[-\alpha\rho^{-\alpha-3}+(\alpha^2+2\alpha+2)\rho^{-\alpha-3}\\
&-\alpha(\alpha+1)\rho^{-\alpha-3}-2\rho^{-\alpha-3}\Big]\\
=&0.
\end{aligned}
$$

Similarly, we can obtain

$$\frac{du}{d\rho}|_{S_3}>0,\quad \frac{d^2u}{d\rho^2}|_{S_3}<0.$$

Since ρ and u are invariant along the contact discontinuities, we consider the projections on the $u-\rho$ plane. We put R_1, R_3, S_1, S_3 and $L: u+\rho^{-\frac{\alpha+1}{2}}=u_l-\rho_l^{-\frac{\alpha+1}{2}}$ together in the phase plane to obtain a picture as in Figure 1, in which R_1, R_3, S_1, S_3 and L divide the phase plane into five parts I, II, III, IV, and V. (i)

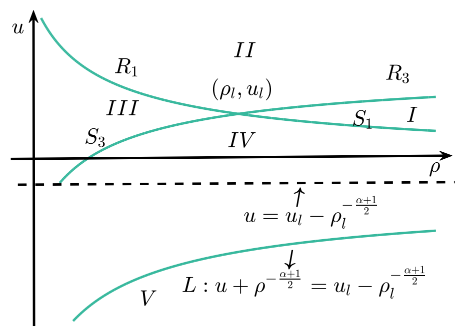

Figure 1

If (ρ_r,u_r,H_r) belongs to areas I, II, III, and IV, the Riemann solutions to the problem defined by (1.1), (1.2) and (1.5) will consist of rarefaction waves, contact discontinuities and shock waves.

(ii) When (ρ_r,u_r,H_r) is located in area V, i.e., $u+\rho^{-\frac{\alpha+1}{2}}\leqslant u_l-\rho_l^{-\frac{\alpha+1}{2}}$, there exists no piecewise smooth and bounded solution to the problem defined by (1.1), (1.2) and (1.5). In this case, we can get a $\delta-$shock wave solution. In order to define the measure solutions, the two-demensional weighted delta function $\omega(s)\delta_L$ supported on a smooth curve L parameterized as $t=t(s), x=x(s)(c\leq s\leq d)$ is defined as

$$\langle\omega(s)\delta_L,\phi(t(s),x(s))\rangle=\int_c^d\omega(s)\phi(t(s),x(s))ds \tag{2.9}$$

for all the test function $\phi\ \epsilon\ C_0^{\infty}([0,\infty)\times(-\infty,+\infty))$.

With this definition, we can get a δ-shock wave solution with the discontinuity $x=x(t)$ in the form:

$$(\rho(t,x),u(t,x),H(t,x))=\begin{cases}(\rho_l,u_l,H_l), x<x(t),\\(\omega(t)\delta(x-x(t)),u_\delta(t),h(t)\delta(x-x(t))), x=x(t),\\(\rho_r,u_r,H_r), x>x(t).\end{cases} \tag{2.10}$$

Moreover, across the discontinuity, the following generalized R-H conditions hold:

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{d\omega(t)}{dt} = u_\delta(t)[\rho] - [\rho u], \\ \dfrac{d\omega(t)u_\delta(t)}{dt} = u_\delta(t)[\rho u] - [\rho u^2 + p], \\ \dfrac{d\Big(\frac{1}{2}\omega(t)u_\delta^2(t) + h(t)\Big)}{dt} = u_\delta(t)\Big[\dfrac{1}{2}\rho u^2 + H\Big] - \Big[(\dfrac{1}{2}\rho u^2 + H + p)u\Big], \end{cases} \tag{2.11}$$

where $x(t)$ and $u_\delta(t)$ denote the location and the velocity of the delta shock wave and $h(t)$ and $\omega(t)$ are the weights of the $\delta-$shock wave on the state variables H and ρ, respectively. Meanwhile, $u_\delta(t)$ satisfies the following condition

$$\lambda_1(\rho_r, u_r) < \lambda_2(\rho_r, u_r) < \lambda_3(\rho_r, u_r) \le u_\delta \le \lambda_1(\rho_l, u_l) < \lambda_2(\rho_l, u_l) < \lambda_3(\rho_l, u_l). \tag{2.12}$$

It was shown in [17] that the delta measure solutions (ρ, u, H) constructed above satisfy

$$\begin{aligned} &\langle \rho, \phi_t \rangle + \langle \rho u, \phi_x \rangle = 0, \\ &\langle \rho u, \phi_t \rangle + \langle \rho u^2 + p, \phi_x \rangle = 0, \\ &\langle \rho u^2/2 + H, \phi_t \rangle + \langle (\rho u^2/2 + H + p)u, \phi_x \rangle = 0, \end{aligned} \tag{2.13}$$

for any $\phi \epsilon\ C_0^\infty((0,\infty) \times (-\infty, +\infty))$, where

$$\langle \rho, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho\phi dxdt + \langle \omega(t)\delta_L, \phi \rangle, \tag{2.14}$$

$$\langle \rho u, \phi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho u\phi dxdt + \langle \omega(t)u_\delta(t)\delta_L, \phi \rangle, \tag{2.15}$$

and H has the similar integral identities as above.

Next we proceed to solve the problem for (2.11) with the initial conditions

$$x(0) = 0, \quad w(0) = 0, \quad H(0) = 0 \ \text{ and } \ u_\delta(0) = u_0. \tag{2.16}$$

Using (2.11) and (2.16), it yields that

$$\begin{cases} \omega(t) = x(t)[\rho] - [\rho u]t, \\ \omega(t)u_\delta(t) = x(t)[\rho u] - [\rho u^2 + p]t, \\ \dfrac{1}{2}\omega(t)u_\delta^2(t) + h(t) = x(t)\Big[\dfrac{1}{2}\rho u^2 + H\Big] - \Big[\Big(\dfrac{1}{2}\rho u^2 + H + p\Big)u\Big]t. \end{cases} \tag{2.17}$$

Utilizing the first and second equations of (2.17), we have

$$[\rho]x^2(t) - 2[\rho u]tx(t) + [\rho u^2 + p]t^2 = 0. \tag{2.18}$$

If $[\rho] = 0$, the equation (2.18) has a unique solution

$$x(t) = \frac{[\rho u^2 + p]}{2[\rho u]}t. \tag{2.19}$$

By (2.11) and (2.17), we obtain that

$$
\begin{aligned}
&u_\delta = \frac{u_r + u_l}{2}, \qquad (2.20)\\
&w(t) = \rho_l(u_l - u_r)t,\\
&h(t) = -\frac{1}{2}\omega(t)u_\delta^2 + x(t)\Big[\frac{1}{2}\rho u^2 + H\Big] - \Big[\Big(\frac{1}{2}\rho u^2 + H + p\Big)u\Big]t.
\end{aligned}
$$

Next, we show that $u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}} < u_\delta(t) < u_l - \sqrt{\frac{\alpha}{\rho_l^{\alpha+1}}}$. In fact, in virtue of $u_r + \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$, it deduces that

$$
\begin{aligned}
u_\delta(t) - \Big(u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big) &= \frac{u_r + u_l}{2} - u_r - \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}} \qquad (2.21)\\
&\geq \frac{1}{2}u_r + \frac{1}{2}\rho_r^{-\frac{\alpha+1}{2}} + \frac{1}{2}\rho_l^{-\frac{\alpha+1}{2}} - \frac{1}{2}u_r - \sqrt{\alpha}\rho_r^{-\frac{\alpha+1}{2}}\\
&= (1-\sqrt{\alpha})\rho_r^{-\frac{\alpha+1}{2}}\\
&> 0,
\end{aligned}
$$

which implies that $u_\delta > u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}$.

Similarly, we can get $u_\delta < u_l - \sqrt{\frac{\alpha}{\rho_l^{\alpha+1}}}$.

If $[\rho] \neq 0$, then

$$
\begin{aligned}
\Delta =&4[\rho u]^2t^2 - 4[\rho][\rho u^2 + p]t^2 \qquad (2.22)\\
=&4t^2\rho_r\rho_l\Big([u]^2 + \Big[\frac{1}{\rho}\Big][p]\Big)\\
=&4t^2\rho_r\rho_l\Big[(u_r - u_l)^2 + \Big(\frac{1}{\rho_r} - \frac{1}{\rho_l}\Big)\Big(-\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_l^\alpha}\Big)\Big]\\
=&4t^2\rho_r\rho_l[(u_r - u_l)^2 - \rho_r^{-(\alpha+1)} + \rho_r^{-1}\rho_l^{-\alpha} + \rho_l^{-1}\rho_r^{-\alpha} - \rho_l^{-(\alpha+1)}]\\
\geq&4t^2\rho_r\rho_l[(u_r - u_l)^2 - \rho_r^{-(\alpha+1)} - \rho_l^{-(\alpha+1)} + 2\rho_r^{-\frac{\alpha+1}{2}}\rho_l^{-\frac{\alpha+1}{2}}]\\
=&4t^2\rho_r\rho_l[(u_r - u_l)^2 - (\rho_r^{-\frac{\alpha+1}{2}} - \rho_l^{-\frac{\alpha+1}{2}})^2]\\
=&4t^2\rho_r\rho_l[(u_r + \rho_r^{-\frac{\alpha+1}{2}}) - (u_l + \rho_l^{-\frac{\alpha+1}{2}})][(u_r - \rho_r^{-\frac{\alpha+1}{2}}) - (u_l - \rho_l^{-\frac{\alpha+1}{2}})]\\
\geq&0.
\end{aligned}
$$

Using equations (2.18) and (2.22), we obtain

$$
x(t) = \frac{[\rho u] \pm \sqrt{\rho_r\rho_l\Big([u]^2 + \Big[\frac{1}{\rho}\Big][p]\Big)}}{[\rho]}t. \qquad (2.23)
$$

Together with this and the first equation of (2.11), we have

$$
u_\delta(t) = \frac{[\rho u] \pm \sqrt{\rho_r\rho_l\Big([u]^2 + \Big[\frac{1}{\rho}\Big][p]\Big)}}{[\rho]}. \qquad (2.24)
$$

In (2.24), selecting the positive sign yields the following conclusion:

$$
\begin{aligned}
&u_\delta - \Big(u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big) \qquad (2.25)\\
=&\frac{1}{[\rho]}\Big\{[\rho u] + \sqrt{\rho_r\rho_l\Big[(u_r-u_l)^2 + \Big(\frac{1}{\rho_r} - \frac{1}{\rho_l}\Big)\Big(-\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_l^\alpha}\Big)\Big]} - [\rho]\Big(u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big)\Big\}\\
=&\frac{1}{[\rho]}\Big\{\sqrt{\rho_r\rho_l\Big[(u_r-u_l)^2 + \Big(\frac{1}{\rho_r} - \frac{1}{\rho_l}\Big)\Big(-\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_l^\alpha}\Big)\Big]}\\
&- \rho_l\Big[u_l - u_r + \Big(\frac{\rho_r}{\rho_l} - 1\Big)\sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big]\Big\}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\Big(\sqrt{\rho_r\rho_l\Big[(u_r-u_l)^2 + \Big(\frac{1}{\rho_r} - \frac{1}{\rho_l}\Big)\Big(-\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_l^\alpha}\Big)\Big]}\Big)^2 \qquad (2.26)\\
&- \Big(\rho_l\Big[u_l - u_r + \Big(\frac{\rho_r}{\rho_l} - 1\Big)\sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big]\Big)^2\\
=&[\rho]\Big[\rho_l(u_r-u_l)^2 + \frac{1}{\rho_r^\alpha} - \frac{1}{\rho_l^\alpha} - \frac{\alpha}{\rho_r^\alpha} + \frac{\alpha\rho_l}{\rho_r^{\alpha+1}} + 2\rho_l(u_r-u_l)\sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big]\\
=&[\rho]\Big[\rho_l\Big(u_r - u_l + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}\Big)^2 - \frac{1}{\rho_l^\alpha} + (1-\alpha)\frac{1}{\rho_r^\alpha}\Big]\\
\geq&[\rho]\Big\{\rho_l\Big[(1-\sqrt{\alpha})\rho_r^{-\frac{\alpha+1}{2}} + \rho_l^{-\frac{\alpha+1}{2}}\Big]^2 - \frac{1}{\rho_l^\alpha} + (1-\alpha)\frac{1}{\rho_r^\alpha}\Big\}\\
=&[\rho]\Big[\rho_l(1-\sqrt{\alpha})^2\rho_r^{-(\alpha+1)} + 2\rho_l(1-\sqrt{\alpha})\rho_r^{-\frac{\alpha+1}{2}}\rho_l^{-\frac{\alpha+1}{2}} + (1-\alpha)\frac{1}{\rho_r^\alpha}\Big]\\
\geq&0.
\end{aligned}
$$

From (2.25) and (2.26), we get

$$u_\delta \geq u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}. \tag{2.27}$$

Similarly, we get

$$u_\delta \leq u_l - \sqrt{\frac{\alpha}{\rho_l^{\alpha+1}}}. \tag{2.28}$$

However, if we take the negative sign in (2.24), we arrive at the following result

$$u_\delta < u_r + \sqrt{\frac{\alpha}{\rho_r^{\alpha+1}}}, \quad u_\delta < u_l - \sqrt{\frac{\alpha}{\rho_l^{\alpha+1}}}. \tag{2.29}$$

From (2.12), we understand that (2.29) leads to a contradiction. Therefore, in (2.24), we should take the positive sign. Consequently, the solution of the equation (2.18) is

$$x(t) = \begin{cases} \dfrac{[\rho u^2 + p]}{2[\rho u]}t, & [\rho] = 0,\\ \dfrac{[\rho u] \pm \sqrt{\rho_r\rho_l([u]^2 + [\frac{1}{\rho}][p])}}{[\rho]}t, & [\rho] \neq 0. \end{cases} \tag{2.30}$$

This together with (2.17) yields that

$$\begin{cases} x(t) = \dfrac{[\rho u]t + \omega(t)}{[\rho]}, \\ u_\delta(t) = \dfrac{[\rho u] + \omega'(t)}{[\rho]}, \\ \omega(t) = \sqrt{\rho_r \rho_l ([u]^2 + [\dfrac{1}{\rho}][p])}t, \\ h(t) = -\dfrac{1}{2}\omega(t) u_\delta^2(t) + x(t)\Big[\dfrac{1}{2}\rho u^2 + H\Big] - \Big[\Big(\dfrac{1}{2}\rho u^2 + H + p\Big)u\Big]t. \end{cases} \tag{2.31}$$

3. Riemann problem of (1.1), (1.2) and (1.4)

For systems (1.1) associated with (1.4), the physically meaningful region under (2.1) is defined by

$$\Omega := \{(\rho, u, H) : \rho > 0, u \in R, H \geq \frac{1}{(\alpha+1)\rho^\alpha} + A\rho \ln \rho\}.$$

And there are three eigenvalues

$$\tilde{\lambda}_1 = u - \sqrt{A + \frac{\alpha}{\rho^{\alpha+1}}}, \quad \tilde{\lambda}_2 = u, \quad \tilde{\lambda}_3 = u + \sqrt{A + \frac{\alpha}{\rho^{\alpha+1}}}, \tag{3.1}$$

with the associated right eigenvectors

$$\begin{aligned} &\vec{\tilde{r}}_1 = (1, -\sqrt{\frac{A}{\rho^2} + \frac{\alpha}{\rho^{\alpha+3}}}, A + \frac{H}{\rho} - \frac{1}{\rho^{\alpha+1}})^T, \quad \vec{\tilde{r}}_2 = (1,0,0)^T, \\ &\vec{\tilde{r}}_3 = (1, \sqrt{\frac{A}{\rho^2} + \frac{\alpha}{\rho^{\alpha+3}}}, A + \frac{H}{\rho} - \frac{1}{\rho^{\alpha+1}})^T. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we deduce that

$$\nabla\tilde{\lambda}_2 \cdot \vec{\tilde{r}}_2 = 0 \ \text{ and } \ \nabla\tilde{\lambda}_i \cdot \vec{\tilde{r}}_i \neq 0 \ \text{ for } \ i = 1, 3,$$

which imply that $\tilde{\lambda}_2$ is linearly degenerate and $\tilde{\lambda}_1$ and $\tilde{\lambda}_3$ are genuinely nonlinear. Therefore, the complete set of elementary waves encompasses all possible rarefaction waves, contact discontinuities, and shock waves.

The rarefaction wave curve $\tilde{R}$ in the space (ρ, u, H), passing through (ρ_l, u_l, H_l), can be expressed as:

$$\tilde{R}_1(\rho_l, u_l, H_l) : \begin{cases} \xi_1 = u_l - \sqrt{A + \dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u + G(\rho) = u_l + G(\rho_l), \\ \quad \dfrac{H}{\rho} - A\ln\rho - \dfrac{1}{(\alpha+1)\rho^{\alpha+1}} \\ = \dfrac{H_l}{\rho_l} - A\ln\rho_l - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \qquad \rho < \rho_l, \tag{3.3}$$

and

$$\tilde{R}_3(\rho_l,u_l,H_l):\begin{cases}\xi_3=u_l+\sqrt{A+\dfrac{\alpha}{\rho_l^{\alpha+1}}},\\ u-G(\rho)=u_l-G(\rho_l),\\ \quad\dfrac{H}{\rho}-A\ln\rho-\dfrac{1}{(\alpha+1)\rho^{\alpha+1}}\\ =\dfrac{H_l}{\rho_l}-A\ln\rho_l-\dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}},\end{cases}\qquad \rho>\rho_l, \tag{3.4}$$

where $G(\rho)=\frac{2\sqrt{A}}{\alpha+1}\ln(\sqrt{A\rho^{\alpha+1}+\alpha}+\sqrt{A}\rho^{\frac{\alpha+1}{2}})-\frac{2}{\alpha+1}\frac{\sqrt{A\rho^{\alpha+1}+\alpha}}{\rho^{\frac{\alpha+1}{2}}}$.

In accordance with equation (3.3), for the 1-rarefaction waves, we find that

$$\frac{du}{d\rho}=-\frac{\sqrt{A+\frac{\alpha}{\rho^{\alpha+1}}}}{\rho}<0,\quad \frac{d^2u}{d\rho^2}=\frac{2A+(\alpha^2+3\alpha)\rho^{-\alpha-1}}{2\sqrt{A+\frac{\alpha}{\rho^{\alpha+1}}}\rho^2}>0. \tag{3.5}$$

Similarly, for 3-rarefaction waves, we obtain

$$\frac{du}{d\rho}>0,\quad \frac{d^2u}{d\rho^2}<0. \tag{3.6}$$

Furthermore, the shock wave curve is characterized as

$$\tilde{S}_1(\rho_l,u_l,H_l):\begin{cases}\sigma_1=u_l-\sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_l}}=\dfrac{\rho u-\rho_l u_l}{\rho-\rho_l},\\ u-u_l=-\sqrt{[A(\rho-\rho_l)-\dfrac{1}{\rho^\alpha}+\dfrac{1}{\rho_l^\alpha}](\dfrac{1}{\rho_l}-\dfrac{1}{\rho})},\\ \quad\dfrac{H}{\rho}+\dfrac{A\rho+A\rho_l-\frac{1}{\rho^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho}\\ =\dfrac{H_l}{\rho_l}+\dfrac{A\rho+A\rho_l-\frac{1}{\rho^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho_l},\end{cases}\qquad \rho>\rho_l, \tag{3.7}$$

and

$$\tilde{S}_3(\rho_l,u_l,H_l):\begin{cases}\sigma_3=u_l+\sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_l}}=\dfrac{\rho u-\rho_l u_l}{\rho-\rho_l},\\ u-u_l=-\sqrt{[A(\rho-\rho_l)-\dfrac{1}{\rho^\alpha}+\dfrac{1}{\rho_l^\alpha}](\dfrac{1}{\rho_l}-\dfrac{1}{\rho})},\\ \quad\dfrac{H}{\rho}+\dfrac{A\rho+A\rho_l-\frac{1}{\rho^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho}\\ =H_l\rho_l+\dfrac{A\rho+A\rho_l-\frac{1}{\rho^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho_l},\end{cases}\qquad \rho<\rho_l. \tag{3.8}$$

Let $a=A(\rho-\rho_l)-\frac{1}{\rho^\alpha}+\frac{1}{\rho_l^\alpha}$, $b=\frac{1}{\rho_l}-\frac{1}{\rho}$,$c=A+\alpha\rho^{-\alpha-1}>0$, then (3.7) can be reformulated as follows:

$$u=u_l-\sqrt{ab}. \tag{3.9}$$

Then, for 1-shock waves, we arrive at the following result:

$$\frac{du}{d\rho} = -\frac{cb + a\rho^{-2}}{2\sqrt{ab}} < 0, \tag{3.10}$$

and

$$\begin{aligned}\frac{d^2u}{d\rho^2} =& -\frac{1}{2}\frac{d\{(ab)^{-\frac{1}{2}}[cb + a\rho^{-2}]\}}{d\rho} \\ =& -\frac{1}{2}\{-\frac{1}{2}(ab)^{-\frac{3}{2}}[cb + a\rho^{-2}]^2 + (ab)^{-\frac{1}{2}}[2c\rho^{-2} - \alpha(\alpha+1)\rho^{-\alpha-2}b - 2a\rho^{-3}]\} \\ =& -\frac{1}{2}(ab)^{-\frac{1}{2}}\{-\frac{1}{2}\frac{b}{a}c^2 - \frac{1}{2}\frac{a}{b}\rho^{-4} + c\rho^{-2} - \alpha(\alpha+1)\rho^{-\alpha-2}b - 2a\rho^{-3}\} \\ \geq& -\frac{1}{2}(ab)^{-\frac{1}{2}}[-c\rho^{-2} + c\rho^{-2} - \alpha(\alpha+1)\rho^{-\alpha-2}b - 2a\rho^{-3}] \\ >&0,\end{aligned} \tag{3.11}$$

which depends on the fact that $a > 0$ and $b > 0$ in (3.7).

Similarly, for 3-shock waves, we obtain

$$\frac{du}{d\rho} = -\frac{cb + a\rho^{-2}}{2\sqrt{ab}} > 0, \quad \frac{d^2u}{d\rho^2} < 0, \tag{3.12}$$

which depends on the fact that $a < 0$ and $b < 0$ in (3.8).

The contact discontinuity curve is defined by

$$J(\rho_l, u_l, H_l) : \begin{cases} u = u_l, \rho = \rho_l, \\ H \neq H_l. \end{cases} \tag{3.13}$$

Since ρ and u remain constant along the contact discontinuities, we focus on their projections in the $u - \rho$ plane. We plot $\tilde{R}_1$, $\tilde{R}_3$, $\tilde{S}_1$, and $\tilde{S}_3$ together in the phase plane to obtain a picture as in Figure 2, in which $\tilde{R}_1$, $\tilde{R}_3$, $\tilde{S}_1$, and $\tilde{S}_3$ divide the phase plane into four distinct regions labeled I, II, III, and IV. If we put R_1, R_3,

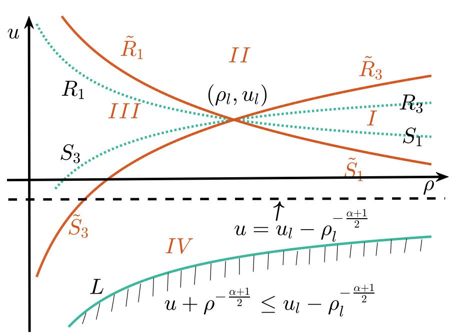

Figure 2

S_1, S_3 and L in Figure 2, the following results are observed:

Lemma 3.1. *R_3 lies below $\tilde{R}_3$, R_1 lies below $\tilde{R}_1$, S_1 lies above $\tilde{S}_1$, and S_3 lies above $\tilde{S}_3$.*

Proof. Firstly, we establish that when $\rho > \rho_l$, the shock wave S_1 is always positioned higher than the wave $\tilde{S}_1$. Based on equations (2.6) and (3.7), it is readily apparent that for $\rho > \rho_l$, $u|_{\tilde{S}_1} < u|_{S_1}$. Similarly, we can confirm that S_3 lies above $\tilde{S}_3$ when $\rho < \rho_l$.

Secondly, we demonstrate that when $\rho < \rho_l$, R_1 is consistently positioned below the wave $\tilde{R}_1$. Based on equations (2.4) and (3.3), we obtain the following relationship:

$$\frac{du}{d\rho}|_{\tilde{R}_1} = -\frac{\sqrt{A + \frac{\alpha}{\rho^{\alpha+1}}}}{\rho} < \frac{du}{d\rho}|_{R_1} = -\sqrt{\frac{\alpha}{\rho^{\alpha+3}}}, \tag{3.14}$$

which leads us to conclude that when $A > 0$, and $\rho < \rho_l$, R_1 is always situated below $\tilde{R}_1$. By the same token, we can establish that R_3 is located below $\tilde{R}_3$ when $\rho > \rho_l$. □

Lemma 3.2. *There exists a positive parameter A_0 such that $0 < A < A_0$, the region $u + \rho^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$ lies below L, $\tilde{S}_1$ and $\tilde{S}_3$.*

Proof. The proof of this theorem will be presented in three parts:

(1) The equation of L is $u + \rho^{-\frac{\alpha+1}{2}} = u_l - \rho_l^{-\frac{\alpha+1}{2}}$, Consequently, it is readily apparent that the area defined by $u + \rho^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$ is situated beneath L.

(2) From (3.7), we have $u|_{\tilde{S}_1} = u_l - \sqrt{[A(\rho - \rho_l) - \frac{1}{\rho^\alpha} + \frac{1}{\rho_l^\alpha}](\frac{1}{\rho_l} - \frac{1}{\rho})}$. In order to prove that the region $u + \rho^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$ lies below $\tilde{S}_1$ when $\rho > \rho_l$, We only need to verify the correctness of the following inequality for any $\rho_r > \rho_l$

$$u_l - \rho_l^{-\frac{\alpha+1}{2}} - \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \sqrt{[A(\rho_r - \rho_l) - \frac{1}{\rho_r^\alpha} + \frac{1}{\rho_l^\alpha}](\frac{1}{\rho_l} - \frac{1}{\rho_r})}, \tag{3.15}$$

which follows that

$$A \leq \frac{(\rho_l^{\frac{1-\alpha}{2}} + \rho_r^{\frac{1-\alpha}{2}})^2}{(\rho_r - \rho_l)^2} =: A_0. \tag{3.16}$$

In virtue of the above inequality, we have the region $u + \rho^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$ lies below $\tilde{S}_1$ when $\rho > \rho_l$.

(3) Similarly, we can prove that the region $u + \rho^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$ lies below $\tilde{S}_3$ when $\rho < \rho_l$.

The proof of Lemma 3.2 is now complete. □

4. The transformation of Riemann solutions as they evolve from the modified Chaplygin gas to the classic Chaplygin gas

In this section, we present the proof of Theorem 1.1 posed in Section 1 and we separate the discussion into two distinct parts.

Proof. (1) Riemann solutions under condition $u_r + \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$.

If $u_r+\rho_r^{-\frac{\alpha+1}{2}}\le u_l-\rho_l^{-\frac{\alpha+1}{2}}$, from Lemma 3.2, we get that the point (ρ_r,u_r) is located within the shaded area, which is part of region IV, as depicted in Figure 2. The solution to the problem defined by equations (1.1), (1.2) and (1.4) can be succinctly written as

$$(\rho_l,u_l,H_l)+\tilde{S}_1+(\rho_{*1},u_{*1},H_{*1})+J+(\rho_{*2},u_{*2},H_{*2})+\tilde{S}_3+(\rho_r,u_r,H_r),$$

where $(\rho_{*1},u_{*1},H_{*1})$ and $(\rho_{*2},u_{*2},H_{*2})$ represent the intermediate states that link the backward shock wave, contact discontinuity, and forward shock wave. According to (3.13), we know $u_{*1}=u_{*2}:=u_*$ and $\rho_{*1}=\rho_{*2}:=\rho_*$. Utilizing equations (3.7) and (3.8), we can derive the following results:

$$\tilde{S}_1(\rho_l,u_l,H_l):\begin{cases}\sigma_1=u_l-\sqrt{\dfrac{[p]\rho_*}{[\rho]\rho_l}}=\dfrac{\rho_*u_*-\rho_lu_l}{\rho_*-\rho_l},\\ u_*-u_l=-\sqrt{[A(\rho_*-\rho_l)-\dfrac{1}{\rho_*^\alpha}+\dfrac{1}{\rho_l^\alpha}](\dfrac{1}{\rho_l}-\dfrac{1}{\rho_*})},\\ \dfrac{H_{*1}}{\rho_*}+\dfrac{A\rho_*+A\rho_l-\frac{1}{\rho_*^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho_*}\\ =\dfrac{H_l}{\rho_l}+\dfrac{A\rho_*+A\rho_l-\frac{1}{\rho_*^\alpha}-\frac{1}{\rho_l^\alpha}}{2\rho_l},\end{cases}\qquad \rho_*>\rho_l, \tag{4.1}$$

and

$$\tilde{S}_3(\rho_r,u_r,H_r):\begin{cases}\sigma_3=u_*+\sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_*}}=\dfrac{\rho_ru_r-\rho_*u_*}{\rho_r-\rho_*},\\ u_r-u_*=-\sqrt{[A(\rho_r-\rho_*)-\dfrac{1}{\rho_r^\alpha}+\dfrac{1}{\rho_*^\alpha}](\dfrac{1}{\rho_*}-\dfrac{1}{\rho_r})},\\ \dfrac{H_r}{\rho_r}+\dfrac{A\rho_r+A\rho_*-\frac{1}{\rho_r^\alpha}-\frac{1}{\rho_*^\alpha}}{2\rho_r}\\ =\dfrac{H_{*2}}{\rho_*}+\dfrac{A\rho_r+A\rho_*-\frac{1}{\rho_r^\alpha}-\frac{1}{\rho_*^\alpha}}{2\rho_*},\end{cases}\qquad \rho_r<\rho_*. \tag{4.2}$$

Lemma 4.1. *Suppose that* $H_l\ge\dfrac{1}{(\alpha+1)\rho_l^\alpha}$, $H_r\ge\dfrac{1}{(\alpha+1)\rho_r^\alpha}$, $u_r+\rho_r^{-\frac{\alpha+1}{2}}\le u_l-\rho_l^{-\frac{\alpha+1}{2}}$, *then*

$$\begin{aligned}&\lim_{A\to0}\rho_*=+\infty,\\ &\lim_{A\to0}H_{*1}=\lim_{A\to0}H_{*2}=+\infty\end{aligned} \tag{4.3}$$

and

$$\lim_{A\to0}A\rho_*=C,\qquad \textit{where}\ \ C\in[0,\infty).$$

Proof. The combination of the second equations of (4.1) and (4.2) results in

$$u_r-u_l=-\sqrt{[A(\rho_*-\rho_l)-\frac{1}{\rho_*^\alpha}+\frac{1}{\rho_l^\alpha}](\frac{1}{\rho_l}-\frac{1}{\rho_*})}$$

$$-\sqrt{[A(\rho_r-\rho_*)-\frac{1}{\rho_r^\alpha}+\frac{1}{\rho_*^\alpha}](\frac{1}{\rho_*}-\frac{1}{\rho_r})}. \tag{4.4}$$

Assuming there is a positive real constant C_0 for which $\lim\limits_{A\to 0}\rho_*=C_0$ and taking the limit on both sides of equation (4.4), we obtain

$$\begin{aligned}\lim_{A\to 0}(u_r-u_l)&=-\sqrt{(-\frac{1}{C_0^\alpha}+\frac{1}{\rho_l^\alpha})(\frac{1}{\rho_l}-\frac{1}{C_0})}-\sqrt{(-\frac{1}{\rho_r^\alpha}+\frac{1}{C_0^\alpha})(\frac{1}{C_0}-\frac{1}{\rho_r})} \\ &\geq -\sqrt{(\rho_l^{-\frac{\alpha+1}{2}}-C_0^{-\frac{\alpha+1}{2}})^2}-\sqrt{(\rho_r^{-\frac{\alpha+1}{2}}-C_0^{-\frac{\alpha+1}{2}})^2} \\ &= -\rho_l^{-\frac{\alpha+1}{2}}+C_0^{-\frac{\alpha+1}{2}}-\rho_r^{-\frac{\alpha+1}{2}}+C_0^{-\frac{\alpha+1}{2}} \\ &> -\rho_l^{-\frac{\alpha+1}{2}}-\rho_r^{-\frac{\alpha+1}{2}},\end{aligned} \tag{4.5}$$

which indicates that $u_r+\rho_r^{-\frac{\alpha+1}{2}}>u_l-\rho_l^{-\frac{\alpha+1}{2}}$, which constitutes a paradox. So $\lim\limits_{A\to 0}\rho_*=+\infty$.

Suppose that $\lim\limits_{A\to 0}A\rho_*=+\infty$, then taking the limit on both sides of equation (4.4), we arrive at $\lim\limits_{A\to 0}(u_r-u_l)=-\infty$, which is a contradiction. So $\lim\limits_{A\to 0}A\rho_*=C$.

Furthermore, employing the third equations from (4.1) and (4.2), we obtain

$$H_{*1}=\frac{\rho_*}{\rho_l}(H_l-\frac{1}{2\rho_l^\alpha})+\frac{\rho_*^{1-\alpha}(A\rho_*\rho_*^\alpha-1)}{2\rho_l}+\frac{1}{2\rho_*^\alpha}+\frac{1}{2\rho_l^\alpha}-\frac{A\rho_l}{2} \tag{4.6}$$

and

$$H_{*2}=\frac{\rho_*}{\rho_r}(H_r-\frac{1}{2\rho_r^\alpha})+\frac{\rho_*^{1-\alpha}(A\rho_*\rho_*^\alpha-1)}{2\rho_r}+\frac{1}{2\rho_*^\alpha}+\frac{1}{2\rho_r^\alpha}-\frac{A\rho_r}{2}. \tag{4.7}$$

Since $0<\alpha\leq 1$, $H_l\geq\frac{1}{(\alpha+1)\rho_l^\alpha}$ and $H_r\geq\frac{1}{(\alpha+1)\rho_r^\alpha}$, we have $H_l\geq\frac{1}{2\rho_l^\alpha}$ and $H_r\geq\frac{1}{2\rho_r^\alpha}$. So $\lim\limits_{A\to 0}H_{*1}=\lim\limits_{A\to 0}H_{*2}=+\infty$.

Up to this point, we have completed the proof of Lemma 4.1. □

Lemma 4.2. *Assume that* $u_r+\rho_r^{-\frac{\alpha+1}{2}}\leq u_l-\rho_l^{-\frac{\alpha+1}{2}}$, *then*

$$\lim_{A\to 0}\sigma_1=\lim_{A\to 0}\sigma_2=\lim_{A\to 0}\sigma_3:=u_\delta. \tag{4.8}$$

Proof. Taking the limit on both sides of the second equation of (4.1) and (4.2), we obtain

$$\lim_{A\to 0}u_*=C, \tag{4.9}$$

where C is a finit constant.

Utilizing (4.3) and invoking the first equation of (4.1) and (4.2), we arrive at

$$\lim_{A\to 0}\sigma_1=\lim_{A\to 0}\sigma_3=\lim_{A\to 0}u_*. \tag{4.10}$$

Given that $\lim\limits_{A\to 0}\sigma_2=\lim\limits_{A\to 0}u_*$, it follows that $\lim\limits_{A\to 0}\sigma_1=\lim\limits_{A\to 0}\sigma_2=\lim\limits_{A\to 0}\sigma_3=\lim\limits_{A\to 0}u_*$.

Taking the limit on both sides of the second equation of (4.1) and (4.2) and considering the statement (4.3), we obtain

$$\lim_{A\to 0}u_*-u_l=-\sqrt{(C+\frac{1}{\rho_l^\alpha})\frac{1}{\rho_l}} \tag{4.11}$$

and

$$u_r - \lim_{A\to 0} u_* = -\sqrt{(C + \frac{1}{\rho_r^\alpha})\frac{1}{\rho_r}}. \tag{4.12}$$

By removing the constant C from (4.11) and (4.12), we are led to the conclusion that

$$[\rho](\lim_{A\to 0} u_*)^2 - 2[\rho u] \lim_{A\to 0} u_* + [\rho u^2 + p] = 0. \tag{4.13}$$

Through direct calculating, (4.13) yields that

$$\lim_{A\to 0} u_* = \begin{cases} \dfrac{[\rho u^2 + p]}{2[\rho u]}, & [\rho] = 0, \\ \dfrac{[\rho u] + \sqrt{\rho_r \rho_l \big([u]^2 + \big[\frac{1}{\rho}\big][p]\big)}}{[\rho]}, & [\rho] \neq 0, \end{cases} \tag{4.14}$$

which is consistent with the expression for u_δ given in (2.30).

To this point, we have completed the demonstration of Lemma 4.2. □

Lemma 4.3. *Assume that* $u_r + \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$, *then*

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \lim\limits_{A\to 0} \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} \rho d\xi = u_\delta[\rho] - [\rho u], \\ \lim\limits_{\varepsilon\to 0} \lim\limits_{A\to 0} \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} \rho u d\xi = u_\delta[\rho u] - [\rho u^2 - \frac{1}{\rho^\alpha}], \\ \lim\limits_{\varepsilon\to 0} \lim\limits_{A\to 0} \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} (\frac{1}{2}\rho u^2 + H) d\xi = u_\delta[\frac{1}{2}\rho u^2 + H] - [(\frac{1}{2}\rho u^2 + H - \frac{1}{\rho^\alpha})u]. \end{cases} \tag{4.15}$$

Proof. Taking $\xi = \frac{x}{t}$, through the application of a self-similar transformation, equation (1.1) can be rewritten as

$$\begin{cases} -\xi\rho_\xi + (\rho u)_\xi = 0, \\ -\xi(\rho u)_\xi + (\rho u^2 + p)_\xi = 0, \\ -\xi(\frac{1}{2}\rho u^2 + H)_\xi + ((\frac{1}{2}\rho u^2 + H + p)u)_\xi = 0. \end{cases} \tag{4.16}$$

Performing an integration on both sides of equations (4.16) with respect to ξ yields

$$\begin{cases} \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} (-\xi\rho_\xi + (\rho u)_\xi) d\xi = 0, \\ \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} (-\xi(\rho u)_\xi + (\rho u^2 + p)_\xi) d\xi = 0, \\ \displaystyle\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} (-\xi(\frac{1}{2}\rho u^2 + H)_\xi + ((\frac{1}{2}\rho u^2 + H + p)u)_\xi) d\xi = 0. \end{cases} \tag{4.17}$$

The third equation of (4.17) can be expressed as

$$\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} (\frac{1}{2}\rho u^2 + H) d\xi = \xi(\frac{1}{2}\rho u^2 + H)\Big|_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon} - (\frac{1}{2}\rho u^3 + Hu + A\rho u - \frac{u}{\rho^\alpha})\Big|_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon}. \tag{4.18}$$

Taking the limit on both sides of (4.18), we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{A\to 0}\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon}(\frac{1}{2}\rho u^2+H)d\xi\\
=&\lim_{\varepsilon\to 0}\lim_{A\to 0}[(\sigma_3+\varepsilon)(\frac{1}{2}\rho_r u_r^2+H_r)-(\sigma_1-\varepsilon)(\frac{1}{2}\rho_l u_l^2+H_l)]\\
&-\lim_{\varepsilon\to 0}\lim_{A\to 0}(\frac{1}{2}\rho_r u_r^3+H_r u_r+A\rho_r u_r-\frac{u_r}{\rho_r^\alpha})\\
&+\lim_{\varepsilon\to 0}\lim_{A\to 0}(\frac{1}{2}\rho_l u_l^3+H_l u_l+A\rho_l u_l-\frac{u_l}{\rho_l^\alpha}).
\end{aligned}$$

By (4.10), we get

$$\lim_{\varepsilon\to 0}\lim_{A\to 0}\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon}(\frac{1}{2}\rho u^2+H)d\xi=u_\delta[\frac{1}{2}\rho u^2+H]-[(\frac{1}{2}\rho u^2+H-\frac{1}{\rho^\alpha})u]. \tag{4.19}$$

Similarly, it is easy to see that

$$\begin{cases}\lim\limits_{\varepsilon\to 0}\lim\limits_{A\to 0}\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon}\rho d\xi=u_\delta[\rho]-[\rho u],\\
\lim\limits_{\varepsilon\to 0}\lim\limits_{A\to 0}\int_{\sigma_1-\varepsilon}^{\sigma_3+\varepsilon}\rho u d\xi=u_\delta[\rho u]-[\rho u^2-\frac{1}{\rho^\alpha}].\end{cases} \tag{4.20}$$

Therefore, if the initial data satisfies the condition $u_r+\rho_r^{-\frac{\alpha+1}{2}}\le u_l-\rho_l^{-\frac{\alpha+1}{2}}$, then the solution of the Riemann problem of (1.1), (1.2) and (1.4) converges to the delta shock wave solution of the Riemann problem of (1.1), (1.2) and (1.5) as A approaches zero. This convergence can be illustrated by Figure 3. □

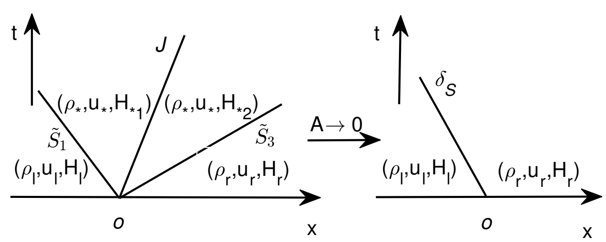

Figure 3

(2) Riemann solutions under condition $u_l-\rho_l^{-\frac{\alpha+1}{2}}<u_r+\rho_r^{-\frac{\alpha+1}{2}}$.

In this part, we explore the topic across four specific cases:

Case 1. $(\rho_r,u_r,H_r)\epsilon$ IV.

Lemma 4.4. *Suppose that $u_l-\rho_l^{-\frac{\alpha+1}{2}}<u_r+\rho_r^{-\frac{\alpha+1}{2}}$ and $(\rho_r,u_r,H_r)\epsilon$ IV. Then, as A approaches 0, $\tilde{S}_1$ and $\tilde{S}_3$ for modified Chaplygin gas transform into the shock waves S_1 and S_3 for the generalized Chaplygin gas, where $\tilde{S}_1$ and $\tilde{S}_3$ are defined in (3.7) and (3.8) and S_1 and S_3 are indicated in (2.6) and (2.7), respectively.*

Proof. Consider the intermediate states (ρ_*,u_*,H_{*1}) and (ρ_*,u_*,H_{*2}) connecting

$\tilde{S}_1$, J and $\tilde{S}_3$, respectively. If $(\rho_r, u_r, H_r)\epsilon$ IV, then

$$\tilde{S}_1(\rho_l, u_l, H_l): \begin{cases} \sigma_1 = u_l - \sqrt{\dfrac{[p]\rho_*}{[\rho]\rho_l}} = \dfrac{\rho_* u_* - \rho_l u_l}{\rho_* - \rho_l}, \\ u_* - u_l = -\sqrt{[A(\rho_* - \rho_l) - \dfrac{1}{\rho_*^\alpha} + \dfrac{1}{\rho_l^\alpha}](\dfrac{1}{\rho_l} - \dfrac{1}{\rho_*})}, \\ \dfrac{H_{*1}}{\rho_*} + \dfrac{A\rho_* + A\rho_l - \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_l^\alpha}}{2\rho_*} \\ = \dfrac{H_l}{\rho_l} + \dfrac{A\rho_* + A\rho_l - \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_l^\alpha}}{2\rho_l}, \end{cases} \qquad \rho_* > \rho_l, \tag{4.21}$$

and

$$\tilde{S}_3(\rho_r, u_r, H_r): \begin{cases} \sigma_3 = u_* + \sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_*}} = \dfrac{\rho_r u_r - \rho_* u_*}{\rho_r - \rho_*}, \\ u_r - u_* = -\sqrt{[A(\rho_r - \rho_*) - \dfrac{1}{\rho_r^\alpha} + \dfrac{1}{\rho_*^\alpha}](\dfrac{1}{\rho_*} - \dfrac{1}{\rho_r})}, \\ \dfrac{H_r}{\rho_r} + \dfrac{A\rho_r + A\rho_* - \frac{1}{\rho_r^\alpha} - \frac{1}{\rho_*^\alpha}}{2\rho_r} \\ = \dfrac{H_{*2}}{\rho_*} + \dfrac{A\rho_r + A\rho_* - \frac{1}{\rho_r^\alpha} - \frac{1}{\rho_*^\alpha}}{2\rho_*}, \end{cases} \qquad \rho_r < \rho_*. \tag{4.22}$$

In virtue of Lemma 4.1, we are aware that $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$ and $(\rho_r, u_r, H_r)\epsilon$ IV holds if and only if $\lim\limits_{A\to 0} \rho_* = C < \infty$. As A tends to 0, by applying (4.21) and (4.22), we observe that

$$\begin{cases} \lim\limits_{A\to 0} \sigma_1 = u_l - \sqrt{\dfrac{[p]\rho_*}{[\rho]\rho_l}}, \\ u_* - u_l = -\sqrt{(\dfrac{1}{\rho_l^\alpha} - \dfrac{1}{\rho_*^\alpha})(\dfrac{1}{\rho_l} - \dfrac{1}{\rho_*})}, \qquad \rho_* > \rho_l, \\ \dfrac{H_{*1}}{\rho_*} - \dfrac{\frac{1}{\rho_*^\alpha} + \frac{1}{\rho_l^\alpha}}{2\rho_*} = \dfrac{H_l}{\rho_l} - \dfrac{\frac{1}{\rho_*^\alpha} + \frac{1}{\rho_l^\alpha}}{2\rho_l}, \end{cases} \tag{4.23}$$

and

$$\begin{cases} \lim\limits_{A\to 0} \sigma_3 = u_* + \sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_*}}, \\ u_r - u_* = -\sqrt{(\dfrac{1}{\rho_*^\alpha} - \dfrac{1}{\rho_r^\alpha})(\dfrac{1}{\rho_*} - \dfrac{1}{\rho_r})}, \qquad \rho_r < \rho_*. \\ \dfrac{H_r}{\rho_r} - \dfrac{\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_*^\alpha}}{2\rho_r} = \dfrac{H_{*2}}{\rho_*} - \dfrac{\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_*^\alpha}}{2\rho_*}, \end{cases} \tag{4.24}$$

In other words, as A approaches 0, $\tilde{S}_1$ for modified Chaplygin gas is transformed into S_1 for the generalized Chaplygin gas, and similarly, $\tilde{S}_3$ for modified Chaplygin gas is transformed into S_3 for the generalized Chaplygin gas. This concludes the demonstration of Lemma 4.4, an illustration of which is provided by Figure 4. $\square$

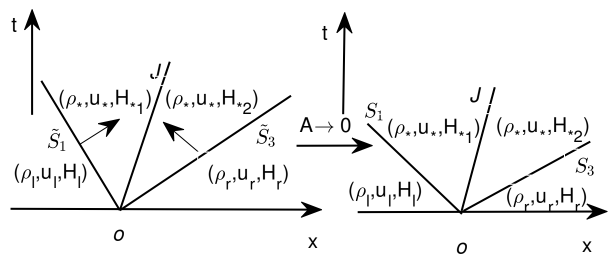

Figure 4

Case 2. $(\rho_r, u_r, H_r)\epsilon$ I.

Lemma 4.5. *Assuming* $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$ *and* $(\rho_r, u_r, H_r)\epsilon$ *I. Then, as A approaches zero,* $\tilde{S}_1$ *and* $\tilde{R}_3$ *for modified Chaplygin gas change into* S_1 *and* R_3 *for the generalized Chaplygin gas, respectively.*

Proof. Let (ρ_*, u_*, H_{*1}) and (ρ_*, u_*, H_{*2}) be the intermediate states connecting $\tilde{S}_1$, J and $\tilde{R}_3$, respectively. If $(\rho_r, u_r, H_r)\epsilon$ I, then
$\tilde{S}_1(\rho_l, u_l, p_l)$:

$$\tilde{S}_1(\rho_l, u_l, H_l) : \begin{cases} \sigma_1 = u_l - \sqrt{\dfrac{[p]\rho_*}{[\rho]\rho_l}} = \dfrac{\rho_* u_* - \rho_l u_l}{\rho_* - \rho_l}, \\ u_* - u_l = -\sqrt{[A(\rho_* - \rho_l) - \dfrac{1}{\rho_*^\alpha} + \dfrac{1}{\rho_l^\alpha}](\dfrac{1}{\rho_l} - \dfrac{1}{\rho_*})}, \\ \dfrac{H_{*1}}{\rho_*} + \dfrac{A\rho_* + A\rho_l - \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_l^\alpha}}{2\rho_*} \\ = \dfrac{H_l}{\rho_l} + \dfrac{A\rho_* + A\rho_l - \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_l^\alpha}}{2\rho_l}, \end{cases} \qquad \rho_* > \rho_l, \tag{4.25}$$

and

$$\tilde{R}_3(\rho_r, u_r, H_r) : \begin{cases} \xi_3 = u_* + \sqrt{A + \dfrac{\alpha}{\rho_*^{\alpha+1}}}, \\ u_r - G(\rho_r) = u_* - G(\rho_*), \\ \dfrac{H_r}{\rho_r} - A\ln\rho_r - \dfrac{1}{(\alpha+1)\rho_r^{\alpha+1}} \\ = \dfrac{H_*}{\rho_*} - A\ln\rho_* - \dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}}, \end{cases} \qquad \rho_r > \rho_*, \tag{4.26}$$

where $G(\rho) = \frac{2\sqrt{A}}{\alpha+1}\ln(\sqrt{A\rho^{\alpha+1}+\alpha} + \sqrt{A}\rho^{\frac{\alpha+1}{2}}) - \frac{2}{\alpha+1}\frac{\sqrt{A\rho^{\alpha+1}+\alpha}}{\rho^{\frac{\alpha+1}{2}}}$. As $A \to 0$, using

(4.25) and (4.26), we see that

$$\begin{cases} \lim\limits_{A\to 0}\sigma_1 = u_l - \sqrt{\dfrac{[p]\rho_*}{[\rho]\rho_l}}, \\ u_* - u_l = -\sqrt{(\dfrac{1}{\rho_l^\alpha} - \dfrac{1}{\rho_*^\alpha})(\dfrac{1}{\rho_l} - \dfrac{1}{\rho_*})}, \qquad \rho_* > \rho_l, \\ \dfrac{H_{*1}}{\rho_*} - \dfrac{\frac{1}{\rho_*^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho_*} = \dfrac{H_l}{\rho_l} - \dfrac{\frac{1}{\rho_*^\alpha}+\frac{1}{\rho_l^\alpha}}{2\rho_l}, \end{cases} \tag{4.27}$$

and

$$\begin{cases} \lim\limits_{A\to 0}\xi_3 = u_* + \sqrt{\dfrac{\alpha}{\rho_*^{\alpha+1}}}, \\ u_r + \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_r^{-\frac{\alpha+1}{2}} = u_* + \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_*^{-\frac{\alpha+1}{2}}, \qquad \rho_r > \rho_*, \\ \dfrac{H_r}{\rho_r} - \dfrac{1}{(\alpha+1)\rho_r^{\alpha+1}} = \dfrac{H_*}{\rho_*} - \dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}}, \end{cases} \tag{4.28}$$

which indicates that $\tilde{S}_1$ for modified Chaplygin gas is converted into S_1 for the generalized Chaplygin gas, and similarly, $\tilde{R}_3$ for modified Chaplygin gas is transformed into R_3 for the generalized Chaplygin gas. This concludes the demonstration of Lemma 4.5, a visualization of which is provided in Figure 5. □

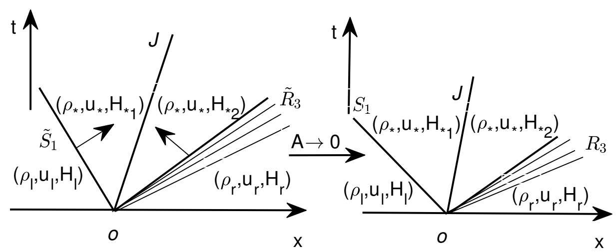

Figure 5

Case 3. $(\rho_r, u_r, H_r)\epsilon$ II.

Lemma 4.6. *Assume that* $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$ *and* $(\rho_r, u_r, H_r)\epsilon$ *II. Then, as* A *tends to zero,* $\tilde{R}_1$ *and* $\tilde{R}_3$ *for modified Chaplygin gas change into* R_1 *and* R_3 *for the generalized Chaplygin gas, respectively.*

Proof. Consider (ρ_*, u_*, H_{*1}) and (ρ_*, u_*, H_{*2}) as the intermediate states that connecting $\tilde{R}_1$, J and $\tilde{R}_3$, respectively. If $(\rho_r, u_r, p_r)\epsilon$ II, then

$$\tilde{R}_1(\rho_l, u_l, H_l): \begin{cases} \xi_1 = u_l - \sqrt{A + \dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u_* + G(\rho_*) = u_l + G(\rho_l), \\ \quad \dfrac{H_*}{\rho_*} - A\ln\rho_* - \dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}} \qquad \rho_* < \rho_l, \\ = \dfrac{H_l}{\rho_l} - A\ln\rho_l - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \tag{4.29}$$

and

$$\tilde{R}_3(\rho_r,u_r,H_r):\begin{cases}\xi_3=u_*+\sqrt{A+\dfrac{\alpha}{\rho_*^{\alpha+1}}},\\ u_r-G(\rho_r)=u_*-G(\rho_*),\\ \quad\dfrac{H_r}{\rho_r}-A\ln\rho_r-\dfrac{1}{(\alpha+1)\rho_r^{\alpha+1}}\\ =\dfrac{H_*}{\rho_*}-A\ln\rho_*-\dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}},\end{cases}\qquad \rho_r>\rho_*, \tag{4.30}$$

where $G(\rho)=\frac{2\sqrt{A}}{\alpha+1}\ln(\sqrt{A\rho^{\alpha+1}+\alpha}+\sqrt{A}\rho^{\frac{\alpha+1}{2}})-\frac{2}{\alpha+1}\frac{\sqrt{A\rho^{\alpha+1}+\alpha}}{\rho^{\frac{\alpha+1}{2}}}$. As $A\to 0$, using (4.29) and (4.30), we see that

$$\begin{cases}\lim\limits_{A\to 0}\xi_1=u_l-\sqrt{\dfrac{\alpha}{\rho_l^{\alpha+1}}},\\ u_*-\dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_*^{-\frac{\alpha+1}{2}}=u_l-\dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_l^{-\frac{\alpha+1}{2}},\\ \dfrac{H_*}{\rho_*}-\dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}}=\dfrac{H_l}{\rho_l}-\dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}},\end{cases}\qquad \rho_*<\rho_l, \tag{4.31}$$

and

$$\begin{cases}\lim\limits_{A\to 0}\xi_3=u_*+\sqrt{\dfrac{\alpha}{\rho_*^{\alpha+1}}},\\ u_r+\dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_r^{-\frac{\alpha+1}{2}}=u_*+\dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_*^{-\frac{\alpha+1}{2}},\\ \dfrac{H_r}{\rho_r}-\dfrac{1}{(\alpha+1)\rho_r^{\alpha+1}}=\dfrac{H_*}{\rho_*}-\dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}},\end{cases}\qquad \rho_r>\rho_*, \tag{4.32}$$

which suggests that $\tilde{R}_1$ for modified Chaplygin gas is transformed into R_1 for the generalized Chaplygin gas, and similarly, $\tilde{R}_3$ for modified Chaplygin gas is transformed into R_3 for the generalized Chaplygin gas. This concludes the demonstration of Lemma 4.6, a visualization of which is provided in Figure 6. □

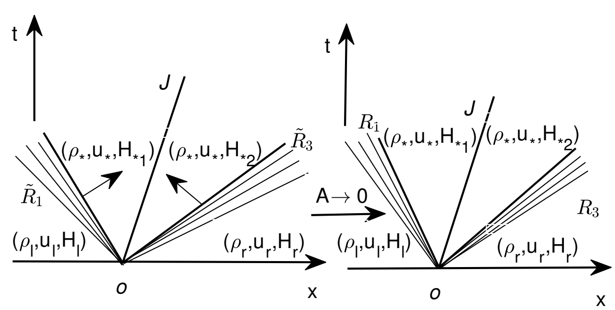

Figure 6

Case 4. $(\rho_r,u_r,H_r)\epsilon$ III.

Lemma 4.7. *Assume that* $u_l-\rho_l^{-\frac{\alpha+1}{2}}<u_r+\rho_r^{-\frac{\alpha+1}{2}}$ *and* $(\rho_r,u_r,H_r)\epsilon$ *III. Then, as A tends to zero,* $\tilde{R}_1$ *and* $\tilde{S}_3$ *for modified Chaplygin gas change into* R_1 *and* S_3 *for the generalized Chaplygin gas, respectively.*

Proof. Consider (ρ_*, u_*, H_{*1}) and (ρ_*, u_*, H_{*2}) as the intermediate states that connecting $\tilde{R}_1$, J and $\tilde{S}_3$, respectively. If $(\rho_r, u_r, H_r)\epsilon$ III, then

$$\tilde{R}_1(\rho_l, u_l, H_l): \begin{cases} \xi_1 = u_l - \sqrt{A + \dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u_* + G(\rho_*) = u_l + G(\rho_l), \\ \quad \dfrac{H_*}{\rho_*} - A\ln\rho_* - \dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}} \\ = \dfrac{H_l}{\rho_l} - A\ln\rho_l - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \qquad \rho_* < \rho_l, \tag{4.33}$$

where $G(\rho) = \frac{2\sqrt{A}}{\alpha+1}\ln(\sqrt{A\rho^{\alpha+1}+\alpha} + \sqrt{A}\rho^{\frac{\alpha+1}{2}}) - \frac{2}{\alpha+1}\frac{\sqrt{A\rho^{\alpha+1}+\alpha}}{\rho^{\frac{\alpha+1}{2}}}$, and

$$\tilde{S}_3(\rho_r, u_r, H_r): \begin{cases} \sigma_3 = u_* + \sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_*}} = \dfrac{\rho_r u_r - \rho_* u_*}{\rho_r - \rho_*}, \\ u_r - u_* = -\sqrt{[A(\rho_r - \rho_*) - \dfrac{1}{\rho_r^\alpha} + \dfrac{1}{\rho_*^\alpha}](\dfrac{1}{\rho_*} - \dfrac{1}{\rho_r})}, \\ \quad \dfrac{H_r}{\rho_r} + \dfrac{A\rho_r + A\rho_* - \frac{1}{\rho_r^\alpha} - \frac{1}{\rho_*^\alpha}}{2\rho_r} \\ = \dfrac{H_{*2}}{\rho_*} + \dfrac{A\rho_r + A\rho_* - \frac{1}{\rho_r^\alpha} - \frac{1}{\rho_*^\alpha}}{2\rho_*}, \end{cases} \qquad \rho_r < \rho_*. \tag{4.34}$$

As $A \to 0$, using (4.33) and (4.34), we see that

$$\begin{cases} \lim\limits_{A\to 0}\xi_1 = u_l - \sqrt{\dfrac{\alpha}{\rho_l^{\alpha+1}}}, \\ u_* - \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_*^{-\frac{\alpha+1}{2}} = u_l - \dfrac{2\sqrt{\alpha}}{\alpha+1}\rho_l^{-\frac{\alpha+1}{2}}, \\ \dfrac{H_*}{\rho_*} - \dfrac{1}{(\alpha+1)\rho_*^{\alpha+1}} = \dfrac{H_l}{\rho_l} - \dfrac{1}{(\alpha+1)\rho_l^{\alpha+1}}, \end{cases} \qquad \rho_* < \rho_l, \tag{4.35}$$

and

$$\begin{cases} \lim\limits_{A\to 0}\sigma_3 = u_* + \sqrt{\dfrac{[p]\rho_r}{[\rho]\rho_*}}, \\ u_r - u_* = -\sqrt{(\dfrac{1}{\rho_*^\alpha} - \dfrac{1}{\rho_r^\alpha})(\dfrac{1}{\rho_*} - \dfrac{1}{\rho_r})}, \\ \dfrac{H_r}{\rho_r} - \dfrac{\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_*^\alpha}}{2\rho_r} = \dfrac{H_{*2}}{\rho_*} - \dfrac{\frac{1}{\rho_r^\alpha} + \frac{1}{\rho_*^\alpha}}{2\rho_*}, \end{cases} \qquad \rho_r < \rho_*, \tag{4.36}$$

which suggests that $\tilde{R}_1$ for modified Chaplygin gas is transformed into R_1 for the generalized Chaplygin gas, and similarly, $\tilde{S}_3$ for modified Chaplygin gas is transformed into S_3 for the generalized Chaplygin gas. This concludes the demonstration of Lemma 4.7, a visualization of which is provided in Figure 7. $\square$

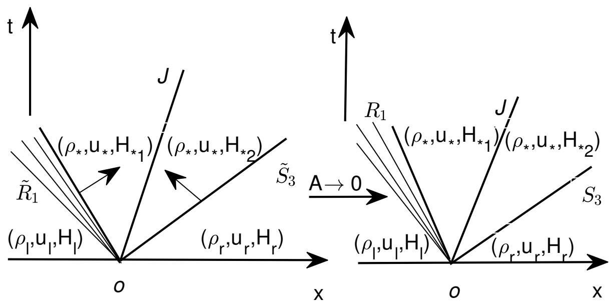

Figure 7

Therefore, if the initial conditions meet the criterion $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$, then the solution of the Riemann problem of (1.1), (1.2) and (1.4) asymptotically approaches the corresponding Riemann solutions of the problem of (1.1), (1.2) and (1.5) as A tends to zero.

To this point, we have completed the proof of Theorem 1.1. $\square$

5. Numerical simulation

In this section, we verify the correctness of our results in Section 5 in virtue of numerical simulation.

Example 5.1. Given the initial data

$$\rho_l = 0.8,\ u_l = 0,\ H_l = 1,\ \rho_r = 1,\ u_r = -2.5,\ H_r = 2,\ A = 8,\ \alpha = 0.5. \tag{5.1}$$

Under conditon $u_r + \rho_r^{-\frac{\alpha+1}{2}} \leq u_l - \rho_l^{-\frac{\alpha+1}{2}}$, the numerical simulations can be conducted using the initial data given in (5.1) (see Figure 8).

Subsequently, we adjust the variable values to $A = 0.001$ and $A = 0.0001$ as in (5.1). The corresponding numerical simulations are depicted in Figure 9 and Figure 10. From the numerical simulation results for a sufficiently small parameter A, it is observed that the solution of the Riemann problem of (1.1), (1.2) and (1.4) tends to converge to the delta shock wave solution of the Riemann problem of (1.1), (1.2) and (1.5) as A tends to zero.

Example 5.2. Given the initial data

$$\rho_l = 0.8,\ u_l = 0,\ H_l = 1,\ \rho_r = 1,\ u_r = 2,\ H_r = 3,\ A = 8,\ \alpha = 0.5. \tag{5.2}$$

Under conditon $u_l - \rho_l^{-\frac{\alpha+1}{2}} < u_r + \rho_r^{-\frac{\alpha+1}{2}}$, the numerical simulations can be conducted using the initial data given in (5.2) (see Figure 11).

Subsequently, we adjust the variable values to $A = 0.001$ and $A = 0.0001$ as in (5.2). The corresponding numerical simulations are depicted in Figure 12 and Figure 13. From the numerical simulation results for a sufficiently small parameter A, it is observed that the solution of the Riemann problem of (1.1), (1.2) and (1.4) tends to converge to the corresponding Riemann solutions of the problem of (1.1), (1.2) and (1.5) as A tends to zero.

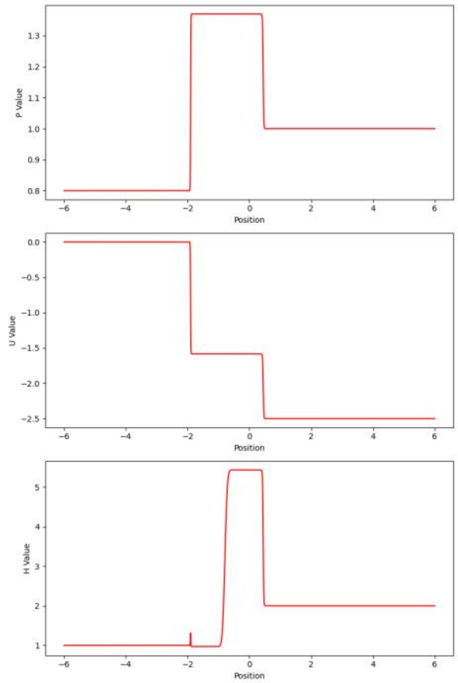

Figure 8

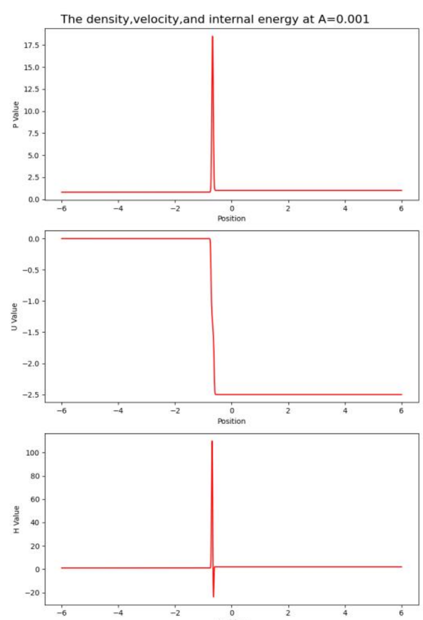

Figure 9

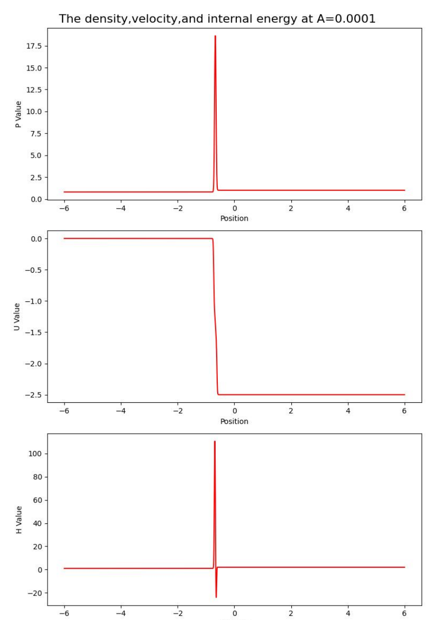

Figure 10

Acknowledgement

I would like to thank the referees for their comments, which improved and clarified the paper.

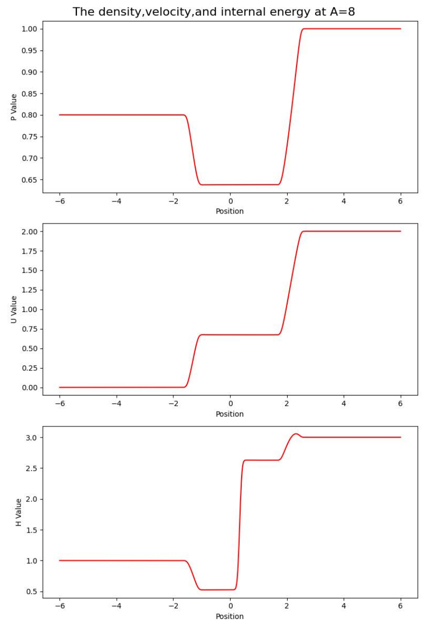

Figure 11

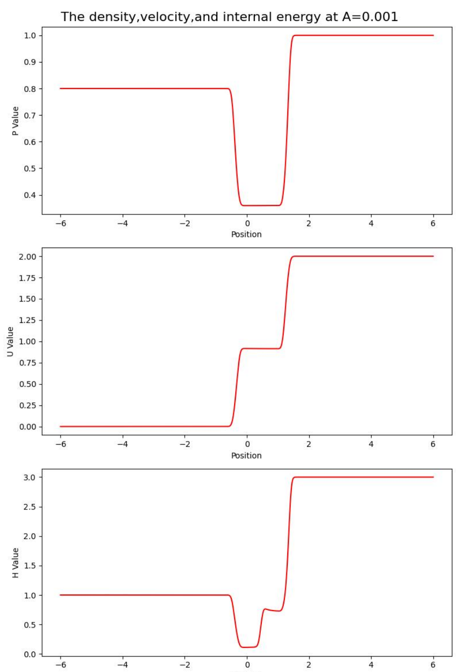

Figure 12

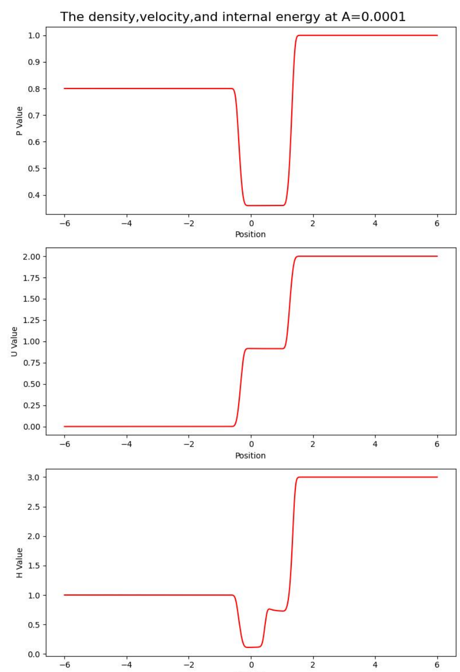

Figure 13

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