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A NEKHOROSHEV TYPE THEOREM FOR THE FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION*

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Abstract We prove a Nekhoroshev type theorem for the fractional nonlinear Schrödinger equation under Dirichlet boundary conditions. More precisely, our findings show that the solutions with ε-small initial data in the Gevrey space remain in their small magnitude over time intervals of order $\varepsilon^{-|\ln \varepsilon|^{\gamma}}$ with $0 < \gamma < 1/10$. The result can be proved by using Birkhoff normal form method and the so-called tame property of the nonlinearity in Gevrey space.

Keywords Hamiltonian PDEs, Nekhoroshev theorem, fractional nonlinear Schrödinger equation, Birkhoff normal form, tame property.

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1. Introduction and main result

We focus on the fractional nonlinear Schrödinger (FNLS) equation

$$\mathbf{i}u_t = (-\Delta)^s u + |u|^2 u \tag{1.1}$$

on the finite x-interval $[0, \pi]$ with Dirichlet boundary conditions

$$u(t, 0) = 0 = u(t, \pi), \quad -\infty < t < +\infty,$$

where $(-\Delta)^s$ denotes the Riesz fractional differentiation defined in [29] with $1/2 < s < 1$.

The fractional Schrödinger equation introduced by Laskin [30,31] derives a fractional version of classical quantum mechanics. By introducing fractional derivatives, it provides a better description of the non-local and non-Markovian behavior of particles in some special systems. Furthermore, the fractional Schrödinger equation provides a new mathematical tool for us to understand the behavior of particles in the microscopic world and some of the exotic phenomena and properties in quantum mechanics. Up to now, there has been a lot of excellent work on the FNLS equation, such as [22,24,28,29,34].

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In this paper, we consider the equation (1.1) with Hamiltonian tools. It is well known that equation (1.1) can be written as a Hamiltonian system

$$\dot{u} = \mathbf{i}\nabla H(u)$$

with the Hamiltonian function

$$H = \langle (-\Delta)^s u, u\rangle + \frac{1}{2}\int_0^\pi |u|^4 dx,$$

where $\langle \cdot, \cdot \rangle$ is the inner product

$$\langle u, v\rangle = \mathbf{Re}\int_0^\pi u\bar{v}dx.$$

The operator $(-\Delta)^s$ under Dirichlet boundary condition has a family of orthonormal eigenfunctions

$$\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin jx \tag{1.2}$$

and the corresponding eigenvalues are

$$\omega_j = j^{2s}, \quad j \geq 1.$$

Consider the Gevrey space

$$\mathcal{H}_{\rho,\theta} := \left\{ u = \sum_{j\geq 1} q_j\phi_j : \|u\|_{\rho,\theta} = 2\|q\|_{\rho,\theta} := 2\sum_{j\geq 1} e^{\rho|j|^\theta}|q_j| < \infty \right\}, \tag{1.3}$$

where $\rho > 0$, $0 < \theta < 1$. The following is our main result:

Theorem 1.1. *For any given $\rho > 0$, $0 < \theta < 1$ and $0 < \gamma < 1/10$, there exists an almost full measure set $\mathcal{F} \subset (1/2, 1)$ and a sufficiently small positive number ε_*, such that for any $s \in \mathcal{F}$ and $0 < \varepsilon < \varepsilon_*$, if the initial value to equation (1.1) fulfills*

$$\|u(0)\|_{\rho,\theta} < \varepsilon/6,$$

then the solution with the initial value $u(0)$ satisfies

$$\|u(t)\|_{\rho,\theta} < \varepsilon/2, \quad \forall\ |t| \leq \varepsilon^{-|\ln\varepsilon|^\gamma}.$$

Remark 1.1. The length of the stability time hinges on the regularity of the initial value, which means that the different topological spaces would yield distinguishing dynamical behavior. Bambusi-Sire [4] showed a polynomial-type stability time for equation (1.1) on the Sobolev space. Explicitly, it was proved in [4] that the solution was stable for arbitrary polynomial times ε^{-r} $(r \gg 1)$ w.r.t. ε-small initial data in the Sobolev norm. In this paper, we engage in the Gevrey space (1.3) and enhance the stability time to a subexponential result for equation (1.1).

Remark 1.2. The proof of Theorem 1.1 can be achieved by using the Birkhoff normal form method and the so-called tame property of the nonlinearity in Gevrey space (1.3). We mention that the tame property in Gevrey space, initially introduced by Cong-Liu-Wang [17], is applied to the nonlinear wave (NLW) equation, in which the indicator θ of space takes a special value 1/2. Motivated by [17], in this paper, we are concerned about the FNLS equation (1.1) and extend the range of indicator θ to $0 < \theta < 1$. It is worth noting that the discrepancies in the choice of parameters and the rate of frequency growth arising from different equations will contribute to some essential distinctions in the construction of non-resonant conditions. Moreover, the space changes also make some of the conclusions related to tame property slightly different from those of [17] (see Section 2 for more details).

Remark 1.3. We can likewise present the stability estimation for FNLS equation (1.1) by establishing similar tame properties in the following two Hilbert spaces.

(1) In the Gevrey space

$$\mathcal{H}_\sigma := \left\{ u = \sum_{j\geq 1} q_j\phi_j : \|u\|_\sigma = 2\|q\|_\sigma := 2\sum_{j\geq 1} e^{\sigma \ln^2 \lfloor j \rceil} |q_j| < \infty \right\}$$

with $\sigma > 0$, $\lfloor j \rceil := \max\{e^5, |j|\}$, we can prove that for almost $s \in (1/2, 1)$, the solutions to (1.1) starting with ε-small initial data maintain stable over time-intervals of length $\varepsilon^{-|\ln \varepsilon|^\gamma}$ with $0 < \gamma < 1/10$.

(2) In the modified Sobolev space

$$\mathcal{H}_p := \left\{ u = \sum_{j\geq 1} q_j\phi_j : \|u\|_p = 2\|q\|_p := 2\sum_{j\geq 1} \lfloor j \rceil^p |q_j| < \infty \right\} \tag{1.4}$$

with $p > 1/2$, $\lfloor j \rceil := \max\{2, |j|\}$, it can be shown that for almost $s \in (1/2, 1)$, if the initial data of equation (1.1) is ε-small, then their evolution remains in a ball of radius 2ε for time intervals of order $e^{C\cdot e^{\frac{1}{\varepsilon}}}$ with $C > 0$.

Remark 1.4. The result of Theorem 1.1 also holds for the following FNLS equation with a more general nonlinearity

$$\mathbf{i}u_t = (-\Delta)^s u + F(|u|^2)u,$$

where $F(z)$ is a real-valued polynomial function in z satisfying $F(0) = 0$. For the sake of simplicity in this paper we take $F(|u|^2)u = |u|^2 u$.

The use of the Birkhoff normal form method to study the long time stability of solutions of Hamiltonian PDEs is a fundamental problem and has been extensively investigated by many authors over the years. The first result was shown by Bourgain in [12], who researched the long time stability for the nonlinear wave (NLW) equation and nonlinear Schrödinger (NLS) equation. Another significant work in this area was given by Bambusi-Grébert in [1], they proved an abstract Birkhoff normal form theorem suited to a large class of PDEs with tame property including d-dimensional NLS equations ($d \geq 1$) and 1-dimensional NLW equations and discussed dynamical consequences on the polynomial long time behavior of the solutions with small initial data in the standard Sobolev space $\mathcal{H}^s \equiv \mathcal{H}^s(\mathbb{T}^d; \mathbb{C})$

$$\mathcal{H}^s := \left\{ q = (q_j)_{j\in\mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d} : \|q\|_s^2 := \sum_{j\in\mathbb{Z}^d} |q_j|^2 |j|^{2s} < \infty \right\},$$

where $j=(j_1,\dots,j_d)$ and $|j|=\sqrt{\sum_{i=1}^d j_i^2}$. More precisely, it was shown that for any $r\geq 1$ and s large enough (depending on r), there exist ε_s and some positive constant C_s such that if the initial datum $q(0)$ fulfills $\varepsilon=\|q(0)\|_s\leq\varepsilon_s$, it holds that

$$\|q(t)\|_s\leq 2\varepsilon,\quad \text{for all } |t|\leq C_s\varepsilon^{-r}.$$

Based on the method introduced by Bambusi-Grébert [1], Bambusi-Sire [4] performed the existence of almost global solutions to the FNLS equation (1.1). Explicitly, for almost $s>1/2$, if the Sobolev norm of the initial value is smaller than ε ($0<\varepsilon\ll 1$), then the corresponding solution is bounded by 2ε over time-intervals of length ε^{-B} (B arbitrary). A large number of similar results have been actualized for water waves equations, Klein-Gordon equations and derivative nonlinear Schrödinger (DNLS) equations, see [9, 10, 18–21, 42, 43] and references therein.

A natural step forward is to consider whether the solutions can be stable in a longer time such as the exponential long time. In the context of finite dimensional Hamiltonian systems of this field, a pioneering work was from Nekhoroshev [35], who exhibited that the evolution of all orbits remained stable over exponentially long time intervals. Since then, a great deal of analogous results have been presented for finite dimensional Hamiltonian systems, see [5, 6, 36], just to mention a few.

Recently, regarding applications to Hamiltonian PDEs, a remarkable work given by Faou-Grébert in [23] was to prove a subexponential long stability time result for d-dimensional NLS equations with analytic initial data. Afterward, Biasco-Massetti-Procesi [11] explored the dynamical behavior for NLS equations in different phase spaces, such as Sobolev space and Gevrey space. Subsequently, Cong-Liu-Wang [17] proved a subexponential result for the NLW equation

$$u_{tt}=u_{xx}-mu-f(u)$$

under Dirichlet boundary conditions in the following Gevrey space

$$\mathcal{H}_\rho:=\left\{u=\sum_{j\geq 1}q_j\phi_j:\|u\|_\rho=2\|q\|_\rho:=2\sum_{j\geq 1}e^{\rho\sqrt{|j|}}|q_j|<\infty\right\}. \tag{1.5}$$

Moreover, we mention that there have been many results in studying the long time stability of solutions of Hamiltonian PDEs by constructing tame property in different spaces, which can refer to [14, 16, 37, 38] for more details.

Lastly, we noticed that there are many results about an accurate description of the long time dynamics of the solutions for Hamiltonian PDEs without external parameters. See [7, 8, 13, 33, 41] for more details. Conversely, an opposite point of view is to establish special orbits for which the Sobolev norms grow as fast as possible, we refer the reader to [2, 3, 15, 25–27].

As far as we know, the periodic or quasi-periodic solutions of the FNLS equation have been obtained by using KAM theory with the following results. Li [32] demonstrated the existence of numerous quasi-periodic solutions for a class of space fractional nonlinear Schrödinger equations using the Riesz fractional derivative. Xu [40] established an infinite dimensional KAM theorem adapted to the FNLS equation with dense normal frequency and obtained a class of small amplitude quasi-periodic solutions with linear stability. Wu-Yuan [39] proved the existence of full dimensional KAM torus for the FNLS equation in the Gevrey space.

Inspired by Bambusi-Sire [4] and Cong-Liu-Wang [17], in the present paper, we would like to study the subexponential long time behavior of the FNLS equation (1.1) in Gevrey space (1.3) by using the Birkhoff normal form method and the so-called tame property.

2. Preliminary

As a preparatory step, we give some basic definitions and introduce the tame property of vector fields.

2.1. The phase space

Given $\rho > 0$, $0 < \theta < 1$, we define the Banach space $\ell_{\rho,\theta}(\mathbb{C})$ of all complex-valued sequences $q = (q_j)_{j\geq 1}$ with

$$\|q\|_{\rho,\theta} := \sum_{j\geq 1} |q_j| e^{\rho |j|^{\theta}} < \infty$$

and the scale of phase spaces

$$(q, \bar{q}) \in \mathcal{P}_{\rho,\theta}(\mathbb{C}) := \ell_{\rho,\theta}(\mathbb{C}) \oplus \ell_{\rho,\theta}(\mathbb{C}).$$

We identify a couple $(q, \bar{q}) \in \mathcal{P}_{\rho,\theta}(\mathbb{C})$ with $z = (z_j)_{j\in\bar{\mathbb{Z}}}$ ($\bar{\mathbb{Z}} := \mathbb{Z}\setminus\{0\}$) via the formula

$$z_{-j} = \bar{q}_j, \quad z_j = q_j, \quad j \geq 1$$

and define

$$\|z\|_{\rho,\theta} := \sum_{j\in\bar{\mathbb{Z}}} |z_j| e^{\rho |j|^{\theta}} < \infty.$$

Given a large $N > 0$, define $\bar{z} = (\bar{z}_j)_{j\in\bar{\mathbb{Z}}}$ by $\bar{z}_j = z_j$ when $|j| \leq N$ and otherwise $\bar{z}_j = 0$. Also let $\hat{z} = z - \bar{z}$.

Moreover, we denote by $B_{\mathbb{C},\rho,\theta}(R)$ the open ball centered at the origin and of radius R in $\mathcal{P}_{\rho,\theta}(\mathbb{C})$. Often, we simply write

$$\mathcal{P}_{\rho,\theta} \equiv \mathcal{P}_{\rho,\theta}(\mathbb{C}), \quad B_{\rho,\theta}(R) \equiv B_{\mathbb{C},\rho,\theta}(R).$$

Definition 2.1. Let $H : \mathcal{P}_{\rho,\theta} \to \mathbb{C}$ be a homogeneous polynomial of degree r by

$$H(z) = \sum_{|k|=r} H_k z^k, \quad k = (k_j)_{j\in\bar{\mathbb{Z}}} \in \mathbb{N}^{\bar{\mathbb{Z}}}, \quad z^k := \prod_{j\in\bar{\mathbb{Z}}} z_j^{k_j},$$

where H_k is the coefficient of the monomial z^k and $|k| := \sum_{j\in\bar{\mathbb{Z}}} k_j$ is the degree of $H(z)$. Then, we define its modulus $\lfloor H \rceil$ by

$$\lfloor H \rceil(z) := \sum_{|k|=r} |H_k| z^k.$$

For convenience, we keep fidelity with the notation and terminology from Bambusi - Grébert [1]. Let $H : \mathcal{P}_{\rho,\theta} \to \mathbb{C}$ be a homogeneous polynomial of degree r. We recall that H is continuous and also analytic if and only if it is bounded, namely if there exists a positive constant C such that

$$|H(z)| \leq C\|z\|^r_{\rho,\theta}, \quad \text{for any } z \in \mathcal{P}_{\rho,\theta}.$$

To the polynomial H it is naturally associated a symmetric r-linear form $\widetilde{H}$ such that

$$\widetilde{H}(z,\ldots,z) = H(z).$$

The r-linear form $\widetilde{H}$ is bounded; that is,

$$|\widetilde{H}(z^{(1)},\ldots,z^{(r)})| \leq C\|z^{(1)}\|_{\rho,\theta}\cdots\|z^{(r)}\|_{\rho,\theta}$$

(and analytic) if and only if H is bounded.

Given a polynomial vector field $X : \mathcal{P}_{\rho,\theta} \to \mathcal{P}_{\rho,\theta}$ homogeneous of degree r, we write it as

$$X(z) = \sum_{j\in\bar{\mathbb{Z}}} X_j(z)\mathbf{e}_j,$$

where $\mathbf{e}_j \in \mathcal{P}_{\rho,\theta}$ is the vector with all components equal to zero but the j-th one, which is equal to 1. Thus $X_j(z)$ is a homogeneous polynomial of degree r. We recall that X is analytic if and only if it is bounded, namely if there exists a positive constant C such that

$$\|X(z)\|_{\rho,\theta} \leq C\|z\|^r_{\rho,\theta}, \quad \text{for any } z \in \mathcal{P}_{\rho,\theta}.$$

Consider the r-linear symmetric form $\widetilde{X}$ and define $\widetilde{X} := \sum_{j\in\bar{\mathbb{Z}}}\widetilde{X}_j\mathbf{e}_j$ with

$$\widetilde{X}(z,\ldots,z) = X(z).$$

Analogously, the r-linear form $\widetilde{X}$ is bounded i.e.

$$\|\widetilde{X}(z^{(1)},\ldots,z^{(r)})\|_{\rho,\theta} \leq C\|z^{(1)}\|_{\rho,\theta}\ldots\|z^{(r)}\|_{\rho,\theta}$$

(and analytic) if and only if X is bounded. Moreover, the modulus of a vector field X is defined by

$$\lfloor X\rceil(z) = \sum_{j\in\bar{\mathbb{Z}}}\lfloor X_j\rceil(z)\mathbf{e}_j.$$

2.2. The tame norm of Hamiltonian vector field

Motivated by Cong-Liu-Wang [17], we will define the tame norm of vector fields and exhibit that the tame property is stable under the Poisson brackets. It is worth noting that in this paper we extend the spatial indicator θ from $1/2$ in [17] to $0 < \theta < 1$, which will give rise to some slight differences about the tame property.

Definition 2.2. Let H be a homogeneous polynomial of degree $r+1$, assume that H satisfies the following two conditions:

1. Tame property, i.e.[1]

$$A := \sup\frac{\|\lfloor\widetilde{X}_H\rceil(w)\|_{\rho,\theta}}{\|w\|^T_{\rho,\theta}} < \infty,$$

[1]The Hamiltonin vector field X_H is defined in (3.7) and $\widetilde{X}_H$ is the r-linear symmetric form of X_H.

where

$$\|w\|_{\rho,\theta}^T = \frac{1}{r}\sum_{l=1}^{r}\Big(\|z^{(1)}\|_{(2^\theta-1)\rho,\theta}\cdots\|z^{(l-1)}\|_{(2^\theta-1)\rho,\theta} \times\|z^{(l)}\|_{\rho,\theta}\|z^{(l+1)}\|_{(2^\theta-1)\rho,\theta}\cdots\|z^{(r)}\|_{(2^\theta-1)\rho,\theta}\Big)$$

and the sup is taken over all the multivectors $w = (z^{(1)}, \ldots, z^{(r)}) \neq 0$;

2. ***Bounded property***, i.e.

$$B := \sup \frac{\|\lfloor \widetilde{X}_H \rceil(w)\|_{(2^\theta-1)\rho,\theta}}{\|w\|_{(2^\theta-1)\rho,\theta}} < \infty,$$

where

$$\|w\|_{(2^\theta-1)\rho,\theta} = \|z^{(1)}\|_{(2^\theta-1)\rho,\theta}\cdots\|z^{(r)}\|_{(2^\theta-1)\rho,\theta}$$

and the sup is taken over all the multivectors $w = (z^{(1)}, \ldots, z^{(r)}) \neq 0$;

Then the tame norm of the vector field X_H can be defined by

$$|H|_{\rho,\theta}^T := \max\{A, B\}.$$

Remark 2.1. Let $f(x) = \sum_{j\geq 1} a_j x^j$ with $x \in \mathbb{R}$ and $a_j \geq 0$, then we have

$$\sup_{|x|\leq R} f(x) = \sup_{0\leq x\leq R} f(x).$$

In view of Remark 2.1, without loss of generality, we can always assume $z = (z_j)_{j\in\bar{\mathbb{Z}}}$ with $z_j \geq 0$ below.

Definition 2.3. Let H be a non-homogeneous polynomial. Consider its Taylor expansion

$$H = \sum_{r\geq 1} H_r,$$

where H_r is homogeneous polynomial of degree r. For $R > 0$, we denote

$$|H|_{\rho,\theta,R}^T := \sum_{r\geq 1} |H_r|_{\rho,\theta}^T \cdot R^{r-1}. \tag{2.1}$$

Such a definition can be naturally extended to the set of analytic functions, i.e., (2.1) is finite. The set of the functions with a finite $|H|_{\rho,\theta,R}^T$-norm is denoted by $T_{\rho,\theta,R}$.

Next, we aim to show that the tame property can be maintained under Poisson brackets. Given two functions $f(z)$ and $g(z)$, we define the Poisson bracket by

$$\{f, g\} = \mathbf{i}\sum_{j\geq 1}\left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial z_{-j}} - \frac{\partial f}{\partial z_{-j}}\frac{\partial g}{\partial z_j}\right).$$

Lemma 2.1. *Let f and g be homogeneous polynomials of degree $n+1$ and $m+1$, respectively, then one has*

$$|\{f, g\}|_{\rho,\theta}^T \leq (n+m)|f|_{\rho,\theta}^T |g|_{\rho,\theta}^T.$$

The proof of this lemma is similar to Lemma 2.6 in [17]. So we omit its proof here. By repeatedly exploiting the lemma 2.1, we obtain the following conclusion.

Corollary 2.1. *Let $f(z)$ and $g(z)$ be homogeneous polynomials of degree $\tilde{r}$ and r, respectively, satisfying $|f|^T_{\rho,\theta}$, $|g|^T_{\rho,\theta} < \infty$. Then for any $\nu \geq 1$, $\nu \in \mathbb{N}$, it follows that*

$$|f_{(\nu,g)}|^T_{\rho,\theta} \leq \frac{1}{\nu!} \cdot \prod_{i=1}^{\nu} (\tilde{r} + i(r-2)) \cdot |f|^T_{\rho,\theta} \left(|g|^T_{\rho,\theta}\right)^{\nu},$$

where

$$f_{(0,g)} := f, \quad f_{(\nu,g)} := \frac{1}{\nu}\{f_{(\nu-1,g)}, g\}.$$

Lastly, we are devoted to discussing the relationship between vector fields and their tame norms. Let us denote

$$\|X_H\|^R_{\rho,\theta} := \sup_{\|z\|_{\rho,\theta} \leq R} \|X_H(z)\|_{\rho,\theta}.$$

Lemma 2.2. *For a given Hamiltonian function H, the following holds:*

$$\|X_H\|^R_{\rho,\theta} \leq |H|^T_{\rho,\theta,R}.$$

The proof of this lemma is similar to Lemma 4.2 in [17]. So we omit its proof here.

Lemma 2.3. *Assume that H has a zero of order three in the variables $\hat{z}$, then one has*

$$\|X_H\|^R_{\rho,\theta} \ \leq \ \frac{4|H|^T_{\rho,\theta,2R}}{e^{(2-2^{\theta})\rho N^{\theta}}}.$$

Proof. Introduce the projector $\overline{\prod}$ on the modes with an index smaller than N and the projector $\widehat{\prod}$ on the modes with a large index. Expand H in the Taylor series (in all the variables); namely, write

$$H = \sum_{j \geq 1} H_j,$$

with H_j homogeneous polynomial of degree j. Consider the vector field of H_j, decompose it into the component on $\bar{z}$ and the component on $\hat{z}$. One has

$$\overline{\prod} X_{H_j} = J_{\bar{z}} \nabla_{\bar{z}} H_j, \tag{2.2}$$

$$\widehat{\prod} X_{H_j} = J_{\hat{z}} \nabla_{\hat{z}} H_j, \tag{2.3}$$

where we denote by $J_{\bar{z}}$ and $J_{\hat{z}}$ the two components of the Poisson tensor. From (2.2) and (2.3) one immediately realizes that $\overline{\prod} X_{H_j}$ has a zero of order three as a function of $\hat{z}$ and $\widehat{\prod} X_{H_j}$ has a zero of order two as a function of $\hat{z}$. Consider $\widehat{\prod} X_{H_j}$ and write $z = \bar{z} + \hat{z}$; one has

$$\begin{aligned} &\widehat{\prod} X_{H_j}(\bar{z} + \hat{z}) \\ =&\widehat{\prod} \widetilde{X}_{H_j}(\underbrace{\bar{z} + \hat{z}, \ldots, \bar{z} + \hat{z}}_{(j-1)-\text{times}}) \end{aligned} \tag{2.4}$$

$$=\sum_{l=2}^{j-1}\begin{pmatrix} j-1 \\ l \end{pmatrix}\widehat{\prod}\widetilde{X}_{H_j}(\underbrace{\hat{z},\ldots,\hat{z}}_{l-\text{times}},\underbrace{\bar{z},\ldots,\bar{z}}_{(j-l-1)-\text{times}}),$$

where the sum starts from 2 since $\widehat{\prod}X_{H_j}$ has a zero of order two as a function of $\hat{z}$. We estimate now a single term of the sum. By the tame property, we have

$$\begin{aligned}&\left\|\widehat{\prod}\widetilde{X}_{H_j}(\underbrace{\hat{z},\ldots,\hat{z}}_{l-\text{times}},\underbrace{\bar{z},\ldots,\bar{z}}_{(j-l-1)-\text{times}})\right\|_{\rho,\theta} \qquad (2.5)\\ &\le |H_j|^T_{\rho,\theta}\cdot\frac{1}{j-1}\Big(l\cdot\|\hat{z}\|_{\rho,\theta}\|\hat{z}\|^{l-1}_{(2^{\theta-1})\rho,\theta}\|\bar{z}\|^{j-l-1}_{(2^{\theta-1})\rho,\theta}\\ &\quad +(j-l-1)\|\hat{z}\|^{l}_{(2^{\theta-1})\rho,\theta}\|\bar{z}\|_{\rho,\theta}\|\bar{z}\|^{j-l-2}_{(2^{\theta-1})\rho,\theta}\Big).\end{aligned}$$

By applying the inequality

$$\|\hat{z}\|_{(2^{\theta-1})\rho,\theta}\le\frac{\|z\|_{\rho,\theta}}{e^{(2-2^\theta)\rho N^\theta}},$$

(2.5) can be bounded by

$$|H_j|^T_{\rho,\theta}\frac{\|z\|^{j-1}_{\rho,\theta}}{e^{(2-2^\theta)(l-1)\rho N^\theta}}.$$

By inserting into (2.4), one gets, for $z\in B_{\rho,\theta}(R)$,

$$\left\|\widehat{\prod}X_{H_j}(z)\right\|_{\rho,\theta}\le 2^j|H_j|^T_{\rho,\theta}\frac{R^{j-1}}{e^{(2-2^\theta)\rho N^\theta}}\le\frac{2|H_j|^T_{\rho,\theta,2R}}{e^{(2-2^\theta)\rho N^\theta}}. \qquad (2.6)$$

Similarly, it holds that

$$\left\|\overline{\prod}X_{H_j}(z)\right\|_{\rho,\theta}\le\frac{2|H_j|^T_{\rho,\theta,2R}}{e^{(2-2^\theta)\rho N^\theta}}. \qquad (2.7)$$

In view of (2.6) and (2.7), one has

$$\|X_{H_j}\|^R_{\rho,\theta}\le\frac{4|H_j|^T_{\rho,\theta,2R}}{e^{(2-2^\theta)\rho N^\theta}}.$$

Summing over j, one gets the thesis. □

3. The results for infinite dimensional Hamiltonian systems

In this section, our task is to perform a Birkhoff normal form theorem for a class of infinite dimensional Hamiltonian systems. The starting point is to introduce some notions and notations. Given $l\ge 2$ and $\mathbf{j}=(j_1,j_2,\cdots,j_l)\in\bar{\mathbb{Z}}^l$, we define

- the monomial associated with $\mathbf{j}$

$$z_{\mathbf{j}}=z_{j_1}z_{j_2}\cdots z_{j_l},$$

- the third largest positive integer associated with $\mathbf{j}$

$$\mu(\mathbf{j}),$$

  i.e. $\mu(\mathbf{j})$ is the third largest integer in $\{|j_1|, \cdots, |j_l|\}$.
- the degree associated with $\mathbf{j}$

$$d(\mathbf{j}) = l,$$

- the divisor associated with $\mathbf{j}$

$$\Omega(\mathbf{j}) = \mathrm{sgn}(j_1) \cdot \omega_{|j_1|} + \cdots + \mathrm{sgn}(j_l) \cdot \omega_{|j_l|}, \tag{3.1}$$

  where $\omega_{|j_i|} = |j_i|^{2s}$.

We denote the set of indices with zero momentum by

$$\mathcal{I}_l := \left\{\mathbf{j} = (j_1, j_2, \cdots, j_l) \in \bar{\mathbb{Z}}^l : j_1 \pm j_2 \pm \cdots \pm j_l = 0\right\}.$$

Moreover, for any $l \geq 3$ and $N \geq 1$, we set

$$\mathcal{J}_l(N) := \{\mathbf{j} \in \mathcal{I}_l : \mu(\mathbf{j}) > N\}.$$

We say that $\mathbf{j} = (j_1, j_2, \cdots, j_l) \in \bar{\mathbb{Z}}^l$ is resonant and write $\mathbf{j} \in \mathcal{N}_l$, if l is even and $\mathbf{j} = \mathbf{i} \cup (\mathbf{-i})$ for some choice of $\mathbf{i} \in \bar{\mathbb{Z}}^{l/2}$. Furthermore, when $\mathbf{j}$ is resonant, the monomial $z_{\mathbf{j}}$ associated with $\mathbf{j}$ only depends on the actions $(I_j)_{j\in\bar{\mathbb{Z}}}$ with $I_j = z_j z_{-j}$.

Next, our aim is to state Birkhoff normal form theorem for infinite dimensional Hamiltonian systems. Let us consider the Hamiltonian systems

$$\begin{cases} \dot{z}_j = \mathbf{i}\dfrac{\partial H}{\partial z_{-j}}, \\ \dot{z}_{-j} = -\mathbf{i}\dfrac{\partial H}{\partial z_j}, \end{cases} \qquad j \geq 1 \tag{3.2}$$

with respect to symplectic structure $\mathbf{i}\sum_{j\geq 1} dz_j \wedge dz_{-j}$. The corresponding Hamiltonian function is

$$H(z) = H_0(z) + P(z), \tag{3.3}$$

where

$$H_0 := \sum_{j\geq 1} \omega_j z_j z_{-j}, \tag{3.4}$$

the real numbers ω_j are frequencies and $P(z)$ has a zero of order at least three at the origin. Here we make some assumptions as follows.

(a) Given two positive numbers r, N, the frequencies ω_j are (r, N)-nonresonant, if ω_j satisfies the following condition

$$|\Omega(\mathbf{j})| \geq \frac{1}{N^{16(d(\mathbf{j}))^6}} \tag{3.5}$$

with $\mathbf{j} \notin \mathcal{N}_{d(\mathbf{j})}$, $\mu(\mathbf{j}) \leq N$ and $d(\mathbf{j}) \leq r$.

(b) The nonlinearity $P(z)$ is of the following form

$$P(z) = \sum_{n\geq 3} P_n(z), \qquad P_n(z) = \sum_{\mathbf{j}\in\mathcal{I}_n} a_{\mathbf{j}} z_{\mathbf{j}}.$$

Moreover, there exists a constant $C > 1$ such that for any $n \geq 3$,

$$|P_n|^T_{\rho,\theta} \leq C^{n-2}. \tag{3.6}$$

Let us denote by

$$X_H = \left(\mathbf{i}\frac{\partial H}{\partial z_{-j}},\ -\mathbf{i}\frac{\partial H}{\partial z_j}\right)_{j\geq 1} \tag{3.7}$$

the Hamiltonian vector field of $H(z)$.

Theorem 3.1. (Birkhoff normal form theorem) *Consider the Hamiltonian system* (3.2) *and suppose that the Hamiltonian* (3.3) *satisfies* (a) *and* (b). *For any $\rho > 0$, $0 < \theta < 1$ and $0 < \gamma < 1/10$, there exists a sufficiently small positive number ε_*, such that for any $0 < \varepsilon < \varepsilon_*$, there exists a canonical transformation $\mathcal{T} : B_{\rho,\theta}(\varepsilon/2) \to B_{\rho,\theta}(\varepsilon)$ changing the Hamiltonian H in* (3.3) *into*

$$\widetilde{H} := H \circ \mathcal{T} = H_0 + Z(z) + W(z) + \mathcal{R}(z),$$

where

(1) *The canonical transformation $\mathcal{T}$ fulfills*

$$\sup_{z\in B_{\rho,\theta}(\varepsilon/2)} \|\mathcal{T} - id\|_{\rho,\theta} \ \leq \ \varepsilon^{\frac{3}{2}}. \tag{3.8}$$

Exactly the same estimate is also true to the inverse of canonical transformation $\mathcal{T}$.

(2) *$Z(z)$ is a polynomial only depending on actions $(I_j)_{j\geq 1}$. $W(z)$ and $\mathcal{R}(z)$ fulfill the following estimations:*

$$\sup_{z\in B_{\rho,\theta}(\varepsilon/2)} \|X_W(z)\|_{\rho,\theta}\,, \quad \sup_{z\in B_{\rho,\theta}(\varepsilon/2)} \|X_{\mathcal{R}}(z)\|_{\rho,\theta} \leq \ \varepsilon^{|\ln\varepsilon|^\gamma + \frac{3}{2}}. \tag{3.9}$$

The proof of Theorem 3.1 is postponed in Section 4. Subsequently, we get an important corollary from Theorem 3.1.

Corollary 3.1. *Consider the Hamiltonian system* (3.2) *and suppose that the Hamiltonian* (3.3) *satisfies* (a) *and* (b). *For any $\rho > 0$, $0 < \theta < 1$ and $0 < \gamma < 1/10$, there exists a sufficiently small positive number ε_*, such that for any $0 < \varepsilon < \varepsilon_*$, if the initial value to equation* (3.2) *fulfills*

$$\|z(0)\|_{\rho,\theta} \leq \varepsilon/6,$$

then the solution with the initial value $z(0)$ satisfies

$$\|z(t)\|_{\rho,\theta} \leq \varepsilon, \quad \forall\ |t| \leq \varepsilon^{-|\ln\varepsilon|^\gamma}.$$

Proof. By applying Theorem 3.1, there exists a canonical transformation $\mathcal{T}$ with $\mathcal{T}(\tilde{z}) = z$, which changes the Hamiltonian system (3.2) into

$$\begin{cases} \dot{\tilde{z}}_j = \mathbf{i}\dfrac{\partial \widetilde{H}}{\partial \tilde{z}_{-j}}, \\ \dot{\tilde{z}}_{-j} = -\mathbf{i}\dfrac{\partial \widetilde{H}}{\partial \tilde{z}_j}, \end{cases} \qquad j \geq 1 \tag{3.10}$$

with

$$\widetilde{H}(\tilde{z}) := H \circ \mathcal{T}(\tilde{z}) = H_0 + Z(\tilde{z}) + W(\tilde{z}) + \mathcal{R}(\tilde{z}).$$

In view of (3.8), one has that the solution $z(t)$ to (3.2) with initial value $\|z(0)\|_{\rho,\theta} \leq \varepsilon/6$ is transformed into $\tilde{z}(t)$ to (3.10) with initial value $\|\tilde{z}(0)\|_{\rho,\theta} \leq \varepsilon/3$. Moreover, for any $\tilde{z}(0) \in B_{\rho,\theta}(\varepsilon/3)$, there exists a time T such that

$$\|\tilde{z}(T)\|_{\rho,\theta} = \varepsilon/2$$

and

$$\tilde{z}(t) \in B_{\rho,\theta}\left(\varepsilon/2\right), \quad \forall\ |t| < T.$$

By taking advantage of (3.9), we get

$$\begin{aligned} \left|\ \|\tilde{z}(T)\|^2_{\rho,\theta} - \|\tilde{z}(0)\|^2_{\rho,\theta}\ \right| &= \left| \int_0^T \frac{d\,\|\tilde{z}(t)\|^2_{\rho,\theta}}{dt} \right| = \left| \int_0^T \left\{ \widetilde{H},\ \|\tilde{z}(t)\|^2_{\rho,\theta} \right\} dt \right| \\ &\leq\ T \sup_{\tilde{z} \in B_{\rho,\theta}(\varepsilon/2)} \left| \left\{ \widetilde{H}(\tilde{z}),\ \|\tilde{z}\|^2_{\rho,\theta} \right\} \right| \\ &\leq\ T \cdot 2 \cdot \varepsilon^{|\ln \varepsilon|^\gamma + \frac{5}{2}}. \end{aligned}$$

Accordingly, for any $|t| \leq T$ $\left(T \geq \varepsilon^{-|\ln \varepsilon|^\gamma}\right)$, one has

$$\left|\ \|\tilde{z}(t)\|^2_{\rho,\theta} - \|\tilde{z}(0)\|^2_{\rho,\theta}\ \right| \leq \frac{5}{36}\varepsilon^2$$

and

$$\tilde{z}(t) \in B_{\rho,\theta}\left(\varepsilon/2\right).$$

Consequently, the solution $z(t)$ to (3.2) with $\tilde{z}(0) \in B_{\rho,\theta}\left(\varepsilon/6\right)$ will satisfy

$$z(t) \in B_{\rho,\theta}(\varepsilon), \quad \forall\ |t| \leq \varepsilon^{-|\ln \varepsilon|^\gamma}.$$

We complete the proof of Corollary 3.1. □

4. The proof of Birkhoff normal form theorem

To perform the proof of Birkhoff normal form theorem, we would like to introduce some related tools. Let us consider an auxiliary Hamiltonian function $\mathcal{X}(z)$ associated with the Hamiltonian equations

$$\dot{z} = X_{\mathcal{X}}(z). \tag{4.1}$$

Denote $\phi^t_{\mathcal{X}}$ as the flow of equation (4.1) and $\phi^t_{\mathcal{X}}|_{t=0}$ is an identity mapping.

Definition 4.1. The transformation $\phi^1_{\mathcal{X}} = \phi^t_{\mathcal{X}}|_{t=1}$ is called a Lie-transformation generated by Hamiltonian $\mathcal{X}(z)$.

Given an analytic function $f(z)$. For any $n \in \mathbb{N}$, it follows that

$$\frac{d^n}{dt^n}\left(f \circ \phi^t_{\mathcal{X}}\right) = \underbrace{\{\{f, \mathcal{X}\}, \ldots \mathcal{X}\}}_{n \text{ times}} \circ \phi^t_{\mathcal{X}}.$$

According to the Taylor expansion, we deduce that

$$f \circ \phi^1_{\mathcal{X}} = \sum_{\nu=0}^{\infty} f_{(\nu,\mathcal{X})},$$

where

$$f_{(0,\mathcal{X})} := f, \quad f_{(\nu,\mathcal{X})} := \frac{1}{\nu}\left\{f_{(\nu-1,\mathcal{X})},\ \mathcal{X}\right\}, \quad \nu \geq 1.$$

Lemma 4.1. (Homological equation) *Let $Q(z) = \sum_{\boldsymbol{j}\in\mathcal{I}_n} Q_{\boldsymbol{j}} z_{\boldsymbol{j}}$ be a homogeneous polynomial of degree n and suppose that the (r, N)-nonresonance condition* (3.5) *is satisfied. Then there exists a unique solution $\mathcal{X}(z)$ such that*

$$\{H_0, \mathcal{X}\} + Q = Z + R,$$

where H_0 is defined in (3.4) *and*

$$Z(z) = \sum_{\substack{\boldsymbol{j}\in\mathcal{N}_n \\ \mu(\boldsymbol{j})\leq N}} Q_{\boldsymbol{j}} z_{\boldsymbol{j}}, \quad R(z) = \sum_{\boldsymbol{j}\in\mathcal{J}_n(N)} Q_{\boldsymbol{j}} z_{\boldsymbol{j}}.$$

Moreover, one has

$$|\mathcal{X}|^T_{\rho,\theta} \leq N^{16n^6} |Q|^T_{\rho,\theta} \quad \textit{and} \quad |Z|^T_{\rho,\theta},\ |R|^T_{\rho,\theta} \leq |Q|^T_{\rho,\theta}. \tag{4.2}$$

Proof. Observing the fact that

$$\{H_0, z_{\mathbf{j}}\} = -\mathbf{i}\Omega(\mathbf{j}) z_{\mathbf{j}},$$

where $\Omega(\mathbf{j})$ is defined in (3.1), we get $\mathcal{X}(z)$, $Z(z)$ and $R(z)$ are all homogeneous polynomials of degree n with the following forms

$$\mathcal{X} = \sum_{\mathbf{j}\in\mathcal{I}_n} \mathcal{X}_{\mathbf{j}} z_{\mathbf{j}}, \quad Z = \sum_{\mathbf{j}\in\mathcal{I}_n} Z_{\mathbf{j}} z_{\mathbf{j}}, \quad R = \sum_{\mathbf{j}\in\mathcal{I}_n} R_{\mathbf{j}} z_{\mathbf{j}}$$

and

$$-\mathbf{i}\Omega(\mathbf{j})\mathcal{X}_{\mathbf{j}} + Q_{\mathbf{j}} = Z_{\mathbf{j}} + R_{\mathbf{j}}, \quad \mathbf{j} \in \mathcal{I}_n.$$

Moreover, the coefficients satisfy

- if $\mu(\mathbf{j}) > N$,

$$\mathcal{X}_{\mathbf{j}} = 0, \quad Z_{\mathbf{j}} = 0, \quad R_{\mathbf{j}} = Q_{\mathbf{j}}, \tag{4.3}$$

- if $\mu(\mathbf{j}) \leq N$ and $\mathbf{j} \in \mathcal{N}_n$,

$$\mathcal{X}_{\mathbf{j}} = 0, \quad Z_{\mathbf{j}} = Q_{\mathbf{j}}, \quad R_{\mathbf{j}} = 0, \tag{4.4}$$

- if $\mu(\mathbf{j}) \leq N$ and $\mathbf{j} \notin \mathcal{N}_n$,

$$\mathcal{X}_{\mathbf{j}} = \frac{Q_{\mathbf{j}}}{\mathbf{i}\Omega(\mathbf{j})}, \qquad Z_{\mathbf{j}} = 0, \qquad R_{\mathbf{j}} = 0. \tag{4.5}$$

In view of (r, N)-nonresonance condition (3.5) and (4.3)-(4.5), we obtain (4.2). □

Subsequently, we are devoted to presenting the iterative lemma, which is the most important component of proving Theorem 3.1. In the statement of the forthcoming iterative lemma, we use the following notation. Given an integer $r \geq 1$ and a sufficiently small positive number ε, set

$$\varepsilon_n := \varepsilon\left(1 - \frac{n}{2r}\right), \quad n \in \mathbb{N}, \quad 0 \leq n \leq r.$$

Lemma 4.2. (Iterative lemma) *Consider the Hamiltonian systems* (3.2) *and suppose that the Hamiltonian* (3.3) *satisfies* (a) *and* (b). *Given an integer $r \geq 1$, $0 < \varepsilon \ll 1$ and $\rho > 0$, $0 < \theta < 1$. For any $0 \leq m \leq r$ and any integer $N > 0$, there exists an analytic canonical transformation $\mathcal{T}^{(m)} : B_{\rho,\theta}(\varepsilon_m) \to B_{\rho,\theta}(\varepsilon)$ changing the Hamiltonian H in* (3.3) *into*

$$H^{(m)} := H \circ \mathcal{T}^{(m)} = H_0 + Z^{(m)}(z) + \mathcal{R}^{N(m)}(z) + \mathcal{R}^{T(m)}(z) \tag{4.6}$$

and the following properties are satisfied:

(1) *the transformation $\mathcal{T}^{(m)}$ satisfies*

$$\sup_{z \in B_{\rho,\theta}(\varepsilon_m)} \left\| \mathcal{T}^{(m)} - id \right\|_{\rho,\theta} \leq \sum_{n=3}^{m+2} N^{16n^6} \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1}. \tag{4.7}$$

(2) *$Z^{(m)}(z)$ is a polynomial of degree $m+2$ with the following form*

$$Z^{(m)}(z) = \sum_{n=3}^{m+2} Z_n^{(m)}(z), \quad Z_n^{(m)}(z) = \sum_{\substack{j \in \mathcal{N}_n \\ \mu(j) \leq N}} \left(Z_n^{(m)}\right)_j z_j.$$

Moreover, for any $3 \leq n \leq m+2$, it holds that

$$\left| Z_n^{(m)} \right|_{\rho,\theta}^{T} \leq \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2}. \tag{4.8}$$

(3) *$\mathcal{R}^{N(m)}(z)$ is a polynomial of degree $m+2$ with the following form*

$$\mathcal{R}^{N(m)}(z) = \sum_{n=3}^{m+2} \mathcal{R}_n^{N(m)}(z), \quad \mathcal{R}_n^{N(m)}(z) = \sum_{j \in \mathcal{J}_n(N)} \left(\mathcal{R}_n^{N(m)}\right)_j z_j.$$

Moreover, for any $3 \leq n \leq m+2$, it follows that

$$\left| \mathcal{R}_n^{N(m)} \right|_{\rho,\theta}^{T} \leq \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2}. \tag{4.9}$$

(4) *The remainder term* $\mathcal{R}^{T(m)}(z)$ *has a zero of order* $m+3$ *as follows*

$$\mathcal{R}^{T(m)}(z) = \sum_{n \geq m+3} \mathcal{R}_n^{T(m)}(z).$$

Moreover, for any $n \geq m+3$*, one has*

$$\left|\mathcal{R}_n^{T(m)}\right|_{\rho,\theta}^{T} \leq \left(2^{5m} N^{16(m+2)^6}\right)^{n-3} C^{n-2}. \tag{4.10}$$

The proof of Lemma 4.2 is postponed in Appendix A.

Lastly, our task is to show the proof of Theorem 3.1 by using the iterative lemma.

Proof. Given $\rho > 0$, $0 < \theta < 1$, $0 < \gamma < 1/10$ and $0 < \varepsilon \ll 1$. Then we are going to take

$$m = r$$

in Lemma 4.2. Let us set

$$r = |\ln \varepsilon|^{\gamma}, \qquad N = \left(\frac{|\ln \varepsilon|^{\gamma+1}}{(2-2^{\theta})\rho}\right)^{\frac{1}{\theta}}. \tag{4.11}$$

If ε is small enough, one has

$$\left(2^{5r} N^{16(r+2)^6} C\right)^{r+3} \varepsilon^{\frac{1}{2}} < 1. \tag{4.12}$$

By applying (1) in Lemma 4.2 and (4.12), the canonical transformation $\mathcal{T}^{(r)}$: $B_{\rho,\theta}(\varepsilon/2) \to B_{\rho,\theta}(\varepsilon)$ satisfies

$$\begin{aligned}
\sup_{z \in B_{\rho,\theta}(\varepsilon/2)} \left\|\mathcal{T}^{(r)} - \mathrm{id}\right\|_{\rho,\theta} &\leq \sum_{n=3}^{r+2} N^{16n^6} \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1} \\
&\leq r \cdot N^{16(r+2)^6} \left(2^{5r} N^{16(r+1)^6}\right)^{r} C^{r} \cdot \varepsilon^{2} \\
&\leq \left(\left(2^{5r} N^{16(r+2)^6} C\right)^{r+1} \varepsilon^{\frac{1}{2}}\right) \cdot \varepsilon^{\frac{3}{2}} \\
&\leq \varepsilon^{\frac{3}{2}},
\end{aligned} \tag{4.13}$$

which implies that we complete the proof of (3.8).

In view of (2) in Lemma 4.2, it can be seen that $Z^{(r)}(z)$ is a polynomial only depending on the actions.

According to (3) in Lemma 4.2, Lemma 2.2, Lemma 2.3, (4.9), (4.11) and (4.12), it follows that

$$\begin{aligned}
\sup_{z \in B_{\rho,\theta}(\varepsilon/2)} \|X_{\mathcal{R}^{N(r)}}(z)\|_{\rho,\theta} &\leq \sum_{n=3}^{r+2} \sup_{z \in B_{\rho,\theta}(\varepsilon/2)} \left\|X_{\mathcal{R}_n^{N(r)}}(z)\right\|_{\rho,\theta} \\
&\leq \sum_{n=3}^{r+2} \frac{4 \cdot \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1}}{e^{(2-2^{\theta})\rho N^{\theta}}} \\
&\leq \frac{4r \cdot \left(2^{5r} N^{16(r+1)^6}\right)^{r} C^{r} \cdot \varepsilon^{2}}{e^{(2-2^{\theta})\rho N^{\theta}}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{e^{(2-2^\theta)\rho N^\theta}} \cdot \left(\left(2^{5r} N^{16(r+2)^6} C\right)^{r+1} \varepsilon^{\frac{1}{2}} \right) \cdot \varepsilon^{\frac{3}{2}} \\
&\leq \frac{1}{e^{(2-2^\theta)\rho N^\theta}} \cdot \varepsilon^{\frac{3}{2}} \\
&= \varepsilon^{|\ln \varepsilon|^\gamma + \frac{3}{2}}. \qquad (4.14)
\end{aligned}
$$

Moreover, by taking advantage of (4) in Lemma 4.2, Lemma 2.2, (4.10)-(4.12), we deduce

$$
\begin{aligned}
\sup_{z \in B_{\rho,\theta}(\varepsilon/2)} \|X_{\mathcal{R}^{T(r)}}(z)\|_{\rho,\theta} &\leq \sum_{n \geq r+3} \sup_{z \in B_{\rho,\theta}(\varepsilon/2)} \left\| X_{\mathcal{R}_n^{T(r)}}(z) \right\|_{\rho,\theta} \\
&\leq \sum_{n \geq r+3} \left(2^{5r} N^{16(r+2)^6}\right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1} \\
&\leq 2 \left(2^{5r} N^{16(r+2)^6} C\varepsilon\right)^{r+2} \\
&\leq \left(\left(2^{5r} N^{16(r+2)^6} C\right)^{r+3} \varepsilon^{\frac{1}{2}} \right) \cdot \varepsilon^{r+\frac{3}{2}} \\
&\leq \varepsilon^{|\ln \varepsilon|^\gamma + \frac{3}{2}}. \qquad (4.15)
\end{aligned}
$$

In view of (4.14) and (4.15), we finish the proof of (3.9).

Consequently, we complete the proof of Theorem 3.1. □

5. The proof of Theorem 1.1

Firstly, our task is to change the equation (1.1) under Dirichlet boundary conditions into an infinite dimensional Hamiltonian system by using Fourier transformation. Let us set

$$u(x,t) = \sum_{j \geq 1} q_j(t) \phi_j(x),$$

where $\phi_j(x)$ is defined in (1.2). Equation (1.1) can be transformed as

$$
\begin{cases}
\dot{q}_j = \mathbf{i} \dfrac{\partial H}{\partial \bar{q}_j}, \\
\dot{\bar{q}}_j = -\mathbf{i} \dfrac{\partial H}{\partial q_j},
\end{cases}
\qquad j \geq 1
$$

with respect to symplectic structure $\mathbf{i} \sum_{j \geq 1} dq_j \wedge d\bar{q}_j$. The corresponding Hamiltonian function is

$$H = H_0 + P(q, \bar{q}), \qquad (5.1)$$

where

$$H_0 := \sum_{j \geq 1} \omega_j q_j \bar{q}_j, \qquad \omega_j := j^{2s} \qquad (5.2)$$

and

$$P(q,\bar{q}) = \frac{1}{2} \int_0^\pi |u|^4 dx = \frac{1}{2} \sum_{j_1 \pm j_2 \pm j_3 \pm j_4 = 0} P_{j_1 j_2 j_3 j_4} q_{j_1} \bar{q}_{j_2} q_{j_3} \bar{q}_{j_4}, \qquad (5.3)$$

with

$$P_{j_1 j_2 j_3 j_4} = \int_0^\pi \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4} dx.$$

Now it suffices to show that the Hamiltonian (5.1) satisfies the assumptions (a) and (b) in Theorem 3.1, then Theorem 1.1 follows from Corollary 3.1.

For almost $s \in (1/2, 1)$, the frequencies $\omega = (\omega_j)_{j\geq 1}$ in (5.2) fulfill the (r, N)-nonresonance condition (3.5), which will be proved in Appendix B. Next, our aim is to show that the nonlinearity P in (5.3) fulfills (3.6), namely, there exists a constant $C > 1$ such that

$$|P|^T_{\rho,\theta} \leq C^2. \tag{5.4}$$

Firstly, we will give a technical lemma.

Lemma 5.1. *Let $0 < \theta < 1$ and $a_1 \geq a_2 \geq \cdots \geq a_N > 0$. Then one has*

$$(a_1 + a_2 + \cdots + a_N)^\theta \leq a_1^\theta + (2^\theta - 1)\sum_{i=2}^N a_i^\theta. \tag{5.5}$$

The proof of (5.5) is very similar with Lemma 5.1 in [17], we omit it here.

Lastly, we present that the nonlinearity P in (5.3) satisfies the tame property. Namely, there exists a constant $A > 0$ such that

$$\left\| \lfloor \widetilde{X}_P \rceil (z^{(1)}, z^{(2)}, z^{(3)}) \right\|_{\rho,\theta} \leq A \|(z^{(1)}, z^{(2)}, z^{(3)})\|^T_{\rho,\theta}. \tag{5.6}$$

In view of (5.3), there exists a constant $\widetilde{C} > 0$ such that

$$\left\| \lfloor \widetilde{X}_P \rceil (z^{(1)}, z^{(2)}, z^{(3)}) \right\|_{\rho,\theta} \leq \widetilde{C} \left\| \widetilde{X}(z^{(1)}, z^{(2)}, z^{(3)}) \right\|_{\rho,\theta}, \tag{5.7}$$

where

$$\widetilde{X}(z^{(1)}, z^{(2)}, z^{(3)}) = \left(\widetilde{X}_j(z^{(1)}, z^{(2)}, z^{(3)}) \right)_{j \in \bar{\mathbb{Z}}}$$

with

$$\widetilde{X}_j(z^{(1)}, z^{(2)}, z^{(3)}) = \sum_\tau \sum_{\substack{j_1 \pm j_2 \pm j_3 = j \\ j_1, j_2, j_3 \in \bar{\mathbb{Z}}}} z_{j_1}^{\tau(1)} z_{j_2}^{\tau(2)} z_{j_3}^{\tau(3)}$$

and τ are all the permutations of the first 3 integers. Moreover, by applying Lemma 5.1, it follows that

$$\begin{aligned}
&\left\| \widetilde{X}(z^{(1)}, z^{(2)}, z^{(3)}) \right\|_{\rho,\theta} \\
&\leq \sum_\tau \sum_{j \in \bar{\mathbb{Z}}} \sum_{j_1 \pm j_2 \pm j_3 = j} |z_{j_1}^{\tau(1)}||z_{j_2}^{\tau(2)}||z_{j_3}^{\tau(3)}| e^{\rho(|j_1|+|j_2|+|j_3|)^\theta} \\
&\leq \sum_\tau \Big(\|z^{\tau(1)}\|_{\rho,\theta} \|z^{\tau(2)}\|_{(2^\theta-1)\rho,\theta} \|z^{\tau(3)}\|_{(2^\theta-1)\rho,\theta} + \|z^{\tau(1)}\|_{(2^\theta-1)\rho,\theta} \|z^{\tau(2)}\|_{\rho,\theta} \\
&\quad \times \|z^{\tau(3)}\|_{(2^\theta-1)\rho,\theta} + \|z^{\tau(1)}\|_{(2^\theta-1)\rho,\theta} \|z^{\tau(2)}\|_{(2^\theta-1)\rho,\theta} \|z^{\tau(3)}\|_{\rho,\theta} \Big) \\
&\leq 18 \cdot \|(z^{(1)}, z^{(2)}, z^{(3)})\|^T_{\rho,\theta}.
\end{aligned} \tag{5.8}$$

By using (5.7) and (5.8), we get (5.6).

Similarly, we obtain that there exists a constant $B > 0$ such that

$$\left\| \lfloor \widetilde{X}_P \rceil (z^{(1)}, z^{(2)}, z^{(3)}) \right\|_{(2^\theta-1)\rho,\theta} \leq B \|(z^{(1)}, z^{(2)}, z^{(3)})\|_{(2^\theta-1)\rho,\theta}. \tag{5.9}$$

In view of (5.6) and (5.9), we complete the proof of (5.4).

Appendix

Appendix A: The proof of Lemma 4.2

Proof. We will prove the thesis inductively. Let us start by rewriting the Hamiltonian H in (3.3) defined on $B_{\rho,\theta}(\varepsilon)$ as follows

$$H^{(0)} := H = H_0 + Z^{(0)}(z) + \mathcal{R}^{N(0)}(z) + \mathcal{R}^{T(0)}(z), \tag{A.1}$$

where H_0 is defined in (3.4), $Z^{(0)}(z) = 0$, $\mathcal{R}^{N(0)}(z) = 0$ and $\mathcal{R}^{T(0)}(z) = P(z)$.

Next, our idea is to search for a Lie-transformation to eliminate the nonnormalized terms of 3-degree polynomials of $\mathcal{R}^{T(0)}(z)$. More precisely, we are devoted to producing a homogeneous polynomial $\mathcal{X}_0(z)$ of degree 3 and denote the time 1 flow $\mathcal{T}_0 := \phi^t_{\mathcal{X}_0}|_{t=1} : B_{\rho,\theta}(\varepsilon_1) \to B_{\rho,\theta}(\varepsilon)$, which changes the Hamiltonian $H^{(0)}$ in (A.1) into the following form

$$\begin{aligned} H^{(1)} :=\ & H^{(0)} \circ \mathcal{T}_0 \\ =\ & \left(H_0 + \mathcal{R}^{T(0)}\right) \circ \mathcal{T}_0 \\ =\ & H_0 \\ & + \{H_0, \mathcal{X}_0\} + P_3 && \text{(A.2)} \\ & + \sum_{\nu\geq 2} (H_0)_{(\nu,\mathcal{X}_0)} + \sum_{\nu\geq 1} (P_3)_{(\nu,\mathcal{X}_0)} + \sum_{\nu\geq 0} \left(\sum_{n\geq 4} P_n\right)_{(\nu,\mathcal{X}_0)}. && \text{(A.3)} \end{aligned}$$

Recalling that (3.6) and Lemma 4.1, we get

$$\text{(A.2)} = Z_0(z) + R_0(z),$$

where

$$Z_0(z) = \sum_{\substack{\mathbf{j}\in\mathcal{N}_3 \\ \mu(\mathbf{j})\leq N}} (P_3)_{\mathbf{j}}\, z_{\mathbf{j}}, \qquad R_0(z) = \sum_{\mathbf{j}\in\mathcal{J}_3(N)} (P_3)_{\mathbf{j}}\, z_{\mathbf{j}}$$

and

$$\sup_{(q,\bar{q})\in B_{\rho,\theta}(\varepsilon)} \|X_{\mathcal{X}_0}(z)\|_{\rho,\theta} \leq |\mathcal{X}_0|^T_{\rho,\theta} \cdot \varepsilon^2 \leq N^{16\cdot 3^6} \cdot C \cdot \varepsilon^2.$$

Furthermore, the transformation $\mathcal{T}_0$ satisfies

$$\sup_{z\in B_{\rho,\theta}(\varepsilon_1)} \|\mathcal{T}_0 - \mathrm{id}\|_{\rho,\theta} \leq \sup_{z\in B_{\rho,\theta}(\varepsilon)} \|X_{\mathcal{X}_0}(z)\|_{\rho,\theta} \leq N^{16\cdot 3^6} \cdot C \cdot \varepsilon^2.$$

Let us set $\mathcal{T}^{(1)} := \mathcal{T}_0 : B_{\rho,\theta}(\varepsilon_1) \to B_{\rho,\theta}(\varepsilon)$, then write

$$H^{(1)} = H^{(0)} \circ \mathcal{T}^{(1)} = H_0 + Z^{(1)}(z) + \mathcal{R}^{N(1)}(z) + \mathcal{R}^{T(1)}(z)$$

defined in $B_{\rho,\theta}(\varepsilon_1)$, where

$$Z^{(1)}(z) := Z^{(0)} + Z_0 = Z_0$$

and

$$\mathcal{R}^{N(1)}(z) := \mathcal{R}^{N(0)} + R_0 = R_0.$$

In view of Lemma 4.1 and (3.6), it follows that

$$|Z^{(1)}|^T_{\rho,\theta} \le C, \qquad |\mathcal{R}^{N(1)}|^T_{\rho,\theta} \le C.$$

Moreover, one has $\mathcal{R}^{T(1)}(z) = $ (A.3) with $\mathcal{R}^{T(1)}(z) = \sum_{n\ge 4} \mathcal{R}^{T(1)}_n(z)$. More precisely, for any $n \ge 4$, we get

$$\mathcal{R}^{T(1)}_n = (H_0)_{(n-2,\mathcal{X}_0)} + (P_3)_{(n-3,\mathcal{X}_0)} + \sum_{k=0}^{n-4} (P_{n-k})_{(k,\mathcal{X}_0)} \,.$$

By applying Corollary 2.1 and (3.6), it holds that

$$\begin{aligned}
\left|\mathcal{R}^{T(1)}_n\right|^T_{\rho,\theta} &\le \left|(H_0)_{(n-2,\mathcal{X}_0)}\right|^T_{\rho,\theta} + \left|(P_3)_{(n-3,\mathcal{X}_0)}\right|^T_{\rho,\theta} + \left|\sum_{k=0}^{n-4}(P_{n-k})_{(k,\mathcal{X}_0)}\right|^T_{\rho,\theta} \\
&\le \frac{1}{(n-2)!}\cdot \prod_{i=1}^{n-3}(3+i)\cdot C\cdot \left(N^{16\cdot 3^6}\cdot C\right)^{n-3} \\
&\quad + \sum_{k=0}^{n-3}\frac{1}{k!}\cdot\prod_{i=1}^{k}(n-k+i)\cdot C^{n-k-2}\cdot\left(N^{16\cdot 3^6}\cdot C\right)^k \\
&\le 2^{n+1}\cdot\left(N^{16\cdot 3^6}\right)^{n-3}\cdot C^{n-2} \\
&\le \left(2^5\cdot N^{16\cdot 3^6}\right)^{n-3}\cdot C^{n-2}.
\end{aligned}$$

Now assume that these statements in Lemma 4.2 are trivially true for $m < r$, then our task is to perform that these propositions which are also valid at rank $m+1$.

Proceeding as before, we are going to create a homogeneous polynomial $\mathcal{X}_m(z)$ of degree $m+3$ and denote the time 1 flow $\mathcal{T}_m := \phi^t_{\mathcal{X}_m}|_{t=1} : B_{\rho,\theta}(\varepsilon_{m+1}) \to B_{\rho,\theta}(\varepsilon_m)$, which changes the Hamiltonian $H^{(m)}$ in (4.6) into the following form

$$\begin{aligned}
H^{(m+1)} :&= H^{(m)}\circ\mathcal{T}_m \\
&= \left(H_0 + Z^{(m)} + \mathcal{R}^{N(m)} + \mathcal{R}^{T(m)}\right)\circ\mathcal{T}_m \\
&= H_0 \;+\; Z^{(m)} \;+\; \mathcal{R}^{N(m)} \\
&\quad + \{H_0,\mathcal{X}_m\} \;+\; \mathcal{R}^{T(m)}_{m+3} && \text{(A.4)} \\
&\quad + \sum_{\nu\ge 2}(H_0)_{(\nu,\mathcal{X}_m)} \;+\; \sum_{\nu\ge 1}\left(Z^{(m)}\right)_{(\nu,\mathcal{X}_m)} + \sum_{\nu\ge 1}\left(\mathcal{R}^{N(m)}\right)_{(\nu,\mathcal{X}_m)} && \text{(A.5)} \\
&\quad + \sum_{\nu\ge 1}\left(\mathcal{R}^{T(m)}_{m+3}\right)_{(\nu,\mathcal{X}_m)} \;+\; \sum_{\nu\ge 0}\left(\sum_{n\ge m+4}\mathcal{R}^{T(m)}_n\right)_{(\nu,\mathcal{X}_m)}. && \text{(A.6)}
\end{aligned}$$

From Lemma 4.1 and (4.10), we get

$$\text{(A.4)} = Z_m(z) + R_m(z),$$

where

$$Z_m(z) = \sum_{\substack{\mathbf{j}\in\mathcal{N}_{m+3}\\ \mu(\mathbf{j})\le N}} \left(\mathcal{R}^{T(m)}_{m+3}\right)_{\mathbf{j}} z_{\mathbf{j}}, \qquad R_m(z) = \sum_{\mathbf{j}\in\mathcal{J}_{m+3}(N)} \left(\mathcal{R}^{T(m)}_{m+3}\right)_{\mathbf{j}} z_{\mathbf{j}} \tag{A.7}$$

and

$$\sup_{z\in B_{\rho,\theta}(\varepsilon_m)} \|X_{\mathcal{X}_m}(z)\|_{\rho,\theta} \le N^{16\cdot(m+3)^6}\cdot\left(2^{5m}N^{16(m+2)^6}\right)^m C^{m+1}\cdot\varepsilon^{m+2}.$$

Moreover, the transformation $\mathcal{T}_m$ satisfies

$$\begin{aligned}
&\sup_{z\in B_{\rho,\theta}(\varepsilon_{m+1})} \|\mathcal{T}_m-\mathrm{id}\|_{\rho,\theta}\\
&\le \sup_{z\in B_{\rho,\theta}(\varepsilon_m)} \|X_{\mathcal{X}_m}(z)\|_{\rho,\theta}\\
&\le N^{16\cdot(m+3)^6}\cdot\left(2^{5m}N^{16(m+2)^6}\right)^m C^{m+1}\cdot\varepsilon^{m+2}.
\end{aligned}\tag{A.8}$$

Let us set $\mathcal{T}^{(m+1)} := \mathcal{T}^{(m)}\circ\mathcal{T}_m : B_{\rho,\theta}(\varepsilon_{m+1})\to B_{\rho,\theta}(\varepsilon)$, which transforms the Hamiltonian H in (3.3) into the following form

$$\begin{aligned}
H^{(m+1)} &= H\circ\mathcal{T}^{(m+1)}\\
&= H^{(m)}\circ\mathcal{T}_m\\
&= H_0 + Z^{(m+1)}(z) + \mathcal{R}^{N(m+1)}(z) + \mathcal{R}^{T(m+1)}(z),
\end{aligned}$$

where

$$Z^{(m+1)}(z) := Z^{(m)}(z) + Z_m(z) \tag{A.9}$$

and

$$\mathcal{R}^{N(m+1)}(z) := \mathcal{R}^{N(m)}(z) + R_m(z). \tag{A.10}$$

More explicitly, recalling that (2) in Lemma 4.2, (4.10), (A.7) and (A.9), one has

$$Z^{(m+1)}(z) = \sum_{n=3}^{m+3} Z_n^{(m+1)}(z),\quad Z_n^{(m+1)}(z) = \sum_{\substack{\mathbf{j}\in\mathcal{N}_n\\ \mu(\mathbf{j})\le N}} \left(Z_n^{(m+1)}\right)_{\mathbf{j}} z_{\mathbf{j}}.$$

For any $3\le n\le m+3$, one has

$$\left|Z_n^{(m+1)}\right|_{\rho,\theta}^{T} \le \left(2^{5n-15}N^{16(n-1)^6}\right)^{n-3} C^{n-2}.$$

Similarly, in view of (3) in Lemma 4.2, (4.10), (A.7) and (A.10), it holds that

$$\mathcal{R}^{N(m+1)}(z) = \sum_{n=3}^{m+3} \mathcal{R}_n^{N(m+1)}(z),\quad \mathcal{R}_n^{N(m+1)}(z) = \sum_{\mathbf{j}\in\mathcal{J}_n(N)} \left(\mathcal{R}_n^{N(m+1)}\right)_{\mathbf{j}} z_{\mathbf{j}}.$$

For any $3\le n\le m+3$, it holds that

$$|\mathcal{R}_n^{N(m+1)}|_{\rho,\theta}^{T} \le \left(2^{5n-15}N^{16(n-1)^6}\right)^{n-3} C^{n-2}.$$

Moreover, $\mathcal{R}^{T(m+1)}(z) = $ (A.5) + (A.6) is of the following form

$$\mathcal{R}^{T(m+1)}(z) = \sum_{n\ge m+4} \mathcal{R}_n^{T(m+1)}(z).$$

More precisely, for any $n \geq m+4$, we get

$$
\begin{aligned}
&\mathcal{R}_n^{T(m+1)} \\
&= (H_0)_{\left(\frac{n-2}{m+1},\mathcal{X}_m\right)} + \sum_{k=\max\left\{\frac{n-m-2}{m+1},1\right\}}^{\frac{n-3}{m+1}} \left(\left(Z_{n-(m+1)k}^{(m)} \right)_{(k,\mathcal{X}_m)} + \left(\mathcal{R}_{n-(m+1)k}^{N(m)} \right)_{(k,\mathcal{X}_m)} \right) \\
&\quad + \left(\mathcal{R}_{m+3}^{T(m)} \right)_{\left(\frac{n-m-3}{m+1},\mathcal{X}_m\right)} + \sum_{k=0}^{\frac{n-m-4}{m+1}} \left(\mathcal{R}_{n-(m+1)k}^{T(m)} \right)_{(k,\mathcal{X}_m)}.
\end{aligned}
$$

By taking advantage of (4.8)-(4.10) and Corollary 2.1, for any $n \geq m+4$, we get

$$
\begin{aligned}
&\left| \mathcal{R}_n^{T(m+1)} \right|_{\rho,\theta}^{T} \\
&\leq \frac{1}{\left(\frac{n-2}{m+1}\right)!} \cdot \prod_{i=1}^{\frac{n-2}{m+1}-1} (m+3+i(m+1)) \cdot \left(2^{5m} \cdot N^{16(m+2)^6} \right)^m \cdot C^{m+1} \\
&\quad \times \left(N^{16(m+3)^6} \cdot \left(2^{5m} \cdot N^{16(m+2)^6} \right)^m \cdot C^{m+1} \right)^{\frac{n-2}{m+1}-1} \\
&\quad + 2 \sum_{k=\max\left\{\frac{n-m-2}{m+1},1\right\}}^{\frac{n-3}{m+1}} \frac{1}{k!} \cdot \left(2^{5m} \cdot N^{16(m+1)^6} \right)^{n-(m+1)k-3} \cdot C^{n-(m+1)k-2} \\
&\quad \times \prod_{i=1}^{k} (n-(m+1)k+i(m+1)) \left(N^{16(m+3)^6} \cdot \left(2^{5m} \cdot N^{16(m+2)^6} \right)^m \cdot C^{m+1} \right)^k \\
&\quad + \sum_{k=0}^{\frac{n-m-3}{m+1}} \frac{1}{k!} \cdot \left(2^{5m} \cdot N^{16(m+2)^6} \right)^{n-(m+1)k-3} \cdot C^{n-(m+1)k-2} \\
&\quad \times \prod_{i=1}^{k} (n-(m+1)k+i(m+1)) \left(N^{16(m+3)^6} \cdot \left(2^{5m} \cdot N^{16(m+2)^6} \right)^m \cdot C^{m+1} \right)^k \\
&\leq \left(2^{5m+5} \cdot N^{16(m+3)^6} \right)^{n-3} C^{n-2}.
\end{aligned}
$$

Lastly, in view of (1) in Lemma 4.2 and (A.8), the transformation

$$
\mathcal{T}^{(m+1)} : B_{\rho,\theta}(\varepsilon_{m+1}) \to B_{\rho,\theta}(\varepsilon)
$$

fulfills

$$
\begin{aligned}
&\sup_{z \in B_{\rho,\theta}(\varepsilon_{m+1})} \left\| \mathcal{T}^{(m+1)} - \mathrm{id} \right\|_{\rho,\theta} \\
&\leq \sup_{z \in B_{\rho,\theta}(\varepsilon_{m+1})} \left\| \mathcal{T}^{(m)} \circ \mathcal{T}_m - \mathcal{T}_m \right\|_{\rho,\theta} + \sup_{z \in B_{\rho,\theta}(\varepsilon_{m+1})} \left\| \mathcal{T}_m - \mathrm{id} \right\|_{\rho,\theta} \\
&\leq \sum_{n=3}^{m+2} N^{16n^6} \cdot \left(2^{5n-15} N^{16(n-1)^6} \right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1} \\
&\quad + N^{16\cdot(m+3)^6} \cdot \left(2^{5m} N^{16(m+2)^6} \right)^m C^{m+1} \cdot \varepsilon^{m+2}
\end{aligned}
$$

$$= \sum_{n=3}^{m+3} N^{16n^6} \cdot \left(2^{5n-15} N^{16(n-1)^6}\right)^{n-3} C^{n-2} \cdot \varepsilon^{n-1}.$$

Consequently, we complete the proof of Lemma 4.2. □

Appendix B: The proof of the nonresonance hypothesis

In this section, our task is to show the proof of non-resonant conditions with the help of Bambusi-Grébert [1].

Lemma B.1. *For any $K \leq r$, consider K indexes $j_1 < \cdots < j_K \leq N$; consider the determinant*

$$D := \begin{vmatrix} \omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\ \dfrac{d\omega_{j_1}}{ds} & \dfrac{d\omega_{j_2}}{ds} & \cdots & \dfrac{d\omega_{j_K}}{ds} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{d^{K-1}\omega_{j_1}}{ds^{K-1}} & \dfrac{d^{K-1}\omega_{j_2}}{ds^{K-1}} & \cdots & \dfrac{d^{K-1}\omega_{j_K}}{ds^{K-1}} \end{vmatrix}.$$

One has

$$|D| \geq \frac{C}{N^{2K^2}}, \tag{B.1}$$

where $0 < C < 1$ is a constant.

Proof. Let us denote

$$\lambda_j := j^2,$$

then one has

$$\frac{d^k \omega_j}{ds^k} = (\ln \lambda_j)^k \, \omega_j.$$

Therefore

$$D = \omega_{j_1} \cdots \omega_{j_K} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{j_1} & x_{j_2} & \cdots & x_{j_K} \\ \vdots & \vdots & \cdots & \vdots \\ x_{j_1}^{K-1} & x_{j_2}^{K-1} & \cdots & x_{j_K}^{K-1} \end{vmatrix} = \omega_{j_1} \cdots \omega_{j_K} \prod_{1 \leq l < k \leq K} \ln \frac{\lambda_{j_k}}{\lambda_{j_l}},$$

where $x_j := \ln \lambda_j$. By applying the fact that

$$\ln \frac{\lambda_{j_k}}{\lambda_{j_l}} \geq \ln \left(1 + \frac{1}{\lambda_{j_l}}\right) \geq \frac{1}{2} \cdot \frac{1}{j_l^2} \geq \frac{C}{N^2},$$

it follows that

$$|D| = \omega_{j_1} \cdots \omega_{j_K} \prod_{1 \leq l < k \leq K} \ln \frac{\lambda_{j_k}}{\lambda_{j_l}} \geq \prod_{1 \leq l < k \leq K} \ln \frac{C}{N^2} \geq \frac{C}{N^{2K^2}}.$$

We complete the proof of (B.1). □

Lemma B.2. ([1]) *Let $u^{(1)}, \cdots, u^{(K)}$ be K independent vectors with $\|u^{(i)}\|_{l^1} \leq 1$. Let $w \in \mathbb{R}^K$ be an arbitrary vector, then there exists $i \in \{1, \cdots, K\}$ such that*

$$\left|u^{(i)} \cdot w\right| \geq \frac{\|w\|_{l^1} \det(u^{(1)}, \cdots, u^{(K)})}{K^{\frac{3}{2}}}.$$

Lemma B.3. ([1]) *Let $w \in \mathbb{Z}^\infty$ be a vector with K component different from zero, namely those with index $j_1, \cdots, j_K$; assume that $K \leq r$ and $j_1 < \cdots < j_K \leq N$. Then for any $s \in (1/2, 1)$, there exists an index $j \in \{0, \cdots, K-1\}$ such that*

$$\left|w \cdot \frac{d^j \omega}{ds^j}(s)\right| \geq C \frac{\|w\|_{l^1}}{N^{2K^2+2}},$$

where ω is the frequency vector.

Lemma B.4. ([1]) *Suppose that $g(s)$ is m times differentiable on an interval $J \subset \mathbb{R}$. Let $J_h := \{s \in J : |g(s)| < h\}$, $h > 0$. If $|g^{(m)}(s)| \geq d > 0$ on J, then $|J_h| \leq M h^{\frac{1}{m}}$, where $M := 2(2 + 3 + \cdots + m + d^{-1})$. Here $|\cdot|$ denotes the Lebesgue measure of set.*

Next, our aim is to prove the following proposition by taking advantage of Lemma B.3 and Lemma B.4. Let us denote $\omega^{(N)} = (\omega_1, \omega_2, \cdots, \omega_N)$.

Proposition 5.1. *For a given positive number N, there exists a set $\mathcal{J}$ satisfying $|(1/2, 1) \setminus \mathcal{J}| \to 0$ as $N \to +\infty$, such that for any $s \in \mathcal{J}$,*

$$\left|\langle k, \omega^{(N)}\rangle + \varepsilon_1 \omega_{j_1} + \varepsilon_2 \omega_{j_2}\right| \geq \frac{1}{N^{16r^6}},$$

where $|k| \leq r+2$, $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$ and $|j_1|, |j_2| > N$.

Proof. Let us define the resonant set $\mathcal{R}$ by

$$\mathcal{R} = \bigcup_{k, j_1, j_2} \widetilde{\mathcal{R}}_{k, j_1, j_2} = \left\{s \in \left(\frac{1}{2}, 1\right) : \left|\langle k, \omega^{(N)}\rangle + \varepsilon_1 \omega_{j_1} + \varepsilon_2 \omega_{j_2}\right| < \frac{1}{N^{16r^6}}\right\},$$

where $|k| \leq r+2$, $\varepsilon_1, \varepsilon_2 \in \{-1, 0, 1\}$ and $|j_1|, |j_2| > N$. Then our task is to prove

$$|\mathcal{R}| \leq \frac{1}{N}. \tag{B.2}$$

Therefore, from (B.2), one has $|\mathcal{R}| \to 0$, as $N \to \infty$. Next, we set

$$\mathcal{J} = \left(\frac{1}{2}, 1\right) \setminus \mathcal{R},$$

which implies that we complete the proof of Proposition 5.1.

We will discuss (B.2) in four cases.

Case 1. When $\varepsilon_1 = \varepsilon_2 = 0$. Let us denote

$$\widetilde{\mathcal{R}}_1 = \bigcup_{|k| \leq r+2} \mathcal{R}_k,$$

where

$$\mathcal{R}_k = \left\{s \in \left(\frac{1}{2}, 1\right) : \left|\langle k, \omega^{(N)}\rangle\right| < \frac{1}{N^{4r^3}}\right\}.$$

In view of Lemma B.3 and Lemma B.4, it holds that

$$\begin{aligned}
|\mathcal{R}_k| &\le 2\left(2+3+\cdots+r+1+C^{-1}N^{2(r+2)^2+2}\right)\cdot\frac{1}{N^{\frac{4r^3}{r+1}}}\\
&\le 3N^{2r^2+8r+11}\cdot\frac{1}{N^{\frac{4r^3}{r+1}}}\\
&\le \frac{3}{N^{\frac{4r^3}{r+1}-2r^2-8r-11}}.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
|\widetilde{\mathcal{R}}_1| &\le \sum_{|k|\le r+2}|\mathcal{R}_k|\\
&\le \frac{3}{N^{\frac{4r^3}{r+1}-2r^2-8r-11}}\cdot N^{r+2}\\
&\le \frac{3}{N^{\frac{4r^3}{r+1}-2r^2-9r-13}}\\
&\le \frac{1}{4N} \qquad (r \text{ is large enough}). \qquad \text{(B.3)}
\end{aligned}$$

Case 2. When $\varepsilon_1=\pm1$, $\varepsilon_2=0$ or $\varepsilon_1=0$, $\varepsilon_2=\pm1$. Without loss of generality, we take $\varepsilon_1=1$, $\varepsilon_2=0$. Let us denote

$$\widetilde{\mathcal{R}}_2=\bigcup_{|k|\le r+2,j_1}\mathcal{R}_{kj_1},$$

where

$$\mathcal{R}_{kj_1}=\left\{s\in\left(\frac{1}{2},1\right):\left|\langle k,\omega^{(N)}\rangle+\omega_{j_1}\right|<\frac{1}{N^{4r^4}}\right\}. \qquad \text{(B.4)}$$

We claim

$$|j_1|\le 2(r+2)N^2.$$

Otherwise, one has

$$\left|\langle k,\omega^{(N)}\rangle+\omega_{j_1}\right|\ge|\omega_{j_1}|-(r+2)N^2>1,$$

which contradicts (B.4). Let us set $\langle\tilde{k},\omega^{(\tilde{N})}\rangle=\langle k,\omega^{(N)}\rangle+\omega_{j_1}$ in place of $\langle k,\omega^{(N)}\rangle$ and $\tilde{N}=2(r+2)N^2$ in place of N. Using the same strategy as **Case** 1, we get

$$\begin{aligned}
|\mathcal{R}_{kj_1}| &\le \frac{3}{N^{\frac{4r^4}{r+2}}}\cdot\left(2(r+2)N^2\right)^{2(r+3)^2+2}\\
&\le \frac{3}{N^{\frac{4r^4}{r+2}-2r^3-18r^2-56r-60}}.
\end{aligned}$$

Therefore, one has

$$\begin{aligned}
|\widetilde{\mathcal{R}}_2| &\le \sum_{|k|\le r+2}\ \sum_{|j_1|\le 2(r+2)N^2}|\mathcal{R}_{kj_1}|\\
&\le \frac{3}{N^{\frac{4r^4}{r+2}-2r^3-18r^2-56r-60}}\cdot N^{r+2}\cdot 2(r+2)N^2
\end{aligned}$$

$$\leq \frac{3}{N^{\frac{4r^4}{r+2}-2r^3-18r^2-58r-68}}$$
$$\leq \frac{1}{4N} \qquad (r \text{ is large enough}). \tag{B.5}$$

Case 3. When $\varepsilon_1\varepsilon_2 = 1$. Let us denote

$$\widetilde{\mathcal{R}}_3 = \bigcup_{|k|\leq r+2, j_1, j_2} \mathcal{R}_{kj_1j_2},$$

where

$$\mathcal{R}_{kj_1j_2} = \left\{ s \in \left(\frac{1}{2}, 1\right) : \left| \langle k, \omega^{(N)} \rangle + \omega_{j_1} + \omega_{j_2} \right| < \frac{1}{N^{4r^4}} \right\}.$$

By applying the same startegy as **Case** 2, it follows that

$$|\widetilde{\mathcal{R}}_3| \leq \frac{1}{4N}. \tag{B.6}$$

Case 4. When $\varepsilon_1\varepsilon_2 = -1$. Without loss of generality, we take $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and $j_1 > j_2 > 0$. Let us denote

$$\widetilde{\mathcal{R}}_4 = \bigcup_{|k|\leq r+2, j_1, j_2} \mathcal{R}_{kj_1j_2},$$

where

$$\mathcal{R}_{kj_1j_2} = \left\{ s \in \left(\frac{1}{2}, 1\right) : \left| \langle k, \omega^{(N)} \rangle + \omega_{j_1} - \omega_{j_2} \right| < \frac{1}{N^{16r^6}} \right\}. \tag{B.7}$$

In view of mean value theorem, we obtain

$$|\omega_{j_1} - \omega_{j_2}| \geq 2s(j_1 - j_2) \cdot |j_2|^{2s-1},$$

thus

$$j_1 - j_2 \leq 4(r+2)N^2, \quad j_2 \leq 2N^{4r^3},$$

otherwise, it holds that

$$\left| \langle k, \omega^{(N)} \rangle + \omega_{j_1} - \omega_{j_2} \right| \geq 1,$$

which contradicts with (B.7). Let us set $\langle \tilde{k}, \omega^{(\tilde{N})} \rangle = \langle k, \omega^{(N)} \rangle + \omega_{j_1} - \omega_{j_2}$ in place of $\langle k, \omega^{(N)} \rangle$ and $\tilde{N} = 2N^{4r^3+1}$ in place of N. Using the same strategy as **Case** 1, we get

$$|\mathcal{R}_{kj_1j_2}| \leq \frac{3}{N^{\frac{16r^6}{r+3}}} \cdot (2N^{4r^3+1})^{2(r+4)^2+3}$$
$$\leq \frac{3}{N^{\frac{16r^6}{r+3}-8r^5-64r^4-140r^3-4r^2-32r-70}}.$$

Therefore, one has

$$|\widetilde{\mathcal{R}}_4| \leq \sum_{|k|\leq r+2} \sum_{|j_1|\leq 2N^{4r^3+1}} \sum_{|j_2|\leq 2N^{4r^3}} |\mathcal{R}_{kj_1j_2}|$$

$$\leq \frac{3}{N^{\frac{16r^6}{r+3}-8r^5-64r^4-140r^3-4r^2-32r-70}} \cdot N^{r+2} \cdot 2N^{4r^3+1} \cdot 2N^{4r^3}$$
$$\leq \frac{3}{N^{\frac{16r^6}{r+3}-8r^5-64r^4-148r^3-4r^2-33r-75}}$$
$$\leq \frac{1}{4N} \qquad (r \text{ is large enough}). \tag{B.8}$$

In view of (B.3), (B.5), (B.6) and (B.8), it holds that

$$|\mathcal{R}| \leq |\widetilde{\mathcal{R}}_1| + |\widetilde{\mathcal{R}}_2| + |\widetilde{\mathcal{R}}_3| + |\widetilde{\mathcal{R}}_4| \leq \frac{1}{N}.$$

We finish the proof of (B.2). □

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