

EXPLORATION OF FRACTIONAL EFFECTS AND MULTI-STABILITY IN THE NONLINEAR (2+1)-DIMENSIONAL DAVEY-STEWARTSON FOKAS SYSTEM

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Abstract This work delves into a profound analysis of the fractional Davey-Stewartson Fokas system, which has applications in analyzing two-dimensional wave packets on water surfaces and modeling pulse propagation in optical fibers. An in-depth comparative study is conducted on the local M and the fractional beta derivatives to elucidate their effects on the system's solutions and to investigate the role of fractional parameters in shaping the system's behavior. Two approaches, the $G'/(bG' + G + a)$ expansion and the Sardar sub-equation approaches, are employed to generate new solitary wave solutions that appear in the form of lump, periodic, singular, bright, and their combo solitons. The solutions are produced under constraint conditions and illustrated using 3D, density, and 2D plots by selecting appropriate values of parameters. Moreover, chaotic phenomena are analyzed through the observation of multi-stability by setting the parameters and varying initial conditions. The extracted solutions validate the effectiveness of the proposed methodologies in deriving exact solutions in nonlinear contexts.

Keywords Davey-Stewartson Fokas system, local M derivative, fractional beta derivative, $G'/(bG' + G + a)$ expansion method, Sardar sub-equation method, Multi-stability.

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1. Introduction

In many disciplines, including physics, engineering, plasma physics, fluid mechanics, control theory, fractional dynamics, biology, and visco-elasticity, among others, fractional calculus has garnered the attention of many analysts [14, 16]. Scholars and scientists are interested in the use of fractional derivatives in the mathematical

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analysis of physical problems. Using the fractional-order derivative helps to address the limitations of the integer-order classical derivative. Fractional-order derivatives are the only way to represent the fading memory effects and crossover observed in various physical systems. The fractional operator model enables a deep understanding of system dynamics. Fractional derivatives are classified on the basis of singular or non-singular kernels [2, 8, 23] and have been the subject of recent literature. Nonetheless, the beta and M fractional derivatives are the operators that are garnering the greatest attention. These operators have been successfully used to express many scientific and non-scientific problems. Different research groups have investigated optical soliton solutions within spatial-time fractional nonlinear optics.

The impact of nonlinear events in applied mathematics, fluid dynamics, nonlinear physics, and ocean engineering has been profound over the past few decades, particularly in calculating analytical and numerical solutions. Of particular importance to researchers are solitary waves, such as periodic, traveling, and soliton waves, which have a broad array of applications. Understanding the applications of nonlinear evolution equations in various fields necessitates a thorough study of solitary wave solutions. Analytical solutions to this kind of equation play a crucial role in clarifying the physical significance of real-world situations by uncovering numerous underlying physical and dynamic properties.

Soliton dynamics is a relatively new field of study in mathematical physics with a wide range of applications, particularly in optics and optical fibers [15]. Such phenomena arise from balancing dispersive and nonlinear effects. Many fields, such as engineering, fluid dynamics, applied mathematics, biology, and hydrodynamics, rely heavily on applications associated with a broad spectrum of methods. A few of these effective techniques for producing exact and numerical solutions are the Backlund transformation [6], Sardar sub-equation [1], $G'/(bG' + G + a)$ expansion [9, 21], the sine-Gordon expansion method [4, 5, 11], modified simple equation method [3, 10], sine-cosine technique [13], Hirota-bilinear method [24, 26], innovative extended algebraic method [7], modified tanh expansion approach [17], Lie group method [12] and direct equation approaches.

The organization of sections follows a logical progression: Section 2 discusses the governing system in detail, while Section 3 introduces the preliminaries of different definitions of fractional. In Section 4, the methodologies are described, and Section 5 explains the system's conversion into an ordinary differential equation (ODE) alongside the application of these methodologies. Section 6 focusing on the multi-stability analysis. Section 7 presents the findings and their interpretations. The research concludes in Section 8, offering a comprehensive study summary.

2. Governing system

The coupled Davey-Stewartson Fokas System (DSFS) provides a robust framework for examining the behavior of nonlinear pulse propagation within monomode optical waveguides and wave packets on water surfaces. The (2+1) dimensional nonlinear DSFS [25] is as follows:

$$\begin{cases} i \frac{\partial \varphi}{\partial t} + w_1 \frac{\partial^2 \varphi}{\partial x^2} + w_2 \varphi \tau = 0, \\ w_3 \frac{\partial \tau}{\partial y} - w_4 \frac{\partial |\varphi|^2}{\partial x} = 0, \end{cases} \quad (2.1)$$

where φ and τ are complex and real-valued potential functions representing the wave packet, the evolution of weakly nonlinear wave packets in a two-dimensional medium is governed by a set of nonlinear partial differential equations (NLPDEs) known as the Davey-Stewartson system. This equation, recognized as one of the most significant mathematical models in fluid dynamics, was first introduced by Andrew Davey and Keith Stewartson, who originally derived it in the context of water waves. By generalizing the nonlinear Schrödinger equation to two spatial dimensions, the DS system can model various physical phenomena, including wave propagation in fluids, plasmas, and optical media. In general, the DSFS describes long-wave and short-wave resonances and other propagating wave structures mathematically.

In earlier studies, various techniques have been adopted to explore DSFS. These encompass the generalized Riccati equation mapping method, the simple equation approach, the sine-gordon expansion technique, the Painlevé procedure, the semi-inverse variational principle, and investigations into bifurcation, chaos, and sensitivity. Given this groundwork, this paper aims to broaden the analysis by conducting a fractional comparison by employing two definitions (namely, the beta and local M derivatives) to resolve DSFS and discover new solitary wave solutions. A secondary aim is to utilize the Sardar subequation technique in conjunction with the $G'/(bG' + G + a)$ expansion method to analyze and resolve the equation. These methodologies, when applied in tandem, afford greater clarity into the intricate interactions within nonlinear optical systems and promote the exploration of previously uncharted soliton solutions.

3. Preliminaries

This section covers the basic definitions of local M [18,22] and fractional beta [18–20] derivatives, along with their associated properties.

3.1. Local M derivative

Let S be a continuous function of t . For $0 < \gamma \leq 1$, the local M derivative (LMD) of order γ of function $S(t)$, denoted by $D_M^{\gamma,\sigma}$ and defined as:

$$D_M^{\gamma,\sigma} [S(t)] = \lim_{h \rightarrow 0} \frac{S(tB_\sigma(ht^{-\gamma})) - S(t)}{h}, \quad \forall t > 0,$$

where $B_\sigma(\cdot)$, $\sigma > 0$ is the Mittag-Leffler function.

Properties: If $S(t)$ and $T(t)$ are two functions of M-differentiable of order γ , which lie in the interval $(0,1]$ and $\sigma > 0$, then the following results can be determined using the concept of LMD:

- I- $D_M^{\gamma,\sigma} [L] = 0$, where L is an arbitrary constant,
- II- $D_M^{\gamma,\sigma} [aS(t) + bT(t)] = aD_M^{\gamma,\sigma} [S(t)] + bD_M^{\gamma,\sigma} [T(t)], \quad \forall a, b \in R,$
- III- $D_M^{\gamma,\sigma} [S(t) * T(t)] = S(t)D_M^{\gamma,\sigma} [T(t)] + T(t)D_M^{\gamma,\sigma} [S(t)],$
- IV- $D_M^{\gamma,\sigma} \left[\frac{S(t)}{T(t)} \right] = \frac{S(t)D_M^{\gamma,\sigma} [T(t)] - T(t)D_M^{\gamma,\sigma} [S(t)]}{(T(t))^2},$
- V- $D_M^{\gamma,\sigma} [S(T(t))] = S'(T(t))D_M^{\gamma,\sigma} [T(t)], \quad D_M^{\gamma,\sigma} \left[S \left(\frac{\Gamma(\sigma+1)t^\gamma}{\gamma} \right) \right] = S' \left(\frac{\Gamma(\sigma+1)t^\gamma}{\gamma} \right)$
 $\times D_M^{\gamma,\sigma} \left[\left(\frac{\Gamma(\sigma+1)t^\gamma}{\gamma} \right) \right], \quad D_M^{\gamma,\sigma} \left[S \left(\frac{\Gamma(\sigma+1)t^\gamma}{\gamma} \right) \right] = S' \left(\frac{\Gamma(\sigma+1)t^\gamma}{\gamma} \right),$ along with $\vartheta =$

$\left(\frac{f\Gamma(\sigma+1)t^\gamma}{\gamma}\right)$, where f is some constant, and eventually we get $D_M^{\gamma,\sigma}[S(\vartheta)] = fS'(\vartheta)$, furthermore, $D_M^{\gamma,\sigma}[S(t)] = \frac{t^{1-\gamma}}{\Gamma(\sigma+1)} \frac{d}{dt} S(t)$, above can also lead us to $D_M^{\gamma,\sigma}\left[\frac{\Gamma(\sigma+1)t^\gamma}{\gamma}\right] = 1$.

3.2. Fractional beta derivative

Let S be a continuous function of t . For $0 < \beta \leq 1$, the beta derivative of function $S(t)$, defined as:

$${}_0^A D_t^\beta [S(t)] = \lim_{h \rightarrow 0} \frac{S(t + h(t + \frac{1}{\Gamma(\beta)})) - S(t)}{h}, \quad \forall t > 0.$$

Properties: If $S(t)$ and $T(t)$ are two functions of beta differentiable of order beta, which lie in the interval $(0,1]$ and $t > 0$, then the following results can be determined using the concept of fractional beta derivative:

- I-** ${}_0^A D_t^\beta [L] = 0$, where L is an arbitrary constant,
- II-** ${}_0^A D_t^\beta [aS(t) + bT(t)] = a{}_0^A D_t^\beta [S(t)] + b{}_0^A D_t^\beta [T(t)], \forall a, b \in R$,
- III-** ${}_0^A D_t^\beta [S(t) * T(t)] = S(t){}_0^A D_t^\beta [T(t)] + T(t){}_0^A D_t^\beta [S(t)],$
- IV-** ${}_0^A D_t^\beta [S(T(t))] = S'(T(t)){}_0^A D_t^\beta [T(t)],$
- V-** ${}_0^A D_t^\beta \left[\frac{S(t)}{T(t)}\right] = \frac{S(t){}_0^A D_t^\beta [T(t)] - T(t){}_0^A D_t^\beta [S(t)]}{(T(t))^2},$
- VI-** ${}_0^A D_t^\beta [S(t)] = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{d}{dt} S(t).$

4. Description of methods

Consider a nonlinear evolution equation for $\varphi = \varphi(x, y, t)$ as follows:

$$I(\varphi, \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial^2 \varphi}{\partial t^2}, \frac{\partial^2 \varphi}{\partial x^2}, \frac{\partial^2 \varphi}{\partial y^2}, \dots) = 0, \quad \forall t \geq 0, \quad (4.1)$$

where I is a polynomial of $\varphi(x, y, t)$ and its nonlinear terms along with its highest-order partial derivatives. To reformulate the above equation into an ODE, the wave transformation is required: $\varphi(x, y, t) = \Xi(\vartheta)$, where $\vartheta = ax + by - ct$. Then we have

$$G(\Xi, \frac{d\Xi}{d\vartheta}, \frac{d^2\Xi}{d\vartheta^2}, \dots) = 0. \quad (4.2)$$

This section provides a detailed description of the $G'/(bG' + G + a)$ expansion method and the Sardar sub-equation method.

4.1. $G'/(bG' + G + a)$ expansion method

Assume that the transformed ODE has a solution as follows:

$$\Xi(\vartheta) = \sum_{e=0}^M f_e R^e(\vartheta), \quad (4.3)$$

where $R = R(\vartheta) = \frac{G'}{bG+G+a}$, for all values of a and b except zero and $f_e, (e = 0, 1, 2, 3, \dots, M)$ are arbitrary constants that will be determined later. The following ODE admits $G = G(\vartheta)$ as its solution:

$$G'' = -\frac{v}{b}G' - \frac{w}{b^2}G - \frac{w}{b^2}a, \quad (4.4)$$

where v and w belong to $\mathbb{R}$. Using the balancing principle, we can determine M' 's value, and $R(\vartheta)$ is the solution of the nonlinear ordinary differential equation (NLODE) which is as follows:

$$R' = \frac{dR(\vartheta)}{d\vartheta} = (v - w - 1)R^2 + \frac{2w - v}{b}R - \frac{w}{b^2}. \quad (4.5)$$

Eqn. (4.5) have two types of solutions shown below:

Type I: If $\varrho = v^2 - 4w > 0$, then $G = -a + k_1 e^{\frac{1}{2b}(-v-\sqrt{\varrho})\vartheta} + k_2 e^{\frac{1}{2b}(-v+\sqrt{\varrho})\vartheta}$, k_1 and k_2 are arbitrary constants that satisfy $a^2 + k_1^2 + k_2^2 \neq 0$.

In this type, $R = R(\vartheta)$ is as:

$$R = \frac{k_1(v + \sqrt{\varrho}) + k_2(v - \sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}}{b k_1(v - 2 + \sqrt{\varrho}) + b k_2(v - 2 - \sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}}.$$

Additionally, $R = R(\vartheta)$ can be written as:

$$R = \frac{(v(k_2 - k_1) - \sqrt{\varrho}(k_2 + k_1)) \sinh(\frac{\sqrt{\varrho}}{2b}\vartheta) + (v(k_2 + k_1) - \sqrt{\varrho}(k_2 - k_1)) \cosh(\frac{\sqrt{\varrho}}{2b}\vartheta)}{b((v-2)(k_2 - k_1) - \sqrt{\varrho}(k_2 + k_1)) \sinh(\frac{\sqrt{\varrho}}{2b}\vartheta) + b((v-2)(k_2 + k_1) - \sqrt{\varrho}(k_2 - k_1)) \cosh(\frac{\sqrt{\varrho}}{2b}\vartheta)}.$$

Furthermore, under some limitations:

If $(v - 2)(k_2 - k_1) - \sqrt{\varrho}(k_2 + k_1) = 0$, then

$$R = \frac{v - 2w}{2b(v - w - 1)} - \frac{\sqrt{\varrho}}{2b(v - w - 1)} \tanh(\frac{\sqrt{\varrho}}{2b}\vartheta).$$

If $(v - 2)(k_2 + k_1) - \sqrt{\varrho}(k_2 - k_1) = 0$, then

$$R = \frac{v - 2w}{2b(v - w - 1)} - \frac{\sqrt{\varrho}}{2b(v - w - 1)} \coth(\frac{\sqrt{\varrho}}{2b}\vartheta).$$

Type II: If $\varrho = v^2 - 4w < 0$, then $G = -a + e^{\frac{-v}{2b}\vartheta} \left(k_1 \cos(\frac{\sqrt{-\varrho}}{2b}\vartheta) + k_2 \sin(\frac{\sqrt{-\varrho}}{2b}\vartheta) \right)$.

In this, $R = R(\vartheta)$ is as:

$$R = \frac{(vk_1 - \sqrt{-\varrho}k_2) \cos(\frac{\sqrt{-\varrho}}{2b}\vartheta) + (vk_2 + \sqrt{-\varrho}k_1) \sin(\frac{\sqrt{-\varrho}}{2b}\vartheta)}{b((v-2)k_1 - \sqrt{-\varrho}k_2) \cos(\frac{\sqrt{-\varrho}}{2b}\vartheta) + b((v-2)k_2 + \sqrt{-\varrho}k_1) \sin(\frac{\sqrt{-\varrho}}{2b}\vartheta)}.$$

Furthermore, under some limitations:

If $(v - 2)k_2 + \sqrt{-\varrho}k_1 = 0$, then

$$R = \frac{v - 2w}{2b(v - w - 1)} + \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \tan(\frac{\sqrt{-\varrho}}{2b}\vartheta).$$

If $(v - 2)k_1 - \sqrt{-\varrho}k_2 = 0$, then

$$R = \frac{v - 2w}{2b(v - w - 1)} - \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \cot(\frac{\sqrt{-\varrho}}{2b}\vartheta).$$

Putting Eqn. (4.3) and (4.5) in Eqn. (4.2) and setting the coefficients of R^e to zero. As a result, a system of equations will be produced. The values of constants or different parameters will be concluded by solving the resulting set of equations. With these values, various exact solutions for Eqn. (4.1) can be obtained.

4.2. Sardar sub-equation method

Assuming the following form for the solution to the Eqn. (4.2)

$$\Xi(\vartheta) = \sum_{e=0}^M g_e R^e(\vartheta), \quad (4.6)$$

here g_e , ($e = 0, 1, 2, 3, \dots, M$) are arbitrary constants to be determined later. The value of M 's can be ascertained by applying the balancing principle, and $R(\vartheta)$ represents the solution to the NLODE.

$$R'(\vartheta) = \sqrt{\omega + cR(\vartheta)^2 + dR(\vartheta)^4}, \quad (4.7)$$

where ω , c , and d are real constants, and the Eqn. (4.7) has solutions as follows:

Case I. If $c > 0$ and $\omega = 0$, then

$$R_1^\pm(\vartheta) = \pm \sqrt{-\frac{rsc}{d}} \operatorname{sech}_{rs}(\sqrt{c}\vartheta), \quad (d < 0),$$

$$R_2^\pm(\vartheta) = \pm \sqrt{\frac{rsc}{d}} \operatorname{csch}_{rs}(\sqrt{c}\vartheta), \quad (d > 0),$$

where $\operatorname{sech}_{rs}(\vartheta) = \frac{2}{re^\vartheta + se^{-\vartheta}}$ and $\operatorname{csch}_{rs}(\vartheta) = \frac{2}{re^\vartheta - se^{-\vartheta}}$.

Case II. If $c < 0$, $d > 0$, and $\omega = 0$, then

$$R_3^\pm(\vartheta) = \pm \sqrt{-\frac{rsc}{d}} \operatorname{sec}_{rs}(\sqrt{-c}\vartheta),$$

$$R_4^\pm(\vartheta) = \pm \sqrt{-\frac{rsc}{d}} \operatorname{csc}_{rs}(\sqrt{-c}\vartheta),$$

where $\operatorname{sec}_{rs}(\vartheta) = \frac{2}{re^{i\vartheta} + se^{-i\vartheta}}$ and $\operatorname{csc}_{rs}(\vartheta) = \frac{2i}{re^{i\vartheta} - se^{-i\vartheta}}$.

Case III. If $c < 0$, $d > 0$ and $\omega = \frac{c^2}{4d}$, then

$$R_5^\pm(\vartheta) = \sqrt{\frac{-c}{2d}} \tanh_{rs}\left(\sqrt{\frac{-c}{2}}\vartheta\right),$$

$$R_6^\pm(\vartheta) = \sqrt{\frac{-c}{2d}} \coth_{rs}\left(\sqrt{\frac{-c}{2}}\vartheta\right),$$

$$R_7^\pm(\vartheta) = \sqrt{\frac{-c}{2d}} \left(\tanh_{rs}(\sqrt{-2c}\vartheta) \pm i\sqrt{rs} \operatorname{sech}_{rs}(\sqrt{-2c}\vartheta) \right),$$

$$R_8^\pm(\vartheta) = \sqrt{\frac{-c}{2d}} \left(\coth_{rs}(\sqrt{-2c}\vartheta) \pm \sqrt{rs} \operatorname{csch}_{rs}(\sqrt{-2c}\vartheta) \right),$$

$$R_9^\pm(\vartheta) = \pm \sqrt{\frac{-c}{8d}} \left(\tanh_{rs}\left(\sqrt{\frac{-c}{8}}\vartheta\right) + \coth_{rs}\left(\sqrt{\frac{-c}{8}}\vartheta\right) \right),$$

where $\tanh_{rs}(\vartheta) = \frac{re^{\vartheta} - se^{-\vartheta}}{re^{\vartheta} + se^{-\vartheta}}$ and $\coth_{rs}(\vartheta) = \frac{re^{\vartheta} + se^{-\vartheta}}{re^{\vartheta} - se^{-\vartheta}}$.

Case IV. If $c > 0$, $d > 0$ and $\omega = \frac{c^2}{4d}$, then

$$\begin{aligned} R_{10}^{\pm}(\vartheta) &= \sqrt{\frac{c}{2d}} \tan_{rs}\left(\sqrt{\frac{c}{2}}\vartheta\right), \\ R_{11}^{\pm}(\vartheta) &= \sqrt{\frac{c}{2d}} \cot_{rs}\left(\sqrt{\frac{c}{2}}\vartheta\right), \\ R_{12}^{\pm}(\vartheta) &= \sqrt{\frac{c}{2d}} \left(\tan_{rs}(\sqrt{2c}\vartheta) \pm \sqrt{rs} \sec_{rs}(\sqrt{2c}\vartheta) \right), \\ R_{13}^{\pm}(\vartheta) &= \sqrt{\frac{c}{2d}} \left(\cot_{rs}(\sqrt{2c}\vartheta) \pm \sqrt{rs} \csc_{rs}(\sqrt{2c}\vartheta) \right), \\ R_{14}^{\pm}(\vartheta) &= \sqrt{\frac{c}{8d}} \left(\tan_{rs}\left(\sqrt{\frac{c}{8}}\vartheta\right) + \cot_{rs}\left(\sqrt{\frac{c}{8}}\vartheta\right) \right), \end{aligned}$$

where $\tan_{rs}(\vartheta) = -i \frac{re^{\vartheta} - se^{-\vartheta}}{re^{\vartheta} + se^{-\vartheta}}$ and $\cot_{rs}(\vartheta) = i \frac{re^{\vartheta} + se^{-\vartheta}}{re^{\vartheta} - se^{-\vartheta}}$.

Put Eqn. (4.6) along with Eqn. (4.7) into Eqn. (4.2), and by setting all coefficients of $R^e(\vartheta)$ equal to zero, that leads us to a set of algebraic equations. By solving that system, we have values of our constants or parameters and then use those values to find several exact solutions for Eqn. (4.1).

5. Computational scrutiny of the governing system

In light of various definitions of derivatives, we now provide the temporal fractional version of the system (2.1).

(1)-In the context of LMD, the representation of our system is as follows:

$$iD_M^{\gamma,\sigma}\varphi + w_1\varphi_{xx} + w_2\varphi\tau = 0, \quad (5.1)$$

$$w_3\tau_y - w_4(|\varphi|^2)_x = 0. \quad (5.2)$$

(2)-In the context of fractional beta derivative,

$$i_0^A D_t^\beta \varphi + w_1\varphi_{xx} + w_2\varphi\tau = 0, \quad (5.3)$$

$$w_3\tau_y - w_4(|\varphi|^2)_x = 0. \quad (5.4)$$

Let's now assume the transformation to change the NLPDE into NLODE, which is as follows:

$$\varphi(x, y, t) = e^{i\psi(x, y, t)}\Xi(\vartheta), \quad \tau(x, y, t) = T(\vartheta). \quad (5.5)$$

In terms of two different definitions of derivatives, ϑ and ψ are defined as:

(i)- For M derivative

$$\vartheta = p_1x + p_2y - p_3 \frac{\Gamma(\sigma+1)}{\gamma} t^\gamma, \quad \psi(x, y, t) = lx + my + n \frac{\Gamma(\sigma+1)}{\gamma} t^\gamma. \quad (5.6)$$

(ii)- For fractional beta derivative

$$\vartheta = p_1x + p_2y - \frac{p_3}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta, \quad \psi(x, y, t) = lx + my + \frac{n}{\beta} \left(t + \frac{1}{\Gamma(\beta)} \right)^\beta. \quad (5.7)$$

Now substitute Eqn. (5.5), (5.6) and (5.7) into Eqn. (5.1) and (5.3); after substituting, we obtain the same following equation:

$$p_1^2 w_1 \Xi''(\vartheta) - (l^2 w_1 + n) \Xi(\vartheta) + w_2 \Xi(\vartheta) T(\vartheta) + i((2lp_1 w_1 - p_3) \Xi'(\vartheta)) = 0. \quad (5.8)$$

The above equation can be expressed in its real and imaginary parts, which are as follows:

$$p_1^2 w_1 \Xi''(\vartheta) - (l^2 w_1 + n) \Xi(\vartheta) + w_2 \Xi(\vartheta) T(\vartheta) = 0, \quad (5.9)$$

$$(2lp_1 w_1 - p_3) \Xi'(\vartheta) = 0, \quad (5.10)$$

from Eqn. (5.10), we have

$$p_3 = 2lp_1 w_1, \quad (5.11)$$

similarly substituting Eqn. (5.5), (5.6), and (5.7) into Eqn. (5.2) and (5.4), then we have

$$w_3 p_2 T'(\vartheta) - 2p_1 w_4 \Xi(\vartheta) \Xi'(\vartheta) = 0, \quad (5.12)$$

then after integrating, we get

$$T(\vartheta) = \frac{p_1 w_4}{w_3 p_2} \Xi^2(\vartheta), \quad (5.13)$$

substituting above eq in Eqn. (5.9), then we get

$$\Xi''(\vartheta) - \frac{l^2 w_1 + n}{p_1^2 w_1} \Xi(\vartheta) + \frac{w_2 w_4}{p_1 p_2 w_3 w_1} \Xi^3(\vartheta) = 0, \quad (5.14)$$

above eq can be rewrite as

$$\Xi''(\vartheta) + \alpha \Xi(\vartheta) + \beta \Xi^3(\vartheta) = 0, \quad (5.15)$$

where $\alpha = -\frac{l^2 w_1 + n}{p_1^2 w_1}$, and $\beta = \frac{w_2 w_4}{p_1 p_2 w_3 w_1}$.

5.1. Execution of $G'/(bG' + G + a)$ expansion method

Getting $M = 1$ by applying the balancing rule on Eqn. (5.15), then from Eqn. (4.3) we have:

$$\Xi(\vartheta) = f_0 + f_1 R(\vartheta). \quad (5.16)$$

By plugging Eqn. (5.16) together with (4.5) into Eqn. (5.15) and by equalizing the coefficients of $R^e(\vartheta)$ to zero, a system of equations emerges. The resolution of this system provides the values of the following constants:

$$f_0 = f_0, \quad f_1 = \frac{2bf_0(1-v+w)}{v-2w}, \quad \alpha = \frac{v^2-4w}{2b^2}, \quad \beta = -\frac{(v-2w)^2}{2b^2 f_0^2}. \quad (5.17)$$

Our findings yield the following solutions:

Type I: For $\varrho = v^2 - 4w > 0$, we have

$$\left\{ \begin{array}{l} \varphi_1(x, y, t) = e^{i\psi(x, y, t)} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{k_1(v+\sqrt{\varrho}) + k_2(v-\sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}}{b k_1(v-2+\sqrt{\varrho}) + b k_2(v-2-\sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}} \right) \Big], \\ \tau_1(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{k_1(v+\sqrt{\varrho}) + k_2(v-\sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}}{b k_1(v-2+\sqrt{\varrho}) + b k_2(v-2-\sqrt{\varrho})e^{\frac{\sqrt{\varrho}}{b}\vartheta}} \right) \Big]^2. \end{array} \right.$$

Furthermore, under some limitations:

If $(v-2)(k_2-k_1) - \sqrt{\varrho}(k_2+k_1) = 0$, then

$$\left\{ \begin{array}{l} \varphi_2(x, y, t) = e^{i\psi(x, y, t)} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{v-2w}{2b(v-w-1)} - \frac{\sqrt{\varrho}}{2b(v-w-1)} \tanh\left(\frac{\sqrt{\varrho}}{2b}\vartheta\right) \right) \Big], \\ \tau_2(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{v-2w}{2b(v-w-1)} - \frac{\sqrt{\varrho}}{2b(v-w-1)} \tanh\left(\frac{\sqrt{\varrho}}{2b}\vartheta\right) \right) \Big]^2. \end{array} \right.$$

If $(v-2)(k_2+k_1) - \sqrt{\varrho}(k_2-k_1) = 0$, then

$$\left\{ \begin{array}{l} \varphi_3(x, y, t) = e^{i\psi(x, y, t)} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{v-2w}{2b(v-w-1)} - \frac{\sqrt{\varrho}}{2b(v-w-1)} \coth\left(\frac{\sqrt{\varrho}}{2b}\vartheta\right) \right) \Big], \\ \tau_3(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ \quad \times \left(\frac{v-2w}{2b(v-w-1)} - \frac{\sqrt{\varrho}}{2b(v-w-1)} \coth\left(\frac{\sqrt{\varrho}}{2b}\vartheta\right) \right) \Big]^2. \end{array} \right.$$

Type II: For $\varrho = v^2 - 4w < 0$, we have

$$\begin{aligned} & \varphi_4(x, y, t) \\ &= e^{i\psi(x, y, t)} \left[f_0 + \frac{2bf_0(1-v+w)}{v-2w} \right. \\ & \quad \times \left(\frac{(vk_1 - \sqrt{-\varrho}k_2) \cos(\frac{\sqrt{-\varrho}}{2b}\vartheta) + (vk_2 + \sqrt{-\varrho}k_1) \sin(\frac{\sqrt{-\varrho}}{2b}\vartheta)}{b((v-2)k_1 - \sqrt{-\varrho}k_2) \cos(\frac{\sqrt{-\varrho}}{2b}\vartheta) + b((v-2)k_2 + \sqrt{-\varrho}k_1) \sin(\frac{\sqrt{-\varrho}}{2b}\vartheta)} \right) \Big], \\ & \tau_4(x, y, t) \end{aligned}$$

$$= \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2b f_0 (1 - v + w)}{v - 2w} \right. \\ \left. \times \left(\frac{(v k_1 - \sqrt{-\varrho} k_2) \cos(\frac{\sqrt{-\varrho}}{2b} \vartheta) + (v k_2 + \sqrt{-\varrho} k_1) \sin(\frac{\sqrt{-\varrho}}{2b} \vartheta)}{b((v - 2)k_1 - \sqrt{-\varrho} k_2) \cos(\frac{\sqrt{-\varrho}}{2b} \vartheta) + b((v - 2)k_2 + \sqrt{-\varrho} k_1) \sin(\frac{\sqrt{-\varrho}}{2b} \vartheta)} \right) \right]^2.$$

Furthermore, under some limitations:

If $(v - 2)k_2 + \sqrt{-\varrho} k_1 = 0$, then

$$\left\{ \begin{array}{l} \varphi_5(x, y, t) = e^{i\psi(x, y, t)} \left[f_0 + \frac{2b f_0 (1 - v + w)}{v - 2w} \right. \\ \quad \times \left(\frac{v - 2w}{2b(v - w - 1)} + \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \tan(\frac{\sqrt{-\varrho}}{2b} \vartheta) \right) \Big], \\ \tau_5(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2b f_0 (1 - v + w)}{v - 2w} \right. \\ \quad \times \left(\frac{v - 2w}{2b(v - w - 1)} + \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \tan(\frac{\sqrt{-\varrho}}{2b} \vartheta) \right) \Big]^2. \end{array} \right.$$

If $(v - 2)k_1 - \sqrt{-\varrho} k_2 = 0$, then

$$\left\{ \begin{array}{l} \varphi_6(x, y, t) = e^{i\psi(x, y, t)} \left[f_0 + \frac{2b f_0 (1 - v + w)}{v - 2w} \right. \\ \quad \times \left(\frac{v - 2w}{2b(v - w - 1)} - \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \cot(\frac{\sqrt{-\varrho}}{2b} \vartheta) \right) \Big], \\ \tau_6(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[f_0 + \frac{2b f_0 (1 - v + w)}{v - 2w} \right. \\ \quad \times \left(\frac{v - 2w}{2b(v - w - 1)} - \frac{\sqrt{-\varrho}}{2b(v - w - 1)} \cot(\frac{\sqrt{-\varrho}}{2b} \vartheta) \right) \Big]^2. \end{array} \right.$$

5.2. Execution of Sardar sub-equation method

After applying the balancing rule to Eqn. (5.15), we obtain $M = 1$ then Eqn. (4.6) becomes as

$$\Xi(\vartheta) = g_0 + g_1 R(\vartheta). \quad (5.18)$$

After inserting Eqn. (4.6) alongside Eqn. (4.7) into Eqn. (5.15), a system of equations is generated by equating the coefficients of $R^e(\vartheta)$ to zero. The values of the following constants are derived by solving this system.

$$g_0 = 0, \quad g_1 = g_1, \quad \alpha = -c, \quad \beta = -\frac{2d}{g_1^2}. \quad (5.19)$$

Using Eqn. (4.7) and (5.5) with (5.19), yields the following solution.

Case I. If $c > 0$ and $\omega = 0$, then

$$\left\{ \begin{array}{l} \varphi_7^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{sech}_{rs}(\sqrt{c}\vartheta) \right) \right], \\ \tau_7^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{sech}_{rs}(\sqrt{c}\vartheta) \right) \right]^2, \end{array} \right. \quad (d < 0),$$

$$\begin{cases} \varphi_8^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{rsc}{d}} \operatorname{csch}_{rs}(\sqrt{c}\vartheta) \right) \right], \\ \tau_8^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{rsc}{d}} \operatorname{csch}_{rs}(\sqrt{c}\vartheta) \right) \right]^2, \quad (d > 0). \end{cases}$$

Case II. If $c < 0$, $d > 0$, and $\omega = 0$, then

$$\begin{cases} \varphi_9^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{sec}_{rs}(\sqrt{-c}\vartheta) \right) \right], \\ \tau_9^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{sec}_{rs}(\sqrt{-c}\vartheta) \right) \right]^2, \\ \varphi_{10}^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{csc}_{rs}(\sqrt{-c}\vartheta) \right) \right], \\ \tau_{10}^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{-\frac{rsc}{d}} \operatorname{csc}_{rs}(\sqrt{-c}\vartheta) \right) \right]^2. \end{cases}$$

Case III. If $c < 0$, $d > 0$ and $\omega = \frac{c^2}{4d}$, then

$$\begin{cases} \varphi_{11}^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} \tanh_{rs} \left(\sqrt{\frac{-c}{2}} \vartheta \right) \right) \right], \\ \tau_{11}^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} \tanh_{rs} \left(\sqrt{\frac{-c}{2}} \vartheta \right) \right) \right]^2, \\ \varphi_{12}^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} \coth_{rs} \left(\sqrt{\frac{-c}{2}} \vartheta \right) \right) \right], \\ \tau_{12}^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} \coth_{rs} \left(\sqrt{\frac{-c}{2}} \vartheta \right) \right) \right]^2, \\ \varphi_{13}^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} (\tanh_{rs}(\sqrt{-2c}\vartheta) \right. \right. \\ \left. \left. \pm i\sqrt{rs} \operatorname{sech}_{rs}(\sqrt{-2c}\vartheta)) \right) \right], \\ \tau_{13}^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} (\tanh_{rs}(\sqrt{-2c}\vartheta) \right. \right. \\ \left. \left. \pm i\sqrt{rs} \operatorname{sech}_{rs}(\sqrt{-2c}\vartheta)) \right) \right]^2, \\ \varphi_{14}^\pm(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} (\coth_{rs}(\sqrt{-2c}\vartheta) \right. \right. \\ \left. \left. \pm \sqrt{rs} \operatorname{csch}_{rs}(\sqrt{-2c}\vartheta)) \right) \right], \\ \tau_{14}^\pm(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{-c}{2d}} (\coth_{rs}(\sqrt{-2c}\vartheta) \right. \right. \\ \left. \left. \pm \sqrt{rs} \operatorname{csch}_{rs}(\sqrt{-2c}\vartheta)) \right) \right]^2, \end{cases}$$

$$\left\{ \begin{array}{l} \varphi_{15}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{-c}{8d}} \left(\tanh_{rs} \left(\sqrt{\frac{-c}{8}} \vartheta \right) \right. \right. \right. \\ \left. \left. \left. + \coth_{rs} \left(\sqrt{\frac{-c}{8}} \vartheta \right) \right) \right) \right], \\ \tau_{15}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{-c}{8d}} \left(\tanh_{rs} \left(\sqrt{\frac{-c}{8}} \vartheta \right) \right. \right. \right. \\ \left. \left. \left. + \coth_{rs} \left(\sqrt{\frac{-c}{8}} \vartheta \right) \right) \right) \right]^2. \end{array} \right.$$

Case IV. If $c > 0$, $d > 0$ and $\omega = \frac{c^2}{4d}$, then

$$\left\{ \begin{array}{l} \varphi_{16}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \tan_{rs} \left(\sqrt{\frac{c}{2}} \vartheta \right) \right) \right], \\ \tau_{16}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \tan_{rs} \left(\sqrt{\frac{c}{2}} \vartheta \right) \right) \right]^2, \\ \varphi_{17}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \cot_{rs} \left(\sqrt{\frac{c}{2}} \vartheta \right) \right) \right], \\ \tau_{17}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \cot_{rs} \left(\sqrt{\frac{c}{2}} \vartheta \right) \right) \right]^2, \\ \varphi_{18}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \left(\tan_{rs}(\sqrt{2c}\vartheta) \right. \right. \right. \\ \left. \left. \left. \pm \sqrt{rs} \sec_{rs}(\sqrt{2c}\vartheta) \right) \right) \right], \\ \tau_{18}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \left(\tan_{rs}(\sqrt{2c}\vartheta) \right. \right. \right. \\ \left. \left. \left. \pm \sqrt{rs} \sec_{rs}(\sqrt{2c}\vartheta) \right) \right) \right]^2, \\ \varphi_{19}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \left(\cot_{rs}(\sqrt{2c}\vartheta) \right. \right. \right. \\ \left. \left. \left. \pm \sqrt{rs} \csc_{rs}(\sqrt{2c}\vartheta) \right) \right) \right], \\ \tau_{19}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{c}{2d}} \left(\cot_{rs}(\sqrt{2c}\vartheta) \right. \right. \right. \\ \left. \left. \left. \pm \sqrt{rs} \csc_{rs}(\sqrt{2c}\vartheta) \right) \right) \right]^2, \\ \varphi_{20}^{\pm}(x, y, t) = e^{i\psi(x, y, t)} \left[g_1 \left(\pm \sqrt{\frac{c}{8d}} \left(\tan_{rs} \left(\sqrt{\frac{c}{8}} \vartheta \right) \right. \right. \right. \\ \left. \left. \left. + \cot_{rs} \left(\sqrt{\frac{c}{8}} \vartheta \right) \right) \right) \right], \\ \tau_{20}^{\pm}(x, y, t) = \frac{p_1 w_4}{w_3 p_2} \left[g_1 \left(\pm \sqrt{\frac{c}{8d}} \left(\tan_{rs} \left(\sqrt{\frac{c}{8}} \vartheta \right) \right. \right. \right. \\ \left. \left. \left. + \cot_{rs} \left(\sqrt{\frac{c}{8}} \vartheta \right) \right) \right) \right]^2. \end{array} \right.$$

6. Multi-stability

In this section, the multi-stability of the system will be evaluated after transforming Eqn. (5.15) using the Galilean transformation and incorporating the perturbed term. The corresponding planar dynamical system is as follows:

$$\begin{cases} \frac{d\Xi}{d\vartheta} = \Upsilon, \\ \frac{d\Upsilon}{d\vartheta} = \varrho_0\Xi + \varrho_1\Xi^3 + \varrho_2\cos(\delta\vartheta), \end{cases} \quad (6.1)$$

where $\varrho_0 = -\alpha$, $\varrho_1 = -\beta$, and the parameters ϱ_2 and δ denote the magnitude and frequency of the external force applied to the system. Multi-stability reflects the adaptability of a system with constrained parameter choices, where varying initial conditions lead to two or more concurrent outcomes. By investigating multi-stability in perturbed dynamical systems, the research reveals how slight disturbances can lead to the formation of multiple stable states, showcasing the system's ability to exhibit diverse behavioral transitions.

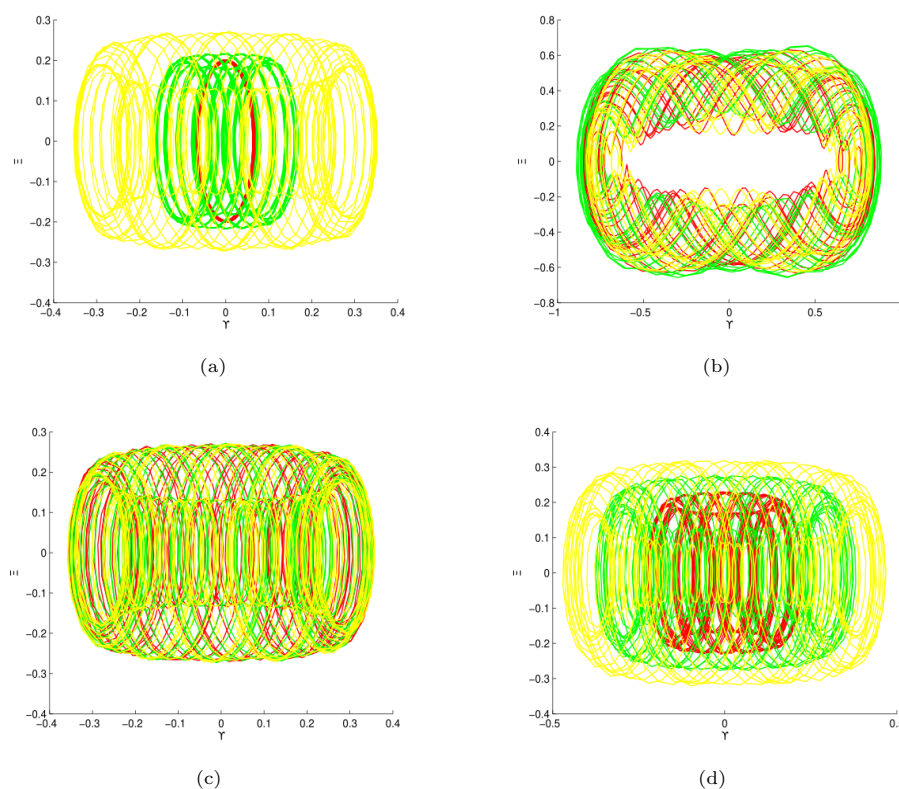


Figure 1. Exploring chaotic tendencies via multi-stability investigation.

The multi-stability of the dynamical system (6.1) is analyzed using fixed pa-

parameter values such as $\varrho_0 = -0.02$, $\varrho_1 = -0.8$, $\varrho_2 = 0.6$, $\delta = 3$, by varying initial conditions as: In **(a)** red $[0.1, 0.1]$, green $[0.2, 0.1]$, yellow $[-0.2, 0.1]$, **(b)** red $[0.2, 0.5]$, green $[0.3, 0.5]$, yellow $[-0.3, 0.5]$, **(c)** red $[0.01, 0.03]$, green $[-0.02, 0.03]$, yellow $[-0.03, 0.03]$, and **(d)** red $[0.1; 0.07]$, green $[-0.2; 0.07]$, yellow $[-0.3; 0.07]$.

7. Findings and their interpretations

Studying the behavior of nonlinear wave phenomena based on their graphical representation is more efficient and beneficial when using the exact solutions produced by mathematical computations. This section furnishes a visual representation comparing the local M and beta derivatives obtained as solutions to the fractional DSFS under the previously outlined approaches. A key distinction between these approaches lies in their methodology and the nature of the solutions they yield. The $G'/(bG' + G + a)$ expansion method, derived from the G'/G technique, is effective in obtaining exact traveling wave solutions of NLPDEs by reducing them to ODEs. In contrast, the Sardar sub-equation method is primarily utilized to extract optical soliton formations in nonlinear light propagation systems. The comparative analysis herein aims to illustrate the distinctions and correlations. Here, 3D, density, and 2D plots are depicted by considering different parameter values.

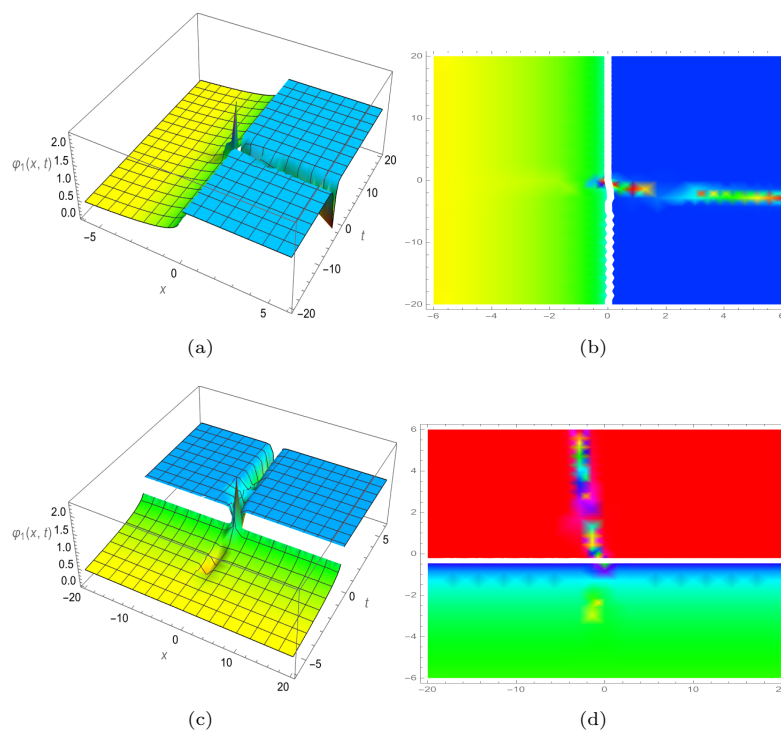


Figure 2. 3D and density plots of $Abs(\varphi_1(x, y, t))$ for γ and $\beta = 0.3$, respectively.

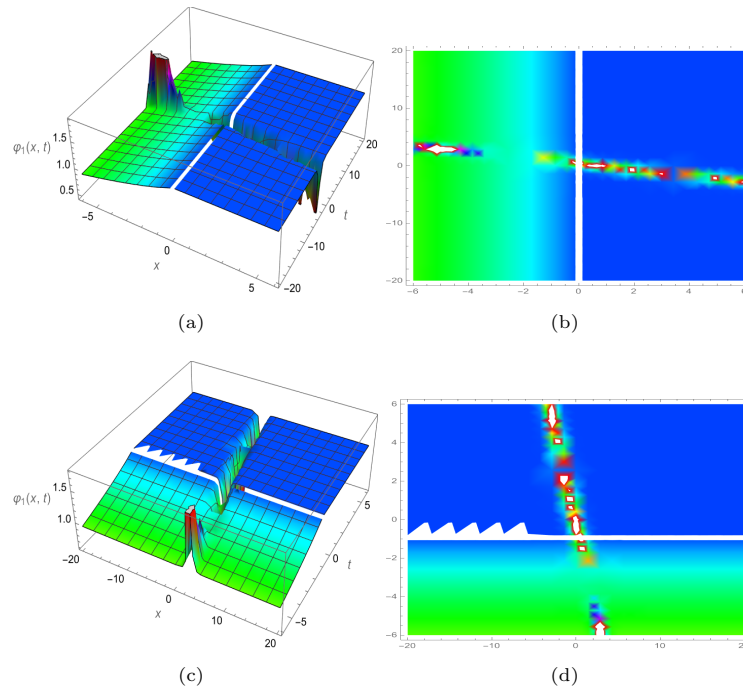


Figure 3. 3D and density plots of $\text{Abs}(\varphi_1(x, y, t))$ for γ and $\beta = 0.9$, respectively.

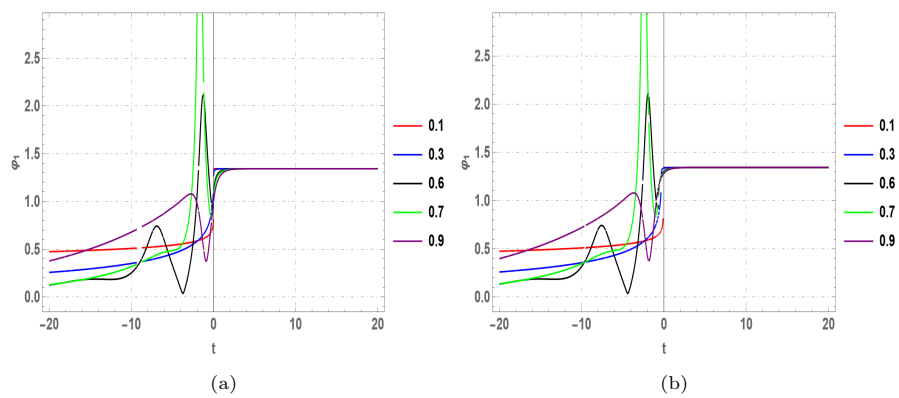


Figure 4. 2D delineation of $\text{Abs}(\varphi_1(x, y, t))$ for different values of γ and β , respectively.

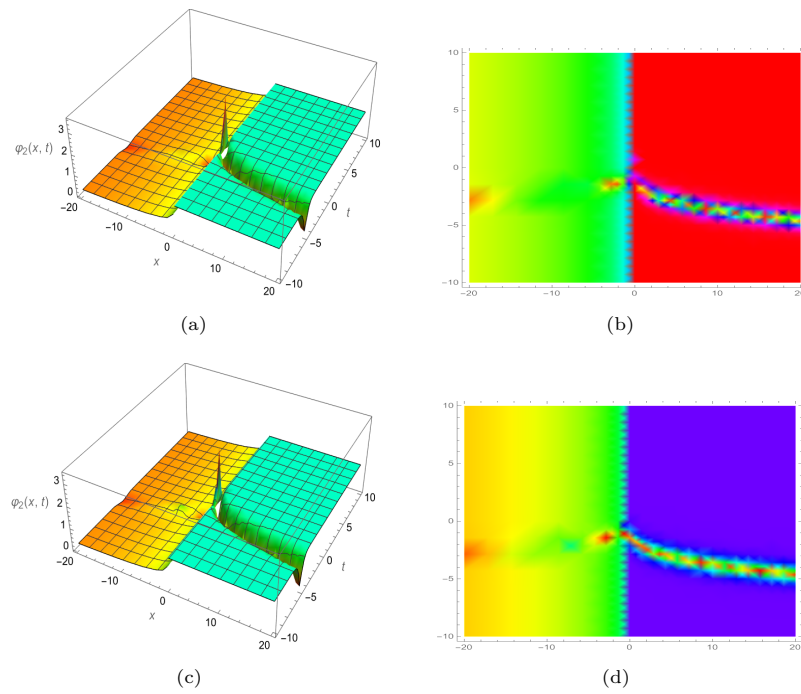


Figure 5. 3D and density plots of $Abs(\varphi_2(x, y, t))$ for γ and $\beta = 0.3$, respectively.

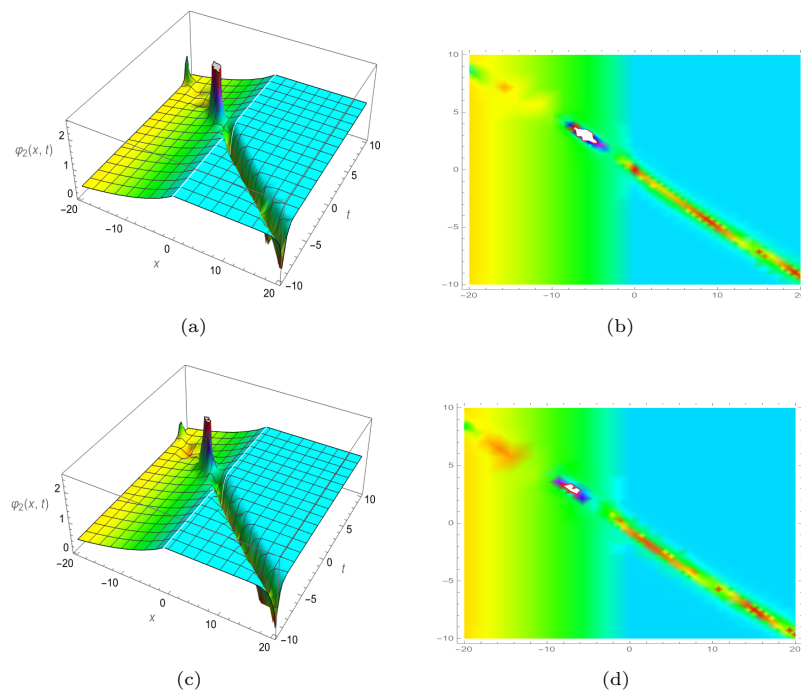


Figure 6. 3D and density plots of $Abs(\varphi_2(x, y, t))$ for γ and $\beta = 0.9$, respectively.

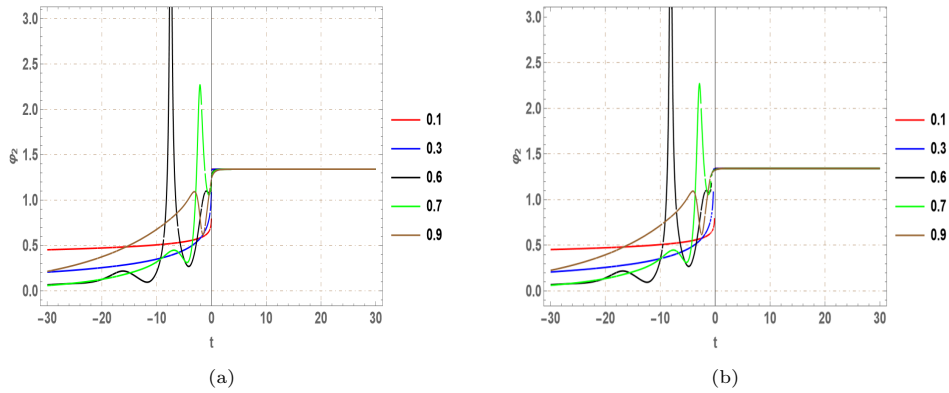


Figure 7. 2D depiction of $Abs(\varphi_2(x, y, t))$ for multiple values of γ and β , respectively.

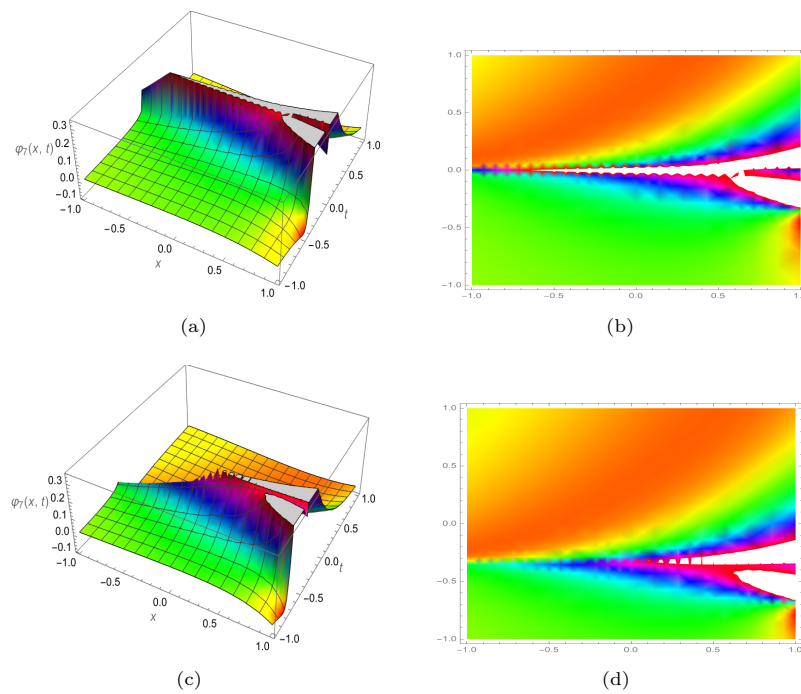


Figure 8. 3D and density plots of $\text{Re}(\varphi_7(x, y, t))$ for γ and $\beta = 0.3$, respectively.

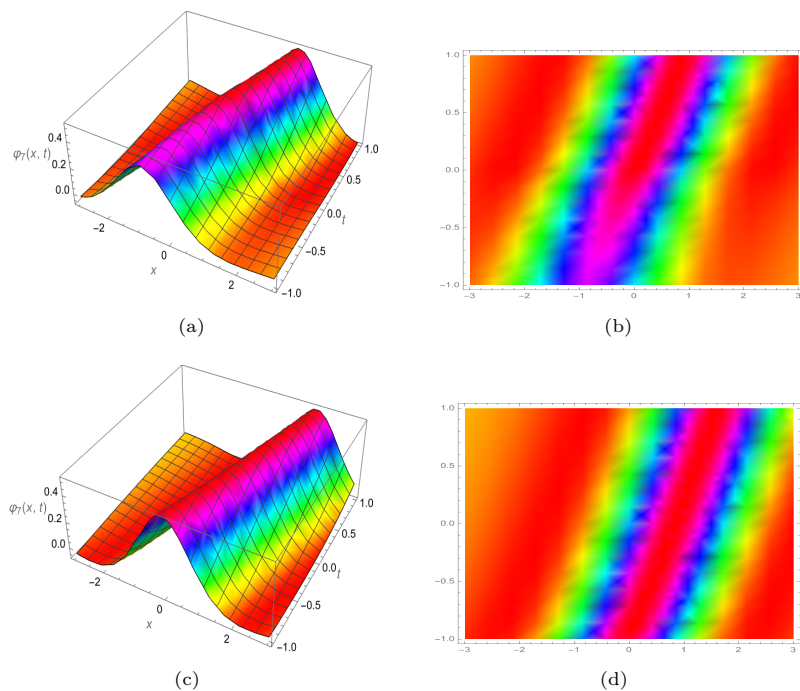


Figure 9. 3D and density plots of $Re(\varphi_7(x, y, t))$ for γ and $\beta = 0.9$, respectively.

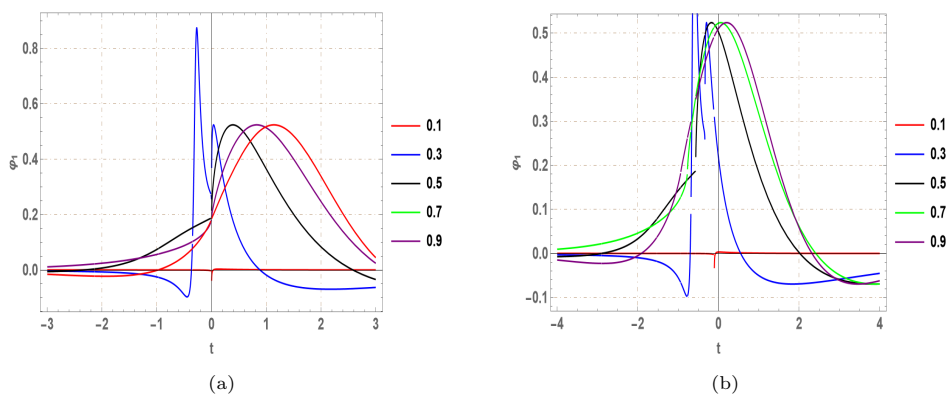


Figure 10. 2D plots of $Abs(\varphi_7(x, y, t))$ for diverse values of γ and β , respectively.

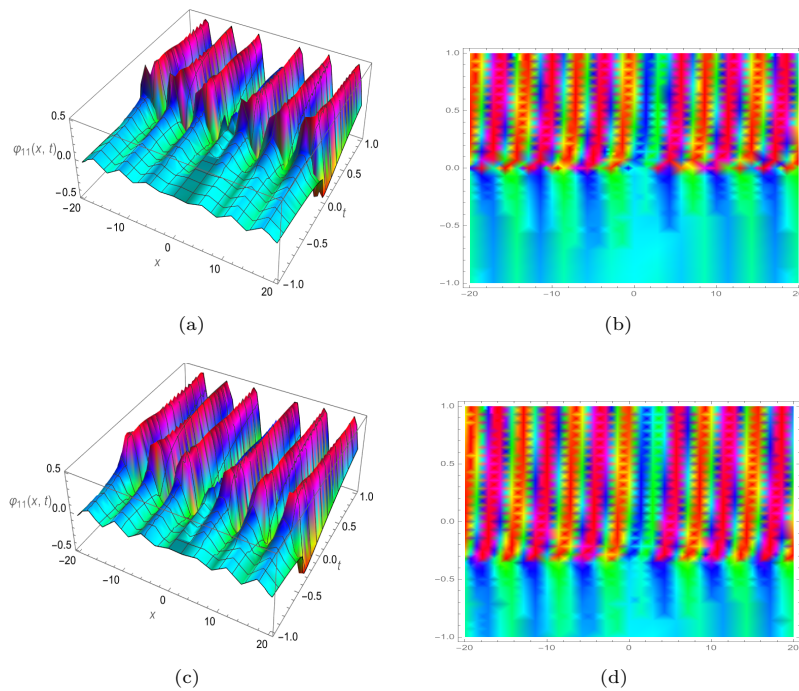


Figure 11. 3D and density plots of $\text{Re}(\varphi_{11}(x, y, t))$ for γ and $\beta = 0.3$, respectively.

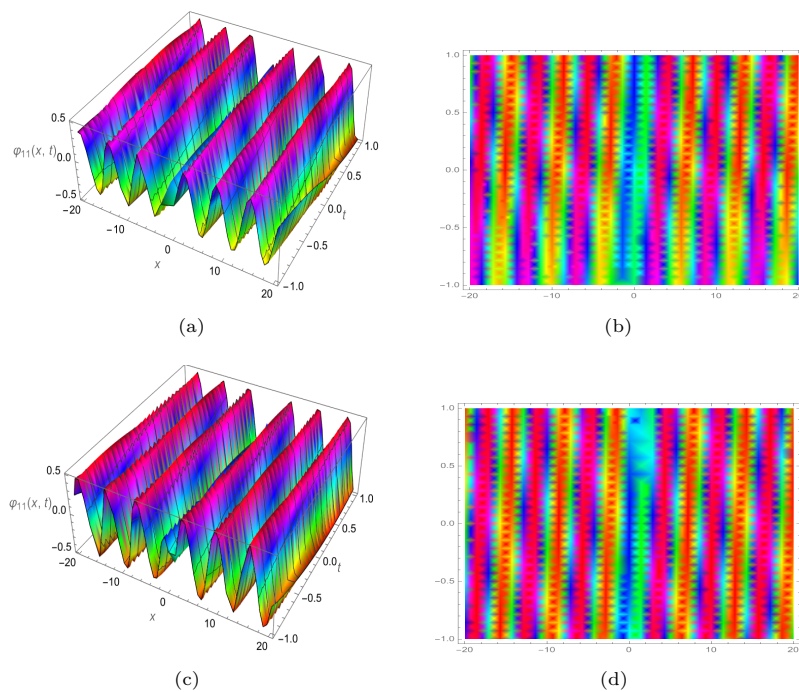


Figure 12. 3D and density plots of $\text{Re}(\varphi_{11}(x, y, t))$ for γ and $\beta = 0.9$, respectively.

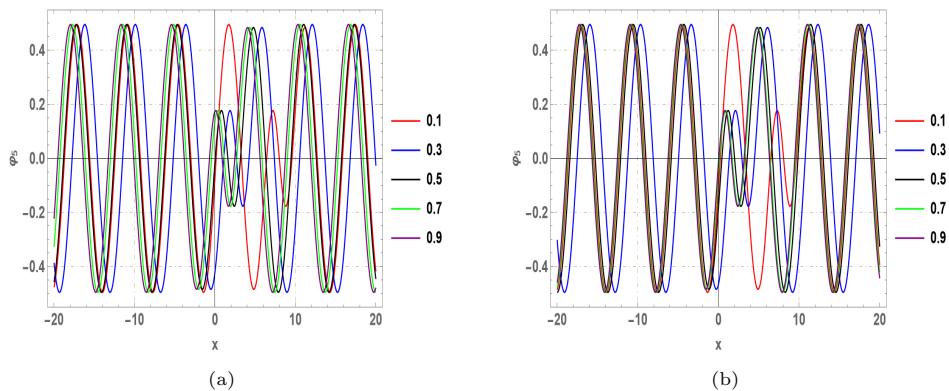


Figure 13. 2D delineation of $Abs(\varphi_{11}(x, y, t))$ for several values of γ and β , respectively.

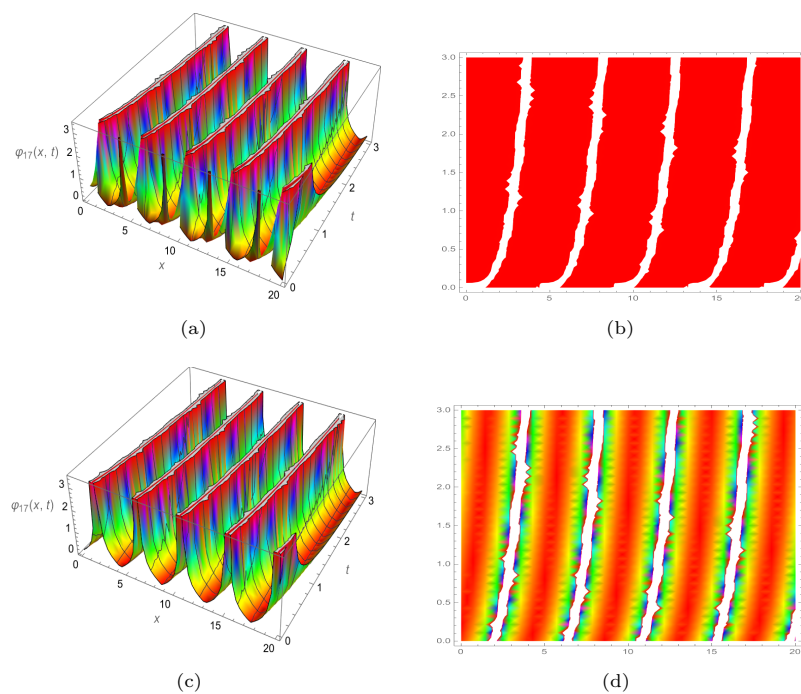


Figure 14. 3D and density plots of $Abs(\varphi_{17}(x, y, t))$ for γ and $\beta = 0.3$, respectively.

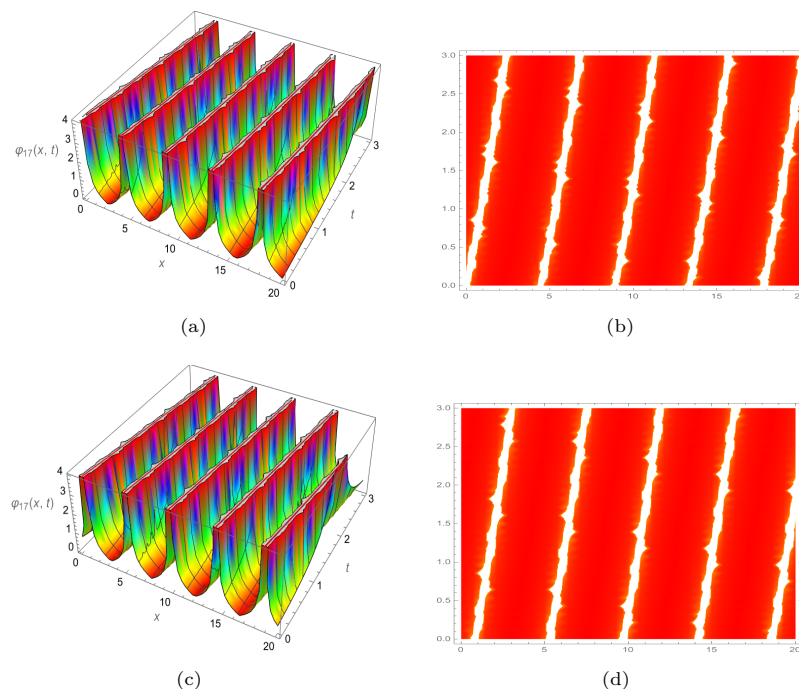


Figure 15. 3D and density plots of $Abs(\varphi_{17}(x, y, t))$ for γ and $\beta = 0.9$, respectively.

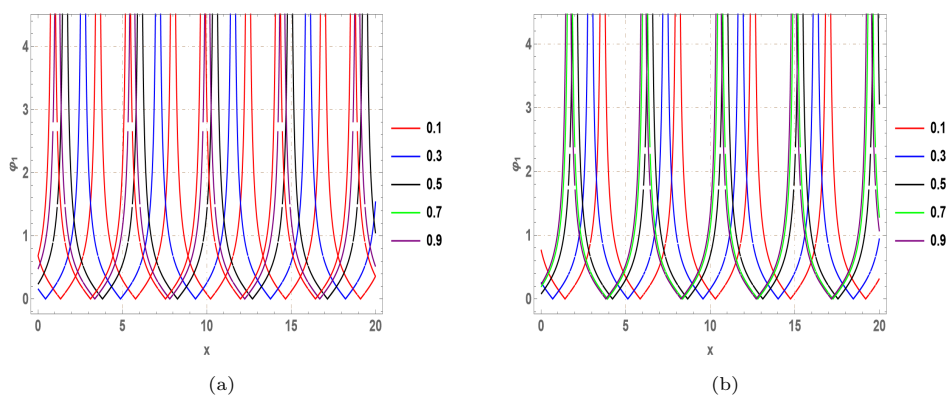


Figure 16. 2D depiction of $Abs(\varphi_{17}(x, y, t))$ for various values of γ and β , respectively.

Figures 2 and 3 represent a visual depiction by adjusting the parameters to ($f_0 = 0.6$; $v = 3$; $w = 1$; $b = 0.7$; $k_1 = -1.25$; $k_2 = 0.9$; $l = 1$; $m = 1$; $n = 0.25$; $y = 0$; $p_1 = 1$; $p_2 = -1$; $p_3 = -0.56$; $\gamma = 0.3, 0.9$; $\beta = 0.3, 0.9$) in 3D and density visualizations, and Figure 4 represents 2D plots of local M and beta for multiple values of the fractional parameter of $Abs(\varphi_1(x, y, t))$, while Figures 5 and 6 showcase by configuring the parameters to ($f_0 = 0.6$; $v = 3$; $w = 1$; $b = 0.7$; $k_1 = -1.25$; $k_2 = 0.9$; $l = 1$; $m = 1$; $n = 0.25$; $y = 0$; $p_1 = 1$; $p_2 = -1$; $p_3 = -0.56$; $\gamma = 0.3, 0.9$; $\beta = 0.3, 0.9$) in 3D and density illustrations and Figure 7 displays the 2D plots of local M and beta across different values of fractional parameter of $Abs(\varphi_2(x, y, t))$

of the solution obtained via the $G'/(bG' + G + a)$ expansion method. Figures 8 and 9 exhibit a pictorial representation by setting the parameters as ($l = -1$; $m = -1$; $r = 1$; $s = 1$; $n = 0.8$; $v = 1$; $w = -1$; $y = 0$; $o = -1$; $p = 1$; $q = -0.8$; $c_1 = 0.7$; $\gamma = 0.3, 0.9$; $\beta = 0.3, 0.9$) in 3D and density delineations, and Figure 10 visualizes the 2D plots of local M and beta using several fractional parameter variations of $Re(\varphi_7(x, y, t))$. Figures 11 and 12 represent by specifying the parameters as ($l = -1$; $m = -1$; $r = 1$; $s = 1$; $n = 0.8$; $c = -1$; $d = 1$; $y = 0$; $o = -1$; $p = 1$; $q = -0.8$; $c_1 = 0.7$; $\gamma = 0.3, 0.9$; $\beta = 0.3, 0.9$) in 3D and density graphical representations, and Figure 13 depicts 2D plots of local M and beta for diverse values of the fractional parameter of $Re(\varphi_{11}(x, y, t))$, while Figures 14 and 15 illustrate By establishing the parameters as ($l = -0.25$; $m = 1$; $r = 1$; $s = 1$; $n = 0.8$; $c = 0.67$; $d = 1.2$; $y = 0$; $p_1 = -0.96$; $p_2 = 1.23$; $p_3 = -1$; $g_1 = 0.7$; $\gamma = 0.3, 0.9$; $\beta = 0.3, 0.9$) in 3D and density depictions, and Figure 16 contains the visualization of 2D plots of local M and beta for various fractional parameter values of $Abs(\varphi_{17}(x, y, t))$ of the solution obtained through the Sardar sub-equation method.

8. Conclusion

This research concluded with an investigation of the fractional (2+1) dimensional DSFS, demonstrating its applications in two-dimensional wave dynamics and optical pulse modeling. We derive several novel optical solitons for the DSFS that were bright, lump, periodic, and singular by using the $G'/(bG + G + a)$ expansion and Sardar sub-equation methods, which differ mainly in their approach and application. The first one is effective for obtaining exact traveling wave solutions. However, the second one is specifically employed in nonlinear optics to identify optical solitons. To show the physical characteristics of comparative analysis of local M and fractional beta derivative using different values for the fractional parameters in the graphic representations of the obtained fractional solutions, 3D, density, and 2D plots were also produced. It was noted that the wave profile was significantly affected by the fractional parameters. Additionally, the analysis of multi-stability further revealed the occurrence of chaotic phenomena within the system. Based on our observation, we have concluded that all of the obtained fractional solutions are accurate, efficient, and adaptable to a wide range of mathematical physics applications. Notably, the optical fractional soliton solutions obtained in this research are newly formulated and are being reported for the first time, setting them apart from earlier literature. Our work sets the stage for future breakthroughs by applying these methods to other nonlinear areas. With more research, exciting discoveries and applications might emerge.

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