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HIGH-ORDER NUMERICAL SCHEME AND THEORETICAL ANALYSIS FOR NONLINEAR TWO-DIMENSIONAL FRACTIONAL VOLTERRA INTEGRAL EQUATIONS WITH INITIAL VALUE SINGULARITY*

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Abstract Various numerical methods have been proposed for solving one-dimensional weakly singular Volterra integral equations (VIEs) with smooth solutions. The main purpose of this paper is to propose and analyze a numerical method for the solution of two-dimensional nonlinear weakly singular VIEs with non-smooth solutions by involving the transformation of variables and modified Block-by-Block method. We rigorously prove that the new scheme can achieve an order of $O(\tau_s^{3+\alpha} + \lambda_t^{3+\beta})$ for non-smooth solutions with step size τ_s, λ_t for $0 < \alpha, \beta < 1$. Some numerical examples are conducted to support the theoretical results and demonstrate the effectiveness of the proposed method.

Keywords Nonlinear Volterra integral equations, non-smooth solution, Block-by-Block method, convergence analysis.

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1. Introduction

Volterra integral equations in one or more variables have a wide range of applications in many fields of applied science and engineering [16]. In general, these equations cannot be solved analytically and therefore require numerical solutions. Various numerical methods have been presented for solving fractional differential and inte-

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gral equations, such as Block-by-Block [2,12,22], fractional collocation method [15], Jacobi spectral tau-collocation method [17], spectral Galerkin methods [10], Sinc collocation approximation [25], product integration approach [18], spline collocation [8], hybrid collocation method [4], reproducing kernel Hilbert space method [1], separation Chebyshev collocation method [21], multi-domain hybrid spectral collocation method [24], pseudo-spectral Galerkin method [13], and Multiquadric quasi-interpolation method [23].

Since VIEs with weakly singular kernels typically have solutions which are non-smooth near the initial point of the interval of integration in [19]. They constructed highly accurate numerical schemes for the non-smooth solution of multi-term weakly singular VIEs based on fractional-order approximation solutions in [20]. In [26], they derived and analyzed an exponentially accurate Jacobi spectral-collocation method for the numerical solution of nonlinear terminal value problems involving the Caputo fractional derivative of rational-order $\alpha \in (0,1)$ with non-smooth solutions. In [14], they provided a rigorous analysis of exponential convergence of the Chebyshev collocation method for third kind linear VIEs with non-smooth solutions. In [27], they provided a rigorous analysis of exponential convergence of an adaptive spectral collocation method for a general nonlinear system of rational-order fractional initial value problems with non-smooth solutions.

Some recent studies provide numerical methods for the two-dimensional weakly singular VIEs [5,6,11]. Nevertheless, there are few research results on the high-order numerical scheme for the two-dimensional VIEs with non-smooth solutions. The aim of this paper is to construct a uniform accuracy high-order numerical scheme for solving nonlinear VIEs with non-smooth solution. In this paper we consider two-dimensional nonlinear weakly singular VIEs with non-smooth solution:

$$u(x,y) = f(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\tau)^{\alpha-1}(y-\lambda)^{\beta-1}\kappa(x,y,\tau,\lambda,u(\tau,\lambda))d\lambda d\tau, \tag{1.1}$$

where $\alpha, \beta \in (0,1)$, $f(\cdot,\cdot)$ and $\kappa(\cdot,\cdot,\cdot,\cdot,u(\cdot,\cdot))$ are known functions defined on $D = [0,X] \times [0,Y]$ and $\Omega = D^2 \times \mathbb{R}$, respectively, $u(x,y)$ is an unknown non-smooth solution defined on D. Furthermore, assume that $\kappa(\cdot,\cdot,\cdot,\cdot,u(\cdot,\cdot))$ satisfies the Lipschitz condition with respect to the fifth variable:

$$|\kappa(\cdot,\cdot,\cdot,\cdot,u_1(\cdot,\cdot)) - \kappa(\cdot,\cdot,\cdot,\cdot,u_2(\cdot,\cdot))| \le L|u_1(\cdot,\cdot) - u_2(\cdot,\cdot)|, \quad L > 0. \tag{1.2}$$

Inspired by the recent works of [12,22], the variable changes $x = s^{\frac{\gamma_1}{\alpha}}$ and $y = t^{\frac{\gamma_2}{\beta}}$ are employed to solve (1.1) with a non-smooth solution in the form of $x^\alpha y^\beta$ multiplying smooth functions. Based on the idea of [12, 22], a new high-order numerical scheme with uniform accuracy is proposed. The convergence of the numerical scheme with the convergence order of $O(\tau_s^{3+\alpha} + \lambda_t^{3+\beta})$ is derived. Some numerical experiments are conducted to validate the results for non-smooth solutions, which are consistent with the theoretical analysis conclusion.

This contribution aims to use suitable smoothing change of variables and modified Block-by-Block method to efficiently obtain high-order numerical scheme with uniform accuracy for the two-dimensional VIEs with weakly singular kernels. The main advantage of the high-order numerical scheme lies in its ability to provide the numerical solutions of two-dimensional VIEs with weakly singular kernels with uniform accuracy. This method proves to be an effective way of obtaining numerical solutions by explicit calculation except for the coupled solution near the two boundary layers.

The structure of this paper is organized as follows. In Section 2, we present Wang-Liu-Cao's scheme for VIEs with smooth solutions. In Section 3, we propose a high-order numerical scheme for nonlinear two-dimensional VIEs with non-smooth solution. The analysis of the truncation error estimation is discussed in Section 4. The results of the convergence analysis are presented in Section 5. In Section 6, numerical examples with non-smoothness are provided to verify the accuracy and effectiveness of the proposed method. Finally, concluding remarks are included in Section 7.

2. Wang-Liu-Cao's high-order scheme and its convergence results

Consider [22], which was used the piecewise biquadratic Lagrange interpolation formula to solve the two-dimensional nonlinear VIEs with weakly singular kernel (1.1), and assume that equation (1.1) has a smooth solution $u(x,y)$, and $\kappa(x,y,s,r,u(s,r))$ satisfies the Lipschitz condition (1.2).

2.1. The existed Wang-Liu-Cao's high-order scheme for smooth solution

The existed Wang-Liu-Cao's high-order scheme for the equation (1.1) with smooth solution can be rewritten in the following form:

$$u_{i_1,j_1} = \begin{cases} f_{i_1,j_1} + h_x^\alpha h_y^\beta \sum_{i=0}^{2}\sum_{j=0}^{2} \Phi_i^{i_1}\tilde{\Phi}_j^{j_1}\kappa_{i_1,j_1}^{i,j}, i_1, j_1 = 1,2, \\ f_{i_1,j_1} + h_x^\alpha h_y^\beta \sum_{i=0}^{i_1}\sum_{j=0}^{2} \Upsilon_i^{i_1}\tilde{\Phi}_j^{j_1}\kappa_{i_1,j_1}^{i,j}, i_1 = 2m+1, 2m+2; j_1 = 1,2, \\ f_{i_1,j_1} + h_x^\alpha h_y^\beta \sum_{i=0}^{2}\sum_{j=0}^{j_1} \Phi_i^{i_1}\aleph_j^{j_1}\kappa_{i_1,j_1}^{i,j}, i_1 = 1,2; j_1 = 2n+1, 2n+2, \\ f_{i_1,j_1} + h_x^\alpha h_y^\beta \sum_{i=0}^{i_1}\sum_{j=0}^{j_1} \Upsilon_i^{i_1}\aleph_j^{j_1}\kappa_{i_1,j_1}^{i,j}, i_1 = 2m+1, 2m+2; j_1 = 2n+1, 2n+2, \end{cases} \tag{2.1}$$

where $\kappa_{i_1,j_1}^{i,j} = \kappa(x_{i_1}, y_{j_1}, x_i, y_j, u_{i,j}), x_i = ih_x, y_j = jh_y, h_x = \frac{X}{2M}, h_y = \frac{Y}{2N}$, $1 \le m \le M-1, 1 \le n \le N-1$, $\Phi_i^{i_1}, \tilde{\Phi}_j^{j_1}, \Upsilon_i^{i_1}$ and $\aleph_j^{j_1}$ can be defined through the following formulas,

$$\Phi_i^1 = \hat{B}_i, \Phi_i^2 = \tilde{B}_i, \tilde{\Phi}_j^1 = \hat{D}_j, \tilde{\Phi}_j^2 = \tilde{D}_j, i,j = 0,1,2,$$
$$\Upsilon_i^{2m+1} = \bar{B}_i^m, i = 0,1,\dots,2m+1, \Upsilon_i^{2m+2} = B_i^m, i = 0,1,\dots,2m+2,$$
$$\aleph_j^{2n+1} = \bar{D}_j^n, j = 0,1,\dots,2n+1, \aleph_j^{2n+2} = D_j^n, j = 0,1,\dots,2n+2,$$
$$\hat{B}_i = \frac{\omega_1^{i,0}}{h_x^\alpha}, \tilde{B}_i = \frac{A_2^{i,0}}{h_x^\alpha}, i = 0,1,2, \bar{B}_0^m = \frac{\omega_{2m+1}^{0,0}}{h_x^\alpha}, \bar{B}_1^m = \frac{\omega_{2m+1}^{1,0} + A_{2m+1}^{0,1}}{h_x^\alpha},$$

$$\bar{B}_2^m = \frac{\omega_{2m+1}^{2,0} + A_{2m+1}^{1,1}}{h_x^\alpha}, B_0^m = \frac{A_{2m+2}^{0,0}}{h_x^\alpha}, B_{2m+2}^m = \frac{A_{2m+2}^{2,m}}{h_x^\alpha},$$

$$B_{2i+1}^m = \frac{A_{2m+2}^{1,i}}{h_x^\alpha}, i = 0, 1, \ldots, m; B_{2i}^m = \frac{A_{2m+2}^{2,i-1} + A_{2m+2}^{0,i}}{h_x^\alpha}, i = 1, \ldots, m,$$

$$\bar{B}_{2i-1}^m = B_{2i}^m, i = 2, 3, \ldots, m+1; \bar{B}_{2i}^m = B_{2i+1}^m, i = 2, 3, \ldots, m,$$

$$\hat{D}_j = \frac{\hat{\omega}_1^{j,0}}{h_y^\beta}, \tilde{D}_j = \frac{\hat{A}_2^{j,0}}{h_y^\beta}, j = 0, 1, 2; \bar{D}_0^n = \frac{\hat{\omega}_{2n+1}^{0,0}}{h_y^\beta}, \bar{D}_1^n = \frac{\hat{\omega}_{2n+1}^{1,0} + \hat{A}_{2n+1}^{0,1}}{h_y^\beta},$$

$$\bar{D}_2^n = \frac{\hat{\omega}_{2n+1}^{2,0} + \hat{A}_{2n+1}^{1,1}}{h_y^\beta}, D_0^n = \frac{\hat{A}_{2n+2}^{0,0}}{h_y^\beta}, D_{2n+2}^n = \frac{\hat{A}_{2n+2}^{2,n}}{h_y^{1-\beta}},$$

$$D_{2j+1}^n = \frac{\hat{A}_{2n+2}^{1,j}}{h_y^\beta}, j = 0, 1, \ldots, n; D_{2j}^n = \frac{\hat{A}_{2n+2}^{2,j-1} + \hat{A}_{2n+2}^{0,j}}{h_y^\beta}, j = 1, 2, \ldots, n,$$

$$\bar{D}_{2j-1}^n = D_{2j}^n, j = 2, 3, \ldots, n+1, \bar{D}_{2j}^n = D_{2j+1}^n, j = 2, 3, \ldots, n,$$

$$\omega_1^{i,0} = \int_a^{x_1} \frac{\varphi_{i,0}(s)}{\varsigma_\alpha^1(s)} ds, A_2^{i,0} = \int_a^{x_2} \frac{\varphi_{i,0}(s)}{\varsigma_\alpha^2(s)} ds, \omega_{2m+1}^{k,0} = \int_a^{x_1} \frac{\varphi_{k,0}(s)}{\varsigma_\alpha^{2m+1}(s)} ds, k = 0, 1, 2,$$

$$A_{2m+1}^{k,i} = \int_{x_{2i-1}}^{x_{2i+1}} \frac{\varphi_{k,2i-1}(s)}{\varsigma_\alpha^{2m+1}(s)} ds, A_{2m+2}^{k,i}$$
$$= \int_{x_{2i}}^{x_{2i+2}} \frac{\varphi_{k,2i}(s)}{\varsigma_\alpha^{2m+2}(s)} ds, k = 0, 1, 2; i = 0, 1, \ldots, m,$$

$$\hat{\omega}_1^{j,0} = \int_c^{y_1} \frac{\phi_{j,0}(r)}{\eta_\beta^1(r)} dr, \hat{A}_2^{j,0} = \int_c^{y_2} \frac{\phi_{j,0}(r)}{\eta_\beta^2(r)} dr, j = 0, 1, 2,$$

$$\hat{\omega}_{2n+1}^{q,0} = \int_c^{y_1} \frac{\phi_{q,0}(r)}{\eta_\beta^{2n+1}(r)} dr, q = 0, 1, 2,$$

$$\hat{A}_{2n+1}^{q,j} = \int_{y_{2j-1}}^{y_{2j+1}} \frac{\phi_{q,2j-1}(r)}{\eta_\beta^{2n+1}(r)} dr, \hat{A}_{2n+2}^{q,j}$$
$$= \int_{y_{2j}}^{y_{2j+2}} \frac{\phi_{q,2j}(r)}{\eta_\beta^{2n+2}(r)} dr, q = 0, 1, 2; j = 0, 1, \ldots, n,$$

where $\varphi_{k,i}(s), \phi_{k,j}(t), k = 0, 1, 2$ are defined as Lagrange basis function of x_i, x_{i+1}, x_{i+2} and y_j, y_{j+1}, y_{j+2}, respectively.

In [22], Wang, Liu and Cao obtained the convergence results for (2.1) with smooth solution.

Theorem 2.1. *Let u and $u_{i,j}$ be the exact solution and numerical solution of equation (1.1), respectively. If $\kappa(\cdot, \cdot, \cdot, \cdot, u(\cdot, \cdot)) \in C^4(D \times D \times \mathbb{R})$ and satisfies (1.2), and the step sizes h_x, h_y meet the following requirements*

$$Lh_x^\alpha h_y^\beta |B_{2m+2}^m||D_{2n+2}^n| < 1, \quad 2Lh_x^\alpha h_y^\beta |B_{2m+2}^m| < 1,$$
$$2Lh_x^\alpha h_y^\beta |D_{2n+2}^n| < 1, \quad CLh_x^\alpha \frac{d^\beta}{\beta} < 1,$$

then the below error estimate holds, namely:

$$|u(x_i, y_j) - u_{i,j}| \le C(h_x^{3+\alpha} + h_y^{3+\beta}), i = 0, 1, \ldots, 2M; j = 0, 1, \ldots, 2N,$$

where C is a positive constant independent of h_x, h_y.

2.2. Existence properties of non-smooth solution for VIEs

Generally speaking, the solution has the initial singularity. In [3, 7], they provided the properties of non-smooth solution of (1.1) in one-dimensional condition.

Lemma 2.1. *Suppose that f is sufficiently smooth, Then there exists a function $\psi \in \mathcal{C}^1[0,T]$ and some $c_1, c_2, \dots, c_J \in \mathbb{R}$, such that the solution of (1) in one-dimensional condition can be expressed in the form*

$$u(t) = \psi(t) + \sum_{j=1}^{J} c_j t^{j\alpha},$$

where $J := \lceil 1/\alpha \rceil - 1$.

It is easy to check that

$$|\frac{d^k u(t)}{dt^k}| \le C(1 + t^{\alpha-k}), \quad k = 1, 2, 3, 4. \tag{2.2}$$

3. A uniform accuracy high-order scheme for two dimensional VIEs with non-smooth solution

Based on the variable transformation $x = s^{\frac{\gamma_1}{\alpha}}$, $y = t^{\frac{\gamma_2}{\beta}}$, $\frac{\gamma_1}{\alpha}, \frac{\gamma_2}{\beta} \ge 4$, one can obtain that:

$$\begin{aligned} u(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}) = f(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{s^{\frac{\gamma_1}{\alpha}}} \int_0^{t^{\frac{\gamma_2}{\beta}}} (s^{\frac{\gamma_1}{\alpha}} - \tau)^{\alpha-1} \\ \times (t^{\frac{\gamma_2}{\beta}} - \lambda)^{\beta-1} \kappa(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}, \tau, \lambda, u(\tau, \lambda)) d\lambda d\tau. \end{aligned} \tag{3.1}$$

We continue variable substitution $\tau = \xi^{\frac{\gamma_1}{\alpha}}, \lambda = \eta^{\frac{\gamma_2}{\beta}}$ on (3.1) and obtain,

$$\begin{aligned} & u(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}) \\ =& f(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{s^{\frac{\gamma_1}{\alpha}}} \int_0^{t^{\frac{\gamma_2}{\beta}}} (s^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \\ & \times (t^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \kappa(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}, \xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, u(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}})) \frac{\gamma_1}{\alpha} \xi^{\frac{\gamma_1}{\alpha}-1} \frac{\gamma_2}{\beta} \eta^{\frac{\gamma_2}{\beta}-1} d\eta d\xi. \end{aligned} \tag{3.2}$$

Let $v(s,t) = u(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}})$, then the equation (3.2) can be written as:

$$\begin{aligned} v(s,t) = f(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}) + \frac{\gamma_1 \gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^s \int_0^t (s^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\ \times (t^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \kappa(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}}, \xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi, \eta)) d\eta d\xi. \end{aligned} \tag{3.3}$$

Similar to Lemma 2.1, we assume that the solution of equation (3.3) has the following form by using the change of variables,

$$v(s,t) = u(s^{\frac{\gamma_1}{\alpha}}, s^{\frac{\gamma_2}{\beta}}) = \psi(s^{\frac{\gamma_1}{\alpha}}, ts^{\frac{\gamma_2}{\beta}}) + \sum_{i=1}^{J_1} \sum_{j=1}^{J_2} c_{i,j} s^{i\frac{\gamma_1}{\alpha}} t^{j\frac{\gamma_2}{\beta}}, \tag{3.4}$$

where $J_1 = \lceil 1/\alpha \rceil - 1, J_2 = \lceil 1/\beta \rceil - 1$.

It is easy to see that

$$|\frac{\partial^{k+l} v(s^{\frac{\gamma_1}{\alpha}}, t^{\frac{\gamma_2}{\beta}})}{\partial s^k \partial t^l}| \leq C(1 + s^{\frac{\gamma_1}{\alpha}-k} t^{\frac{\gamma_2}{\beta}-l}), \quad \text{for } k, l = 1, 2, 3, 4. \tag{3.5}$$

Divide the interval $D = [0, X] \times [0, Y]$ into $2M \times 2N$ parts, the interval $[0, X^{\frac{\alpha}{\gamma_1}}]$ into $2M$ parts, and $[0, Y^{\frac{\beta}{\gamma_2}}]$ into $2N$ parts for positive M and N. The step sizes are $\tau_s = \frac{X^{\frac{\alpha}{\gamma_1}}}{2M}$ and $\lambda_t = \frac{Y^{\frac{\beta}{\gamma_2}}}{2N}$. Let $s_i = i\tau_s, i \in \{0, 1, 2, \ldots, 2M\}$, $t_j = j\lambda_t, j \in \{0, 1, 2, \ldots, 2N\}$,$f_{i,j} = f(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}})$ and $\kappa_{i,j}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi, \eta)) = \kappa(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, \xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi, \eta))$. The numerical solution of the equation (3.3) at (s_i, t_j) is denoted as $v_{i,j}$, where $v_{i,0} = f(s_i^{\frac{\gamma_1}{\alpha}}, 0)$ and $v_{0,j} = f(0, t_j^{\frac{\gamma_2}{\beta}})$. Let $\varphi_{i,k}(\xi)$ and $\phi_{j,q}(\eta)$, $i, j = 0, 1, 2$ be the quadratic interpolation basis polynomials at points s_k, s_{k+1}, s_{k+2} and t_q, t_{q+1}, t_{q+2}, respectively.

We first estimate the value of $v(s_1, t_1)$ at the point (s_1, t_1), which has the following form:

$$\begin{aligned}
&v(s_1, t_1) \\
&= f(s_1^{\frac{\gamma_1}{\alpha}}, t_1^{\frac{\gamma_2}{\beta}}) + \frac{\gamma_1 \gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{s_1} \int_0^{t_1} (s_1^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\
&\quad \times (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \kappa(s_1^{\frac{\gamma_1}{\alpha}}, t_1^{\frac{\gamma_2}{\beta}}, \xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi, \eta)) d\eta d\xi \\
&\approx f_{1,1} + \frac{\gamma_1 \gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{s_1} \int_0^{t_1} (s_1^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \\
&\quad \times \sum_{i=0}^{2} \sum_{j=0}^{2} \varphi_{i,0}(\xi) \phi_{j,0}(\eta) \kappa_{1,1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) d\eta d\xi \\
&= f_{1,1} + \sum_{i=0}^{2} \sum_{j=0}^{2} A_1^{i,0} B_1^{j,0} \kappa_{1,1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}),
\end{aligned} \tag{3.6}$$

where

$$A_1^{i,0} = \frac{\gamma_1}{\Gamma(\alpha+1)} \int_0^{s_1} (s_1^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,0}(\xi) d\xi, \quad i = 0, 1, 2, \tag{3.7}$$

$$B_1^{j,0} = \frac{\gamma_2}{\Gamma(\beta+1)} \int_0^{t_1} (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \phi_{j,0}(\eta) d\eta, \quad j = 0, 1, 2. \tag{3.8}$$

Similarly, $v(s_1, t_2), v(s_2, t_1)$, and $v(s_2, t_2)$ are defined as

$$v(s_1, t_2) \approx f_{1,2} + \sum_{i=0}^{2} \sum_{j=0}^{2} A_1^{i,0} B_2^{j,0} \kappa_{1,2}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}), \tag{3.9}$$

$$v(s_2, t_p) \approx f_{2,p} + \sum_{i=0}^{2} \sum_{j=0}^{2} A_2^{i,0} B_p^{j,0} \kappa_{2,p}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}), p = 1, 2, \tag{3.10}$$

where

$$B_2^{j,0} = \frac{\gamma_2}{\Gamma(\beta+1)} \int_0^{t_2} (t_2^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \phi_{j,0}(\eta) d\eta, \quad j = 0, 1, 2, \tag{3.11}$$

$$A_2^{i,0} = \frac{\gamma_1}{\Gamma(\alpha+1)} \int_0^{s_2} (s_2^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,0}(\xi) d\xi, \quad i = 0, 1, 2. \tag{3.12}$$

We further estimate $v(s_{2m+l_1}, t_{p_1}), m = 1, 2, \ldots, M-1$ and $v(s_{l_1}, t_{2n+p_1}), l_1, p_1 = 1, 2; n = 1, 2, \ldots, N-1$. Assuming $v_{i,p_1}, i \le 2m$ and $v_{l_1,j}, j \le 2n$ are known. For $v(s_{2m+1}, t_1)$, we have

$$\begin{aligned}
& v(s_{2m+1}, t_1) \\
&= f_{2m+1,1} + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{s_{2m+1}} \int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\
&\quad \times (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \kappa_{2m+1,1}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta)) d\eta d\xi \\
&\approx f_{2m+1,1} + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^{s_1} \int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\
&\quad \times (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \sum_{i=0}^{2}\sum_{j=0}^{2} \varphi_{i,0}(\xi)\phi_{j,0}(\eta) \kappa_{2m+1,1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) d\eta d\xi \\
&\quad + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} \int_{s_{2k-1}}^{s_{2k+1}} \int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\
&\quad \times (t_1^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \varphi_{i,2k-1}(\xi)\phi_{j,0}(\eta) \kappa_{2m+1,1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}) d\eta d\xi \\
&= f_{2m+1,1} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_1^{j,0} \kappa_{2m+1,1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_1^{j,0} \kappa_{2m+1,1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}),
\end{aligned} \tag{3.13}$$

where the definition of $B_1^{j,0}$ is given in equation (3.8), and

$$A_{2m+1}^{i,0} = \frac{\gamma_1}{\Gamma(\alpha+1)} \int_0^{s_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,0}(\xi) d\xi, \quad i = 0, 1, 2, \tag{3.14}$$

$$A_{2m+1}^{i,k} = \frac{\gamma_1}{\Gamma(\alpha+1)} \int_{s_{2k-1}}^{s_{2k+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,2k-1}(\xi) d\xi, \quad i = 0, 1, 2; k = 1, \ldots, m. \tag{3.15}$$

We can use the following approximations to estimate $v(s_{2m+2}, t_1)$ and $v(s_{2m+l_1}, t_2)$, $l_1 = 1, 2$,

$$\begin{aligned}
v(s_{2m+1}, t_2) &\approx f_{2m+1,2} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_2^{j,0} \kappa_{2m+1,2}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_2^{j,0} \kappa_{2m+1,2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}),
\end{aligned} \tag{3.16}$$

$$v(s_{2m+2}, t_1) \approx f_{2m+2,1} + \sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+2}^{i,k} B_1^{j,0} \kappa_{2m+2,1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k+i,j}), \tag{3.17}$$

$$v(s_{2m+2},t_2)\approx f_{2m+2,2}+\sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_2^{j,0}\kappa_{2m+2,2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{2k+i,j}), \quad (3.18)$$

where $B_2^{j,0}$ is defined in equation (3.11), $B_1^{j,0}$ is in equation (3.8), $A_{2m+1}^{i,0}$ and $A_{2m+1}^{i,k}$ are respectively defined in equations (3.14) and (3.15), and the definition of $A_{2m+2}^{i,k}$ is as follows:

$$\begin{aligned}A_{2m+2}^{i,k}=&\frac{\gamma_1}{\Gamma(\alpha+1)}\int_{s_{2k}}^{s_{2k+2}}(s_{2m+2}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\varphi_{i,2k}(\xi)d\xi,\\&i=0,1,2;k=0,1,\ldots,m.\end{aligned} \quad (3.19)$$

Next, we estimate $v(s_{l_1},t_{2n+p_1}),l_1,p_1=1,2$. Using the same method, and directly obtain

$$\begin{aligned}v(s_1,t_{2n+1})&=f_{1,2n+1}+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_1}\int_0^{t_{2n+1}}(s_1^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\&\quad\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\&\approx f_{1,2n+1}+\sum_{i=0}^{2}\sum_{j=0}^{2}A_1^{i,0}B_{2n+1}^{j,0}\kappa_{1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})\\&\quad+\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_1^{i,0}B_{2n+1}^{j,q}\kappa_{1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v_{i,2q-1+j}),\end{aligned} \quad (3.20)$$

$$\begin{aligned}v(s_2,t_{2n+1})&=f_{2,2n+1}+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_2}\int_0^{t_{2n+1}}(s_2^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\&\quad\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\&\approx f_{2,2n+1}+\sum_{i=0}^{2}\sum_{j=0}^{2}A_2^{i,0}B_{2n+1}^{j,0}\kappa_{2,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})\\&\quad+\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_2^{i,0}B_{2n+1}^{j,q}\kappa_{2,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v_{i,2q-1+j}),\end{aligned} \quad (3.21)$$

$$\begin{aligned}v(s_1,t_{2n+2})&=f_{1,2n+2}+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_1}\int_0^{t_{2n+2}}(s_1^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\&\quad\times(t_{2n+2}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{1,2n+2}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\&\approx f_{1,2n+2}+\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_1^{i,0}B_{2n+2}^{j,q}\kappa_{1,2n+2}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{i,2q+j}),\end{aligned} \quad (3.22)$$

$$\begin{aligned}v(s_2,t_{2n+2})&=f_{2,2n+2}+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_2}\int_0^{t_{2n+2}}(s_2^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\&\quad\times(t_{2n+2}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2,2n+2}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\&\approx f_{2,2n+2}+\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_2^{i,0}B_{2n+2}^{j,q}\kappa_{2,2n+2}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{i,2q+j}),\end{aligned} \quad (3.23)$$

where $A_1^{i,0}$ and $A_2^{i,0}$ are defined in equations (3.7) and (3.12), respectively, and $B_{2n+1}^{j,0}$, $B_{2n+1}^{j,q}$, and $B_{2n+2}^{j,q}$ are as follows:

$$B_{2n+1}^{j,0}=\frac{\gamma_2}{\Gamma(\beta+1)}\int_0^{t_1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\phi_{j,0}(\eta)d\eta,\quad j=0,1,2, \tag{3.24}$$

$$B_{2n+1}^{j,q}=\frac{\gamma_2}{\Gamma(\beta+1)}\int_{t_{2q-1}}^{t_{2q+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\phi_{j,2q-1}(\eta)d\eta, \tag{3.25}$$
$$j=0,1,2;q=1,2,\ldots,n,$$

$$B_{2n+2}^{j,q}=\frac{\gamma_2}{\Gamma(\beta+1)}\int_{t_{2q}}^{t_{2q+2}}(t_{2n+2}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\phi_{j,2q}(\eta)d\eta, \tag{3.26}$$
$$j=0,1,2;q=0,1,2,\ldots,n.$$

Next, we approximate $v(s_i,t_j)$ for $i,j>2$. Assume $v_{i,j},v_{i,2n+q},v_{2m+p,j}$, $p,q=1,2;0\le i\le 2m,0\le j\le 2n,1\le m\le M-1,1\le n\le N-1$ are known, and then further construct the approximation for $v(s_{2m+p},t_{2n+q}),p,q=1,2$ as follows. For $v(s_{2m+1},t_{2n+1})$, we have

$$\begin{aligned}
&v(s_{2m+1},t_{2n+1})\\
=&f_{2m+1,2n+1}+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big[\int_0^{s_1}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\
&+\sum_{q=1}^{n}\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\\
&\times\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\
&+\sum_{k=1}^{m}\int_{2k-1}^{2k+1}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\\
&+\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))d\eta d\xi\Big]\\
\doteq&f_{2m+1,2n+1}+E_1+E_2+E_3+E_4,
\end{aligned}\tag{3.27}$$

where E_1 is the integral over the subdomain $[0,s_1]\times[0,t_1]$, which can be calculated using the following biquadratic Lagrange interpolation,

$$\begin{aligned}
E_1\approx&\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_1}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\\
&\times\sum_{i=0}^{2}\sum_{j=0}^{2}\varphi_{i,0}(\xi)\phi_{j,0}(\eta)\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})d\eta d\xi\\
=&\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_{2n+1}^{j,0}\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j}),
\end{aligned}\tag{3.28}$$

where $A_{2m+1}^{i,0}$ is as shown in equation (3.14), and $B_{2n+1}^{j,0}$ is defined in equation (3.24).

For E_2, we have

$$\begin{aligned} E_2 =& \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{q=1}^{n} \int_0^{s_1} \int_{t_{2q-1}}^{t_{2q+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \\ & \times \eta^{\frac{\gamma_2}{\beta}-1} \kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta)) d\eta d\xi \\ \approx & \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q-1+j}), \end{aligned} \tag{3.29}$$

where $A_{2m+1}^{i,0}$ is as shown in equation (3.14), and $B_{2n+1}^{j,q}$ is defined in equation (3.25).

For E_3, we have

$$\begin{aligned} E_3 =& \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=1}^{m} \int_{s_{2k-1}}^{s_{2k+1}} \int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \\ & \times \eta^{\frac{\gamma_2}{\beta}-1} \kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta)) d\eta d\xi \\ \approx & \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,0} \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}), \end{aligned} \tag{3.30}$$

where $A_{2m+1}^{i,k}$ is as shown in equation (3.15), and $B_{2n+1}^{j,0}$ is defined in equation (3.24).

The same approach is applied to compute E_4, and its form is as follows:

$$\begin{aligned} E_4 =& \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{k=1}^{m}\sum_{q=1}^{n} \int_{s_{2k-1}}^{s_{2k+1}} \int_{t_{2q-1}}^{t_{2q+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \\ & \times (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1} \kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta)) d\eta d\xi \\ \approx & \sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1} \\ & \times (s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,2q-1+j}), \end{aligned} \tag{3.31}$$

where $A_{2m+1}^{i,k}$ is as shown in equation (3.15), and $B_{2n+1}^{j,q}$ is defined in equation (3.25).

Substituting equations (3.28)-(3.31) into equation (3.27), we obtain

$$\begin{aligned} & v_{2m+1,2n+1} \\ =& f_{2m+1,2n+1} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+1}^{j,0} \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \\ & + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q-1+j}) \\ & + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,0} \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}) \\ & + \sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,2q-1+j}). \end{aligned} \tag{3.32}$$

Similar to $v(s_{2m+1}, t_{2n+1})$, we compute the approximation for $v(s_{2m+2}, t_{2n+1})$. The integral region is divided into segments, and piecewise biquadratic interpolation is used for approximation. Therefore, the approximation for $v(s_{2m+2}, t_{2n+1})$ can be given by the following expression

$$\begin{aligned}
&v(s_{2m+2}, t_{2n+1})\\
=&f_{2m+2,2n+1} + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_{2m+2}}\int_0^{t_{2n+1}}(s_{2m+2}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\\
&\times \xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+2,2n+1}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta))d\eta d\xi\\
\approx&f_{2m+2,2n+1} + \sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+1}^{j,0}\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k+i,j})\\
&+\sum_{k=0}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+1}^{j,q}\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{2k+i,2q-1+j}), \quad (3.33)
\end{aligned}$$

where $A_{2m+2}^{i,k}$, $B_{2n+1}^{j,0}$, and $B_{2n+1}^{j,q}$ are defined in equations (3.15), (3.24), and (3.25), respectively.

The approximations for $v(s_{2m+1}, t_{2n+2})$ and $v(s_{2m+2}, t_{2n+2})$ can be similarly given as follows

$$\begin{aligned}
&v(s_{2m+1}, t_{2n+2})\\
=&f_{2m+1,2n+2} + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_{2m+1}}\int_0^{t_{2n+2}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times (t_{2n+2}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+1,2n+2}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta))d\eta d\xi\\
\approx&f_{2m+1,2n+2} + \sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_{2n+2}^{j,q}\kappa_{2m+1,2n+2}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q+j})\\
&+\sum_{k=1}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_{2n+2}^{j,q}\kappa_{2m+1,2n+2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,2q+j}), \quad (3.34)\\
&v(s_{2m+2}, t_{2n+2})\\
=&f_{2m+2,2n+2} + \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_{2m+2}}\int_0^{t_{2n+2}}(s_{2m+2}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times (t_{2n+2}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\kappa_{2m+2,2n+2}(\xi^{\frac{\gamma_1}{\alpha}}, \eta^{\frac{\gamma_2}{\beta}}, v(\xi,\eta))d\eta d\xi \quad (3.35)\\
\approx&f_{2m+2,2n+2} + \sum_{k=0}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+2}^{j,q}\kappa_{2m+2,2n+2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{2k+i,2q+j}),
\end{aligned}$$

where $A_{2m+1}^{i,0}$, $A_{2m+1}^{i,k}$, $B_{2n+2}^{j,q}$, and $A_{2m+2}^{i,k}$ are defined in equations (3.14), (3.15), (3.26), and (3.19), respectively.

Finally, the high-order numerical scheme for equation (1.1) can be rewritten in the following form, for $p_1, q_1 = 1, 2$,

$$v_{p_1,q_1} = f_{p_1,q_1} + \sum_{i=0}^{2}\sum_{j=0}^{2}A_{p_1}^{i,0}B_{q_1}^{j,0}\kappa_{p_1,q_1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}),$$

$$\begin{aligned}
v_{2m+1,q_1} &= f_{2m+1,q_1} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{q_1}^{j,0} \kappa_{2m+1,q_1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{q_1}^{j,0} \kappa_{2m+1,q_1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}), \\
v_{2m+2,q_1} &= f_{2m+2,q_1} + \sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+2}^{i,k} B_{q_1}^{j,0} \kappa_{2m+2,q_1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k+i,j}), \\
v_{p_1,2n+1} &= f_{p_1,2n+1} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{p_1}^{i,0} B_{2n+1}^{j,0} \kappa_{p_1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \\
&\quad + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{p_1}^{i,0} B_{2n+1}^{j,q} \kappa_{p_1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q-1+j}), \\
v_{p_1,2n+2} &= f_{p_1,2n+2} + \sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{p_1}^{i,0} B_{2n+2}^{j,q} \kappa_{p_1,2n+2}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q+j}), \\
v_{2m+1,2n+1} &= f_{2m+1,2n+1} + \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+1}^{j,0} \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j}) \qquad (3.36) \\
&\quad + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q-1+j}) \\
&\quad + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,0} \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j}) \\
&\quad + \sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+1}^{j,q} \kappa_{2m+1,2n+1} \\
&\quad \times (s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,2q-1+j}), \\
v_{2m+1,2n+2} &= f_{2m+1,2n+2} + \sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0} B_{2n+2}^{j,q} k_{2m+1,2n+2}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q+j}) \\
&\quad + \sum_{k=1}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k} B_{2n+2}^{j,q} \kappa_{2m+1,2n+2} \\
&\quad \times (s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_{2q+j}^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,2q+j}), \\
v_{2m+2,2n+1} &= f_{2m+2,2n+1} + \sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+2}^{i,k} B_{2n+1}^{j,0} \kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k+i,j}) \\
&\quad + \sum_{k=0}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+2}^{i,k} B_{2n+1}^{j,q} \kappa_{2m+2,2n+1} \\
&\quad \times (s_{2k+i}^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{2k+i,2q-1+j}), \\
v_{2m+2,2n+2} &= f_{2m+2,2n+2}
\end{aligned}$$

$$+\sum_{k=0}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+2}^{j,q}\kappa_{2m+2,2n+2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{2k+i,2q+j}).$$

4. Truncation errors analysis

In this section, we will analyze the truncation error for (3.36). Let $r_{i,j}$ be the truncation error at the point (s_i,t_j) as

$$r_{i,j}:=v(s_i,t_j)-\hat{v}_{i,j}, \tag{4.1}$$

where $\hat{v}_{i,j}$ is an approximation of $v(s_i,t_j)$, and replace $v_{i,j}$ with $v(s_i,t_j)$ in (3.36), such as

$$\begin{aligned}
&\hat{v}_{2m+1,2n+1}\\
&=f_{2m+1,2n+1}+\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_{2n+1}^{j,0}\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_i,t_j))\\
&\quad+\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_{2n+1}^{j,q}\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_i,t_{2q-1+j}))\\
&\quad+\sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_{2n+1}^{j,0}\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_j))\\
&\quad+\sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_{2n+1}^{j,q}\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_{2q-1+j})),
\end{aligned} \tag{4.2}$$

for $i=2m+1, j=2n+1$.

For ease of notation, let $\frac{\partial^3\kappa}{\partial s^3}\doteq\partial_s^3\kappa$. Based on the idea of [22], we can obtain that $r_{i,j}$ satisfies the following Lemma.

Lemma 4.1. *If $\kappa(\cdot,\cdot,\cdot,\cdot,u(\cdot,\cdot))\in C^4(D\times D\times\mathbb{R})$, then it holds that*

$$|r_{i,j}|\le C(\tau_s^{3+\alpha}+\lambda_t^{3+\beta}),$$

where C is a positive constant and independent of τ_s and λ_t.

Proof. For $r_{2m+1,2n+1}$, using (3.32), (4.1) and (4.2), it is obvious that

$$\begin{aligned}
&r_{2m+1,2n+1}\\
&=v(s_{2m+1},t_{2n+1})-\hat{v}_{2m+1,2n+1}\\
&=\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big\{\int_0^{s_1}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\\
&\quad\times[\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))\\
&\quad-\sum_{i=0}^{2}\sum_{j=0}^{2}\varphi_{i,0}(\xi)\phi_{j,0}(\eta)\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_i,t_j))]d\eta d\xi\\
&\quad+\sum_{q=1}^{n}\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}
\end{aligned}$$

$$
\begin{aligned}
&\times [\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta)) - \sum_{i=0}^{2}\sum_{j=0}^{2}\varphi_{i,0}(\xi)\phi_{j,2q-1}(\eta)\\
&\times \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_i,t_{2q-1+j})]d\eta d\xi\\
&+\sum_{k=1}^{m}\int_{s_{2k-1}}^{s_{2k+1}}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\\
&\times [\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta)) - \sum_{i=0}^{2}\sum_{j=0}^{2}\varphi_{i,2k-1}(\xi)\phi_{j,0}(\eta)\\
&\times \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_j))]d\eta d\xi\\
&+\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\\
&\times [\kappa_{2m+1,2n+1}(\xi^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\xi,\eta))\sum_{i=0}^{2}\sum_{j=0}^{2}\varphi_{i,2k-1}(\xi)\phi_{j,2q-1}(\eta)\\
&-\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_{2q-1+j}))]d\eta d\xi\Big\}\\
=&\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\times\Big\{\int_0^{s_1}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}R_1 d\eta d\xi\\
&+\sum_{q=1}^{n}\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}R_2 d\eta d\xi\\
&+\sum_{k=1}^{m}\int_{s_{2k-1}}^{s_{2k+1}}\int_0^{t_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}R_3 d\eta d\xi\\
&+\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}R_4 d\eta d\xi\Big\}\\
\doteq&\sum_{i=0}^{4}r_{2m+1,2n+1}^{(i)}. \qquad (4.3)
\end{aligned}
$$

Using the Taylor's formula, the approximation of the function $\kappa_{2m+1,2n+1}$ corresponding to the integral interval $[0,s_1]\times[0,t_1]$ can be obtained as

$$
\begin{aligned}
R_1 =& \frac{1}{3!}\partial_s^3\kappa_{2m+1,2n+1}(\theta_1^{\frac{\gamma_1}{\alpha}}(\xi),\eta^{\frac{\gamma_2}{\beta}},v(\theta_1(\xi),\eta))\prod_{i=0}^{2}(\xi-s_i)\\
&+\sum_{i=0}^{2}\frac{\varphi_{i,0}(\xi)}{3!}\partial_t^3\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},\varepsilon_1^{\frac{\gamma_2}{\beta}}(\eta),v(s_i,\varepsilon_1(\eta)))\prod_{j=0}^{2}(\eta-t_j),
\end{aligned}
$$

where $(\theta_1(\xi),\varepsilon_1(\eta))\in[0,s_1]\times[0,t_1]$.

For $(\xi,\eta)\in[0,s_1]\times[t_{2q-1},t_{2q+1}]$, there exists $(\theta_2(\xi),\varepsilon_q(\eta))\in[0,s_1]\times[t_{2q-1},$

$t_{2q+1}]$ to satisfy that

$$R_2 = \frac{1}{3!}\partial_s^3 \kappa_{2m+1,2n+1}(\theta_2^{\frac{\gamma_1}{\alpha}}(\xi), \eta^{\frac{\gamma_2}{\beta}}, v(\theta_2(\xi),\eta)) \prod_{i=0}^{2}(\xi - s_i)$$
$$+ \sum_{i=0}^{2} \frac{1}{3!}\varphi_{i,0}(\xi)\partial_t^3 \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, \varepsilon_q^{\frac{\gamma_2}{\beta}}(\eta), v(s_i, \varepsilon_q(\eta))) \prod_{j=0}^{2}(\eta - t_{2q-1+j}).$$

Similarly, for all $(\xi,\eta) \in [s_{2k-1}, s_{2k+1}] \times [0,t_1]$, there exists $(\theta_k(\xi), \varepsilon_2(\eta)) \in [s_{2k-1}, s_{2k+1}] \times [0,t_1]$ such that

$$R_3 = \frac{1}{3!}\partial_s^3 \kappa_{2m+1,2n+1}(\theta_k^{\frac{\gamma_1}{\alpha}}(\xi), \eta^{\frac{\gamma_2}{\beta}}, v(\theta_k(\xi),\eta)) \prod_{i=0}^{2}(\xi - s_{2k-1+i})$$
$$+ \sum_{i=0}^{2} \frac{1}{3!}\varphi_{i,2k-1}(\xi)\partial_t^3 \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, \varepsilon_2^{\frac{\gamma_2}{\beta}}(\eta), v(s_{2k-1+i}, \varepsilon_2(\eta))) \prod_{j=0}^{2}(\eta - t_j).$$

Additionally, for all $(\xi,\eta) \in [s_{2k-1}, s_{2k+1}] \times [t_{2q-1}, t_{2q+1}]$, there exists $(\theta_{k_1}(\xi), \varepsilon_{q2}(\eta)) \in [s_{2k-1}, s_{2k+1}] \times [t_{2q-1}, t_{2q+1}]$ such that

$$R_4 = \frac{1}{3!}\partial_s^3 \kappa_{2m+1,2n+1}(\theta_{k_1}^{\frac{\gamma_1}{\alpha}}(\xi), \eta^{\frac{\gamma_2}{\beta}}, v(\theta_{k_1}(\xi),\eta)) \prod_{i=0}^{2}(\xi - s_{2k-1+i})$$
$$+ \sum_{i=0}^{2} \frac{1}{3!}\varphi_{i,2k-1}(\xi)\partial_t^3 \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, \varepsilon_{q_2}^{\frac{\gamma_2}{\beta}}(\eta), v(s_{2k-1+i}, \varepsilon_{q_2}(\eta)))$$
$$\times \prod_{j=0}^{2}(\eta - t_{2q-1+j}).$$

For $r_{2m+1,2n+1}^{(1)}$, a direct computation shows that

$$|r_{2m+1,2n+1}^{(1)}|$$
$$\leq \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Big| \int_0^{s_1}\int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}$$
$$\times \eta^{\frac{\gamma_2}{\beta}-1} \frac{1}{3!}\partial_s^3 \kappa_{2m+1,2n+1}(\theta_1^{\frac{\gamma_1}{\alpha}}(\xi), \eta^{\frac{\gamma_2}{\beta}}, v(\theta_1(\xi),\eta)) \prod_{i=0}^{2}(\xi - s_i) d\eta d\xi \Big|$$
$$+ \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)} \Big| \int_0^{s_1}\int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \eta^{\frac{\gamma_2}{\beta}-1}$$
$$\times \sum_{i=0}^{2} \frac{\varphi_{i,0}(\xi)}{3!} \partial_t^3 \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, \varepsilon_1^{\frac{\gamma_2}{\beta}}(\eta), v(s_i, \varepsilon_1(\eta))) \prod_{j=0}^{2}(\eta - t_j) d\eta d\xi \Big|$$
$$\doteq R_1^{(1)} + R_1^{(2)}. \tag{4.4}$$

For $R_1^{(1)}$ and $R_1^{(2)}$ of (4.4), by using integral mean value theorem and monotonicity of function, we have

$$R_1^{(1)} \leq \frac{C\gamma_1\gamma_2\tau_s^3}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + s_1^{\frac{\gamma_1}{\alpha}-3})$$

$$\times \int_0^{s_1}\int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta d\xi$$

$$\le \frac{C\gamma_1\gamma_2\tau_s^4\lambda_t}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})$$

$$\times (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}, \tag{4.5}$$

$$R_1^{(2)} \le \frac{C\gamma_1\gamma_2\lambda_t^3}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_1^{\frac{\gamma_2}{\beta}-3})$$

$$\times \int_0^{s_1}\int_0^{t_1} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\alpha-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta d\xi$$

$$\le \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_1^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}. \tag{4.6}$$

We substitute (4.5) and (4.6) back into (4.4) to obtain $r_{2m+1,2n+1}^{(1)}$,

$$|r_{2m+1,2n+1}^{(1)}| \le \frac{C\gamma_1\gamma_2\tau_s^4\lambda_t}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})$$

$$\times (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}$$

$$+ \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_1^{\frac{\gamma_2}{\beta}-3})$$

$$\times (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}. \tag{4.7}$$

Similar to $r_{2m+1,2n+1}^{(1)}$, for $r_{2m+1,2n+1}^{(2)}$, it is easily seen that

$$|r_{2m+1,2n+1}^{(2)}|$$

$$\le \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{q=1}^{n}\Big|\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}$$

$$\times (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\frac{1}{3!}\partial_s^3\kappa_{2m+1,2n+1}(\theta_2^{\frac{\gamma_1}{\alpha}}(\xi),\eta^{\frac{\gamma_2}{\beta}},v(\theta_2(\xi),\eta))$$

$$\times \prod_{i=0}^{2}(\xi - s_i)d\eta d\xi\Big|$$

$$+ \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{q=1}^{n}\Big|\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_1}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}$$

$$\times \eta^{\frac{\gamma_2}{\beta}-1}\sum_{i=0}^{2}\frac{\varphi_{i,0}(\xi)}{3!}\partial_t^3\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}},\varepsilon_q^{\frac{\gamma_2}{\beta}}(\eta),v(s_i,\varepsilon_q(\eta)))\prod_{j=0}^{2}(\eta - t_{2q-1+j})d\eta d\xi\Big|$$

$$\doteq R_2^{(1)} + R_2^{(2)}. \tag{4.8}$$

For $R_2^{(1)}$ in (4.8), by using integral mean value theorem and monotonicity of function, one can obtain

$$R_2^{(1)} \le \frac{C\gamma_1\gamma_2\tau_s^3}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})\sum_{q=1}^{n}\Big|\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}$$

$$
\begin{aligned}
&\times (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta d\xi\big| \\
&\le \frac{C\gamma_1\gamma_2\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1} \\
&\times \sum_{q=1}^{n}\int_{t_{2q-1}}^{t_{2q+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta \\
&\le \frac{C\gamma_1\gamma_2\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3}) \\
&\times (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\int_{t_1}^{t_{2n+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta \\
&= \frac{C\gamma_1\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta}. \quad (4.9)
\end{aligned}
$$

For $R_2^{(2)}$, using the triangle inequality, yields

$$
\begin{aligned}
R_2^{(2)} \le\ & \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{q=1}^{n}\Big|\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \\
&\times \eta^{\frac{\gamma_2}{\beta}-1}\sum_{i=0}^{2}\frac{\varphi_{i,0}(\xi)}{3!}[\partial_t^3\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, \varepsilon_q^{\frac{\gamma_2}{\beta}}(\eta), v(s_i,\varepsilon_q(\eta))) \\
&- \partial_t^3\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q}^{\frac{\gamma_2}{\beta}}, v(s_i,t_{2q}))]\prod_{j=0}^{2}(\eta - t_{2q-1+j})d\eta d\xi\Big| \\
&+ \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{q=1}^{n}\Big|\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1} \\
&\times \eta^{\frac{\gamma_2}{\beta}-1}\sum_{i=0}^{2}\frac{\varphi_{i,0}(\xi)}{3!}\partial_t^3\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q}^{\frac{\gamma_2}{\beta}}, v(s_i,t_{2q}))\prod_{j=0}^{2}(\eta - t_{2q-1+j})d\eta d\xi\Big| \\
\doteq\ & A_1 + A_2. \quad (4.10)
\end{aligned}
$$

For A_1, by using integral mean value theorem and monotonicity of function, one get

$$
\begin{aligned}
A_1 \le\ & \frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{q=1}^{n}\int_0^{s_1}\int_{t_{2q-1}}^{t_{2q+1}}\big|(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1} \\
&\times \sum_{i=0}^{2}\frac{\varphi_{i,0}(\xi)}{3!}\times C\partial_t^4\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, (\varepsilon_q^*)^{\frac{\gamma_2}{\beta}}, v(s_i,\varepsilon_q^*))\lambda_t\prod_{j=0}^{2}(\eta - t_{2q-1+j})\big|d\eta d\xi \\
\le\ & \frac{C\gamma_1\gamma_2(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_0^{s_1}\Big|\sum_{i=0}^{2}\frac{\varphi_{\bar{i},0}(\xi)}{3!}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\Big|d\xi \\
&\times \sum_{q=1}^{n}\int_{t_{2q-1}}^{t_{2q+1}}\big|(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\prod_{j=0}^{2}(\eta - t_{2q-1+j})\big|d\eta \\
\le\ & \frac{C\gamma_1\gamma_2(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}
\end{aligned}
$$

$$\times \sum_{q=1}^{n} \int_{t_{2q-1}}^{t_{2q+1}} \big|(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\prod_{j=0}^{2}(\eta - t_{2q-1+j})\big|d\eta$$
$$\leq \frac{C\gamma_1\gamma_2(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}$$
$$\times \int_{t_1}^{t_{2n+1}} (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta$$
$$= \frac{C\gamma_1(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_1^{\frac{\gamma_2}{\beta}})^{\beta}. \tag{4.11}$$

We subdivide A_2 into A_{21} and A_{22}: A_{21} when $1 \leqslant q \leqslant n-1$, and A_{22} when $q = n$.

For A_{21}, $1 \leqslant q \leqslant n-1$, we have the following estimate:

$$A_{21}$$
$$\leq \frac{C\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})\int_0^{s_1} \frac{|\sum_{i=0}^{2}\varphi_{i,0}(\xi)|}{3!}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}d\xi$$
$$\times \sum_{q=1}^{n-1}\int_{t_{2q-1}}^{t_{2q+1}} \big|(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\prod_{j=0}^{2}(\eta - t_{2q-1+j})\big|d\eta$$
$$\leq \frac{C\gamma_1\gamma_2\tau_s}{2\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\sum_{q=1}^{n-1}|\,(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \tilde{t}^{\frac{\gamma_2}{\beta}})^{\beta-1}\tilde{t}^{\frac{\gamma_2}{\beta}-1}$$
$$\times \int_{t_{2q-1}}^{t_{2q}}\prod_{j=0}^{2}(\eta - t_{2q-1+j})d\eta + (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \tilde{\tilde{t}}^{\frac{\gamma_2}{\beta}})^{\beta-1}\tilde{\tilde{t}}^{\frac{\gamma_2}{\beta}-1}\int_{t_{2q}}^{t_{2q+1}}\prod_{j=0}^{2}(\eta - t_{2q-1+j})d\eta|$$
$$= \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\sum_{q=1}^{n-1}[(t_{2n+1}^{\frac{\gamma_2}{\beta}} - \tilde{\tilde{t}}^{\frac{\gamma_2}{\beta}})^{\beta-1}\tilde{\tilde{t}}^{\frac{\gamma_2}{\beta}-1}$$
$$- (t_{2n+1}^{\frac{\gamma_2}{\beta}} - \tilde{t}^{\frac{\gamma_2}{\beta}})^{\beta-1}\tilde{t}^{\frac{\gamma_2}{\beta}-1}]$$
$$\leq \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\sum_{q=1}^{n-1}[(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_{2q+1}^{\frac{\gamma_2}{\beta}})^{\beta-1}t_{2q+1}^{\frac{\gamma_2}{\beta}-1}$$
$$- (t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_{2q-1}^{\frac{\gamma_2}{\beta}})^{\beta-1}t_{2q-1}^{\frac{\gamma_2}{\beta}-1}]$$
$$\leq \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}} - t_{2n-1}^{\frac{\gamma_2}{\beta}})^{\beta-1}t_{2n-1}^{\frac{\gamma_2}{\beta}-1}$$
$$\leq \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_2}{\alpha}-1}$$
$$\times [\frac{\gamma_2}{\beta}\xi_1^{\frac{\gamma_2}{\beta}-1}(t_{2n+1} - t_{2n-1})]^{\beta-1}t_{2n-1}^{\frac{\gamma_2}{\beta}-1} \tag{4.12}$$
$$\leq \frac{C\gamma_1\gamma_2\lambda_t^4\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta}t_{2n+1}^{\frac{\gamma_2}{\beta}-1})^{\beta-1}t_{2n-1}^{\frac{\gamma_2}{\beta}-1}(2\lambda_t)^{\beta-1}$$
$$\leq \frac{C\gamma_1\gamma_2\lambda_t^{3+\beta}\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1 + t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{2\gamma_2}{\beta})^{\beta-1}t_{2n+1}^{(\frac{\gamma_2}{\beta}-1)(\beta-1)}t_{2n-1}^{\frac{\gamma_2}{\beta}-1},$$

where $\tilde{t} \in (t_{2q-1}, t_{2q}), \tilde{\tilde{t}} \in (t_{2q}, t_{2q+1}), \xi_1 \in (t_{2n-1}, t_{2n+1})$.

For A_{22}, $q=n$, the estimate is following:

$$
\begin{aligned}
&A_{22}\\
\le& \frac{C\gamma_1\gamma_2\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\Big|\int_{t_{2n-1}}^{t_{2n+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\\
&\times\ \eta^{\frac{\gamma_2}{\beta}-1}\prod_{j=0}^{2}(\eta-t_{2q-1+j})d\eta\Big|\\
\le& \frac{C\gamma_1\gamma_2\lambda_t^3\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\\
&\times\int_{t_{2n-1}}^{t_{2n+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta\\
\le& \frac{C\gamma_1\lambda_t^3\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta}\xi_2^{\frac{\gamma_2}{\beta}-1}(t_{2n+1}-t_{2n-1}))^{\beta}\\
\le& \frac{C\gamma_1\lambda_t^3\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta}t_{2n+1}^{\frac{\gamma_2}{\beta}-1})^{\beta}\cdot 2^{\beta}\cdot\lambda_t^{\beta}\\
\le& \frac{C\gamma_1\lambda_t^{3+\beta}\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot 2^{\beta}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta},
\end{aligned}\tag{4.13}
$$

where $\xi_2\in(t_{2n-1},t_{2n+1})$.

Combining (4.12) and (4.13) gives the desired estimate A_2:

$$
\begin{aligned}
A_2\le&\frac{C\gamma_1\gamma_2\lambda_t^{3+\beta}\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\\
&\times s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{2\gamma_2}{\beta})^{\beta-1}t_{2n+1}^{(\frac{\gamma_2}{\beta}-1)(\beta-1)}t_{2n-1}^{\frac{\gamma_2}{\beta}-1}\\
&+\frac{C\gamma_1\lambda_t^{3+\beta}\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{2\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta}.
\end{aligned}\tag{4.14}
$$

Further, by substituting (4.14) and (4.11) into (4.10) leads to

$$
\begin{aligned}
R_2^{(2)}\le&\frac{C\gamma_1(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}\\
&+\frac{C\gamma_1\gamma_2\lambda_t^{3+\beta}\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot 2^{\beta-1}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\\
&\times s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta})^{\beta-1}t_{2n+1}^{(\frac{\gamma_2}{\beta}-1)(\beta-1)}t_{2n-1}^{\frac{\gamma_2}{\beta}-1}\\
&+\frac{C\gamma_1\lambda_t^{3+\beta}\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot 2^{\beta}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta}.
\end{aligned}\tag{4.15}
$$

Finally, combining (4.15) with (4.9) yields $r_{2m+1,2n+1}^{(2)}$:

$$
\begin{aligned}
&|r_{2m+1,2n+1}^{(2)}|\\
\le&R_2^{(1)}+R_2^{(2)}\\
\le&\frac{C\gamma_1\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_1^{\frac{\gamma_1}{\alpha}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C\gamma_1(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})\lambda_t^4\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}\\
&+\frac{C\gamma_1\gamma_2\lambda_t^{3+\beta}\tau_s}{8\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot 2^{\beta-1}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}\\
&\times(\frac{\gamma_2}{\beta})^{\beta-1}t_{2n+1}^{(\frac{\gamma_2}{\beta}-1)(\beta-1)}t_{2n-1}^{\frac{\gamma_2}{\beta}-1}\\
&+\frac{C\gamma_1\lambda_t^{3+\beta}\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}\cdot 2^{\beta}\cdot(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1}s_1^{\frac{\gamma_1}{\alpha}-1}(\frac{\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta}.
\end{aligned}\tag{4.16}
$$

Similar to the proof of $r^{(2)}_{2m+1,2n+1}$, we have

$$
\begin{aligned}
&|r^{(3)}_{2m+1,2n+1}|\\
\le&\frac{C\gamma_2\tau_s^4\lambda_t}{\Gamma(\alpha+1)\Gamma(\beta+1)}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-4})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}\\
&+\frac{C\gamma_1\gamma_2\tau_s^{3+\alpha}\lambda_t}{4\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}(\frac{2\gamma_1}{\alpha})^{\alpha-1}s_{2m+1}^{(\frac{\gamma_1}{\alpha}-1)(\alpha-1)}s_{2m-1}^{\frac{\gamma_1}{\alpha}-1}\\
&+\frac{C\gamma_2\tau_s^{3+\alpha}\lambda_t}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}(\frac{\gamma_1}{\alpha})^{\alpha}s_{2m+1}^{\gamma_1-\alpha}\cdot 2^{\alpha}\\
&+\frac{C\gamma_2\lambda_t^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta-1}t_1^{\frac{\gamma_2}{\beta}-1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}.
\end{aligned}\tag{4.17}
$$

Similarly, for $r^{(4)}_{2m+1,2n+1}$, we have the following form by using directly calculating:

$$
\begin{aligned}
&|r^{(4)}_{2m+1,2n+1}|\\
\le&\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}|\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\eta^{\frac{\gamma_2}{\beta}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\frac{1}{3!}\partial_s^3\kappa_{2m+1,2n+1}(\theta_{k_1}^{\frac{\gamma_1}{\alpha}}(\xi),\eta^{\frac{\gamma_2}{\beta}},v(\theta_{k_1}(\xi),\eta))\\
&\times\prod_{i=0}^{2}(\xi-s_{2k-1+i})d\eta d\xi|+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\times|\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\\
&\times\eta^{\frac{\gamma_2}{\beta}-1}\sum_{i=0}^{2}\frac{1}{3!}\varphi_{i,2k-1}(\xi)\partial_t^3\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},\varepsilon_{q_2}^{\frac{\gamma_2}{\beta}}(\eta),v(s_{2k-1+i},\varepsilon_{q_2}(\eta)))\\
&\times\prod_{j=0}^{2}(\eta-t_{2q-1+j})d\eta d\xi|\\
\doteq&R_4^{(1)}+R_4^{(2)}.
\end{aligned}\tag{4.18}
$$

For $R_4^{(1)}$ in (4.18), one can obtain

$$
R_4^{(1)}\le\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}|\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}
$$

$$
\begin{aligned}
&\times \xi^{\frac{\gamma_1}{\alpha}-1}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\frac{1}{3!}\eta^{\frac{\gamma_2}{\beta}-1}[\partial_s^3\kappa_{2m+1,2n+1}(\theta_{k_1}^{\frac{\gamma_1}{\alpha}}(\xi),\eta^{\frac{\gamma_2}{\beta}},v(\theta_{k_1}(\xi),\eta))\\
&-\partial_s^3\kappa_{2m+1,2n+1}(\theta_{2k}^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\theta_{2k},\eta))]\prod_{i=0}^{2}(\xi-s_{2k-1+i})d\eta d\xi\,|\\
&+\frac{\gamma_1\gamma_2}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,|\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}\frac{1}{3!}\partial_s^3\kappa_{2m+1,2n+1}(\theta_{2k}^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\theta_{2k},\eta))\\
&\times\prod_{i=0}^{2}(\xi-s_{2k-1+i})d\eta d\xi\,|\\
\doteq&R_{44}^{(1)}+R_{44}^{(2)}. \qquad (4.19)
\end{aligned}
$$

Here, we obtain $R_{44}^{(1)}$ through the following estimations,

$$
\begin{aligned}
R_{44}^{(1)}\leq&\frac{C\gamma_1\gamma_2\tau_s}{\Gamma(\alpha+1)\Gamma(\beta+1)}\,|\sum_{k=1}^{m}\sum_{q=1}^{n}\int_{s_{2k-1}}^{s_{2k+1}}\int_{t_{2q-1}}^{t_{2q+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\\
&\times(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\frac{1}{3!}\eta^{\frac{\gamma_2}{\beta}-1}\partial_s^4\kappa_{2m+1,2n+1}((\theta_{k_1}^*)^{\frac{\gamma_1}{\alpha}},\eta^{\frac{\gamma_2}{\beta}},v(\theta_{k1}^*,\eta))\\
&\times\prod_{i=0}^{2}(\xi-s_{2k-1+i})d\eta d\xi\,|\\
\leq&\frac{C\gamma_1\gamma_2\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-4})\sum_{k=1}^{m}|\int_{s_{2k-1}}^{s_{2k+1}}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}d\xi|\\
&\times\sum_{q=1}^{n}|\int_{t_{2q-1}}^{t_{2q+1}}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-\eta^{\frac{\gamma_2}{\beta}})^{\beta-1}\eta^{\frac{\gamma_2}{\beta}-1}d\eta|\\
=&\frac{C\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-4})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}. \qquad (4.20)
\end{aligned}
$$

Similar to $R_{44}^{(1)}$ and $R_2^{(2)}$, we have

$$
\begin{aligned}
R_{44}^{(2)}\leq&\frac{C\gamma_1^{\alpha}\tau_s^{3+\alpha}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(\frac{\alpha}{2})^{1-\alpha}s_{2m-1}^{\gamma_1-\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}\\
&+\frac{C\tau_s^{3+\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(\frac{2\gamma_1}{\alpha})^{\alpha}s_{2m+1}^{\gamma_1-\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}. \qquad (4.21)
\end{aligned}
$$

Similar to $R_4^{(1)}$, for $R_4^{(2)}$, we have:

$$
\begin{aligned}
R_4^{(2)}\leq&\frac{C\lambda_t^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}\\
&+\frac{C\gamma_2^{\beta}\lambda_t^{3+\beta}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(\frac{2}{\beta})^{\beta-1}t_{2n-1}^{\gamma_2-\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma}{\alpha}})^{\alpha}\\
&+\frac{C\lambda t^{3+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(\frac{2\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}. \qquad (4.22)
\end{aligned}
$$

Finally, based on the proven inequalities mentioned above, it can be concluded that

$$\begin{aligned}
&|r^{(4)}_{2m+1,2n+1}| \\
\le &\frac{C\tau_s^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-4})(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta} \\
&+\frac{C\gamma_1^{\alpha}\tau_s^{3+\alpha}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(\frac{\alpha}{2})^{1-\alpha}s_{2m-1}^{\gamma_1-\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta} \\
&+\frac{C\tau_s^{3+\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+s_{2m+1}^{\frac{\gamma_1}{\alpha}-3})(\frac{2\gamma_1}{\alpha})^{\alpha}s_{2m+1}^{\gamma_1-\alpha}(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta} \\
&+\frac{C\lambda_t^4}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-4})(t_{2n+1}^{\frac{\gamma_2}{\beta}}-t_1^{\frac{\gamma_2}{\beta}})^{\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha} \\
&+\frac{C\gamma_2^{\beta}\lambda_t^{3+\beta}}{4\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(\frac{2}{\beta})^{\beta-1}t_{2n-1}^{\gamma_2-\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma}{\alpha}})^{\alpha} \\
&+\frac{C\lambda t^{3+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1+t_{2n+1}^{\frac{\gamma_2}{\beta}-3})(\frac{2\gamma_2}{\beta})^{\beta}t_{2n+1}^{\gamma_2-\beta}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}.
\end{aligned} \tag{4.23}$$

Therefore, we have

$$|r_{2m+1,2n+1}| \le C(\tau_s^{3+\alpha}+\lambda_t^{3+\beta}). \tag{4.24}$$

We can use the method of proving the truncation error at (s_{2m+1}, t_{2n+1}) to similarly prove the truncation errors at other layers. Therefore, we can conclude that the truncation error $r_{i,j}$ satisfies

$$|r_{i,j}| \le C(\tau_s^{3+\alpha}+\lambda_t^{3+\beta}), \text{ for } i=0,1,\ldots,2M; j=0,1,\ldots,2N.$$

Lemma 4.1 have been completed the proof. □

5. Convergence analysis

We present some lemmas that are helpful for the convergence analysis.

Lemma 5.1. [9] *Let* $\{a_i > 0, 0 \le i \le N_{step}\}$ *and*

$$a_i \le b_i + M_f h^{\sigma+1-\gamma_0\beta_0}\sum_{j=0}^{i-1}\frac{j^{\sigma}}{(i^{\beta_0}-j^{\beta_0})^{\gamma_0}}a_j, \quad 1 \le i \le N_{step}, \tag{5.1}$$

where $0 < \gamma_0 < 1$, $1 \le \beta_0 \le \sigma+1$, $\sigma \ge 0$, M_f *is a positive constant, and* $\{b_i, 0 \le i \le N_{step}\}$ *is a non-decreasing sequence, then,*

$$a_i \le b_i \sum_{n=0}^{\infty}(\frac{M_f(ih)^{\sigma+1-\gamma_0\beta_0}}{\beta_0})^n \hat{B}_n(\gamma_0,\beta_0,\sigma), \quad 0 \le i \le N_{step}, \tag{5.2}$$

where

$$\hat{B}_n(\gamma_0,\beta_0,\sigma) = \begin{cases} 1, & n=0, \\ \prod_{l=1}^{n} B(\frac{l}{\beta_0}(\sigma+1-\alpha_0\beta_0)+\alpha_0,(1-\alpha_0)), & n \ge 1, \end{cases} \tag{5.3}$$

and $B(p,q)$ is the Beta function defined for $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$ *as*

$$B(p,q)=\int_0^1 s^{p-1}(1-s)^{q-1}ds. \tag{5.4}$$

When $\sigma+1-\beta_0=0$,

$$a_i\le b_iE_{1-\gamma_0}(\frac{M_f\Gamma(1-\gamma_0)}{\beta_0}(ih)^{\beta_0(1-\gamma_0)}),\quad 0\le i\le N_{step}. \tag{5.5}$$

Lemma 5.2. *These coefficients of (3.36) satisfy the following:*

$$\begin{aligned}
&|A_p^{i,0}|\le C\tau_s^{\gamma_1},\\
&|A_{2m+1}^{i,k}|\le C\tau_s^{\gamma_1}(2k+1)^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}}-(2k+1)^{\frac{\gamma_1}{\alpha}}]^{\alpha-1},\\
&|A_{2m+2}^{i,k}|\le C\tau_s^{\gamma_1}(2k+2)^{\frac{\gamma_1}{\alpha}-1}[(2m+2)^{\frac{\gamma_1}{\alpha}}-(2k+2)^{\frac{\gamma_1}{\alpha}}]^{\alpha-1},\\
&|A_{2m+1}^{i,m}|\le C\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}}-(2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha},\\
&|A_{2m+2}^{i,m}|\le C\tau_s^{\gamma_1}[(2m+2)^{\frac{\gamma_1}{\alpha}}-(2m)^{\frac{\gamma_1}{\alpha}}]^{\alpha},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&|B_p^{j,0}|\le C\lambda_t^{\gamma_2},\\
&|B_{2n+1}^{j,q}|\le C\lambda_t^{\gamma_2}(2q+1)^{\frac{\gamma_2}{\beta}-1}[(2n+1)^{\frac{\gamma_2}{\beta}}-(2q+1)^{\frac{\gamma_2}{\beta}}]^{\beta-1},\\
&|B_{2n+2}^{j,q}|\le C\lambda_t^{\gamma_2}(2q+2)^{\frac{\gamma_2}{\beta}-1}[(2n+2)^{\frac{\gamma_2}{\beta}}-(2q+2)^{\frac{\gamma_2}{\beta}}]^{\beta-1},\\
&|B_{2n+1}^{j,n}|\le C\lambda_t^{\gamma_2}[(2n+1)^{\frac{\gamma_2}{\beta}}-(2n-1)^{\frac{\gamma_2}{\beta}}]^{\beta},\\
&|B_{2n+2}^{j,n}|\le C\lambda_t^{\gamma_2}[(2n+2)^{\frac{\gamma_2}{\beta}}-(2n)^{\frac{\gamma_2}{\beta}}]^{\beta},
\end{aligned} \tag{5.7}$$

where A and B are coefficients of numerical scheme (3.36).

Proof. By using the definition of $A_1^{0,0}$, we have

$$\begin{aligned}
|A_1^{0,0}|&=|\frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{s_1}(s_1^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\varphi_{0,0}(\xi)d\xi|\\
&=|\frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{s_1}(s_1^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\frac{(\xi-s_1)(\xi-s_2)}{(s_0-s_1)(s_0-s_2)}d\xi|\\
&\le\frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{s_1}(s_1^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}d\xi\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{s_1^{\frac{\gamma_1}{\alpha}}}(s_1^{\frac{\gamma_1}{\alpha}}-z)^{\alpha-1}dz\\
&=\frac{1}{\Gamma(\alpha+1)}(s_1^{\frac{\gamma_1}{\alpha}})^{\alpha}\\
&\le C\tau_s^{\gamma_1}.
\end{aligned} \tag{5.8}$$

Similarly, we can arrive at $|A_1^{i,0}|\le C\tau_s^{\gamma_1}$, $|A_2^{i,0}|\le C\tau_s^{\gamma_1}$, for $i=0,1,2$.
When $k=0$, by using the mean value theorem, we can obtain

$$\begin{aligned}
|A_{2m+1}^{0,0}|&=|\frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{s_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\varphi_{0,0}(\xi)d\xi|\\
&\le\frac{\gamma_1}{\Gamma(\alpha+1)}\int_0^{s_1}(s_{2m+1}^{\frac{\gamma_1}{\alpha}}-\xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1}\xi^{\frac{\gamma_1}{\alpha}-1}\frac{(\xi-s_1)(\xi-s_2)}{(s_0-s_1)(s_0-s_2)}d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\gamma_1 \tau_s}{\Gamma(\alpha+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_1^{\frac{\gamma_1}{\alpha}})^{\alpha-1} s_1^{\frac{\gamma_1}{\alpha}-1} \\
&\leq \frac{\gamma_1 \tau_s^{\gamma_1}}{\Gamma(\alpha+1)}[(2m+1)^{\frac{\gamma_1}{\alpha}} - 1]^{\alpha-1} \\
&\leq C\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - 1]^{\alpha-1}. \qquad (5.9)
\end{aligned}
$$

Similarly, we can obtain

$$|A_{2m+1}^{i,0}| \leq C\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - 1]^{\alpha-1}, i = 0, 1, 2.$$

When $k \in \{1, 2, \ldots, m-1\}$, by directly calculating we have

$$
\begin{aligned}
|A_{2m+1}^{i,k}| &= |\frac{\gamma_1}{\Gamma(\alpha+1)} \int_{s_{2k-1}}^{s_{2k+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,2k-1}(\xi) d\xi| \\
&\leq \frac{\gamma_1}{\Gamma(\alpha+1)} \int_{s_{2k-1}}^{s_{2k+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} d\xi \\
&\leq \frac{2\gamma_1 \tau_s}{\Gamma(\alpha+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_{2k+1}^{\frac{\gamma_1}{\alpha}})^{\alpha-1} s_{2k+1}^{\frac{\gamma_1}{\alpha}-1} \\
&\leq \frac{2\gamma_1 \tau_s^{\gamma_1}}{\Gamma(\alpha+1)}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2k+1)^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}(2k+1)^{\frac{\gamma_1}{\alpha}-1} \\
&\leq C\tau_s^{\gamma_1}(2k+1)^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2k+1)^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}. \qquad (5.10)
\end{aligned}
$$

Similarly, we also have

$$|A_{2m+2}^{i,k}| \leq C\tau_s^{\gamma_1}(2k+2)^{\frac{\gamma_1}{\alpha}-1}[(2m+2)^{\frac{\gamma_1}{\alpha}} - (2k+2)^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}, k = 1, 2, \ldots, m-1.$$

When $k = m$,

$$
\begin{aligned}
|A_{2m+1}^{i,m}| &= |\frac{\gamma_1}{\Gamma(\alpha+1)} \int_{s_{2m-1}}^{s_{2m+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} \varphi_{i,2m-1}(\xi) d\xi| \\
&\leq \frac{\gamma_1}{\Gamma(\alpha+1)} \int_{s_{2m-1}}^{s_{2m+1}} (s_{2m+1}^{\frac{\gamma_1}{\alpha}} - \xi^{\frac{\gamma_1}{\alpha}})^{\alpha-1} \xi^{\frac{\gamma_1}{\alpha}-1} d\xi \\
&= \frac{1}{\Gamma(\alpha+1)}(s_{2m+1}^{\frac{\gamma_1}{\alpha}} - s_{2m-1}^{\frac{\gamma_1}{\alpha}})^{\alpha} \\
&= \frac{\tau_s^{\gamma_1}}{\Gamma(\alpha+1)}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha} \\
&\leq C\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha}. \qquad (5.11)
\end{aligned}
$$

Similarly, we can obtain $|A_{2m+2}^{i,m}| \leq C\tau_s^{\gamma}[(2m+2)^{\frac{\gamma_1}{\alpha}} - (2m)^{\frac{\gamma_1}{\alpha}}]^{\alpha}$.

Using a similar proof method, we can prove formula (5.7), which is omitted here. The proof is then completed. □

Theorem 5.1. *Let $v(s_i, t_j)$ and $v_{i,j}$ be the exact solution and numerical solution of (3.36) for $i \in \{0, 1, \ldots, 2M\}, j \in \{0, 1, \ldots, 2N\}$, respectively. If there exist positive constants τ_s^* and λ_t^* such that*

$$\tau_s \leq \tau_s^* \leq \frac{1}{N\sqrt[\alpha]{CL}} \quad \text{and} \quad \lambda_t \leq \lambda_t^* \leq \frac{1}{M\sqrt[\beta]{CL}}, \qquad (5.12)$$

then the estimate is given by

$$|v(s_i,t_j)-v_{i,j}|\leq C(\tau_s^{3+\alpha}+\lambda_t^{3+\beta}),\ \textit{for } i=0,1,\dots,2M; j=0,1,\dots,2N, \quad (5.13)$$

where C is a constant and independent of τ_s,λ_t.

Proof. Let $e_{i,j}=v(s_i,t_j)-v_{i,j}$, for $i=0,1,2,\dots,2M; j=0,1,2,\dots,2N$. When $i,j=1,2$, we have

$$e_{k,l}=\sum_{i=0}^{2}\sum_{j=0}^{2}A_k^{i,0}B_l^{j,0}[\kappa_{k,l}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_i,t_j))-\kappa_{k,l}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})]+r_{k,l},$$

where $k,l=1,2$. $A_1^{i,0}$, $A_2^{i,0}$, and $B_1^{j,0}$, $B_2^{j,0}$ are given in (3.7), (3.12), (3.8), and (3.11), respectively. According to Lemma 5.2, we have the following result:

$$|e_{k,l}|\leq CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}\sum_{i=0}^{2}\sum_{j=0}^{2}|e_{i,j}|+|r_{k,l}|,\quad k,l=1,2.$$

By combining these four inequalities, we obtain

$$|e_{i,j}|\leq CL(|r_{1,1}|+|r_{2,1}|+|r_{1,2}|+|r_{2,2}|),i,j=1,2. \quad (5.14)$$

Applying the Lemma 4.1, we can get the results of convergence for $i,j=1,2$. When $i\geq 3$, $j=1,2$, $e_{i,j}$ is

$$\begin{aligned}
e_{2m+1,1}=&\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_1^{j,0}[\kappa_{2m+1,1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_i,t_j))-\kappa_{2m+1,1}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})]\\
&+\sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_1^{j,0}[\kappa_{2m+1,1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_j))\\
&-\kappa_{2m+1,1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{2k-1+i,j})]+r_{2m+1,1},\\
e_{2m+1,2}=&\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_2^{j,0}[\kappa_{2m+1,2}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_i,t_j))-\kappa_{2m+1,2}(s_i^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{i,j})]\\
&+\sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_2^{j,0}[\kappa_{2m+1,2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_j))\\
&-\kappa_{2m+1,2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{2k-1+i,j})]+r_{2m+1,2},\\
e_{2m+2,1}=&\sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_1^{j,0}[\kappa_{2m+2,1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k+i},t_j))\\
&-\kappa_{2m+2,1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{2k+i,j})]+r_{2m+2,1},\\
e_{2m+2,2}=&\sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_2^{j,0}[\kappa_{2m+2,2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v(s_{2k+i},t_j))\\
&-\kappa_{2m+2,2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_j^{\frac{\gamma_2}{\beta}},v_{2k+i,j})]+r_{2m+2,2},
\end{aligned} \quad (5.15)$$

where $A_{2m+1}^{i,0}$, $A_{2m+1}^{i,k}$, and $B_1^{j,0}$, $B_2^{j,0}$ are defined in (3.14), (3.15), (3.8), and (3.11), respectively. Define $\|\bar{e}_i\| = \max\{|e_{i,1}|, |e_{i,2}|, i \in \{0, 1, \ldots, 2M\}\}$ and $\|\bar{r}_i\| = \max\{|r_{i,1}|, |r_{i,2}|, i \in \{0, 1, \ldots, 2M\}\}$, we can obtain:

$$\begin{aligned} \|\bar{e}_{2m+1}\| &\leq CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}\sum_{i=0}^{2m} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|\bar{e}_i\| \\ &\quad + LA_{2m+1}^{2,m}\|\bar{e}_{2m+1}\| + \|\bar{r}_{2m+1}\|, \qquad (5.16) \\ \|\bar{e}_{2m+2}\| &\leq CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}\sum_{i=0}^{2m+1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+2)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|\bar{e}_i\| \\ &\quad + LA_{2m+2}^{2,m}\|\bar{e}_{2m+2}\| + \|\bar{r}_{2m+2}\|, \quad m = 1, 2, \ldots, N-1. \qquad (5.17) \end{aligned}$$

For (5.16), applying Lemma 4.1, we have

$$\|\bar{e}_{2m+1}\| \leq CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}\sum_{i=0}^{2m} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|\bar{e}_i\| + C\|\bar{r}_{2m+1}\|. \qquad (5.18)$$

For inequality (5.18), using Gronwall inequality [9], we have

$$\|\bar{e}_{2m+1}\| \leq C\|\bar{r}_{2m+1}\|E_\alpha(\frac{CL\lambda_t^{\gamma_2}\Gamma(\alpha)\alpha}{\gamma_1}((2m+1)\tau_s)^{\gamma_1}) \leq C\tau_s^{3+\alpha}. \qquad (5.19)$$

Using the same line for (5.17), we can get

$$\|\bar{e}_{2m+2}\| \leq C\|\bar{r}_{2m+2}\|E_\alpha(\frac{CL\lambda_t^{\gamma_2}\Gamma(\alpha)\alpha}{\gamma_1}((2m+2)\tau_s)^{\gamma_1}) \leq C\tau_s^{3+\alpha}. \qquad (5.20)$$

Similarly, by using Lemma 4.1, we can obtain

$$\begin{aligned} |e_{i,j}| &\leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}), \quad i \geq 3; j = 1, 2, \qquad (5.21) \\ |e_{i,j}| &\leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}), \quad i = 1, 2; j \geq 3. \qquad (5.22) \end{aligned}$$

Next, for $e_{i,j}, i, j \geq 3$, we have

$$\begin{aligned} &e_{2m+1,2n+1} \\ &= \sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0}B_{2n+1}^{j,0}[\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v(s_i, t_j)) \\ &\quad - \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{i,j})] \\ &\quad + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,0}B_{2n+1}^{j,q}[\kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v(s_i, t_{2q-1+j})) \\ &\quad - \kappa_{2m+1,2n+1}(s_i^{\frac{\gamma_1}{\alpha}}, t_{2q-1+j}^{\frac{\gamma_2}{\beta}}, v_{i,2q-1+j})] \\ &\quad + \sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2} A_{2m+1}^{i,k}B_{2n+1}^{j,0}[\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v(s_{2k-1+i}, t_j)) \\ &\quad - \kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}}, t_j^{\frac{\gamma_2}{\beta}}, v_{2k-1+i,j})] \end{aligned}$$

$$+\sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_{2n+1}^{j,q}[\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_{2q-1+j}))$$
$$-\kappa_{2m+1,2n+1}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v_{2k-1+i,2q-1+j})]+r_{2m+1,2n+1},$$

$$e_{2m+2,2n+1}$$
$$=\sum_{k=0}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+1}^{j,0}[\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{j}^{\frac{\gamma_2}{\beta}},v(s_{2k+i},t_j))$$
$$-\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{j}^{\frac{\gamma_2}{\beta}},v_{2k+i,j})]$$
$$+\sum_{k=0}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+1}^{j,q}[\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v(s_{2k+i},t_{2q-1+j}))$$
$$-\kappa_{2m+2,2n+1}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{2q-1+j}^{\frac{\gamma_2}{\beta}},v_{2k+i,2q-1+j})]+r_{2m+2,2n+1},$$

$$e_{2m+1,2n+2}$$
$$=\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,0}B_{2n+2}^{j,q}[\kappa_{2m+1,2n+2}(s_{i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v(s_i,t_{2q+j}))$$
$$-\kappa_{2m+1,2n+2}(s_{i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{i,2q+j})]$$
$$+\sum_{k=1}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+1}^{i,k}B_{2n+2}^{j,q}[\kappa_{2m+1,2n+2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v(s_{2k-1+i},t_{2q+j}))$$
$$-\kappa_{2m+1,2n+2}(s_{2k-1+i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{2k-1+i,2q+j})]+r_{2m+1,2n+2},$$

$$e_{2m+2,2n+2}$$
$$=\sum_{k=0}^{m}\sum_{q=0}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}A_{2m+2}^{i,k}B_{2n+2}^{j,q}[\kappa_{2m+2,2n+2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v(s_{2k+i},t_{2q+j}))$$
$$-\kappa_{2m+2,2n+2}(s_{2k+i}^{\frac{\gamma_1}{\alpha}},t_{2q+j}^{\frac{\gamma_2}{\beta}},v_{2k+i,2q+j})]+r_{2m+2,2n+2}.$$

We define $\|e_i\|=\max_{0\le j\le 2N}|e_{i,j}|$, $\|r_i\|=\max_{0\le j\le 2N}|r_{i,j}|$, from which we can obtain

$$|e_{2m+1,2n+1}|$$
$$\le\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,0}|\cdot|B_{2n+1}^{j,0}|\cdot L|e_{ij}|$$
$$+\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,0}|\cdot|B_{2n+1}^{j,q}|\cdot L|e_{i,2q-1+j}|$$
$$+\sum_{k=1}^{m}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,k}|\cdot|B_{2n+1}^{j,0}|\cdot L|e_{2k-1+i,j}|$$
$$+\sum_{k=1}^{m}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,k}|\cdot|B_{2n+1}^{j,q}|\cdot L|e_{2k-1+i,2q-1+j}|+r_{2m+1,2n+1}$$

$$
\begin{aligned}
&\le \sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,0}|\cdot|B_{2n+1}^{j,0}|\cdot L\|e_i\| + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,0}|\cdot|B_{2n+1}^{j,q}|\cdot L\|e_i\| \\
&\quad + \sum_{k=1}^{m-1}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,k}|\cdot|B_{2n+1}^{j,0}|\cdot L\|e_{2k-1+i}\| \\
&\quad + \sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,m}|\cdot|B_{2n+1}^{j,0}|\cdot L\|e_{2m-1+i}\| \\
&\quad + \sum_{k=1}^{m-1}\sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,k}|\cdot|B_{2n+1}^{j,q}|\cdot L\|e_{2k-1+i}\| \\
&\quad + \sum_{q=1}^{n}\sum_{i=0}^{2}\sum_{j=0}^{2}|A_{2m+1}^{i,m}|\cdot|B_{2n+1}^{j,q}|\cdot L\|e_{2m-1+i}\| + r_{2m+1,2n+1} \\
&\le CL\sum_{i=0}^{2m}\sum_{j=0}^{2n}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\lambda_t^{\gamma_2} j^{\frac{\gamma_2}{\beta}-1}[(2n+1)^{\frac{\gamma_2}{\beta}} - j^{\frac{\gamma_2}{\beta}}]\|e_i\| \\
&\quad + CL\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha}\sum_{j=0}^{2n}\lambda_t^{\gamma_2} j^{\frac{\gamma_2}{\beta}-1}[(2n+1)^{\frac{\gamma_2}{\beta}} - j^{\frac{\gamma_2}{\beta}}]\|e_{2m+1}\| \\
&\quad + CL\lambda_t^{\gamma_2}[(2n+1)^{\frac{\gamma_2}{\beta}-1} - 1]^{\beta-1}\sum_{i=0}^{2m}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| + \|r_{2m+1}\| \\
&\le CL\sum_{i=0}^{2m}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\|\int_{t_0}^{t_{2n+1}}(t_{2n+1}-y)^{\beta-1}dy \\
&\quad + CL\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha}\|e_{2m+1}\|\int_{t_0}^{t_{2n+1}}(t_{2n+1}-y)^{\beta-1}dy \\
&\quad + CL\lambda_t^{\gamma_2}(2n)^{\beta-1}\sum_{i=0}^{2m}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| + \|r_{2m+1}\| \\
&= CL\sum_{i=0}^{2m}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\|\frac{t_{2n+1}^{\beta}}{\beta} \\
&\quad + CL\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha}\|e_{2m+1}\|\frac{t_{2n+1}^{\beta}}{\beta} \\
&\quad + CL\lambda_t^{\gamma_2}(2n)^{\beta-1}\sum_{i=0}^{2m}\tau_s^{\gamma_1} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| + \|r_{2m+1}\|.
\end{aligned}
\tag{5.23}
$$

Since the above formula (5.23) applies to $n = 1, 2, \ldots, N-1$, we have

$$
\begin{aligned}
\|e_{2m+1}\| \le & CL\frac{Y^{\beta}}{\beta}\tau_s^{\gamma_1}\sum_{i=0}^{2m} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| \\
& + CL\frac{Y^{\beta}}{\beta}\tau_s^{\gamma_1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - (2m-1)^{\frac{\gamma_1}{\alpha}}]^{\alpha}\|e_{2m+1}\|
\end{aligned}
$$

$$+ CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}(2n)^{\beta-1}\sum_{i=0}^{2m} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| + \|r_{2m+1}\|. \tag{5.24}$$

Therefore, from (5.24), we obtain

$$\begin{aligned}\|e_{2m+1}\| \leq & [CL\frac{Y^\beta}{\beta}\tau_s^{\gamma_1} + CL\tau_s^{\gamma_1}\lambda_t^{\gamma_2}(2n)^{\beta-1}] \\ & \times \sum_{i=0}^{2m} i^{\frac{\gamma_1}{\alpha}-1}[(2m+1)^{\frac{\gamma_1}{\alpha}} - i^{\frac{\gamma_1}{\alpha}}]^{\alpha-1}\|e_i\| + C\|r_{2m+1}\|. \end{aligned} \tag{5.25}$$

By the discrete Gronwall inequality [9] and (5.25), one can obtain

$$\begin{aligned}\|e_{2m+1}\| &\leq C\|r_{2m+1}\|E_\alpha\{\frac{(CL\frac{Y^\beta}{\beta} + CL\lambda_t^{\gamma_2}(2n)^{\beta-1})\Gamma(\alpha)}{\frac{\gamma_1}{\alpha}}((2m+1)\tau_s)^{\gamma_1}\} \\ &\leq C\|r_{2m+1}\|E_\alpha\{\frac{\alpha}{\gamma_1}CL\Gamma(\alpha)X^{\gamma_1}[\frac{Y^\beta}{\beta} + \lambda_t^{\gamma_2}(2n)^{\beta-1}]\}. \end{aligned} \tag{5.26}$$

Combining the results of the above estimates (5.26) with Lemma 4.1, we obtain

$$|r_{2m+1,2n+1}| \leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}), \tag{5.27}$$

where C is a constant and independent of τ_s, λ_t.

Similar to $r_{2m+1,2n+1}$, we have

$$\begin{aligned}&|r_{2m+2,2n+1}| \leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}), \quad |r_{2m+1,2n+2}| \leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}), \\ &|r_{2m+2,2n+2}| \leq C(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}). \end{aligned} \tag{5.28}$$

Combining (5.27) and (5.28), we complete the proof of Theorem 5.1. □

6. Numerical examples

In this section, we use the constructed high-order uniform accuracy numerical scheme to solve the two-dimensional nonlinear VIEs with three numerical examples to test the convergence of the proposed method and measure the accuracy of the error. For these comparisons, the maximum error is applied. All numerical examples are implemented using MATLAB software with an 11th Gen Intel® Core™ i7-11800H @ 2.30GHz CPU and 8.00 GB RAM.

Example 6.1. For this example, we solve the following two-dimensional linear VIEs with a smooth solution using the Wang-Liu-Cao's scheme from [22]:

$$\begin{aligned}u(x,y) =& f(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-\tau)^{\alpha-1}(y-\lambda)^{\beta-1} \\ & \times (xy+s+t)u(\tau,\lambda)\,d\lambda\,d\tau, \quad (x,y) \in [0,1]^2, \end{aligned}$$

where

$$f(x,y) = x^4y^4 - \frac{576}{\Gamma(\alpha+5)\Gamma(\beta+5)}x^{5+\alpha}y^{5+\beta} - \frac{2880}{\Gamma(\alpha+6)\Gamma(\beta+5)}x^{5+\alpha}y^{4+\beta}$$

$$- \frac{2880}{\Gamma(\alpha+5)\Gamma(\beta+6)} x^{4+\alpha} y^{5+\beta},$$

and $u(x,y) = x^4 y^4$.

In this example, we test different values of α, β, and choose step sizes $h_x = \frac{X^{\frac{\alpha}{\gamma_1}}}{2M}$ and $h_y = \frac{Y^{\frac{\beta}{\gamma_2}}}{2N}$ with $\gamma_1 = \gamma_2 = 1, N = 2M$, defining the error $e^u_{h_x,h_y}$ and convergence order (C.O.) as:

$$e^u_{h_x,h_y} = \max_{\substack{i=1,\cdots,2M \\ j=1,\cdots,2N}} |u(x_i, y_j) - u_{i,j}|, \text{ C.O.} = \log_2 \left(\frac{e^u_{2h_x,2h_y}}{e^u_{h_x,h_y}} \right).$$

Table 1. Comparison of maximum error, convergence order for different parameters.

N	$\alpha = 0.4, \beta = 0.6$	C.O.	$\alpha = 0.3, \beta = 0.5$	C.O.
2^3	$1.80435e-4$	-	$4.09253e-4$	-
2^4	$1.82768e-5$	3.303398	$4.40888e-5$	3.214511
2^5	$1.79929e-6$	3.344514	$4.61842e-6$	3.254943
2^6	$1.74270e-7$	3.368027	$4.76330e-7$	3.277351
2^7	$1.67200e-8$	3.381956	$4.86900e-8$	3.290138

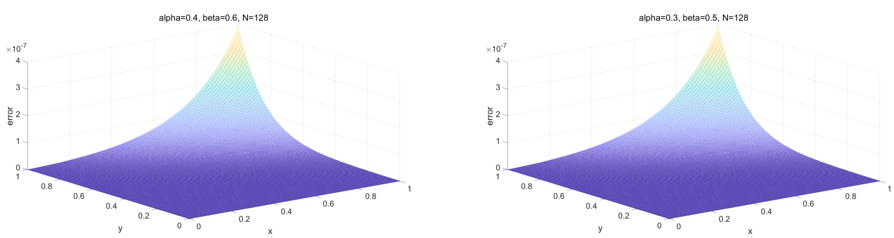

Figure 1. Error distribution for $\alpha = 0.4, \beta = 0.6$ (left) and $\alpha = 0.3, \beta = 0.5$ (right) with $N = 128$.

From Table 1, we can see that the maximum error and convergence order for different N, α, β values. When $\alpha = 0.4, \beta = 0.6$, the observed order is approximate to 3.4. When $\alpha = 0.3, \beta = 0.5$, the observed order is about 3.3. This is basically consistent with the theoretical prediction, approaching the theoretical order of $3+\alpha$ when h_y is relatively small compared to h_x, $O(h_x^{3+\alpha} + h_y^{3+\beta}) = O(h_x^{3+\alpha})$. Figure 1 shows the error distribution with $N = 2M$ and $N = 128$, for $\alpha = 0.4, \beta = 0.6$, and $\alpha = 0.3, \beta = 0.5$,respectively. Both Table 1 and Figure 1 suggest that the proposed scheme provides very accurate approximation for the VIEs with smooth solution.

Example 6.2. We apply the proposed method to solve the following two–dimensional linear VIEs with non-smooth solution:

$$u(x,y) = f(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\tau)^{\alpha-1} (y-\lambda)^{\beta-1}$$
$$\times (xy + s + t) u(\tau, \lambda) d\lambda d\tau, \quad (x,y) \in [0,1] \times [0,1].$$

The right hand side function $f(x,y)$ is

$$f(x,y) = x^{2+\alpha} y^{2+\beta} - \frac{1}{\Gamma(\alpha)\Gamma(B)} x^{2\alpha+3} B(3+\alpha, \alpha) y^{2\beta+3} B(3+\beta, \beta)$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{2\alpha+3}B(4+\alpha,\alpha)y^{2\beta+2}B(3+\beta,\beta)\\
&-\frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{2\alpha+2}B(3+\alpha,\alpha)y^{2\beta+3}B(4+\alpha,\alpha).
\end{aligned}
$$

The exact solution is $u(x,y)=x^{2+\alpha}y^{2+\beta}$ and the definition of the $B(\cdot,\cdot)$ function is given in (5.4). In this example and subsequent examples, different values of α,β are tested, with step sizes $\tau_s=\frac{X^{\frac{\alpha}{\gamma_1}}}{2M}$ and $\lambda_t=\frac{Y^{\frac{\beta}{\gamma_2}}}{2N}$ with $\gamma_1=\gamma_2=\gamma, N=2M$.

In Table 2, assume that λ_t is sufficiently small relative to τ_s, then $O(\tau_s^{3+\alpha}+\lambda_t^{3+\beta})=O(\tau_s^{3+\alpha})$. These tables show the maximum errors and convergence orders for different values of N,α,β,γ. In Tables 2, with $\alpha=0.6,\beta=0.8$ and $\gamma=4$, the tested orders are close to 3.6. and with $\alpha=0.5,\beta=0.7$ and $\gamma=3$, the tested orders are close to 3.5. It is easy to see that the theoretical order is close to $3+\alpha$.

Table 2. Comparison of maximum error, convergence order for different parameters.

N	$\alpha=0.6,\beta=0.8,\gamma=4$	C.O.	$\alpha=0.5,\beta=0.7,\gamma=3$	C.O.
2^3	9.18037e-3	-	9.12070e-3	-
2^4	8.44753e-4	3.44195	8.90016e-4	3.35724
2^5	7.18857e-5	3.55475	8.10651e-5	3.45668
2^6	5.86942e-6	3.61441	7.08155e-6	3.51694
2^7	4.68563e-7	3.64690	6.03780e-7	3.55198

The Figure 2 shows the error distribution with $N=2M$ and $N=128$ for $\alpha=0.6,\beta=0.8,\gamma=4$ and $\alpha=0.5,\beta=0.7,\gamma=3$, respectively. From Figure 2, it proves that the errors are close to 10^{-7}.

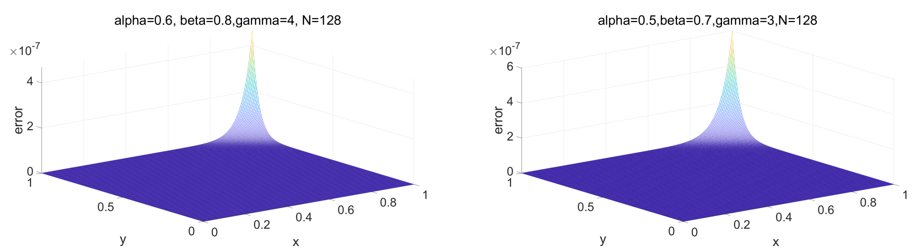

Figure 2. Error distribution for $\alpha=0.6,\beta=0.8,\gamma=4$ (left) and $\alpha=0.2,\beta=0.5,\gamma=3$ (right) with $N=128$.

Example 6.3. Consider the following two-dimensional nonlinear VIEs with non-smooth solution:

$$u(x,y)=f(x,y)+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y(x-s)^{\alpha-1}(y-t)^{\beta-1}(xy+s+t)u^2(s,t)dtds,$$

and the right-hand side is

$$
\begin{aligned}
f(x,y)=&x^{2+\alpha}y^{2+\beta}-\frac{x^{5+3\alpha}y^{5+3\alpha}}{\Gamma(\alpha)\Gamma(\beta)}B(5+2\alpha,\alpha)B(5+2\beta,\beta)\\
&-\frac{x^{5+3\alpha}y^{4+3\beta}}{\Gamma(\alpha)\Gamma(\beta)}B(6+2\alpha,\alpha)B(5+2\beta,\beta)
\end{aligned}
$$

$$-\frac{x^{4+3\alpha}y^{5+3\beta}}{\Gamma(\alpha)\Gamma(\beta)}B(5+2\alpha,\alpha)B(6+2\beta,\beta),$$

and the exact solution is $u(x,y) = x^{2+\alpha}y^{2+\beta}$.

It is easy to prove that $O(\tau_s^{3+\alpha} + \lambda_t^{3+\beta}) = O(\tau_s^{3+\alpha})$ when λ_t is sufficiently small compared to τ_s. Table 3 provides the maximum errors and convergence orders for different values of N, α, β, γ. In Table 3, with $\alpha = 0.6, \beta = 0.8$ and $\gamma = 4$, the tested orders are close to 3.6, and with $\alpha = 0.5, \beta = 0.7$ and $\gamma = 3$, the tested orders are close to 3.5. Figure 3 shows the error distribution with $N = 2M$ and $N = 128$, for $\alpha = 0.6, \beta = 0.8, \gamma = 4$ and $\alpha = 0.5, \beta = 0.7, \gamma = 3$, respectively. Both Table 3 and Figure 3 suggest that the proposed scheme provides very accurate approximation for the VIEs with non-smooth solution.

Table 3. Comparison of maximum error and convergence order for different parameters.

N	$\alpha = 0.6, \beta = 0.8, \gamma = 4$	C.O.	$\alpha = 0.5, \beta = 0.7, \gamma = 3$	C.O.
2^3	2.02043e-4	-	8.02853e-3	-
2^4	1.70662e-5	3.56545	7.98257e-4	3.33021
2^5	1.39517e-6	3.61262	7.44388e-5	3.42272
2^6	1.11790e-7	3.64155	6.61161e-6	3.49298
2^7	8.82100e-9	3.66371	5.70100e-7	3.53572

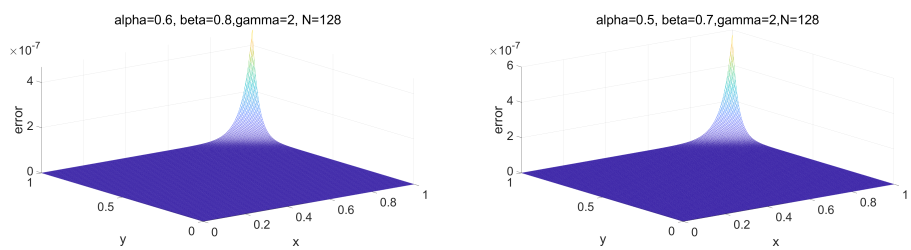

Figure 3. Error distribution for $\alpha = 0.6, \beta = 0.8, \gamma = 4$ (left) and $\alpha = 0.5, \beta = 0.7, \gamma = 3$ (right) with $N = 128$.

7. Concluding remarks

Since VIEs with weakly singular kernels typically have solutions that are non-smooth near the initial point of the interval of integration, it is important to provide a way to recover the accuracy of solutions as well as to obtain high-order accuracy at the approximation level. In this paper, we introduce a new uniform accuracy high-order scheme for nonlinear VIEs with non-smooth solution through suitable variable transformation. We demonstrate that the uniform accuracy numerical scheme achieves convergence of order $(\tau_s^{3+\alpha} + \lambda_t^{3+\beta})$ for nonlinear VIEs with non-smoothness. Taking into account the regularity of the problems, we have obtained the schemes and convergence results. The numerical examples validate our theoretical findings. In the future, we aim to apply this method to develop higher-order schemes for solving high-dimensional partial differential equations and systems of equations.

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