Journal of Applied Analysis and Computation
Volume 15, Number 6, December 2025, 3403–3413
Website:http://www.jaac-online.com
DOI:10.11948/20250005

NONEXISTENCE OF COEXISTING STEADY-STATE SOLUTIONS FOR A REACTION-DIFFUSION COMPETING SYSTEM WITH FRACTIONAL TYPE CROSS-DIFFUSION*

Ningning Zhu[1] and Fanwei Meng[1,†]

Abstract We discuss a system of two competing species with fractional type cross-diffusion. The basic idea is to make a link among the extreme values of steady-state solutions according to the maximum principle. Then by introducing a proper discriminant function, which is monotonically decreasing, we establish sufficient conditions such that the system has no coexisting steady-state solutions.

Keywords Fractional type cross-diffusion, steady-state, coexisting solution.

MSC(2010) 35J60, 35A01.

1. Introduction

In this work, we consider the nonexistence of coexisting steady-state solutions of the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta \left[\left(d_1 + \alpha v + \delta u + \dfrac{\beta}{\gamma + v} \right) u \right] + u(1 - u - a_1 v), & x \in \Omega \times (0, T), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + v(1 - a_2 u - v), & x \in \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$; $T \in (0, +\infty]$; u and v are the densities of the two competing species, respectively. α, β, γ and δ are nonnegative numbers; α stands for the cross-diffusion pressures; δ represents the self-diffusion pressures; and the nonlinear term of fractional form $\Delta \left(\frac{\beta u}{\gamma + v} \right)$ means that the population pressure of the species u weakens in high-density areas

†The corresponding author.
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
*The authors were supported by Natural Science Foundation of Shandong Province (No. ZR2021QA103).
Email: znn199192@163.com(N. Zhu), fwmeng@qfnu.edu.cn(F. Meng)

of the species v. The parameters a_i and $d_i (i = 1, 2)$ are all positive constants; a_1, a_2 describe the inter-specific competitions; and d_1, d_2 are their diffusion rates.

When $\beta = 0$, system (1.1) becomes the simplified SKT model, which was first proposed by Shigesada, Kawasaki and Teramoto [16] in 1979 when considering a nonlinear dispersive force and an environmental potential function. Since the system was proposed, a large number of experts have studied it from different aspects (see [1–4, 11–14, 17, 19]). For example, the references [1, 3, 4, 11, 17] obtained the global existence of smooth solutions of the SKT model under different hypotheses. By using the maximum principle, Harnack inequality and a prior estimates, Lou and Ni [12, 13] obtained sufficient conditions on the existence and nonexistence of non-constant steady-state solutions of the SKT model. By constructing auxiliary functions, Lou et al. [14] established specific parameter ranges such that there are no nonconstant steady-state solutions with and without self-diffusion in the first equation of the SKT model, respectively. Moreover, many researchers have done a lot of work on the SKT model with Dirichlet boundary conditions (see, for example, [7–10,15,18,20,21]). By using the fixed point index method, sufficient conditions for the existence of positive steady-state solutions under Dirichlet boundary conditions were given in [15] for fixed or sufficiently large cross diffusion coefficients. Wu [18] studied the existence of steady-state solutions for a one-dimensional system under Dirichlet boundary conditions based on the singular perturbation method.

We notice that there are several papers on the SKT model with fractional type diffusion. Jia et al. [5] proved the boundedness of positive steady-state solutions for a class of competitive systems with fractional cross diffusion terms. They also used monotone iterative methods to prove the existence and nonexistence of positive steady-state solutions. Using bifurcation theory, Kadota and Kuto [6] discussed the existence problem of positive steady-state solutions for a predator-prey system with fractional cross diffusion terms under Dirichlet boundary conditions. But on the whole, there is not much study in this area. The purpose of this work is to establish sufficient conditions such that the following system has no coexisting solutions:

$$\begin{cases} \Delta\left[\left(d_1 + \alpha v + \delta u + \dfrac{\beta}{\gamma + v}\right) u\right] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ d_2 \Delta v + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In the following, we always assume that:

$$\alpha, \beta, \gamma, \delta \geq 0, \text{ and } a_i, d_i > 0 \text{ for } i = 1, 2.$$

Considering that u and v represent species densities, we focus on the nonnegative classical solution (u, v) of (1.2), which means that $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, $u, v \geq 0$ in $\overline{\Omega}$, and (u, v) satisfies (1.2) in the pointwise sense.

The remainder of this work is organized as follows. In section 2, we show that the nonnegative classical solutions are strictly positive if they are not identically equal to zero. By rewriting system (1.2), we obtain properties of the exterme values of u and v. Section 3 constructs a new function that is monotonically decreasing in the domain of definition. Then Theorem 3.1 the main result, gives the parameter ranges for nonexistence of coexisting solutions.

2. Preliminaries

First of all, we obtain the positivity of u and v in $\overline{\Omega}$, which ensures that the fractional type cross-diffusion in system (1.2) does make sense.

Proposition 2.1. *et (u, v) be a nonnegative classical solution of (1.2) with $u \not\equiv 0$ and $v \not\equiv 0$. Then $u > 0$ and $v > 0$ in $\overline{\Omega}$.*

Proof. (i) We first prove $v > 0$ in $\overline{\Omega}$ by contradiction.

Suppose that there is $x' \in \overline{\Omega}$ such that $v(x') = 0$. Then $v(x') = \min_{x\in\overline{\Omega}} v(x)$.

Let

$$\mathcal{L}_1 v = -d_2 \Delta v + cv \quad \text{with} \quad c = a_2 u + v.$$

Then

$$c \geq 0 \ \text{ and } \ \mathcal{L}_1 v = v \geq 0 \ \text{ in } \overline{\Omega}.$$

If $x' \in \Omega$, by using the strong maximum principle, we can see that v is constant in $\overline{\Omega}$, and therefore $v = 0$, which contradicts $v \not\equiv 0$.

If $x' \in \partial\Omega$, then $v(x) > v(x')$ for $x \in \Omega$. Applying Hopf's boundary lemma, we get $\frac{\partial v}{\partial \nu}(x') < 0$, which is impossible.

(ii) Now, we aim to investigate the positivity of u. Let $w = \left(d_1 + \alpha v + \delta u + \frac{\beta}{\gamma+v}\right) u$. Due to $d_1 > 0, \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0, v > 0$ and $u \geq 0$ in $\overline{\Omega}$, we can see $w \geq 0$ in $\overline{\Omega}$. So we only need to prove $w > 0$ in $\overline{\Omega}$. Otherwise, there is $x'' \in \overline{\Omega}$ such that $w(x'') = \min_{x\in\overline{\Omega}} w(x) = 0$.

It follows from (1.2) that

$$\Delta w + u(1 - u - a_1 v) = \Delta w + \frac{1 - u - a_1 v}{d_1 + \alpha v + \delta u + \frac{\beta}{\gamma+v}} w = 0.$$

Let

$$\mathcal{L}_2 w = -\Delta w + cw \quad \text{with} \quad c = \frac{u + a_1 v}{d_1 + \alpha v + \delta u + \frac{\beta}{\gamma+v}}.$$

Then

$$c > 0 \ \text{ and } \ \mathcal{L}_2 w = \frac{w}{d_1 + \alpha v + \delta u + \frac{\beta}{\gamma+v}} \geq 0 \ \text{ in } \overline{\Omega}.$$

If $x'' \in \Omega$, the strong maximum principle shows that w is constant in Ω, and thus $w = 0$. By $d_1 > 0$, we have $u = 0$, a contradiction to $u \not\equiv 0$.

If $x'' \in \partial\Omega$, we have $w(x) > w(x'')$ for $x \in \Omega$. Thus, it comes from Hopf's boundary lemma that

$$\begin{aligned}
\frac{\partial w}{\partial \nu}(x'') = &\left(d_1 + \alpha v(x'') + \delta u(x'') + \frac{\beta}{\gamma + v(x'')}\right) \frac{\partial u}{\partial \nu}(x'') \\
&+ u(x'') \left[\left(\alpha - \frac{\beta}{(\gamma + v(x''))^2}\right) \frac{\partial v}{\partial \nu}(x'') + \delta \frac{\partial u}{\partial \nu}(x'')\right] \\
<&0,
\end{aligned}$$

a contradiction to the Neumann boundary conditions. This completes the proof. □

Next, we cite Lemma 3.1 in [14] which will be useful to describe the relationships among the extreme values of solutions of (1.2). One can find its proof in Proposition 2.2 of [12].

Lemma 2.1. *Let $A(x) \in C^0(\overline{\Omega})$ be positive in $\overline{\Omega}$, $B(x) \in C^0(\overline{\Omega}, \mathbb{R}^n)$, $C(x) \in C^0(\overline{\Omega})$, and $\tilde{u} \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfy*

$$\begin{cases} A(x)\Delta\tilde{u}(x) + B(x)\cdot\nabla\tilde{u}(x) + C(x) = 0, & x \in \Omega, \\ \dfrac{\partial\tilde{u}}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Then there exist $\overline{x}, \underline{x} \in \overline{\Omega}$ such that

$$\tilde{u}(\overline{x}) = \max_{x\in\overline{\Omega}} \tilde{u}(x), \qquad C(\overline{x}) \geq 0$$

and

$$\tilde{u}(\underline{x}) = \min_{x\in\overline{\Omega}} \tilde{u}(x), \qquad C(\underline{x}) \leq 0.$$

Now, we rewrite system (1.2) as the form of (2.1):

$$\begin{cases} \left(d_1 + \alpha v + 2\delta u + \dfrac{\beta}{\gamma+v}\right)\Delta u + \nabla u\cdot\left(2\delta\nabla u + 2\alpha\nabla v - \dfrac{2\beta\nabla v}{(\gamma+v)^2}\right) \\ + u\left[\left(\alpha - \dfrac{\beta}{(\gamma+v)^2}\right)\dfrac{v(-1+a_2u+v)}{d_2}\right. \\ \left. + \dfrac{2\beta|\nabla v|^2}{(\gamma+v)^3} + 1 - u - a_1v\right] = 0, & x \in \Omega, \\ d_2\Delta v + v(1 - a_2u - v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Applying Lemma 2.1 to the first equation of system (2.2), we can directly derive the following two lemmas without proofs.

Lemma 2.2. *Suppose that (u, v) is a nonnegative classical solution of (2.2) with $u \not\equiv 0$ and $v \not\equiv 0$. Then*

$$\begin{aligned} &\left(\alpha - \frac{\beta}{(\gamma+v(x_1))^2}\right)\frac{v(x_1)(-1+a_2M_1+v(x_1))}{d_2} \\ &+ \frac{2\beta|\nabla v(x_1)|^2}{(\gamma+v(x_1))^3} + 1 - M_1 - a_1v(x_1) \geq 0, \end{aligned}$$

where $x_1 \in \overline{\Omega}$, and $M_1 = u(x_1) = \max_{\overline{\Omega}} u(x)$.

Lemma 2.3. *Suppose that (u, v) is a nonnegative classical solution of (2.2), $u \not\equiv 0$ and $v \not\equiv 0$. Then*

$$\left(\alpha - \frac{\beta}{(\gamma+v(x_2))^2}\right)\frac{v(x_2)(-1+a_2m_1+v(x_2))}{d_2} + 1 - m_1 - a_1v(x_2) \leq 0,$$

where $x_2 \in \overline{\Omega}$, and $m_1 = u(x_2) = \min_{\overline{\Omega}} u(x)$.

Similarly, an application of Lemma 2.1 to the second equation of system (2.2) leads to the following result.

Lemma 2.4. *If* (u,v) *is a nonnegative classical solution of* (2.2) *with* $u \not\equiv 0$ *and* $v \not\equiv 0$, *then*

$$1 - a_2 M_1 \le m_2 \le v(x) \le M_2 \le 1 - a_2 m_1 < 1$$

for all $x \in \overline{\Omega}$, *where* $M_1 = \max_{\overline{\Omega}} u(x)$, $m_1 = \min_{\overline{\Omega}} u(x)$, $M_2 = \max_{\overline{\Omega}} v(x)$ *and* $m_2 = \min_{\overline{\Omega}} v(x)$.

Proof. Set $C(x) = v(1 - a_2 u - v)$. By Lemma 2.1 again, there exist $x_3, x_4 \in \overline{\Omega}$ such that

$$v(x_3) = \max_{\overline{\Omega}} v(x) \triangleq M_2,\ C(x_3) \ge 0,$$

and

$$v(x_4) = \min_{\overline{\Omega}} v(x) \triangleq m_2,\ C(x_4) \le 0.$$

Therefore,

$$1 - a_2 u(x_3) - M_2 \ge 0 \quad \text{and} \quad 1 - a_2 u(x_4) - m_2 \le 0.$$

Combining these two inequalities, we have

$$1 - a_2 M_1 \le 1 - a_2 u(x_4) \le m_2 \le v(x) \le M_2 \le 1 - a_2 u(x_3) \le 1 - a_2 m_1 < 1$$

for any $x \in \overline{\Omega}$. □

Unless otherwise specified, we assume that

$$\begin{aligned} M_1 &= \max_{\overline{\Omega}} u(x) = u(x_1), \qquad m_1 = \min_{\overline{\Omega}} u(x) = u(x_2), \\ M_2 &= \max_{\overline{\Omega}} v(x) = v(x_3), \quad \text{and} \quad m_2 = \min_{\overline{\Omega}} v(x) = v(x_4), \end{aligned}$$

where $x_i \in \overline{\Omega}$ $(i = 1, 2, 3, 4)$.

3. Nonexistence of coexisting solutions

In this section, we concentrate on system (2.2) and aim to explore sufficient conditions on the absence of coexisting solutions of system (2.2). First, we define the following function on $(0,1)$, which will play an important role in our subsequent analysis.

Lemma 3.1. *Let*

$$\begin{aligned} \phi(s) &= 1 - \left(\alpha - \frac{\beta}{(\gamma+s)^2}\right) \frac{a_2 s}{d_2}, \quad s \in (0,1), \\ \varphi(s) &= \left(\alpha - \frac{\beta}{(\gamma+s)^2}\right) \frac{s(-1+s)}{d_2} + 1 - a_1 s, \quad s \in (0,1), \end{aligned}$$

where $\alpha, \beta, \gamma, a_1, a_2$ *and* d_2 *are the same as those in* (2.2). *Suppose that* $a_2 > 1$, $\alpha a_2 (a_2 - 1) < d_2 (a_1 a_2 - 1)$, *and*

$$\begin{cases} \alpha a_2 < d_2 < 2\alpha a_2, \\ d_2(a_2 - 1) < \beta a_2, \\ d_2^2 a_2 (1 - a_1)^2 + \beta(2\alpha a_2 - d_2 - a_2 d_2) < 0, \\ d_2^2 (1 - a_1 a_2)^2 + 2\alpha\beta a_2^2 + d_2^2 (1 - a_1 a_2)(a_2 - 1) - \beta a_2 d_2 > 0. \end{cases} \tag{3.1}$$

Define $f(s) = \frac{\varphi(s)}{\phi(s)}$, $s \in (0,1)$. *Then* $f'(s) \le 0$ *on* $(0,1)$.

Proof. Firstly, due to $\alpha a_2 < d_2$, it is easy to see that $\phi(s) > 0$ for all $s \in (0,1)$. Then direct calculations imply that

$$\begin{aligned} f(s) &= \frac{\left(\alpha - \frac{\beta}{(\gamma+s)^2}\right)\frac{s(-1+s)}{d_2} + 1 - a_1 s}{1 - \left(\alpha - \frac{\beta}{(\gamma+s)^2}\right)\frac{a_2 s}{d_2}} \\ &= \frac{[\alpha(\gamma+s)^2 - \beta]s(s-1) + d_2(\gamma+s)^2(1-a_1 s)}{d_2(\gamma+s)^2 - a_2 s[\alpha(\gamma+s)^2 - \beta]} \\ &= \frac{\alpha s(s-1)(\gamma+s)^2 - \beta s(s-1) + d_2(1-a_1 s)(\gamma+s)^2}{d_2(\gamma+s)^2 - \alpha a_2 s(\gamma+s)^2 + \beta a_2 s} \\ &\triangleq \frac{p(s)}{q(s)}, \end{aligned}$$

which has its derivative f' given by

$$f'(s) = \frac{p'(s)q(s) - p(s)q'(s)}{q^2(s)}, \quad s \in (0,1).$$

Furthermore,

$$\begin{aligned} &p'(s)q(s) - p(s)q'(s) \\ =&\big[\alpha(\gamma+s)^2(s-1) + 2\alpha s(s-1)(\gamma+s) + \alpha s(\gamma+s)^2 - \beta(s-1) - \beta s \\ &- a_1 d_2(\gamma+s)^2 + 2d_2(1-a_1 s)(\gamma+s)\big]\big[d_2(\gamma+s)^2 - \alpha a_2 s(\gamma+s)^2 + \beta a_2 s\big] \\ &- \big[\alpha s(s-1)(\gamma+s)^2 - \beta s(s-1) + d_2(1-a_1 s)(\gamma+s)^2\big] \\ &\times \big[2d_2(\gamma+s) - \alpha a_2(\gamma+s)^2 - 2\alpha a_2 s(\gamma+s) + \beta a_2\big] \\ \triangleq& f_1(s)(\gamma+s)^4 + f_2(s)(\gamma+s)^3 + f_3(s)(\gamma+s)^2 + f_4(s)(\gamma+s) + f_5(s), \end{aligned}$$

where

$$\begin{aligned} f_1(s) =&[\alpha(s-1) + \alpha s - a_1 d_2](d_2 - \alpha a_2 s) + \alpha a_2[\alpha s(s-1) + d_2(1-a_1 s)] \\ =& -\alpha^2 a_2 s^2 + 2\alpha d_2 s + (\alpha a_2 d_2 - \alpha d_2 - a_1 d_2^2), \\ f_2(s) =&[2\alpha s(s-1) + 2d_2(1-a_1 s)](d_2 - \alpha a_2 s) \\ &- 2(d_2 - \alpha a_2 s)[\alpha s(s-1) + d_2(1-a_1 s)] \\ =&0, \\ f_3(s) =&(d_2 - \alpha a_2 s)(-2\beta s + \beta) + \beta a_2 s[\alpha(s-1) + \alpha s - a_1 d_2] \\ &- \beta a_2[\alpha s(s-1) + d_2(1-a_1 s)] - \alpha\beta a_2 s(s-1) \\ =&\beta[2\alpha a_2 s^2 - 2d_2 s + (d_2 - a_2 d_2)], \\ f_4(s) =&\beta a_2 s[2\alpha s(s-1) + 2d_2(1-a_1 s)] + \beta s(s-1)(2d_2 - 2\alpha a_2 s) \\ =&2\beta d_2 s[s(1-a_1 a_2) + (a_2 - 1)], \\ f_5(s) =&\beta a_2 s(-2\beta s + \beta) + \beta^2 a_2 s(s-1) \\ =& -\beta^2 a_2 s^2. \end{aligned}$$

Next we shall divide our proof into three parts.

(i) $f_1(s) < 0$, $s \in (0,1)$.

In fact, it follows from the condition

$$\alpha a_2(a_2-1) < d_2(a_1a_2-1)$$

that

$$4\alpha^2 d_2^2 + 4\alpha^2 a_2(\alpha a_2 d_2 - \alpha d_2 - a_1 d_2^2) < 0.$$

Clearly, when $s \in (0,1)$, $f_1(s) < 0$.

(ii) $f_3(s) < 0$, $s \in (0,1)$.

Let $f_3(s) = 0$. We can solve that

$$s_1 = \frac{2d_2 - \sqrt{4d_2^2 - 8\alpha a_2(d_2 - a_2d_2)}}{4\alpha a_2}, \quad s_2 = \frac{d_2 + \sqrt{d_2^2 - 2\alpha a_2(d_2 - a_2 d_2)}}{2\alpha a_2}.$$

As $a_2 > 1$, we have $s_1 < 0$. Moreover, $a_2d_2 > 2\alpha a_2 - d_2 > 0$ entails that

$$d_2^2 - 2\alpha a_2(d_2 - a_2d_2) > (2\alpha a_2 - d_2)^2,$$

which implies that $s_2 > 1$, and thus $f_3(s) < 0$, $s \in (0,1)$.

(iii) $f_3(s)(\gamma+s)^2 + f_4(s)(\gamma+s) + f_5(s) < 0$, $s \in (0,1)$.

Given that $f_3(s) < 0$, $s \in (0,1)$, we next prove that $f_4^2(s) - 4f_3(s)f_5(s) < 0$, $s \in (0,1)$. A simple calculation shows that

$$\begin{aligned}
&f_4^2(s) - 4f_3(s)f_5(s)\\
=&4\beta^2 d_2^2 s^2[s(1-a_1a_2) + (a_2-1)]^2 + 4\beta^3 a_2 s^2[2\alpha a_2 s^2 - 2d_2 s + (d_2 - a_2 d_2)]\\
=&4\beta^2 s^2\Big\{d_2^2[s(1-a_1a_2)+(a_2-1)]^2 + \beta a_2[2\alpha a_2 s^2 - 2d_2 s + d_2(1-a_2)]\Big\}\\
=&4\beta^2 s^2\Big\{d_2^2(1-a_1a_2)^2 s^2 + 2d_2^2(1-a_1a_2)(a_2-1)s + d_2^2(a_2-1)^2\\
&+ 2\alpha\beta a_2^2 s^2 - 2\beta a_2 d_2 s + \beta a_2 d_2(1-a_2)\Big\}\\
=&4\beta^2 s^2\Big\{\big[d_2^2(1-a_1a_2)^2 + 2\alpha\beta a_2^2\big]s^2 + \big[2d_2^2(1-a_1a_2)(a_2-1) - 2\beta a_2 d_2\big]s\\
&+ \big[d_2^2(a_2-1)^2 + \beta a_2 d_2(1-a_2)\big]\Big\}\\
\triangleq&4\beta^2 s^2(As^2 + Bs + C),
\end{aligned}$$

where

$$\begin{aligned}
A &= d_2^2(1-a_1a_2)^2 + 2\alpha\beta a_2^2 > 0,\\
B &= 2d_2^2(1-a_1a_2)(a_2-1) - 2\beta a_2 d_2 < 0,\\
C &= d_2^2(a_2-1)^2 + \beta a_2 d_2(1-a_2).
\end{aligned}$$

In order to have $f_4^2(s) - 4f_3(s)f_5(s) < 0$ for all $s \in (0,1)$, we hope that

$$B^2 - 4AC > 0, \quad \frac{-B - \sqrt{B^2-4AC}}{2A} < 0, \quad \text{and} \quad \frac{-B + \sqrt{B^2-4AC}}{2A} > 1.$$

In fact, on the basis of $d_2(a_2-1)<\beta a_2$, one can easily see that $C<0$, and thus $B^2-4AC>0$, which leads to $\frac{-B-\sqrt{B^2-4AC}}{2A}<0$. On the other hand, in view of $d_2^2(1-a_1a_2+a_2-1)^2+\beta a_2(2\alpha a_2-d_2-a_2d_2)<0$, we have

$$\begin{aligned}
&A+B+C\\
=&d_2^2(1-a_1a_2)^2+2\alpha\beta a_2^2+2d_2^2(1-a_1a_2)(a_2-1)-2\beta a_2d_2+d_2^2(a_2-1)^2\\
&+\beta a_2d_2(1-a_2)\\
=&d_2^2(1-a_1a_2+a_2-1)^2+2\alpha\beta a_2^2-2\beta a_2d_2+\beta a_2d_2(1-a_2)\\
=&d_2^2a_2^2(1-a_1)^2+2\alpha\beta a_2^2-\beta a_2d_2-\beta a_2^2d_2\\
=&d_2^2a_2^2(1-a_1)^2+\beta a_2(2\alpha a_2-d_2-a_2d_2)\\
<&0,
\end{aligned}$$

which establish that

$$B^2-4AC>(2A+B)^2.$$

Finally, the last inequality of (3.1) ensures that

$$2A+B=2d_2^2(1-a_1a_2)^2+4\alpha\beta a_2^2+2d_2^2(1-a_1a_2)(a_2-1)-2\beta a_2d_2>0.$$

Consequently,

$$\sqrt{B^2-4AC}>2A+B,$$

that is,

$$\frac{-B+\sqrt{B^2-4AC}}{2A}>1.$$

In conclusion, we obtain that $f'(s)<0$ for all $s\in(0,1)$. □

Remark 3.1. It should be noted that no matter whether $a_1>1$ or $a_1<1$, the conditions (3.1) are likely to remain valid. For example, conditions (3.1) are valid with either

$$a_1=a_2=1.5,\ \alpha=\beta=1,\ d_2=2,$$

or

$$a_1=0.9,\ a_2=1.3,\ \alpha=1,\ \beta=0.6,\ d_2=2.4.$$

We leave the elementary validation to the interested readers.

In the following, we give a further favorable property of the function f, which can be verified in a straightforward manner.

Lemma 3.2. *Assume that $a_2>1$, $a_1a_2>1$, and $\alpha a_2<d_2$. Then the function f defined in Lemma 3.1 satisfies*

$$f(s)>\frac{1-s}{a_2}\quad \textit{for all } s\in\left(0,\min\left\{1,\frac{a_2-1}{a_1a_2-1}\right\}\right).$$

Proof. Obviously, we only need to verify that

$$\frac{\left(\alpha-\frac{\beta}{(\gamma+s)^2}\right)\frac{s(-1+s)}{d_2}+1-a_1s}{1-\left(\alpha-\frac{\beta}{(\gamma+s)^2}\right)\frac{a_2s}{d_2}}>\frac{1-s}{a_2},$$

which is equivalent to

$$a_2(1 - a_1 s) > 1 - s. \tag{3.2}$$

When $s \in \left(0, \min\left\{1, \frac{a_2-1}{a_1 a_2 - 1}\right\}\right)$, (3.2) is valid. This completes the proof. $\square$

Now, based on the above lemmas, we can establish the nonexistence result on coexisting solutions as follows.

Theorem 3.1. *Let $a_1 < 1$ and $a_2 > 1$. Suppose that* (3.1) *holds and (u, v) is a nonnegative classical solution of* (2.2). *Then (u, v) is not a coexisting solution, that is, at least one of the two species is extinct.*

Proof. Let us assume on the contrary that $u \not\equiv 0$ and $v \not\equiv 0$. Due to Lemma 2.3, we can see that

$$\left(\alpha - \frac{\beta}{(\gamma + v(x_2))^2}\right) \frac{v(x_2)(-1 + a_2 m_1 + v(x_2))}{d_2} + 1 - m_1 - a_1 v(x_2) \le 0,$$

which is equivalent to

$$\begin{aligned} & m_1 \left[1 - \left(\alpha - \frac{\beta}{(\gamma + v(x_2))^2}\right) \frac{a_2 v(x_2)}{d_2}\right] \\ \ge & \left(\alpha - \frac{\beta}{(\gamma + v(x_2))^2}\right) \frac{v(x_2)(-1 + v(x_2))}{d_2} + 1 - a_1 v(x_2), \end{aligned}$$

where $m_1 = \min_{\overline{\Omega}} u(x) = u(x_2)$. Now if $\alpha a_2(a_2 - 1) < d_2(a_1 a_2 - 1)$, which implies $a_1 a_2 > 1$, then Lemmas 2.4 and 3.1 indicate that

$$m_1 \ge f(v(x_2)) \ge f(1 - a_2 m_1),$$

that is,

$$\begin{aligned} & \left[\beta a_2(1 - a_2 m_1) + d_2(\gamma + 1 - a_2 m_1)^2 - \alpha a_2(1 - a_2 m_1)(\gamma + 1 - a_2 m_1)^2\right] m_1 \\ \ge & \beta a_2 m_1(1 - a_2 m_1) - \alpha a_2 m_1(1 - a_2 m_1)(\gamma + 1 - a_2 m_1)^2 \\ & + d_2(\gamma + 1 - a_2 m_1)^2(1 - a_1 + a_1 a_2 m_1). \end{aligned}$$

This in turn implies that

$$(1 - a_1 a_2) m_1 \ge 1 - a_1.$$

But it is impossible because of $a_1 < 1$ and $a_1 a_2 > 1$. Hence

$$\alpha a_2(a_2 - 1) \ge d_2(a_1 a_2 - 1).$$

If $a_1 a_2 > 1$, then $1 < \frac{a_2 - 1}{a_1 a_2 - 1}$. According to Lemma 3.2, we know that $f(s) > \frac{1-s}{a_2}$ for all $s \in (0, 1)$. If $a_1 a_2 \le 1$, by using arguments similar to those in the proof of Lemma 3.2, one can easily derive that $f(s) > \frac{1-s}{a_2}$ for all $s > 0$. Therefore no matter what the sign of $a_1 a_2 - 1$ is, we always have

$$m_1 \ge f(v(x_2)) > \frac{1 - v(x_2)}{a_2},$$

which is absurd, for once more due to Lemma 2.4 we know that

$$m_1 \le \frac{1 - v(x_2)}{a_2}.$$

This completes the proof. $\square$

Declarations

Conflict of interest. The authors state that there is no conflict of interest.

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

References

[1] H. Amann, *Dynamic theory of quasilinear parabolic systems, III: Global existence*, Math Z., 1989, 202(2), 219–250.

[2] L. Desvillettes, *About the triangular Shigesada-Kawasaki-Teramoto reaction cross diffusion system*, Ric. Mat., 2024, 73(1), S105–S114.

[3] J. Guerand, A. Menegaki and A. Trescases, *Global smooth solutions for triangular reaction-cross diffusion systems*, Bull. Sci. Math., 2023, 189, Paper No. 103342, 27 pp.

[4] L. T. Hoang, T. V. Nguyen and T. V. Phan, *Gradient estimates and global existence of smooth solutions for a system of cross-diffusion equations*, SIAM J. Math. Anal., 2015, 47(3), 2122–2177.

[5] Y. Jia, J. Wu and H. Xu, *Positive solutions of a Lotka-Volterra competition model with cross-diffusion*, Comput. Math. Appl., 2014, 68(10), 1220–1228.

[6] T. Kadota and K. Kuto, *Positive steady states for a prey-predator model with some nonlinear diffusion terms*, J. Math. Anal. Appl., 2006, 323(2), 1387–1401.

[7] J. Kang, *Uniqueness of steady state positive solutions to a general elliptic system with Dirichlet boundary conditions*, J. Appl. Anal. Comput., 2022, 12(6), 2370–2385.

[8] K. Kuto and H. Sato, *Morse Index of steady-states to the SKT model with Dirichlet boundary conditions*, SIAM J. Math. Anal., 2024, 56(4), 5386–5408.

[9] K. Kuto and Y. Yamada, *Multiple coexistence states for a prey-predator system with cross-diffusion*, J. Differential Equations, 2004, 197(2), 315–348.

[10] K. Kuto and Y. Yamada, *Positive solutions for Lotka-Volterra competition systems with large cross-diffusion*, Appl. Anal., 2010, 89(7), 1037–1066.

[11] D. Le, *Global existence for a class of strongly coupled parabolic systems*, Ann. Mat. Pura Appl., 2006, 185(1), 133–154.

[12] Y. Lou and W.-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, 1996, 131(1), 79–131.

[13] Y. Lou and W.-M. Ni, *Diffusion vs cross-Diffusion: An elliptic approach*, J. Differential Equations, 1999, 154(1), 157–190.

[14] Y. Lou, Y. Tao and M. Winkler, *Nonexistence of nonconstant steady-state solutions in a triangular cross-diffusion model*, J. Differential Equations, 2017, 262(10), 5160–5178.

[15] W. Ruan, *Positive steady-state solutions of a competing reaction-diffusion system with large cross-diffusion coefficients*, J. Math. Anal. Appl., 1996, 197(2), 558–578.

[16] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biol., 1979, 79(1), 83–99.

[17] P. V. Tuoc, *Global existence of solutions to Shigesada-Kawasaki-Keramoto cross-diffusion systems on domains of arbitrary dimensions*, Proc. Amer. Math. Soc., 2007, 135(12), 3933–3941.

[18] Y. Wu, *Existence of stationary solutions with transition layers for a class of cross-diffusion systems*, P. Roy. Soc. Edinb. A, 2002, 132(6), 1493–1511.

[19] S. Zhang, Z. Yu and Y. Meng, *Stability of traveling wave fronts for nonlocal diffusive systems*, J. Appl. Anal. Comput, 2024, 14(4), 2063–2081.

[20] N. Zhu, *Existence of steady-state solutions for a class of competing systems with cross-diffusion and self-diffusion*, Electron. J. Qual. Theory Differ. Equ., 2021, 2, 1–10.

[21] N. Zhu and D. Tang, *Nonexistence of coexisting steady-state solutions of Dirichlet problem for a cross-diffusion model*, Math. Method. Appl. Sci., 2019, 42(1), 346–353.

Received January 2025; Accepted April 2025; Available online May 2025.