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A FRACTIONAL PREDATOR-PREY MODEL WITH ALLEE EFFECT AND CONSTRUCTIVE IMPACT ON PREY CARRYING CAPACITY

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Abstract This paper explores a prey-predator model that incorporates several biological phenomena, with a focus on the positive feedback that certain prey species have on their own carrying capacity. Traditional models treat carrying capacity as a constant; however, this study assumes a variable carrying capacity influenced by the prey population. To account for the memory effect and hereditary properties within biological systems, we employ fractional differential equations using the Caputo fractional derivative. Additionally, we incorporate the Allee effect, which plays a critical role in population dynamics, especially at low population densities. Through numerical analysis, the model's stability and dynamic behavior are examined, providing insights into species coexistence, population cycles, and extinction risks. This framework aims to enrich existing models and offer a more comprehensive understanding of prey-predator interactions with prey species impacting their carrying capacity.

Keywords Prey-predator, carrying capacity, Allee effect, Caputo fractional derivative, Hopf bifurcation.

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1. Introduction

Ecological modeling has become an indispensable tool in understanding the intricate web of interactions within ecosystems, enabling researchers to simulate and analyze how various biological processes and environmental factors affect species dynamics. One of the most urgent challenges these models seek to address is the rapid loss of biodiversity, which has accelerated due to factors such as habitat destruction, climate change, and overexploitation of natural resources [33]. Among the key contributors to ecosystem resilience are ecosystem engineers -organisms like beavers, termites, and earthworms- that play an active role in shaping their environment. These species modify their habitats in ways that benefit not only themselves but also other species, creating rich, complex ecosystems [21, 22].

The survival of most species in ecosystems and the richness of biodiversity are fundamentally rooted in prey-predator dynamics. In mathematical biology, prey-predator models have long served as essential tools for exploring species interactions. Over time, these models have been refined to include increasingly complex biological phenomena, such as adaptive behaviors and fluctuating environmental factors.

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The ultimate aim is to uncover the conditions that lead to stability or instability, population cycles, and long-term survival strategies. These models advanced significantly after the original Lotka-Volterra framework, with key refinements based on observed ecological dynamics. A well-established concept is that prey populations grow logistic when predators are absent where resource limitations regulate their expansion. Logistic growth highlights two key factors: The intrinsic growth rate and the carrying capacity; the latter is a term used to denote limiting population supported by the environment [31]. Traditionally, carrying capacity has been treated as a fixed parameter. However, this paper shifts the focus to models with variable carrying capacity, as some studies have suggested that fluctuating environmental conditions may alter resource availability over time [23, 24, 29, 31, 38]. Additionally, in [23, 24, 38], researchers have conducted studies based on the assumption that prey populations influence their carrying capacity. Our work seeks to expand on these approaches by introducing several refinements.

The first enhancement involves the use of fractional differential equations (FDEs). In previous studies [23, 24, 38], researchers suggested that the effect of prey populations on their carrying capacity takes time to manifest, incorporating delayed terms to reflect this. In contrast, we introduce the Caputo fractional derivative (CFD) to capture a more generalized memory for the system. FDEs, with their non-local properties, are widely used to account for the hereditary and memory characteristics of systems, and they have proven effective in capturing the complete dynamics of the system [5, 15, 32]. FDEs have been applied across various fields, including epidemics modeling [16,34], biochemical processes [36,37], ecological modeling [15,35], environmental modeling [8] etc. Their application in biological systems is particularly relevant, where memory refers to the ability to retain information from past events and use it to influence future behaviors [10, 30].

In this article, we assume that the prey's impact on their carrying capacity is positive, referring to it as a constructive impact. To illustrate this, let us consider a biological scenario involving an important ecosystem engineer beaver (Castor canadensis). Beavers serve as both ecosystem engineers and keystone species, significantly influencing the landscape and biodiversity through their actions [22]. By building dams, they alter the flow of water, creating large wetlands [3, 26]. As herbivores, they enhance their food sources, such as aquatic plants, as often they form a substantial part of their diet, depending on the habitat [20]. Moreover, beavers are prey for species like wolves, and in habitats where their ranges coincide, wolves can be the main predators, with beavers playing a crucial role in the wolves' diet. The beaver's environmental engineering and interactions with predators, such as wolves, have a noteworthy influence on the broader ecological processes [2, 9].

In this study, we introduce a novel prey-predator model incorporating dynamic carrying capacity influenced by prey populations and a memory-dependent ecological interaction via fractional derivatives. The key highlights of our work are as follows:

- We extend classical prey-predator models by considering a variable carrying capacity that depends on prey density, incorporating ecological memory effects.
- We employ FDEs using the Caputo fractional derivative to better capture the hereditary and memory properties of ecological interactions.
- We introduce the Allee effect to explore how cooperative behaviors impact

prey survival and carrying capacity dynamics.

- We analyze the stability properties of the system under different parameter conditions, providing insights into the long-term population dynamics.

The following section is dedicated to constructing the studied model from scratch. We will also add the Allee effect into our system after stating its importance for population ecology and explaining how it increases the original aspect of this study. The inclusion of the fractional derivative will also be explained in the next section, along with some basic properties of the CFD and FDEs. We conclude the paper with numerical simulations to illustrate the model's dynamics, followed by a discussion in the Conclusions section.

2. Model construction and fractional version

First, we present a basic predator-prey model, after which we will introduce some enhancements. Let u and v represent the density of prey and predator populations.

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{K}\right) - f(u)v, \\ \dfrac{dv}{dt} = cf(u)v - dv. \end{cases} \tag{2.1}$$

Here, r is the prey growth rate, and K is the carrying capacity, the two main defining terms of logistic growth. The function $f(u)$ is called functional response, which basically characterizes the change in the density of the prey attacked per unit time per predator as the prey density varies [14]. Predator growth solely depends on prey's presence, where c is the conversion rate reflecting the ability of predators to turn prey into an additional per capita growth rate for the predator population. Predators also experience a natural mortality with a rate of d.

Carrying capacity is a crucial ecological factor determining the maximum population size a species can sustain based on available resources such as food, water, and habitat. It plays a significant role in shaping predator-prey interactions, including the well-known paradox of enrichment [1]. Traditionally, predator-prey models assume a constant carrying capacity, but ecological systems are inherently dynamic, influenced by environmental fluctuations and species interactions. As a result, variable carrying capacities have been introduced in mathematical models to capture more realistic ecosystem dynamics. Several approaches have been proposed to model varying carrying capacities, including time-dependent functions linked to a biotic resources [29], predator-dependent carrying capacities [13], and logistic forms incorporating environmental variability [23,24,31,38]. These formulations acknowledge that carrying capacity is not a fixed parameter but a function of ecological interactions, species behavior, and habitat conditions. Following this perspective, instead of using a constant carrying capacity K, we introduce a prey-dependent carrying capacity formulated as

$$\kappa(t) = K + \beta u(t). \tag{2.2}$$

We define β as the environmental impact parameter, representing how prey's activities influence their carrying capacity. Since we focus on species like ecosystem engineers or those that positively affect their environment, we assume $\beta > 0$ and call this parameter the constructive impact parameter for the rest of the article. As mentioned earlier, certain species enhance biodiversity through actions like seed dispersal and nutrient cycling, which in turn can increase resource availability and raise their carrying capacity. Following the explanation in [23], we also assume $\beta < 1$, as values greater than one would result in unbounded prey growth. Additionally, we adopt a Holling type I functional response for $f(u)$, i.e. $f(u) = \alpha u$. Under these assumptions, we obtain the following modified predator-prey model:

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{K + \beta u}\right) - \alpha uv, \\ \dfrac{dv}{dt} = c\alpha uv - dv. \end{cases} \tag{2.3}$$

The Allee effect is a key ecological phenomenon characterized by a positive relationship between an individual's fitness and the population size of their own species. This effect plays a significant role in the extinction risk of low-density populations, as factors like mate limitation, cooperative defense, and environmental conditioning become more challenging at smaller population sizes. Unlike logistic growth models, which typically focus on resource limitations, the Allee effect highlights how social and biological factors can drive population decline. In particular, difficulties in finding mates, social dysfunction, and predator avoidance contribute to its impact. Due to its critical role in population dynamics, the Allee effect has gained increasing attention in studies of predator-prey systems, with ecologists and mathematicians investigating its implications for species survival. Several studies have explored the connection between ecosystem engineers and the Allee effect. Some of these investigations have concluded that ecosystem engineers, through their beneficial impact on the environment, can reduce the critical Allee threshold for other species. In this study, we assume that species like ecosystem engineers, which positively influence their own carrying capacity, also experience an Allee threshold. We incorporate this idea by defining a multiplicative Allee effect as a factor in the prey growth term within the system, as follows:

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \dfrac{u}{K + \beta u}\right)(u - m) - \alpha uv, \\ \dfrac{dv}{dt} = c\alpha uv - dv. \end{cases} \tag{2.4}$$

Here, m represents the Allee parameter, where $m < 0$ corresponds to the weak Allee effect, and $m > 0$ indicates the strong Allee effect. In the latter case, m can be considered as a critical threshold, below which the population is likely to decline toward extinction if the population size falls too low. We also assume $-K < m < K$ because it is biologically reasonable.

Now, we incorporate the CFD into (2.4) following the work [19]. The system's formulation using fractional order is presented below:

$$\begin{cases} {}_{t_0}^{C}D_t^{\zeta} u(t) = ru\left(1 - \dfrac{u}{K + \beta u}\right)(u - m) - \alpha uv, \\ {}_{t_0}^{C}D_t^{\zeta} v(t) = c\alpha uv - dv. \end{cases} \tag{2.5}$$

Here, ${}_{t_0}^{C}D_t^{\zeta}$ represents CFD with order $\zeta \in (0, 1)$, and it is defined as

$${}_{t_0}^{C}D_t^{\zeta} g(t) = \frac{1}{\Gamma(1-\zeta)} \int_{t_0}^{t} \frac{g'(\nu)}{(t-\nu)^{\zeta}} d\nu.$$

The introduction of fractional-order derivatives in model (2.5) enhances the modeling framework by incorporating memory effects and long-range dependence, which are often present in ecological systems. Unlike the integer-order model (2.4), which assumes that the rate of change in a population depends only on the current state, the fractional-order model accounts for past states, leading to more realistic dynamics. This is particularly relevant in ecosystems where the impact of past population sizes and environmental conditions persists over time. Moreover, fractional models have been shown to exhibit richer dynamical behaviors, including long-term stability shifts and more accurate predictions of transient dynamics.

Using fractional derivatives in ecological modeling introduces the challenge of dimensional homogeneity. The Caputo fractional derivative, commonly used in fractional-order models, has dimensions of time$^{-\alpha}$, which differs from the classical derivative's time^{-1}. As highlighted in recent studies [12, 28], this discrepancy can lead to inconsistencies if parameters are not appropriately adjusted. Some works propose modifications to ensure dimensional consistency in fractional-order ecological models [6, 28]. While our model follows the standard approach of replacing ordinary derivatives with fractional derivatives without additional scaling, we acknowledge this issue and emphasize that future studies could explore dimensionally homogeneous formulations for a more rigorous representation of fractional ecological interactions.

As ζ approaches 1, the influence of previous events on future outcomes decreases, resulting in a short memory for the system. In contrast, as ζ moves closer to 0, the impact of past events becomes substantial. Incorporating fractional order and memory effects into the system enhances its stability, which is formalized in the following theorem that will be applied in the next section.

Theorem 2.1. [25] *Consider the system with CFD*

$$ {}^{C}_{t_0}D^{\zeta}_{t}Y(t) = H(Y(t)), \quad Y(0) = Y_0 \in \mathbb{R}^n, \quad \zeta \in (0,1), \tag{2.6}$$

where $Y(t) = (y_1(t), \dots, y_n(t)) \in \mathbb{R}^n$ *and* $H : [h_1, \dots, h_n] : \mathbb{R}^n \to \mathbb{R}^n$. *Let*

$$J_{Y*} = \frac{\partial H}{\partial Y} = \frac{\partial(h_1, \dots, h_n)}{\partial(y_1, \dots, y_n)}$$

be the Jacobian matrix at the equilibrium point Y^* *of* (2.6). *If the eigenvalues* λ_i, $i \in \{1, \dots, n\}$ *of* J_{Y*} *satisfy following conditions*

$$|arg(\lambda_i)| > \frac{\zeta\pi}{2}, \ i \in \{1, \dots, n\}.$$

Y^* *is locally asymptotically stable (LAS).*

The following theorems will be used in the next section to prove the positivity and existence-uniqueness of the solutions of the system (2.5).

Lemma 2.1. [4] *Let* $g \in C^q([t_0, T], \mathbb{R})$. *Suppose that for any* $t_1 \in (t_0, T]$, *one has* $g(t_1) = 0$ *and* $g(t) < 0$ *for* $t_0 \leq t < t_1$; *then it follows that* ${}^{C}_{t_0}D^{\zeta}_{t}g(t_1) > 0$.

Lemma 2.2. [17] *For the system*

$${}^{C}_{0}D^{\zeta}_{t}Y(t) = H(t, Y), \, t \geq 0$$

with initial condition $Y(0) = (Y_1(0), \ldots, Y_n(0))$, *where* $0 < \zeta \leq 1$, $H : [0, \infty) \times \Delta \to \mathbb{R}^n$, $\Delta \subseteq \mathbb{R}^n$. *If* $H(t, Y)$ *fulfills the local Lipschitz condition with respect to* $Y \in \mathbb{R}^n$, *i.e.,*

$$\|H(t, Y) - H(t, \tilde{Y})\| \leq L\|Y - \tilde{Y}\|,$$

then there exists a unique solution of the above system on $[0, \infty) \times \Delta$, *where*

$$\|Y(u_1, u_2, \ldots, u_n) - \tilde{Y}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)\| = \sum_{i=1}^{n} |u_i - \tilde{u}_i|,\ u_i, \tilde{u}_i \in \mathbb{R}.$$

3. Dynamical analysis of the system (2.5)

In this section, we first establish the existence and uniqueness of solutions, as well as their non-negativeness, followed by an analysis of local stability and Hopf bifurcation for the system (2.5). The existence and uniqueness of the solution ensure that the model provides well-defined and predictable population dynamics for given initial conditions. Non-negativeness guarantees biologically meaningful solutions, ensuring that population sizes remain non-negative over time. Local stability analysis helps understand how small perturbations affect species persistence, indicating whether populations return to equilibrium after minor disturbances. The Hopf bifurcation suggests that the system can transition from a stable state to sustained population cycles, which aligns with observed predator-prey oscillations in ecological systems.

3.1. Existence-uniquness of solutions

Here, we study the existence-uniqueness of the solutions of the system (2.5) in the region $\Omega \times [t_0, T)$ where

$$\Omega = \{(u, v) \in \mathbb{R}_+^2 : \max\{|u|, |v|\} < \xi\},$$

$T < \infty$, ξ is large, and $\mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$.

Theorem 3.1. *For any non-negative initial conditions, the Caputo fractional order system* (2.5) *admits a unique solution.*

Proof. Let $Y_0, \tilde{Y}_0 \in \Omega$ and $Y(t), \tilde{Y}(t)$ be two solutions of the system $D^\zeta Y = H(Y)$, starting from $Y_0, \tilde{Y}_0$. Here, $Y = (u, v)^T$ and $H(Y) = (H_1(Y), H_2(Y))^T$ with

$$\begin{aligned} H_1(Y) &= ru\left(1 - \frac{u}{K + \beta u}\right)(u - m) - \alpha uv, \\ H_2(Y) &= c\alpha uv - dv. \end{aligned} \tag{3.1}$$

Then, we have

$$\begin{aligned} &\|H(Y) - H(\tilde{Y})\| \\ =&|H_1(Y) - H_1(\tilde{Y})| + |H_2(Y) - H_2(\tilde{Y})| \\ =&\left|ru\left(1 - \frac{u}{K + \beta u}\right)(u - m) - r\tilde{u}\left(1 - \frac{\tilde{u}}{K + \beta\tilde{u}}\right)(\tilde{u} - m) - \alpha uv + \alpha\tilde{u}\tilde{v}\right| \\ &+ |c\alpha uv - c\alpha\tilde{u}\tilde{v} - dv + d\tilde{v}| \\ =&\Big|r(u^2 - \tilde{u}^2) + rm(\tilde{u} - u) + \frac{r}{(K + \beta u)(K + \beta\tilde{u})}(\tilde{u}^3 - u^3) \end{aligned}$$

$$
\begin{aligned}
&- \frac{r}{(K+\beta u)(K+\beta\tilde{u})} m(u^2-\tilde{u}^2) + \alpha(\tilde{u}\tilde{v} - uv)| + |c\alpha(uv - \tilde{u}\tilde{v}) + d(\tilde{v} - v)| \\
\leq& 2r\xi|u-\tilde{u}| + rm|u-\tilde{u}| + 3\frac{r}{K^2}\xi^2|u-\tilde{u}| + 2\frac{r}{K^2}m\xi|u-\tilde{u}| + \alpha\xi|u-\tilde{u}| \\
&+ \alpha\xi|v-\tilde{v}| + c\alpha\xi|u-\tilde{u}| + c\alpha\xi|v-\tilde{v}| + d|v-\tilde{v}| \\
=&[2r\xi + rm + \frac{r\xi}{K^2}(3\xi+2m) + \alpha\xi(1+c)]|u-\tilde{u}| + [\alpha\xi(1+c)+d]|v-\tilde{v}| \\
=&L_1|u-\tilde{u}| + L_2|v-\tilde{v}| \\
\leq& L\|Y-\tilde{Y}\|
\end{aligned}
$$

where $L_1 = 2r\xi + rm + \frac{r\xi}{K^2}(3\xi + 2m) + \alpha\xi(1 + c)$, $L_2 = \alpha\xi(1 + c) + d$, $L = \max\{L_1, L_2\}$. Thus, $H(Y)$ satisfies the local Lipschitz condition, and the conclusion follows directly from Lemma 2.2. □

Theorem 3.2. *If the initial conditions* $u(t_0) \geq 0$, $v(t_0) \geq 0$, *then all solutions of the system* (2.5) *are non-negative.*

Proof. Assume that $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ be the solutions of the system (2.5) for $t > t_0$ starting from $\begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix}$. Suppose that the assumption is false, and there exists $t_1 > t_0$ such that $u(t) > 0$ for $t_0 \leq t < t_1$, $u(t_1) = 0$, and $u(t) < 0$ for $t > t_1$. From the system (2.5), we obtain ${}_{t_0}^{C}D_t^{\zeta} u(t)|_{t=t_1} = 0$. Lemma 2 implies that $u(t) = 0$ for $t > t_1$, which contradicts with $u(t) < 0$ for $t > t_1$. Therefore, we conclude $u(t) \geq 0$ for all $t \geq t_0$. We can similarly prove $v(t) \geq 0$, for $\forall t \geq t_0$. Hence, the solutions remains non-negative. □

3.2. Equilibrium points

We set both equations of (2.5) equal to zero to obtain equilibrium points:

$$
\begin{cases}
D^{\zeta}u(t) = 0, \\
D^{\zeta}v(t) = 0.
\end{cases}
$$

System (2.5) allows one extinction and two boundary equilibriums. Besides, we can also have one coexistence equilibrium depending on conditions.

i. $E_0 = (0, 0)$.
ii. $E_m = (m, 0)$.
iii. $E_K = \left(\frac{K}{1-\beta}, 0\right)$. Since we assume $0 \leq \beta < 1$, we have $\frac{K}{1-\beta} > 0$.
iv. The coexistence equilibrium $E^* = (u^*, v^*)$ with

$$
u^* = \frac{d}{c\alpha},\; v^* = \frac{r(d - cm\alpha)(cK\alpha + d(-1+\beta))}{c\alpha^2(cK\alpha + d\beta)}.
$$

We have the coexistence equilibrium (i.e., $v^* > 0$) if one of the followings is satisfied:

1. $m < \dfrac{d}{c\alpha} < K$ (i.e., $m < u^* < K$),
2. $m < K \leq \dfrac{d}{c\alpha} < \dfrac{K}{1-\beta}$ (i.e., $m < K \leq u^* < \dfrac{K}{1-\beta}$).

For the both cases the Allee parameter m should be smaller than K, which is biologically logical. The first condition is commonly encountered in predator-prey models with Allee effects. The enviromental carrying capacity K must exceed the prey component u^* of the coexistence equilibrium E^* and the Alle threshold must remain below the prey component. The second case is more interesting in the context of variable carrying capacity, original aspect of this study. The enviromental carrying capacity K can be smaller than prey component u^*, but then the own-improved carrying capacity of prey $\frac{K}{1-\beta}$ should exceed u^*.

3.3. Stability analysis

Theorem 3.3. *i.* $E_0 = (0, 0)$ *is always LAS.*

ii. $E_m = (m, 0)$ *is unstable for* $m < K$*, and it is a saddle if* $m < \dfrac{d}{c\alpha}$.

iii. $E_K = \left(\frac{K}{1-\beta}, 0\right)$ *is LAS if* $\frac{K}{1-\beta} < \frac{d}{c\alpha}$*, othwerwise it is a saddle. Note that (comparing with existence conditions of* P_c*), if* E^* *exists,* E_K *is a saddle.*

Proof. (i.) Since the eigenvalues of J_{E_0} are negative real numbers $-d$,$-mr$; E_0 always LAS. (ii) The eigenvalues of J_{E_m} are $\lambda_1 = -d + cm\alpha$, $\lambda_2 = mr(1 - \frac{m}{K+m\beta})$. For $m < K$, λ_2 is a positive real number and it does not satisfy LAS condition $|\arg(\lambda_2)| > \frac{\zeta\pi}{2}$, for any $\zeta \in (0, 1]$. The eigenvalue λ_1 is a negative real number for $m < \frac{d}{c\alpha}$. (iii.) The Jacobian matrix J_{E_K} evaluated at E_K are negative numbers $\lambda_1 = -d + \frac{cK\alpha}{1-\beta}$, $\lambda_2 = -r(K + m(-1 + \beta))$. Since $m < K$ and $\beta \in [0, 1)$, λ_2 is negative real number and satisfy LAS condition. Lastly, if $\frac{K}{1-\beta} < \frac{d}{c\alpha}$, λ_1 is also negative real number and E_K is LAS. □

Here, we assume that the existence conditions of E^* are satisfied. Then, system (2.5) allows one positive equilibrium $E^* = (u^*, v^*)$. The Jacobian matrix J_{E^*} evaluated at E^* can be written as

$$J_{E^*} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

with

$$j_{11} = \frac{dr(c^2K(K+m)\alpha^2 + 2cdK\alpha(\beta-1) + d^2(\beta-1)\beta)}{c\alpha(cK\alpha + d\beta)^2}, \quad j_{12} = \frac{-d}{c},$$
$$j_{21} = -\frac{r(d - cm\alpha)(cK\alpha + d(\beta-1))}{\alpha(cK\alpha + d\beta)}, \; j_{22} = 0.$$

The characteristic equation is given by

$$p(\lambda) = \lambda^2 + p_1\lambda + p_0 = 0 \tag{3.2}$$

with $p_1 = -(j_{11}+j_{22}) = -j_{11}$, $p_0 = j_{11}j_{22} - j_{12}j_{21} = -j_{12}j_{21}$. j_{12} is always negative for positive parameter values and j_{21} is always positive under positivity conditions of E^*. This implies p_0 is also positive. Then, we get the following theorem.

Theorem 3.4. *The coexistence equilibrium $E^* = (u^*, v^*)$ is LAS if one of the following conditions satisfied:*

- *(a)* $p_1 \geq 0$,
- *(b)* $p_1 < 0$, $p_1^2 - 4p_0 < 0$, $\left|\tan^{-1}(\frac{\sqrt{4p_0 - p_1^2}}{p_1})\right| > \frac{\zeta\pi}{2}$.

Proof. The roots of the the characteristic equation (3.2) is given by following formula:

$$\lambda_{1,2} = \frac{-p_1 \pm \sqrt{p_1^2 - 4p_0}}{2}.$$

(a) Firstly, $p_1 = 0$ implies that $\lambda_{1,2} = \pm i\sqrt{p_0}$ and $|arg(\lambda_{1,2})| = \frac{\pi}{2} > \frac{\zeta\pi}{2}$. For $p_1 > 0$ and $p_1^2 - 4p_0 \geq 0$, both $\lambda_{1,2} \in \mathbb{R}$ with $\lambda_{1,2} < 0$, and the condition $|arg(\lambda_{1,2})| = \pi > \frac{\zeta\pi}{2}$ is satisfied. Then, the conditions $p_1 > 0$ and $p_1^2 - 4p_0 < 0$ implies that both eigenvalues $\lambda_{1,2}$ are complex numbers with $Re(\lambda_{1,2}) < 0$, and we have $|arg(\lambda_{1,2})| > \frac{\zeta\pi}{2}$.

(b) For $p_1 < 0$ and $p_1^2 - 4p_0 < 0$, both eigenvalues are complex conjugates with strictly positive real parts. Then, $|tan^{-1}(\frac{\sqrt{4p_0 - p_1^2}}{p_1})| > \frac{\zeta\pi}{2}$ ensures $|arg(\lambda_{1,2})| > \frac{\zeta\pi}{2}$. □

Note that, two bistability situation present for the system (2.5). The first one between equilibriums E_0 and E_K. The other one is between E_0 and E^*. Bistability can represent important biological scenario which can explain how populations might settle into different stable states, such as extinction or survival, based on initial conditions.

3.4. Hopf bifurcation

Theorem 3.5. [18] *Consider the fractional order system*

$$D^{\zeta}Y(t) = H(Y), \quad Y(0) = Y_0 \in \mathbb{R}^2 \tag{3.3}$$

with $\zeta \in (0, 1)$. The system (3.3) undergoes a Hopf bifurcation around the equilibrium point Y^* of (3.3), if the followings are satisfied:

1) *The Jacobian matrix evaluated at Y^* has a pair of complex conjugate eigenvalues $\lambda_{1,2} = a \pm ib$ (where $a > 0$).*
2) *There exist a critical $\zeta = \zeta_h$ value satisfying $m(\zeta_h) = 0$ where $m(\zeta) = \frac{\zeta\pi}{2} - min_{1\leq j\leq 2}|arg(\lambda_j)|$.*
3) $\left.\frac{dm(\zeta)}{d\zeta}\right|_{\zeta=\zeta_h} \neq 0$.

Theorem 3.6. *Assume that the existence conditions of E^* hold. Defining p_0 and p_1 as coefficients of the characteristic equation (3.2), suppose that the inequalities $p_1 < 0$, $p^2 < 4p_0$ are satisfied. A Hopf bifurcation exists around E^* as ζ passes through the value $\zeta_h = \frac{2}{\pi}\left|tan^{-1}\left(\frac{\sqrt{4p_0 - p_1^2}}{p_1}\right)\right|$.*

Proof. The eigenvalues $\lambda_{1,2}$ of the Jacobian J_{P_c} are complex conjugates with $Re(\lambda_{1,2}) > 0$ under the conditions $p_1 < 0$ and $p_1^2 < 4p_0$. So, the stability of E^* is depending on $\zeta \in (0, 1)$. We also have $min_{1\leq j\leq 2}|arg(\lambda_i)| = \left|tan^{-1}\left(\frac{\sqrt{4p_0 - p_1^2}}{p_1}\right)\right|$ for

Table 1. Parameter values for numerical analysis.

r	K	β	m	α	c	d
0.3	2	0.15	0.35	0.35	0.20	0.08

$\zeta = \zeta_h = \frac{2}{\pi}\left|tan^{-1}\left(\frac{\sqrt{4p_0 - p_1^2}}{p_1}\right)\right|$and $m(\zeta_h) = 0$. Last condition for the existence of the Hopf bifurcation also holds since

$$\frac{dm(\zeta)}{d\zeta}\Big|_{\zeta=\zeta_h} = \frac{\pi}{2} \neq 0.$$

$\square$

4. Numerical simulations

In this part, we provide numerical examples to support previously discussed theoretical findings. To perform simulations, we apply the predictor-corrector method. This approach, firstly suggested by [7], utilizes a combination of product integration techniques [11].

Given that variable carrying capacity is a central aspect of this study, the parameters associated with it, K and β, will be the primary focus of this section. In particular, we will examine the environmental impact parameter, β, and its dynamic relationship with other system parameters in detail. The Allee parameter, m, is also critical when considering the survival or extinction of both species. In cases of the strong Allee effect ($m > 0$), the presence and magnitude of oscillatory solutions significantly affect population dynamics. Additionally, the memory effect introduced through the fractional order derivative has a profound influence on the nature of these oscillations. As noted in the introduction, instead of using delayed terms, we incorporate fractional dynamics, making the Caputo fractional derivative parameter, ζ, another focus of our numerical investigation. Although the topics are interconnected, we will examine this section under four main headings for a more organized analysis. Unless otherwise specified, parameter values from relevant studies, listed in Table 1, are applied.

(i) Bistability for the system (2.5)

Bistability refers to a situation where a system can converge to two distinct equilibrium points within the same parametric region, depending on the initial conditions. In our system (2.5), we observe two cases of bistability. The first occurs between the extinction equilibrium E_0 and the boundary equilibrium E_K (Figure 1(a)). In scenarios with a higher environmental carrying capacity K and a lower predator death rate d, the system may either converge to the extinction equilibrium E_0 or the coexistence equilibrium E^*, resulting in bistability between E_0 and E^* (Figure 1(b)). If the initial predator population is significantly larger than the prey population, the system tends toward extinction. However, if the initial conditions favor the prey, their population can persist, while the predator's survival depends on the availability of resources or a lower death rate.

(ii) Impact of the fractional order parameter ζ

In Figure 2 and Figure 3, we present time series solutions for different values of the fractional order ζ. In Figure 2, we use the same parameters as in Figure 1(a),

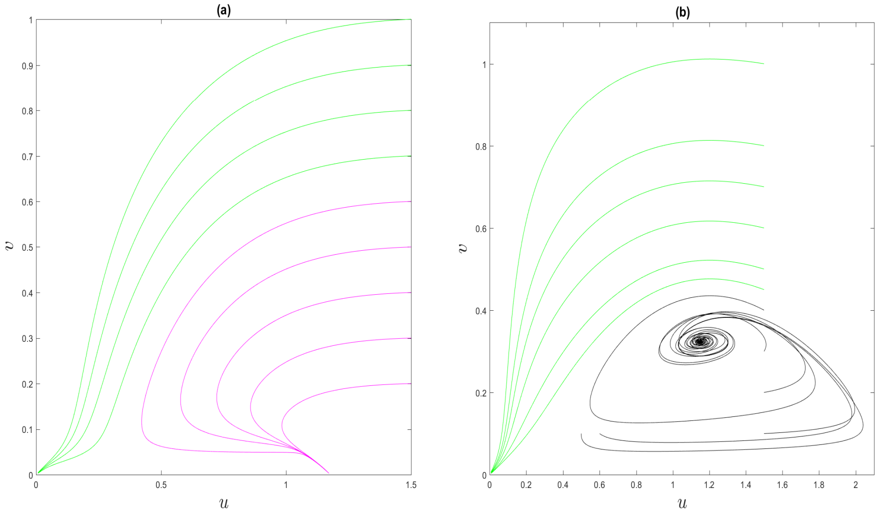

Figure 1. Phase diagrams for different inital conditions with $\zeta = 0.80$; (a) $K = 1$, $d = 0.12$; (b) $K = 2$, $d = 0.08$.

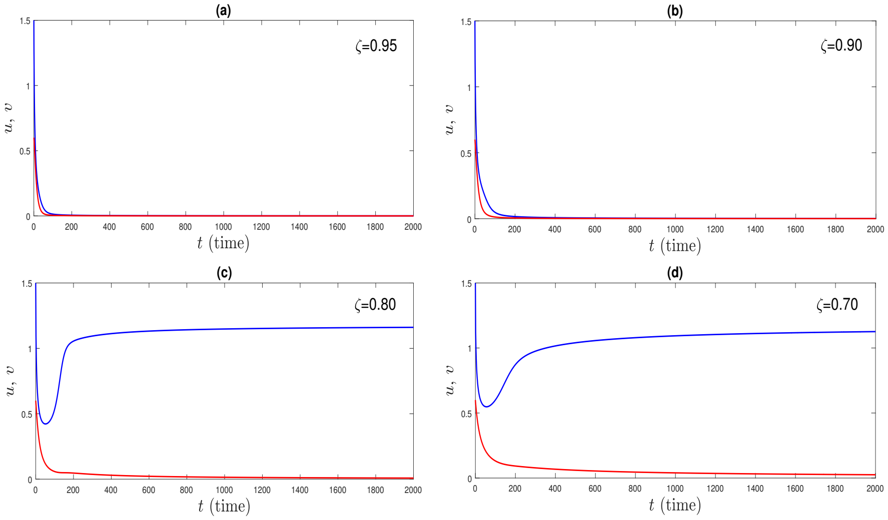

Figure 2. Time series solution of the system (2.5) for different fractional order parameter.

where no coexistence equilibrium is present. In this case, increasing the memory effect allows the prey population to survive, while the predator population eventually goes extinct, though it persists for a longer period. In Figure 3, parameter values are selected to allow for a coexistence equilibrium. However, when the memory effect is weaker (i.e., ζ is closer to 1), both species go extinct. As the fractional order ζ decreases, the populations survive in an oscillatory mode and eventually converge to the locally asymptotically stable (LAS) coexistence equilibrium E^*. It should be noted that the Allee parameter plays a crucial role in this behavior, as oscillations with larger amplitudes, coupled with higher critical Allee values, can lead to the extinction of both populations. In Figure 4, we present bifurcation diagrams with respect to ζ, which align with the earlier discussions. As ζ approaches 1, the memory effect weakens, causing the equilibrium point E^* to lose stability through a Hopf bifurcation. This destabilization leads to oscillations in both populations, with fluctuations increasing until both species eventually go extinct. Biologically speaking, it is evident that each species relies on previous interactions in the ecosystem to achieve a more stable balance.

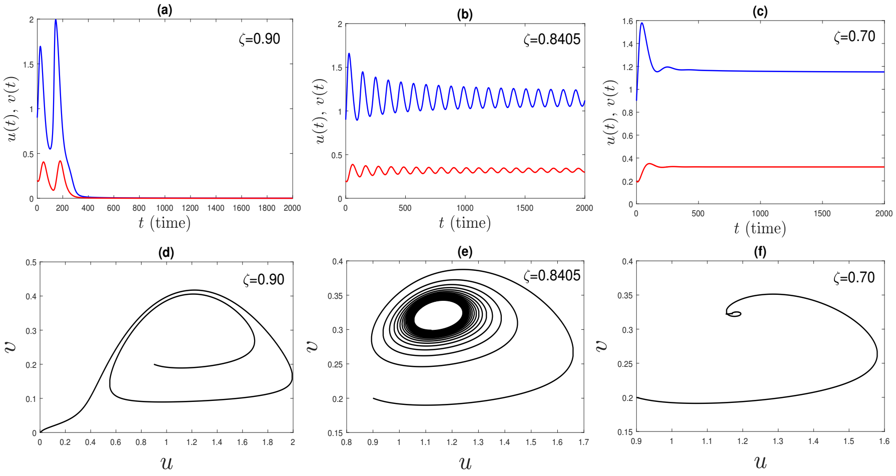

Figure 3. Time series solution of the system (2.5) for different fractional order parameter.

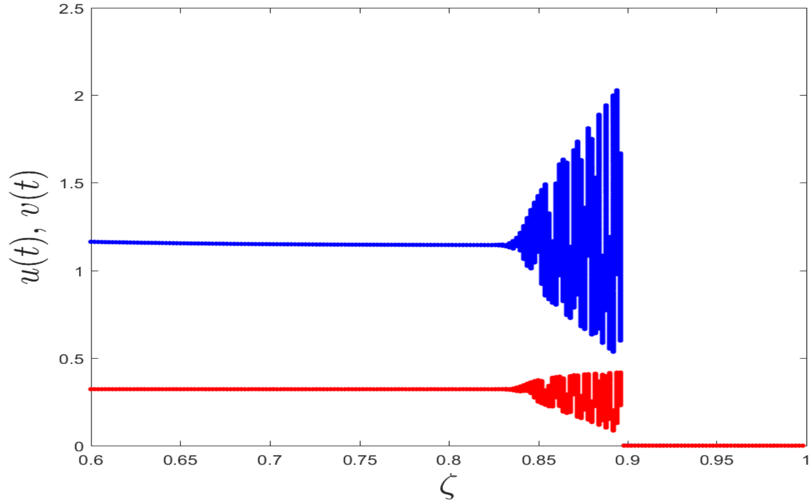

Figure 4. Bifurcation diagrams for the system (2.5) with respect to ζ.

(iii) The investigation on the parameters of the variable carrying capacity term

In Figure 7, the time-series solutions for the system (2.5) are displayed for varying values of K and β. As both parameters increase, we observe the "paradox of enrichment", a biological phenomenon where an abundance of food for the prey

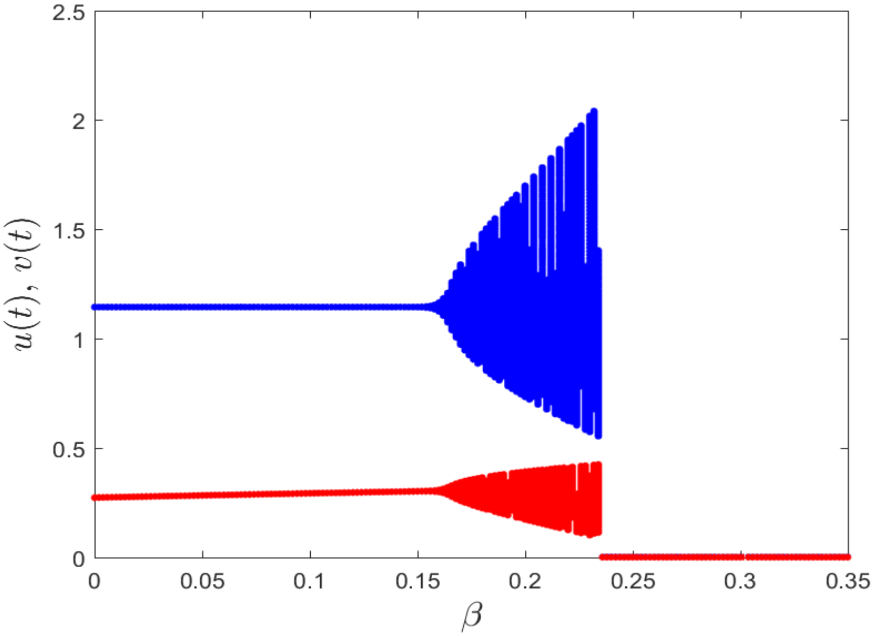

Figure 5. Bifurcation diagrams for the system (2.5) with respect to constructive impact parameter β.

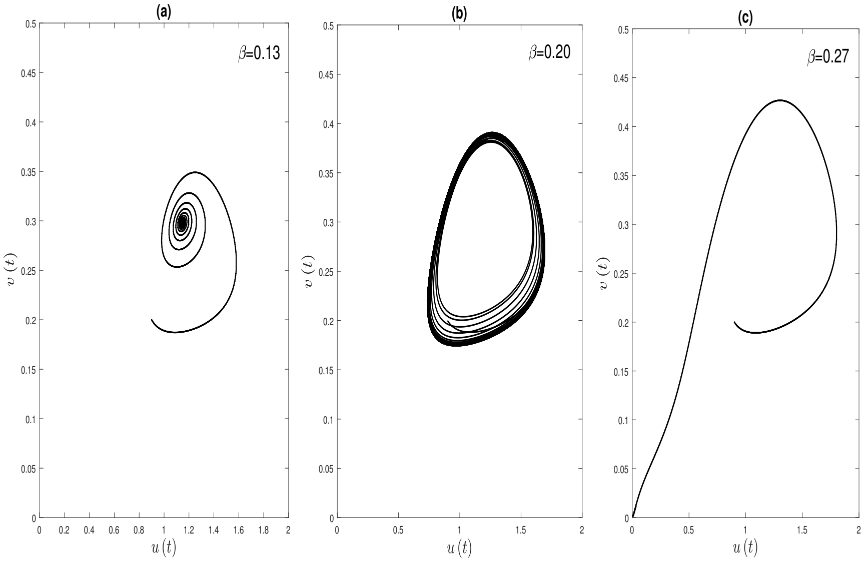

Figure 6. Phase diagrams for three different values of the constructive impact parameter β.

leads to destabilization of the predator population [27]. The increase in K and β enhances environmental resources for the prey, resulting in oscillatory population dynamics (Figure 5, Figure 6, Figure 7, (b),(c)). Further increases can eventually drive both prey and predator populations to extinction (Figure 6(c), Figure 7(d)). Additionally, introducing a strong Allee effect ($m = 0.35$) creates a critical threshold for prey extinction, accelerating the process of ending these oscillations with extinction.

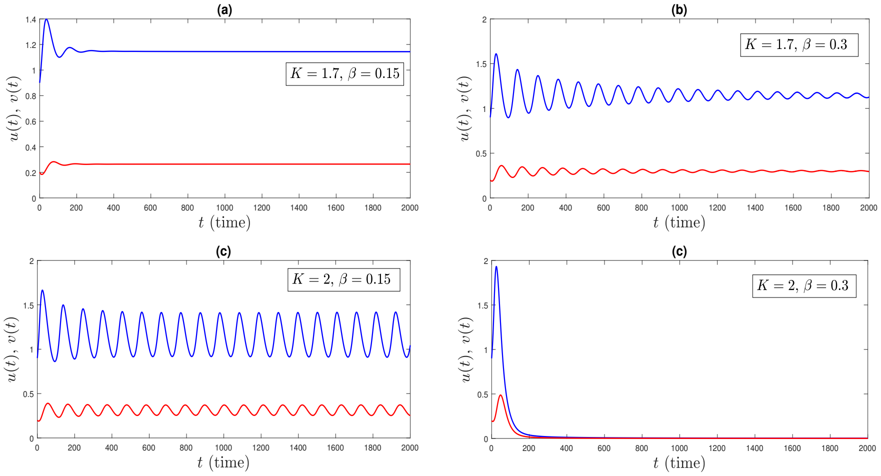

Figure 7. Time series solution of the system (2.5) for different values of K and β with $\zeta = 0.85$.

In Figure 8, the schematic bifurcation diagram for $K \in [0.8, 2.8]$ and $\beta \in [0, 1)$ is shown to explore the dynamic relationship between the parameters of the variable carrying capacity term, $1/\kappa(t)$ where $\kappa(t) = K + \beta t$. Firstly, The vertical light green line represents $K = u^*$, indicating that if the constructive impact of prey is excluded (i.e., $\beta = 0$), there is no positive equilibrium for $K \leq u^*$. The light blue line marks the transcritical bifurcation, above which the LAS coexistence equilibrium $E^* = (u^*, v^*)$ emerges, and the boundary equilibrium E_K becomes unstable. The solid orange curve represents the Hopf bifurcation curve for $\zeta = 0.85$, above which the coexistence equilibrium E^* becomes unstable. The dashed orange curve indicates where the eigenvalues of the Jacobian matrix evaluated at E^* become complex with a positive real part, signaling the point where the fractional order begins to alter the stability of E^*. In other words, the dashed orange curve would be Hopf bifurcation curve if we use ordinary differential equations (i.e., $\zeta = 1$). Essentially, the region between those orange curves represents the enlargement of the region of stability of the coexistence equilibrium E^* of the system (2.5) for $\zeta = 0.85$.

(iv) Influence of the Allee parameter m

In Figure 9(a) we present the bifurcation diagram with varying $m \in [-0.5, 1]$. In Figure 9(b), we present the bifurcation diagram with varying β while there is no Allee effect (i.e., $m = 0$). For both figures, we incorporate a memory effect with $\zeta = 0.80$. The figure in (b) is just the counterpart of the bifurcation diagram in Figure 5, where we plotted the bifurcation diagram for varying β values with $m = 0.35$. With this comparison, we emphasize that with the inclusion of the Allee effect, these oscillatory behaviors lead to the collapse of both populations. Without a strong Allee effect, the populations exhibit oscillatory patterns with increasing amplitudes depending on the increase of the constructive impact parameter β.

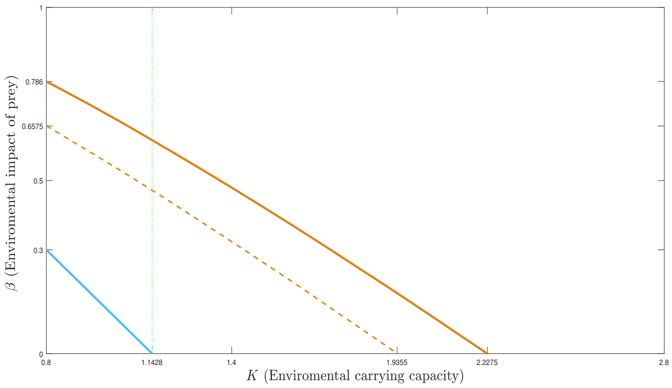

Figure 8. The schematic bifurcation diagram of parameters K and β with $\zeta = 0.85$.

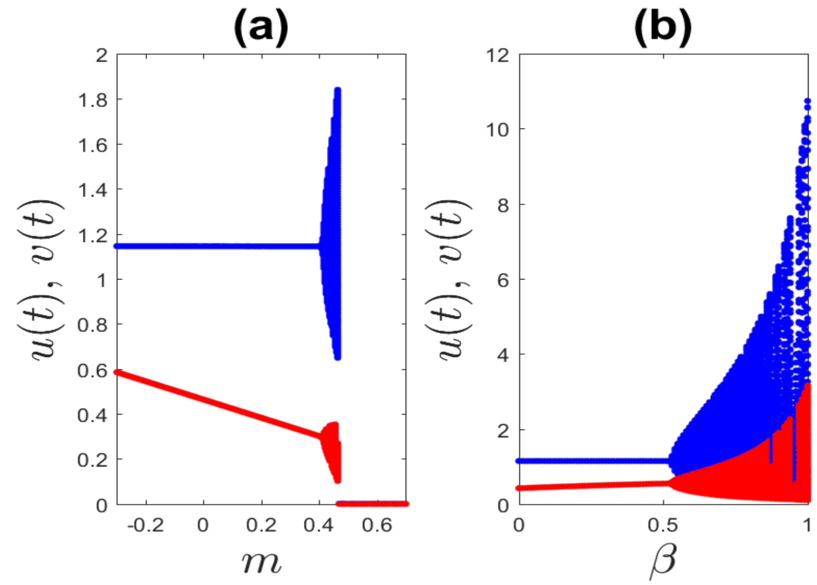

Figure 9. (a) Bifurcation diagram with respect to Allee parameter m (b) Bifurcation diagram with respect to constructive impact parameter β while $m = 0$.

In Figure 9(a), we observe that the increase in m causing a decrease in predator component y^* of E^* while prey component x^* of E^* do not change. For $m \approx 0.4131$, the system undergoes a Hopf bifurcation, and we observe periodic behaviors (Figure 9(a), Figure 10). As the Allee parameter increases, prey growth becomes more constrained, and the prey population density begins to influence the stability of the predator population. A further rise in the m parameter intensifies the Allee effect, leading to larger oscillations and raising the critical threshold the prey population must surpass. Over time, as the Allee effect intensifies, both populations ultimately face extinction due to the heightened pressure (Figure 9(a), Figure 10(c)).

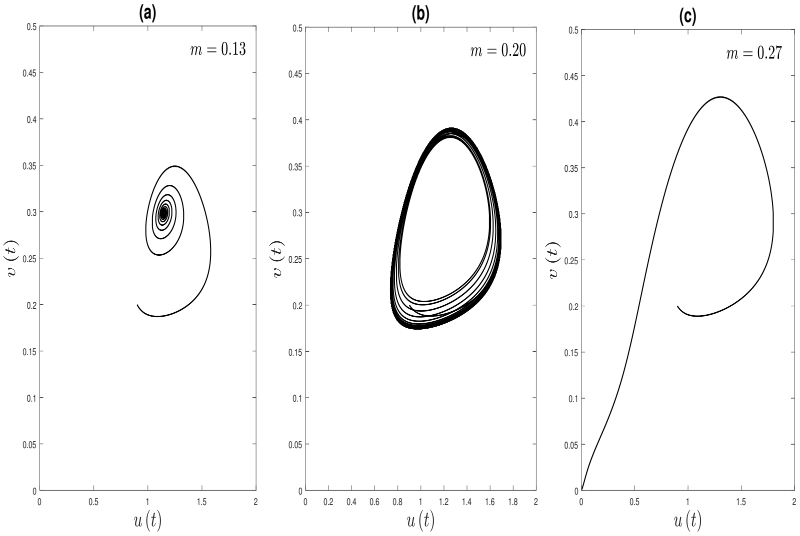

Figure 10. Phase diagrams for three different values of the constructive impact parameter β.

5. Conclusions

In this work, we work on a prey-predator model with prey species that are able to positively impact their own carrying capacity and are subject to the Allee effect. We also adapt fractional differentiation in the system (2.5) to incorporate the memory effect depending on the fractional order ζ. We proved the existence-uniqueness of the solutions. We discuss the structure of nonnegative equilibria and their local stability. The system (2.5) can have positive equilibria depending on conditions. We showed that two bistability situations can exist for the system depending on the parameters. Afterward, the existence of Hopf bifurcation concerning fractional order parameter ζ is also proved. Finally, we support theoretical findings through detailed numerical examples with biological interpretations.

In conclusion, varying carrying capacity is a significant biological phenomenon that exerts broad direct and indirect effects on ecological systems. In this study, we focus on the constructive impact of prey species on their own carrying capacity, drawing inspiration from ecosystem engineers. We also explore the relationship between the constructive impact parameter, β, and the Allee parameter, m. Given the importance of the Allee effect in biological systems, its presence notably influences our model, as prey density can directly affect the availability of resources in their environment. Future research could further integrate varying carrying capacity into predator-prey models, investigating its interactions with other biological factors. This approach could uncover hidden patterns and emergent behaviors in ecosystems, offering deeper insights into the dynamics of predator-prey relationships.

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