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THE $\bar{\partial}$-DRESSING METHOD FOR THE TWO-DIMENSIONAL HARRY DYM EQUATION

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Abstract The (2+1)-dimensional Harry Dym (HD) equation is solved via the $\bar{\partial}$-dressing method in this paper. By introducing long derivatives E_x, E_y and E_t and new expressions for the kernel functions K of $\bar{\partial}$-problem, a type of general solution of the HD equation is obtained. Under the reality of the solution u of the HD equation, several classes of exact explicit solutions of the HD equation, including the solutions with functional parameters, line solitons and rational solutions, are constructed by the $\bar{\partial}$-dressing method.

Keywords HD equation, $\bar{\partial}$-dressing method, line soliton, rational solution.

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1. Introduction

The inverse scattering transform method was proposed by Gardner, Greene, Kruskal, and Miura in 1967 to study initial value problems related to the famous Korteweg-de Vries (KdV) equation [22]. The modern history of integrable equations began with the famous work of Martin Kruskal and his colleagues on the Cauchy problem of KdV, using what later became known as the inverse scattering transform method [33]. The inverse spectral transform method has been generalized and successfully applied to the computation of a wide class of exact solutions of various (2+1)-dimensional nonlinear evolution equations. Many experts and scholars have investigated different equations via the inverse spectral transform method [4, 5, 23, 30]. For example, Beals and Coifman has been recently studied $n \times n$ AKNS which extends the AKNS problem to systems of n equations [1]. Zakharov and Shabat has solved soliton solutions, as well as multi-soliton solutions and high-order soliton solutions [29]. At the same time, the problems of solving the modified KdV equation [31], Schrödinger equation [21], Degasperis-Procesi equation [14], Camassa-Holm equation [13] and other equations [2, 3, 20, 28, 36] were studied through inverse scattering transform method. The basic tools for solving (2+1)-dimensional integrable nonlinear equations via IST are now the non-local Riemann-Hilbert problem [15], the $\bar{\partial}$-problem [7] and the $\bar{\partial}$-dressing method [8, 9]. Different kinds of exact solutions of some (2+1)-dimensional integrable nonlinear evolution equations were studied by the $\bar{\partial}$-dressing method [6, 10, 12, 17]. Such as

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the plane solitons of mKP equation [26], the line solitons and line rational lumps of the Kaup-Kuperschmidt (KK) and Savada-Kotera (SK) equations [18], the multiple pole solutions of the Kadomtsev-Petviashvili (KP) equation, the modified Kadomtsev-Petviashvili (mKP) equation and the Davey-Stewartson (DS) system of equations [16]. The study of exact solutions of integrable nonlinear equations, such as soliton solutions, multi-pole solutions, rational solutions, etc. is an important topic in the field of mathematical physics.

In this paper, we introduce a $\bar{\partial}$-dressing method for solving several classes of the exact explicit solutions of the HD equation, including the solutions with functional parameters, line solitons and rational solutions. Introducing the (2+1)-dimensional integrable generalization of the Harry Dym equation(see [25])

$$u_t - u^3 u_{xxx} - 3u^{-1}(u^2 \partial_x^{-1} u_y / u^2)_y = 0, \tag{1.1}$$

which is the only known two-dimensional generalization of the so-called WKI (Wadati, Konno and Ichikawa) equations, and it describes the propagation of certain nonlinear waves, such as soliton solutions or singular waves. Solutions to such equations are often localized and stable, similar to soliton phenomena in fluids or optics. Where u depends on time variable t and space variables x, y. At the same time, we need know that ∂_x^{-1} is an inverse operator of ∂_x, $\partial_x^{-1}\partial_x = \partial_x \partial_x^{-1} = 1$.

Equation (1.1) has the following Lax pair

$$\begin{aligned} L\Phi =& u^2 \Phi_{2x} + \Phi_y, \\ T\Phi =& \Phi_t - 4\Phi_{3x} + 6u^2 u_x \Phi_x - 6u^2 \partial_x^{-1} u_y / u^2 \Phi_x, \end{aligned} \tag{1.2}$$

where the commutativity condition $L[T(\Phi)] = T[L(\Phi)]$ is equivalent to Equation (1.1).

We assume that the asymptotic value of u at infinity is constant as follows

$$u(x, y, t) = \tilde{u}(x, y, t) - \varepsilon, \quad \tilde{u}(x, y, t) \xrightarrow[x^2+y^2 \to \infty]{} -\varepsilon \neq 0. \tag{1.3}$$

The complete introduction of the spectral parameter λ into the linear problem (1.2) is achieved through the function χ

$$\Phi := \chi(\lambda) exp[\frac{i}{\lambda} x + (\frac{\varepsilon}{\lambda})^2 y + 4i(\frac{\varepsilon}{\lambda})^3 t], \tag{1.4}$$

which $\chi \to 1$ as $\lambda \to \infty$, the function Φ defined by (1.4) obeys system (1.2). Thus the problem of characterizing the inverse problem data does not arise.

In this paper, we study the HD equation (1.1) by means of the inverse spectral transform method. It is organized as follows: Section 2 explores the $\bar{\partial}$-dressing method for HD equation. Section 3 utilizes this method to introduce the solutions with functional parameters. Sections 4 and 5 construct line solitons and rational solutions in detail, respectively.

2. $\bar{\partial}$-dressing method for the HD equation

For the HD equation, we consider this case of $u(x, y, t) \to -\varepsilon$ as $x^2 + y^2 \to \infty$ by the $\bar{\partial}$-dressing method. Then based on the above approach we introduce a $\bar{\partial}$-problem

$$\frac{\partial \chi(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} = (\chi * \lambda)(\lambda, \bar{\lambda}) = \int\int_C \chi(\xi, \bar{\xi}) K(\xi, \bar{\xi}; \lambda, \bar{\lambda}) d\xi \wedge d\bar{\xi}. \tag{2.1}$$

Where χ and K are the scalar complex-valued functions. We assume that $\chi \to 1$ as $\lambda \to \infty$, and the problem (2.1) is uniquely solvable.

In the following, in order to give a relation between problem (2.1) and x, y, t, we introduce the following relation for K

$$\frac{\partial K}{\partial \eta} = \begin{pmatrix} \frac{i}{\xi} & \frac{i}{\lambda} & 0 \\ 0 & -(\frac{\varepsilon}{\xi})^2 & -(\frac{\varepsilon}{\lambda})^2 \\ 4i(\frac{\varepsilon}{\xi})^3 & 4i(\frac{\varepsilon}{\lambda})^3 & 0 \end{pmatrix} \begin{pmatrix} K(\xi,\bar{\xi};\lambda,\bar{\lambda};x,y,t) \\ -K(\xi,\bar{\xi};\lambda,\bar{\lambda};x,y,t) \\ K(\xi,\bar{\xi};\lambda,\bar{\lambda};x,y,t) \end{pmatrix}, \tag{2.2}$$

with

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial t} \end{pmatrix}^T. \tag{2.3}$$

Then we further get that

$$K(\xi,\bar{\xi};\lambda,\bar{\lambda};x,y,t) = K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) exp(\triangle\nu(x,y,t,\xi,\lambda)), \tag{2.4}$$

where

$$\triangle\nu(x,y,t,\xi,\lambda) := i(\frac{1}{\xi} - \frac{1}{\lambda})x + \varepsilon^2(\frac{1}{\xi^2} - \frac{1}{\lambda^2})y - 4i\varepsilon^3(\frac{1}{\xi^3} - \frac{1}{\lambda^3})t = \nu(\xi) - \nu(\lambda). \tag{2.5}$$

Some long derivative operators E_x, E_y and E_t are introduced

$$\begin{aligned} E_x &= \partial_x + \frac{i}{\lambda}, \\ E_y &= \partial_y + \varepsilon^2 \frac{1}{\lambda^2}, \\ E_t &= \partial_t + 4i\varepsilon^3 \frac{1}{\lambda^3}. \end{aligned} \tag{2.6}$$

For the equations

$$\begin{aligned} \Gamma_1 &= u^2 E_x^2 + E_y, \\ \Gamma_2 &= E_t + 4u^3 E_x^3 + 6uu_x E_x - 6u^2 \partial_x^{-1} u_y / u^2 E_x, \end{aligned} \tag{2.7}$$

we can find that

$$\Gamma_1 \chi = 0, \quad \Gamma_2 \chi = 0. \tag{2.8}$$

Then we consider (2.8) for the series expansion of χ in the neighborhood of the points $\lambda = 0$ and $\lambda = \infty$: when $\lambda = 0$, $\chi = \chi_0 + \lambda\chi_1 + \lambda^2\chi_2 + \ldots$; when $\lambda = \infty$, $\chi = \tilde{\chi}_0 + \chi_{-1}/\lambda + \chi_{-1}/\lambda^2 + \ldots$. Due to canonical normalization $\tilde{\chi}_0 = 1$, in the neighbourhood of $\lambda = \infty$, the system (2.8) imply

$$u = (\frac{-\chi_{-1y}}{i + \chi_{-1x}})^{\frac{1}{2}}. \tag{2.9}$$

In the case of the normalized $\tilde{\chi}_0 = 1$, the solution of the $\bar{\partial}$-problem (2.1) is equivalent to the solution of the integral equation

$$\chi(\lambda,\bar{\lambda}) = 1 + \int\int_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda' - \lambda)} \int\int_C \chi(\xi,\bar{\xi}) K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) e^{(\nu(\xi)-\nu(\lambda'))} d\xi \wedge d\bar{\xi}. \tag{2.10}$$

Equation (2.10) implies that

$$\chi_0 = 1 + \int\int_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i \lambda} \int\int_C \chi(\xi,\bar{\xi}) K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) e^{(\nu(\xi)-\nu(\lambda))} d\xi \wedge d\bar{\xi}, \tag{2.11}$$

and

$$\chi_{-1} = -\int\int_C \frac{d\lambda \wedge d\bar{\lambda}}{2\pi i} \int\int_C \chi(\xi,\bar{\xi}) K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) e^{(\nu(\xi)-\nu(\lambda))} d\xi \wedge d\bar{\xi}, \tag{2.12}$$

where K_0 is an arbitrary function.

We know that the solutions of the HD equations constructed using the $\bar{\partial}$-dressing method are all complex valued, and finding real-valued solutions to the equations of this equation is the next major part of our study. From (2.9), (2.12) and the reality condition $u = \overline{u}$, we obtain the following constraints on the kernel K of $\bar{\partial}$-problem (in the weak limit)

$$K_0(\xi,\bar{\xi};\lambda,\overline{\lambda}) = \overline{K_0(-\overline{\xi},-\xi;-\overline{\lambda},-\lambda)}, \tag{2.13}$$

$$K_0(\xi,\bar{\xi};\lambda,\overline{\lambda}) = \overline{K_0(\overline{\lambda},\lambda;\overline{\xi},\xi)}. \tag{2.14}$$

Below we construct different exact solutions of the HD equation (1.1) based on two different constraints (2.13) and (2.14).

3. Solutions with functional parameters

Consider the kernel function K of problem (2.1) as follows

$$K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) = \pi \sum_{m=1}^{N} w_m(\xi,\bar{\xi}) v_m(\lambda,\bar{\lambda}), \tag{3.1}$$

where w_m and v_m are arbitrary functions. For such kernel K, the function χ can be obtained that

$$\chi(\lambda) = 1 + \pi \sum_{m=1}^{N} \ell_m(x,y,t) \int\int_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda'-\lambda)} v_m(\lambda',\bar{\lambda}') e^{-\nu(\lambda')}, \tag{3.2}$$

where $\ell_m(x,y,t)$ is defined by

$$\ell_m(x,y,t) = \int\int_C w_m(\lambda,\bar{\lambda}) \chi(\lambda,\bar{\lambda}) e^{\nu(\lambda)} d\lambda \wedge d\bar{\lambda}. \tag{3.3}$$

The quantities ℓ_m are computed from the following algebraic system

$$H_{m1}\ell_1 + H_{m2}\ell_2 + \cdots + H_{mN}\ell_N = \zeta_m, \quad m = 1,\ldots,N, \tag{3.4}$$

where

$$\zeta_m(x,y,t) = \int\int_C w_m(\lambda,\bar{\lambda}) e^{\nu(\lambda)} d\lambda \wedge d\bar{\lambda}, \tag{3.5}$$

and

$$H_{mj} = \delta_{mj} + \pi \int\int_C d\lambda \wedge d\bar{\lambda} \int\int_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda'-\lambda)} e^{\nu(\lambda)-\nu(\lambda')} w_m(\lambda,\bar{\lambda}) v_j(\lambda',\bar{\lambda}'). \tag{3.6}$$

From equation (2.12), we find χ_{-1} satisfy

$$\chi_{-1} = -\frac{1}{2i}\sum_{m=1}^{N} \ell_m \vartheta_{mx} = -\frac{1}{2i}\sum_{m,j=1}^{N} \vartheta_m H_{mj}^{-1}\zeta_j = i\partial_x \ln\det H, \tag{3.7}$$

where $\vartheta_m(x,y,t)$ is defined by

$$\vartheta_m(x,y,t) = \int\int_C v_m(\lambda,\bar{\lambda}) e^{-\nu(\lambda)} d\lambda \wedge d\bar{\lambda}. \tag{3.8}$$

The solutions (2.9) of the HD equation are parametrized by the $2N$ arbitrary complex-valued functions w_m and v_m $(m = 1,\dots,N)$. Then the solutions (2.9) with functional parameters can be represented in the equivalent form

$$u = (\frac{-\partial_{xy}^2 \ln\det H}{1+\partial_x^2 \ln\det H})^{\frac{1}{2}}, \tag{3.9}$$

where the matrix H is defined by

$$H_{mj} = \delta_{mj} + \frac{1}{2}\partial_x^{-1}(\zeta_m \vartheta_j). \tag{3.10}$$

The reality conditions (2.13) and (2.14) restrict the functions w_m and v_m. For the reality conditions (2.13), we find that

$$\overline{w_m(\xi,\bar{\xi})} = w_m(-\bar{\xi},-\xi), \quad \overline{v_m(\xi,\bar{\xi})} = v_m(-\bar{\xi},-\xi). \tag{3.11}$$

For the reality conditions (2.14), we get that

$$\overline{w_m(\xi,\bar{\xi})} = \gamma_m v_m(\bar{\xi},\xi), \quad \overline{\gamma_m} = \gamma_m. \tag{3.12}$$

From conditions (3.11) we further obtain the relationship

$$\overline{\vartheta_m} = -\vartheta_m, \quad \overline{\zeta_m} = -\zeta_m. \tag{3.13}$$

We find that ζ_m and ϑ_m satisfy

$$\begin{aligned} &\varepsilon^2 Z_{xx} + Z_y = 0, \\ &Z_t + 4\varepsilon^3 Z_{xxx} = 0. \end{aligned} \tag{3.14}$$

4. Two types of specific exact solutions

In this section, under two constraints (2.13) and (2.14) on the kernel function K, we study two physically meaningful exact solutions of the two-dimensional HD equation-line solitons and rational solutions. These solutions are constructed by $\bar{\partial}$-dressing methods, and they have potential applications in integral system, fluid dynamics, and condensed matter physics. The results reveal new phenomena of the two-dimensional HD equation in high-dimensional nonlinear wave dynamics and provide a theoretical basis for understanding wave propagation and interaction in complex media.

4.1. Line solitons of the HD equation

Line solitons are localized wave solutions propagating in a specific direction in a two-dimensional nonlinear fluctuation equation with properties such as stability and absence of scattering (elastic collisions). For the HD equation (1.1), line solitons may describe bending waves or surface tension dominated fluctuations whose amplitudes propagate along a straight line without dispersion. In two-dimensional shallow water waves or rotating fluids, the HD equation may be similar to the Kadomtsev-Petviashvili equation, with line solitons corresponding to obliquely propagating shallow water soliton waves.

In this section, the line solitons of the HD equation correspond to the special choices of the functions w_m and v_m or ζ_m and ϑ_m in the solutions (3.9). Let us consider the following two cases. For the real-valued line solitons of equation (1.1) the appropriate choice is

$$\begin{aligned} w_m(\xi,\bar{\xi}) &= \delta(\xi - i\beta_m), \\ v_m(\lambda,\bar{\lambda}) &= \gamma_m \delta(\lambda - i\alpha_m), \end{aligned} \tag{4.1}$$

where α_m, β_m and γ_m are arbitrary real constants. Bringing equation (4.1) into equations (3.5) and (3.8) yields

$$\zeta_m(x,y,t) = -2ie^{\nu(i\beta_m)}, \tag{4.2}$$

and

$$\vartheta_m(x,y,t) = -2i\gamma_m e^{-\nu(i\alpha_m)}, \tag{4.3}$$

where

$$\nu(\lambda) = \frac{i}{\lambda}x + (\frac{\varepsilon}{\lambda})^2 y + 4i(\frac{\varepsilon}{\lambda})^3 t. \tag{4.4}$$

The simplest solutions (3.9) of corresponding to this choice of kernel K (3.1) is

$$u = [\frac{ab\Delta}{(1+\Delta)^2 - a^2\Delta}]^{\frac{1}{2}}, \tag{4.5}$$

where the $N \times N$ matrix H is

$$H_{mj} = \delta_{mj} + \frac{2\gamma_m}{\beta_m - \alpha_j} exp[\nu(i\beta_m) - \nu(i\alpha_j)], \tag{4.6}$$

with

$$\Delta = \frac{2\gamma_1}{\beta_1 - \alpha_1} e^{\triangle \nu}, \quad a = i(\frac{1}{\beta_1^2} - \frac{1}{\alpha_1^2}), \quad b = 2i\varepsilon^3(\frac{1}{\alpha_1^3} - \frac{1}{\beta_1^3}), \tag{4.7}$$

$$\triangle \nu = \nu(i\beta_1) - \nu(i\alpha_1) = (\frac{1}{\beta_1} - \frac{1}{\alpha_1})x - \varepsilon^2(\frac{1}{\beta_1^2} - \frac{1}{\alpha_1^2})y - 4\varepsilon^3(\frac{1}{\beta_1^3} - \frac{1}{\alpha_1^3})t. \tag{4.8}$$

For the complex-valued line solitons of equation (1.1) the correspond to the choice

$$\begin{aligned} w_m(\xi,\bar{\xi}) &= \delta(\xi - \rho_m), \\ v_m(\lambda,\bar{\lambda}) &= \gamma_m \delta(\lambda - \overline{\rho_m}). \end{aligned} \tag{4.9}$$

where γ_m are arbitrary real constants, and ρ_m are the complex numbers $\rho_m = \rho_{mR} + i\rho_{mI}$. From equations (3.5), (3.8) and (4.9), we get that

$$\zeta_m(x,y,t) = -2ie^{\nu(\rho_m)}, \tag{4.10}$$

and

$$\vartheta_m(x,y,t) = -2i\gamma_m e^{-\nu(\bar{\rho}_m)}. \tag{4.11}$$

Then the simplest solution (3.9) of corresponding to the choice of kernel K (3.1) has

$$u = \Big[\frac{\tilde{a}\tilde{b}\widetilde{\Delta}}{(1+\widetilde{\Delta})^2 + \tilde{a}^2\widetilde{\Delta}}\Big]^{\frac{1}{2}}, \tag{4.12}$$

where the $N \times N$ matrix H presents

$$H_{mj} = \delta_{mj} + \frac{2i\gamma_m}{\rho_m - \bar{\rho}_j} e^{[\nu(\rho_m) - \nu(\bar{\rho}_j)]}, \tag{4.13}$$

with

$$\widetilde{\Delta} = \frac{2i\rho_1}{\rho_1 - \overline{\rho}_1} e^{\triangle\tilde{\nu}}, \quad \tilde{a} = -\frac{4\rho_{1R}\rho_{1I}}{|\rho_1|^4}, \quad \tilde{b} = \frac{4i\varepsilon^2(3\rho_{1R}^2\rho_{1I} - \rho_{1I}^3)}{|\rho_1|^6}, \tag{4.14}$$

$$\triangle\,\tilde{\nu} = \nu(\rho_1) - \nu(\overline{\rho}_1) = i(\frac{1}{\rho_1} - \frac{1}{\overline{\rho}_1})x + \varepsilon^2(\frac{1}{\rho_1^2} - \frac{1}{\overline{\rho}_1^2})y - 4i\xi^3(\frac{1}{\rho_1^3} - \frac{1}{\overline{\rho}_1^3})t. \tag{4.15}$$

4.2. Rational solutions of the HD equation

Rational Solutions in the field of mathematical physics are a class of exact solutions with a special structure and physical meaning. Such solutions are of great value in areas such as nonlinear integrable systems, quantum field theory, fluid dynamics, etc.

In this section, we construct the rational solutions of the HD equation via the $\bar{\partial}$-dressing method. Depending on the various constraints on the kernel K, we have different choices for the kernel K of $\bar{\partial}$-problem (2.1). For the reality (2.13), we choose the kernel K of the $\bar{\partial}$-problem

$$K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) = \frac{\pi}{2}\sum_{m=1}^{N} X_m\delta(\xi - ip_m)\delta(\lambda - ip_m), \tag{4.16}$$

where $\delta(\lambda - ip_m)$ is the complex Dirac delta function, p_m $(m = 1,\ldots,N)$ is the set of isolated points distinct from the origin. $X_m(\xi,\lambda)$ are arbitrary functions with the property $\overline{X_m(\xi,\lambda)} = X_m(-\bar{\xi},-\bar{\lambda})$.

In what follows, using the dressing method, let us take the kernel K_0 (4.16) as an example and give the detailed computational steps for constructing the rational solutions of equation (1.1). Putting equation (4.16) into equation (2.1), we can obtain

$$\frac{\partial\chi}{\partial\bar{\lambda}} = -\frac{1}{2}\sum_{m=1}^{N}\chi(ip_m)e^{\nu(ip_m)-\nu(\lambda)}X_m(ip_m,\lambda)\delta(\lambda - ip_m), \tag{4.17}$$

where

$$\nu(\lambda) = \frac{i}{\lambda}x + (\frac{\varepsilon}{\lambda})^2 y + 4i(\frac{\varepsilon}{\lambda})^3 t. \tag{4.18}$$

Then we further get equation (2.10) has the following form

$$\chi(\lambda,\bar{\lambda}) = 1 + i\sum_{m=1}^{N}\frac{\chi(ip_m)X_m(ip_m,ip_m)}{\lambda - ip_m}, \tag{4.19}$$

where $\lambda \neq i\alpha_m$, $(m = 1,\ldots,N)$, when taking the $\lim \lambda = ip_m$, utilizing (2.10) and (4.17) one can get $\chi(ip_m)(m = 1,\ldots,N)$

$$\chi(ip_m) = 1 - \frac{1}{2}\int\int_C \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - ip_m}\sum_{j=1}^{N}\chi(ip_j)e^{\nu(ip_j)-\nu(\lambda)}X_j(ip_j,\lambda)\delta(\lambda - ip_j). \tag{4.20}$$

The quantities $\chi(ip_m)$ obey the system of equations

$$\begin{aligned}&\chi(ip_m)[1 + iX'(ip_m,ip_m) - iX(ip_m,ip_m)\nu'(ip_m)] + \sum_{j\neq m}^{N}\frac{\chi(ip_j)X_j(ip_j,ip_j)}{p_m - p_j}\\ =&0,\quad m = 1,\ldots,N.\end{aligned} \tag{4.21}$$

In order to verify equation (4.21), equation (4.20) is discussed below in two cases. When $j = m$, the corresponding term in (4.20) is

$$\begin{aligned}&\int\int_C \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - ip_m}\chi(\Lambda_m)e^{\nu(ip_m)-\nu(\lambda)}X_m(ip_m,\lambda)\delta(\lambda - ip_m)\\ =&2i\chi(ip_m)[X'(ip_m,ip_m) - X(ip_m,ip_m)\nu'(ip_m)],\end{aligned} \tag{4.22}$$

with the identity

$$\begin{aligned}\int\int_C \frac{d\lambda\wedge d\bar{\lambda}}{(\lambda-\lambda_0)^{n-1}}\varphi(\lambda)\delta(\lambda-\lambda_0) &= 2i\mathop{Res}_{\lambda=\lambda_0}\frac{\varphi(\lambda)}{(\lambda-\lambda_0)^n}\\ &= 2i\lim_{\lambda\to\lambda_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}(\varphi(\lambda)),\end{aligned} \tag{4.23}$$

where

$$\nu'(ip_m) = \frac{\partial\nu(\lambda)}{\partial\lambda}|_{\lambda=ip_m},\quad X'(ip_m) = \frac{\partial X(ip_m,\lambda)}{\partial\lambda}|_{\lambda=ip_m},\quad X_m = X(ip_m,ip_m). \tag{4.24}$$

When $j \neq m$, the terms in (4.20) are equal to

$$\int\int_C \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - ip_m}\sum_{j\neq m}^{N}\chi(ip_m)e^{\nu(ip_m)-\nu(\lambda)}X_m(ip_m,\lambda)\delta(\lambda - ip_m) = 2\sum_{j\neq m}^{N}\frac{\chi(ip_j)X_j(ip_j)}{\alpha_m - ip_j}. \tag{4.25}$$

Thus, equation (4.21) is derived from (4.22) and (4.25). Solving the system (4.21) with respect to $\chi(ip_m)$, one then finds

$$\chi(ip_m) = \sum_{j=1}^{N}A_{mj}^{-1}X_j^{-1}, \tag{4.26}$$

where the $N \times N$ matrix A is defined by

$$A_{mj} = \theta_m \delta_{mj} - \frac{1 - \delta_{mj}}{p_m - p_j}, \tag{4.27}$$

and

$$\theta_m = \frac{1}{p_m^2} x + \frac{2\varepsilon^2}{p_m^3} y - \frac{12\varepsilon^3}{p_m^4} t + \tau_m, \quad \tau_m = \frac{1 + iX'(ip_m)}{X_m} = \overline{\tau_m}. \tag{4.28}$$

Using equation (4.19), the coefficients χ_0 and χ_{-1} are generated to have the following expressions

$$\chi_0 = 1 - \sum_{m=1}^{N} \frac{X_m}{p_m} \chi(ip_m), \tag{4.29}$$

and

$$\chi_{-1} = i \sum_{m=1}^{N} X_m \chi(ip_m). \tag{4.30}$$

Then the solutions (2.9) can be represented in the equivalent form

$$u = \Big[\frac{\det(A^{-1}BA^{-1})}{1 - \det(A^{-1}CA^{-1})}\Big]^{\frac{1}{2}}, \tag{4.31}$$

where the $N \times N$ matrixs B and C are

$$B_{em} = -\frac{2\varepsilon^2}{p_m^3}, \quad C_{em} = \frac{1}{p_m^2}. \tag{4.32}$$

Corresponding to the kernel K_0 (4.16), the simplest solution of this type takes the form

$$u = \Big[\frac{2\varepsilon^2}{p_1(1 - p_1^2\theta_1^2)}\Big]^{\frac{1}{2}}, \tag{4.33}$$

with

$$\theta_1 = \frac{1}{p_1^2} x + \frac{2\varepsilon^2}{p_1^3} y - \frac{12\varepsilon^3}{p_1^4} t + \tau_1, \quad \tau_1 = \frac{1 + iX'(ip_1)}{X_1}. \tag{4.34}$$

Then we choose the more complex kernel K of the $\bar{\partial}$-problem as follows

$$K_0(\xi, \bar{\xi}; \lambda, \bar{\lambda}) = \frac{\pi}{2} \sum_{m=1}^{N} [X_m(\xi, \lambda)\delta(\xi - \rho_m)\delta(\lambda - \rho_m) + \overline{X_m}(\xi, \lambda)\delta(\xi + \overline{\rho_m})\delta(\lambda + \overline{\rho_m})], \tag{4.35}$$

which can be seen as a generalized form of kernel K_0 (4.16). For convenience, we introduce some symbols: Υ and Θ are the sets of complex constants Υ_m and Θ_m, respectively. Ω is the set of quantities Ω_m, $(m = 1, \ldots, N)$

$$\Upsilon := (\tau_1, \overline{\tau_1}, \ldots, \tau_N, \overline{\tau_N}), \tag{4.36}$$

$$\Theta := (\rho_1, -\overline{\rho_1}, \ldots, \rho_N, -\overline{\rho_N}), \tag{4.37}$$

$$\Omega := (X_1\chi(\rho_1), \overline{X_1}\chi(-\overline{\rho_1}), \ldots, X_N\chi(\rho_N), \overline{X_N}\chi(-\overline{\rho_N})). \tag{4.38}$$

Using equations (2.11), (2.12) and (4.35), the coefficients χ_0 and χ_{-1} are yielded

$$\chi_0 = 1 - i\sum_{m=1}^{2N} \frac{\Omega_m}{\Theta_m}, \tag{4.39}$$

and

$$\chi_{-1} = i\sum_{m=1}^{2N} \Omega_m. \tag{4.40}$$

The system of equations for Ω_m satisfies the form

$$\sum_{j=1}^{2N} A_{mj}\Omega_j = 1, \tag{4.41}$$

where the $2N \times 2N$ matrix A is defined by

$$A_{mj} = \theta_m \delta_{mj} - \frac{i(1-\delta_{mj})}{\Theta_m - \Theta_j}, \tag{4.42}$$

and

$$\theta_m = -\frac{1}{\Theta_m^2}x + \frac{2i\varepsilon^2}{\Theta_m^3}y - \frac{12\varepsilon^3}{\Theta_m^4}t + \Upsilon_m. \tag{4.43}$$

The solutions (2.9) of the HD equation are representable as

$$u = \Big[\frac{\det(A^{-1}BA^{-1})}{1+\det(A^{-1}CA^{-1})}\Big]^{\frac{1}{2}}, \tag{4.44}$$

where the $2N \times 2N$ matrixs B is defined by

$$B_{em} = \frac{2i\varepsilon^2}{\Theta_m^3}, \tag{4.45}$$

and the matrixs $C_{2N\times 2N}$ has

$$C_{em} = \frac{1}{\Theta_m^2}. \tag{4.46}$$

The simplest solution which correspond to one term in the kernel K_0 (4.35) has

$$u = \left[\frac{2\varepsilon^2(\rho_1^3 - \overline{\rho_1}^3)}{|\rho_1|^6\Lambda - |\rho_1|^2(\rho_1^2 + \overline{\rho_1}^2)}\right]^{\frac{1}{2}}, \tag{4.47}$$

where

$$\Lambda = |\theta_1|^2 - \frac{1}{4\rho_{1R}}, \quad \rho_1 = \rho_{1R} + i\rho_{1I}, \tag{4.48}$$

and

$$\theta_1 = -\frac{1}{\Theta_1^2}x + \frac{2i\varepsilon^2}{\Theta_1^3}y - \frac{12\varepsilon^3}{\Theta_1^4}t + \Upsilon_1. \tag{4.49}$$

The reality condition (2.14) of u corresponds to the kernel K of the $\bar{\partial}$-problem in the following form

$$K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) = \frac{\pi}{2}\sum_{m=1}^{N} X_m(\xi,\lambda)\delta(\xi-\alpha_m)\delta(\lambda-\alpha_m), \tag{4.50}$$

where X_m are arbitrary functions with $\overline{X_m} = X_m$, and $\overline{\alpha_m} = \alpha_m$. With the use of equations (2.11), (2.12) and (4.50), the coefficients χ_0 and χ_{-1} present

$$\chi_0 = 1 - i\sum_{m=1}^{N}\frac{X_m}{\alpha_m}\chi(\alpha_m), \tag{4.51}$$

and

$$\chi_{-1} = i\sum_{m=1}^{N} X_m\chi(\alpha_m). \tag{4.52}$$

Then the quantities $\chi(\alpha_m)$ satisfy the system of equations

$$\sum_{j=1}^{N} A_{mj}\chi(\alpha_j)X_j = 1, \tag{4.53}$$

where the $N\times N$ matrix A is defined by

$$A_{mj} = \theta_m\delta_{mj} - \frac{i(1-\delta_{mj})}{\alpha_m-\alpha_j}, \tag{4.54}$$

and

$$\theta_m = -\frac{1}{\alpha_m^2}x + \frac{2i\varepsilon^2}{\alpha_m^3}y - \frac{12\varepsilon^3}{\alpha_m^4}t + \tau_m. \tag{4.55}$$

The solutions (2.9) can be constructed by equations (4.51)-(4.53)

$$u = \Big[\frac{i\det(A^{-1}BA^{-1})}{1+\det(A^{-1}CA^{-1})}\Big]^{\frac{1}{2}}, \tag{4.56}$$

where the matrixs $B_{N\times N}$ and $C_{N\times N}$ are

$$B_{em} = \frac{2\varepsilon^2}{\alpha_m^3}, \quad C_{em} = \frac{1}{\alpha_m^2}. \tag{4.57}$$

The simplest solution of this type is of the form

$$u = \Big[\frac{2i\varepsilon^2}{\alpha_1(1+\alpha_1^2\theta_1^2)}\Big]^{\frac{1}{2}}, \tag{4.58}$$

with

$$\theta_1 = -\frac{1}{\alpha_1^2}x + \frac{2i\varepsilon^2}{\alpha_1^3}y - \frac{12\varepsilon^3}{\alpha_1^4}t + \tau_1. \tag{4.59}$$

Then we choose the more complex kernel K of the $\bar{\partial}$-problem

$$K_0(\xi,\bar{\xi};\lambda,\bar{\lambda}) = \frac{\pi}{2}\sum_{m=1}^{N}[X_m(\xi,\lambda)\delta(\xi-\rho_m)\delta(\lambda-\rho_m) + \overline{X_m}(\xi,\lambda)\delta(\xi-\overline{\rho_m})\delta(\lambda-\overline{\rho_m})], \tag{4.60}$$

where K_0 satisfy the reality condition (2.14) of u. For convenience, several symbols are introduced

$$\tilde{\Upsilon} := (\tau_1, \overline{\tau_1}, \ldots, \tau_N, \overline{\tau_N}), \tag{4.61}$$

$$\tilde{\Theta} := (\rho_1, \overline{\rho_1}, \ldots, \rho_N, \overline{\rho_N}), \tag{4.62}$$

$$\tilde{\Omega} := (X_1\chi(\rho_1), \overline{X_1}\chi(\overline{\rho_1}), \ldots, X_N\chi(\rho_N), \overline{X_N}\chi(\overline{\rho_N})), \tag{4.63}$$

where $\tilde{\Upsilon}$ and $\tilde{\Theta}$ are the sets of complex constants $\tilde{\Upsilon}_m$ and $\tilde{\Theta}_m$, respectively. $\tilde{\Omega}$ is the set of quantities $\tilde{\Omega}_m$, $(m = 1, \ldots, N)$. Combining equations (2.11), (2.12) and (4.60), the coefficients χ_0 and χ_{-1} of the series expansion of χ near $\lambda = 0$ and $\lambda = \infty$ are obtained as follows

$$\chi_0 = 1 - i\sum_{m=1}^{2N} \frac{\tilde{\Omega}_m}{\tilde{\Theta}_m}, \tag{4.64}$$

and

$$\chi_{-1} = i\sum_{m=1}^{2N} \tilde{\Omega}_m. \tag{4.65}$$

We further find the system of equations for $\tilde{\Omega}_m$ satisfies

$$\sum_{j=1}^{2N} A_{mj}\tilde{\Omega}_j = 1, \tag{4.66}$$

where the matrix $A_{2N\times 2N}$ is defined by

$$A_{mj} = \theta_m\delta_{mj} - i\frac{(1-\delta_{mj})}{\tilde{\Theta}_m - \tilde{\Theta}_j}, \tag{4.67}$$

and

$$\theta_m = -\frac{1}{\tilde{\Theta}_m^2}x + \frac{2i\varepsilon^2}{\tilde{\Theta}_m^3}y - \frac{12\varepsilon^3}{\tilde{\Theta}_m^4}t + \tilde{\Upsilon}_m. \tag{4.68}$$

By organizing equations (4.64)-(4.68), the solutions (2.9) of the HD equation are constructed

$$u = \Big[\frac{\det(A^{-1}BA^{-1})}{1+\det(A^{-1}CA^{-1})}\Big]^{\frac{1}{2}}, \tag{4.69}$$

where the matrixs $B_{2N\times 2N}$ is defined by

$$B_{em} = \frac{2i\varepsilon^2}{\tilde{\Theta}_m^3}, \tag{4.70}$$

and the matrixs $C_{2N\times 2N}$ has

$$C_{em} = \frac{1}{\tilde{\Theta}_m^2}. \tag{4.71}$$

The simplest solution which correspond to one term in the kernel K_0 (4.60) has

$$u = \left[\frac{2\varepsilon^2(\rho_1^3 + \overline{\rho_1}^3)}{|\rho_1|^6\tilde{\Lambda} - i|\rho_1|^2(\rho_1^2 + \overline{\rho_1}^2)}\right]^{\frac{1}{2}}, \tag{4.72}$$

where

$$\tilde{\Lambda} = |\theta_1|^2 + \frac{1}{4\lambda_{1I}}, \quad \theta_1 = -\frac{1}{\tilde{\Theta}_1^2}x + \frac{2i\varepsilon^2}{\tilde{\Theta}_1^3}y - \frac{12\varepsilon^3}{\tilde{\Theta}_1^4}t + \tilde{\Upsilon}_1, \tag{4.73}$$

and ρ_1 is the complex number $\rho = \rho_{1R} + i\rho_{1I}$.

5. Conclusions

The $\bar{\partial}$-method introduced in this paper constructs a variety of solutions to the HD equation (1.1) under two constraints (2.13) and (2.14) on the kernel K, including the solutions with functional parameters, line solitons and rational solutions. In this paper, the $\bar{\partial}$-dressing method corresponds to bare operators of linear auxiliary problems (1.2) with constant asymptotic value of u at infinity, i.e.$u|_{x^2+y^2\to\infty} \to -\varepsilon \neq 0$.

In the future, we will consider the construction of other types of exact solutions with constant asymptotic values at infinity of two-dimensional HD equation via the $\bar{\partial}$-dressing method, such as periodic solutions, multiple-pole solutions solutions and so on. At the same time, we will research different exact solutions of other types of nonlinear equations [11, 19, 24, 27, 32, 34, 35] especially high-dimensional equations, which play an important role in many areas of mathematical physics. The methods and results presented in this paper may provide a good inspiration for dealing with similar high-dimensional nonlinear equations.

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