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MULTIPLE SOLUTIONS FOR P-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS WITH ψ-CAPUTO DERIVATIVE AND IMPULSIVE EFFECTS*

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Abstract In this study, we consider the multiplicity of solutions for a new class of p-Laplacian fractional differential equations involving ψ-Caputo fractional derivative with instantaneous and non-instantaneous impulses. Since the p-Laplacian operator, ψ-Caputo fractional derivative and impulsive effects exist at the same time, it is difficult to establish the variational structure of the considered problem. By virtue of critical point theorems and variational methods, we give some new criteria to guarantee that the problem has at least two weak solutions and infinitely many weak solutions. Some recent results are improved and extended.

Keywords Fractional differential equations, ψ-Caputo fractional derivative, instantaneous impulses, non-instantaneous impulses, variational methods.

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1. Introduction

Fractional calculus is an active research field focusing on derivatives and integrals of arbitrary order. To date, it has broadly applied in many areas of science and engineering, such as viscoelastic mechanics, control theory, epidemiology, neural network, etc. For the recent developments and applications about it, the readers can see the monographs [15, 25, 30] and the papers [2, 9, 29, 36].

In recent decades, there are a wide variety of definitions of fractional differential and integral operators in the literature [13, 15, 31], such as the classical Caputo, Riemann-Liouville, Hadamard and Erdelyi-Kober versions. Various forms of definitions are all widely employed in different types of fractional initial and boundary value problems. To overcome the inconvenience caused by the vast number of definitions, the ψ-Caputo fractional derivative is presented in [1, 12, 15, 24], which unifies

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different forms of definitions of fractional derivatives into a whole expression by depending upon a kernel function. This is a more general fractional differential operator and the classical fractional derivatives can be obtained by choosing suitable kernels. Moreover, it can greatly improved the accuracy of the mathematical models established by practical problems. For instance, Almeida [1] considered a population growth model and obtained a better accuracy on the model by selecting different kernels. Recently, the study of fractional differential equations (FDEs) with ψ-Caputo fractional derivative has attracted the attention of many scholars. Some methods [3, 8, 14, 21], such as fixed point theorems and variational approach, have been used to investigate the existence results for the ψ-Caputo-type FDEs. Especially, in [14], Khaliq et al. first used variational methods to study the following ψ-Caputo fractional boundary value problem and obtain the existence of at least one weak solution.

$$\begin{cases} {}^C D_{T^-}^{\alpha,\psi(t)} \left(\psi'(t) {}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right) = \nabla G(t, x(t)), \quad a.e.\ t \in [0, T], \\ x(0) = x(T) = 0, \end{cases}$$

where ${}^C D_{0^+}^{\alpha,\psi(t)}$, ${}^C D_{T^-}^{\alpha,\psi(t)}$ denote the left and right ψ-Caputo fractional derivatives of order $0 < \alpha \leq 1$ respectively, $\nabla G(t, x)$ is gradient of a function $G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$, $\psi(t) \in C^1[0, T]$ is an increasing function with $\psi'(t) \neq 0$ for $t \in [0, T]$.

On the other hand, the boundary value problems for FDEs with impulsive effects have been studied extensively in recent years. Such equations appear in a number of mathematical models in pharmacology, automatic control systems, population dynamics, and so forth [16, 27]. The existence and multiplicity results of solutions of impulsive problems for FDEs are enriched by some classical tools [7,17,34,35,39,40], such as fixed point theorems, coincidence degree theory and variational methods. Recently, inspired by Khaliq et al. [14], some authors tried to use variational approach to investigate FDEs with both impulsive effects and ψ-Caputo fractional derivative, and obtained some existence and multiplicity results of solutions. In [18], Li et al. studied the following FDEs involving ψ-Caputo fractional derivative with instantaneous and non-instantaneous impulses, and obtained at least three distinct classical solutions via three critical points theorem of Bonanno et al. [6].

$$\begin{cases} \quad {}^C D_{T^-}^{\alpha,\psi(t)} \left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right) \\ = \lambda f_i(t, x(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, 1, ..., m, \\ \Delta \left({}^C D_{T^-}^{\alpha,\psi(t)} \left(I_{0^+}^{1-\alpha,\psi(t)} x\right)\right)(t_i) = I_i(x(t_i)), \quad i = 1, 2, ..., m, \\ \quad {}^C D_{T^-}^{\alpha,\psi(t)} \left(I_{0^+}^{1-\alpha,\psi(t)} x\right)(t) \\ = {}^C D_{T^-}^{\alpha,\psi(t)} \left(I_{0^+}^{1-\alpha,\psi(t)} x\right)(t_i^+), \quad t \in (t_i, s_i],\ i = 1, 2, ..., m, \\ \quad {}^C D_{T^-}^{\alpha,\psi(t)} \left(I_{0^+}^{1-\alpha,\psi(t)} x\right)(s_i^-) \\ = {}^C D_{T^-}^{\alpha,\psi(t)} \left(I_{0^+}^{1-\alpha,\psi(t)} x\right)(s_i^+), \quad i = 1, 2, ..., m, \\ x(0) = x(T) = 0, \end{cases} \tag{1.1}$$

where ${}^C D_{0^+}^{\alpha,\psi(t)}$, ${}^C D_{T^-}^{\alpha,\psi(t)}$ denote the left and right ψ-Caputo fractional derivatives of order $0 < \alpha \leq 1$ respectively, $I_{0^+}^{1-\alpha,\psi(t)}$ denotes the left ψ-Riemann-Liouville fractional integral of order $1-\alpha$, $\lambda>0$, the function $\psi(t)\in C^1[0,T]$ increases with $\psi'(t)\neq 0$ for $t\in[0,T]$.

In [19], Li et al. studied a class of (p,q)-Laplacian-type FDEs involving ψ-Caputo fractional derivative and ψ-Riemann-Liouville fractional integral with instantaneous impulses and obtained at least one nontrivial ground state solution by means of the Nehari manifold method. In [20], Li et al. considered a class of fractional differential system involving ψ-Caputo fractional derivative with instantaneous impulses and obtained infinitely many classical solutions using critical point theory.

To our knowledge, there is very limited work studying FDEs with impulsive effects and the ψ-Caputo fractional differential operators via variational approach. No published literature has studied the multiplicity results of solutions for p-Laplacian FDEs involving the ψ-Caputo fractional derivative with instantaneous and non-instantaneous impulses. Therefore, the objective of this article is to use variational methods to investigate the following impulsive boundary value problem:

$$\begin{cases} \quad {}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right)\right)+|x(t)|^{p-2}x(t) \\ =\lambda f_i(t,x(t)), \quad t\in(s_i,t_{i+1}],\ i=0,1,...,m, \\ \Delta\left(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)\right)(t_i)=I_i(x(t_i)), \quad i=1,2,...,m, \\ \quad I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t) \\ =I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t_i^+), \quad t\in(t_i,s_i],\ i=1,2,...,m, \\ \quad I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(s_i^-) \\ =I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(s_i^+), \quad i=1,2,...,m, \\ x(0)=x(T)=0, \end{cases} \tag{1.2}$$

where $\Phi_p(x)=|x|^{p-2}x$, $p>1$, $\lambda>0$, ${}^C D_{0^+}^{\alpha,\psi(t)}$, ${}^C D_{T^-}^{\alpha,\psi(t)}$ denote the left and right ψ-Caputo fractional derivatives of order $\alpha\in(\frac{1}{p},1]$ respectively, $I_{T^-}^{1-\alpha,\psi(t)}$ denotes the right ψ-Riemann-Liouville fractional integral of order $1-\alpha$, $0=s_0<t_1<s_1<t_2<s_2<...<t_m<s_m<t_{m+1}=T$, the function $\psi(t)\in C^1[0,T]$ increases with $\psi'(t)\neq 0$ for $t\in[0,T]$, $f_i\in C((s_i,t_{i+1}]\times\mathbb{R},\mathbb{R})$, $I_i\in C(\mathbb{R},\mathbb{R})$, the instantaneous impulses occur at the points t_i and the non-instantaneous impulses continue during the intervals $(t_i,s_i]$.

$$\begin{aligned} &\quad \Delta\left(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)\right)(t_i) \\ &=I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t_i^+)-I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t_i^-), \\ &I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t^\pm)=\lim_{t\to t^\pm} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t), \\ &I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(s^\pm)=\lim_{t\to s^\pm} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x\right)(t). \end{aligned}$$

Two major contributions of the present article are as below: Firstly, we propose a new model on ψ-Caputo-type FDEs with two types of impulses, which is different from the problem (1.1). We also establish its variational structure and obtain some new results concerning the multiplicity of solutions by depending on the parameter λ. Secondly, since the classical fractional differential and integral operators can be obtained by selecting different kernel functions, some research results in existing literature with the classical fractional operators are generalized and supplemented. If the kernel function $\psi(t) = t$, the ψ-Caputo fractional derivative and ψ-Riemann-Liouville fractional integral reduce to the classical Caputo fractional derivative and Riemann-Liouville fractional integral respectively. At the same time, the problem (1.2) becomes a class of p-Laplacian FDEs with instantaneous and non-instantaneous impulses. If $\psi(t) = t$ and $p = 2$, the problem (1.2) becomes a class of FDEs involving instantaneous and non-instantaneous impulses. If $\psi(t) = t$, $p = 2$ and $\alpha = 1$, the problem (1.2) becomes a class of second order instantaneous and non-instantaneous impulsive differential equations. In addition, if $t_i = s_i$, $i = 1, 2, ..., m$, every interval $(t_i, s_i]$ reduces to a point. This means the non-instantaneous impulses are not generated. Meanwhile, combining with $\psi(t) = t$, the problem (1.2) becomes a class of p-Laplacian FDEs involving only instantaneous impulses. Based on the above analysis, these studies, such as [10, 26, 32, 33, 38], are special cases of this paper, that is, the problem we investigate is more general.

2. Preliminaries

In this section, some definitions of fractional integrals and differentials are first introduced. Let $-\infty < \zeta < \eta < +\infty$, $t \in [\zeta, \eta]$. $\psi(t) \in C^1[\zeta, \eta]$ is an increasing function with $\psi'(t) \neq 0$. The left and right ψ-Riemann-Liouville fractional integrals of a function $x(t)$ (see [1, 15]) are defined as follows:

$$\begin{aligned} I_{\zeta^+}^{\alpha,\psi(t)} x(t) &= \frac{1}{\Gamma(\alpha)} \int_\zeta^t x(s)(\psi(t) - \psi(s))^{\alpha-1} \psi'(s) ds, \quad \alpha > 0, \\ I_{\eta^-}^{\alpha,\psi(t)} x(t) &= \frac{1}{\Gamma(\alpha)} \int_t^\eta x(s)(\psi(s) - \psi(t))^{\alpha-1} \psi'(s) ds, \quad \alpha > 0, \end{aligned} \tag{2.1}$$

where $x(t)$ is integrable on $[\zeta, \eta]$.

The left and right ψ-Riemann-Liouville fractional derivatives and ψ-Caputo fractional derivatives of a function $x(t)$ (see [1, 15]) are defined as follows:

$$\begin{aligned} D_{\zeta^+}^{\alpha,\psi(t)} x(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) I_{\zeta^+}^{1-\alpha,\psi(t)} x(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) \int_\zeta^t x(s)(\psi(t) - \psi(s))^{-\alpha} \psi'(s) ds, \quad 0 < \alpha < 1, \\ D_{\eta^-}^{\alpha,\psi(t)} x(t) &= \left(\frac{-1}{\psi'(t)} \frac{d}{dt} \right) I_{\eta^-}^{1-\alpha,\psi(t)} x(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{-1}{\psi'(t)} \frac{d}{dt} \right) \int_t^\eta x(s)(\psi(s) - \psi(t))^{-\alpha} \psi'(s) ds, \quad 0 < \alpha < 1, \end{aligned} \tag{2.2}$$

$$\begin{aligned} {}^C D_{\zeta^+}^{\alpha,\psi(t)} x(t) &= I_{\zeta^+}^{1-\alpha,\psi(t)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) x(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_\zeta^t (\psi(t) - \psi(s))^{-\alpha} x'(s) ds, \quad 0 < \alpha < 1, \end{aligned} \tag{2.3}$$

$$ {}^C D_{\eta^-}^{\alpha,\psi(t)} x(t) = I_{\eta^-}^{1-\alpha,\psi(t)} \left(\frac{-1}{\psi'(t)} \frac{d}{dt} \right) x(t)$$

$$= \frac{-1}{\Gamma(1-\alpha)} \int_t^\eta (\psi(s) - \psi(t))^{-\alpha} x'(s) ds, \quad 0 < \alpha < 1,$$

where $x(t) \in C^1[\zeta, \eta]$.

On the other hand, if $x(t) \in C^n[\zeta, \eta]$, the following relationships hold (see [1]):

$$\begin{aligned}
&{}^C D_{\zeta^+}^{\alpha,\psi(t)} x(t) \\
=&D_{\zeta^+}^{\alpha,\psi(t)} \left(x(t) - \sum_{j=0}^{n-1} \frac{1}{j!} (\psi(t) - \psi(\zeta))^j \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^j x(\zeta) \right), \quad \alpha > 0, \qquad (2.4) \\
&{}^C D_{\eta^-}^{\alpha,\psi(t)} x(t) \\
=&D_{\eta^-}^{\alpha,\psi(t)} \left(x(t) - \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} (\psi(\eta) - \psi(t))^j \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^j x(\eta) \right), \ \alpha > 0, \qquad (2.5)
\end{aligned}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$; $n = \alpha$ for $\alpha \in \mathbb{N}$.

In particular, using the definitions of (2.4) and (2.5), and boundary condition $x(0) = x(T) = 0$, the following equalities

$$\begin{aligned}
{}^C D_{0^+}^{\alpha,\psi(t)} x(t) &= D_{0^+}^{\alpha,\psi(t)} x(t), \\
{}^C D_{T^-}^{\alpha,\psi(t)} x(t) &= D_{T^-}^{\alpha,\psi(t)} x(t)
\end{aligned} \qquad (2.6)$$

hold for every $0 < \alpha < 1$ (see [18, Lemma 3]).

Lemma 2.1 ([1, Theorem 4])**.** *Let $\alpha > 0$, $x(t) \in C^n[\zeta, \eta]$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$, then*

$$I_{\zeta^+}^{\alpha,\psi(t)} {}^C D_{\zeta^+}^{\alpha,\psi(t)} x(t) = x(t) - \sum_{j=0}^{n-1} \frac{\left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^j x(\zeta)}{j!} (\psi(t) - \psi(\zeta))^j,$$

$$I_{\eta^-}^{\alpha,\psi(t)} {}^C D_{\eta^-}^{\alpha,\psi(t)} x(t) = x(t) - \sum_{j=0}^{n-1} (-1)^j \frac{\left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^j x(\eta)}{j!} (\psi(\eta) - \psi(t))^j.$$

Especially, if $0 < \alpha < 1$, then

$$\begin{aligned}
I_{\zeta^+}^{\alpha,\psi(t)} {}^C D_{\zeta^+}^{\alpha,\psi(t)} x(t) &= x(t) - x(\zeta), \\
I_{\eta^-}^{\alpha,\psi(t)} {}^C D_{\eta^-}^{\alpha,\psi(t)} x(t) &= x(t) - x(\eta).
\end{aligned}$$

From Lemma 2.1 and $x(0) = x(T) = 0$, we obtain

$$\begin{aligned}
I_{0^+}^{\alpha,\psi(t)} {}^C D_{0^+}^{\alpha,\psi(t)} x(t) &= x(t), \\
I_{T^-}^{\alpha,\psi(t)} {}^C D_{T^-}^{\alpha,\psi(t)} x(t) &= x(t).
\end{aligned} \qquad (2.7)$$

As we all know, for $1 < \nu < +\infty$ and any fixed $t \in [0, T]$,

$$\|x\|_\infty = \max_{t \in [0,T]} |x(t)|, \ \|x\|_{L^\nu([0,t])} = \left(\int_0^t |x(s)|^\nu ds \right)^{\frac{1}{\nu}},$$

and

$$\|x\|_{L^\nu} = \left(\int_0^T |x(t)|^\nu dt\right)^{\frac{1}{\nu}}.$$

Lemma 2.2 ([14, Lemma 2.1]). *Let $0 < \alpha \leq 1$, $1 \leq p < +\infty$. For $x \in L^p([0,T],\mathbb{R})$, we have*

$$\|I_\delta^{\alpha,\psi(t)} x\|_{L^p([0,t])} \leq \frac{(\psi(t))^\alpha \max_{0\leq t\leq T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\|x\|_{L^p([0,t])}, \quad \delta \in [0,t],\ t\in[0,T].$$

Definition 2.1. Let $\frac{1}{p} < \alpha \leq 1$, $1 < p < +\infty$. The ψ-Caputo fractional derivative space $E_0^{\alpha,\psi,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with weighted norm

$$\|x\|_{\alpha,\psi,p} = \left(\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt + \int_0^T \psi'(t)|x(t)|^p dt\right)^{\frac{1}{p}}.$$

Obviously, the space $E_0^{\alpha,\psi,p}$ is separable and reflexive (see [14]). We also define the following norm

$$\|x\| = \left(\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \psi'(t)|x(t)|^p dt\right)^{\frac{1}{p}}.$$

Lemma 2.3. *For $x \in E_0^{\alpha,\psi,p}$, the norm $\|\cdot\|_{\alpha,\psi,p}$ is equivalent to the norm $\|\cdot\|$, namely, there exist two positive constants ϱ_1 and ϱ_2 such that*

$$\varrho_1\|x\|_{\alpha,\psi,p} \leq \|x\| \leq \varrho_2\|x\|_{\alpha,\psi,p}.$$

Proof. By choosing $\varrho_2 = 1$, we have $\|x\| \leq \varrho_2\|x\|_{\alpha,\psi,p}$. On the other hand, from Lemma 2.2 and (2.7), one has

$$\begin{aligned}
&\int_0^T \psi'(t)|x(t)|^p dt \\
&= \int_0^T \psi'(t)|I_{0^+}^{\alpha,\psi(t)}{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt \\
&\leq \max_{0\leq t\leq T}\{\psi'(t)\}\left(\frac{(\psi(T))^\alpha \max_{0\leq t\leq T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\right)^p \int_0^T |{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt \\
&\leq \left(\frac{(\psi(T))^\alpha \max_{0\leq t\leq T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\right)^p \frac{\max_{0\leq t\leq T}\{\psi'(t)\}}{\min_{0\leq t\leq T}\{\psi'(t)\}} \int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt \\
&\leq \left(\frac{(\psi(T))^\alpha \max_{0\leq t\leq T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\right)^p \frac{\max_{0\leq t\leq T}\{\psi'(t)\}}{\min_{0\leq t\leq T}\{\psi'(t)\}}\|x\|^p.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x\|_{\alpha,\psi,p}^p &= \int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt + \int_0^T \psi'(t)|x(t)|^p dt \\
&\leq \left(1 + \left(\frac{(\psi(T))^\alpha \max_{0\leq t\leq T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\right)^p \frac{\max_{0\leq t\leq T}\{\psi'(t)\}}{\min_{0\leq t\leq T}\{\psi'(t)\}}\right)\|x\|^p.
\end{aligned}$$

So take

$$\varrho_1 = \left(1 + \left(\frac{(\psi(T))^{\alpha} \max_{0\le t\le T}\{\psi'(t)\}}{\Gamma(\alpha+1)}\right)^{p} \frac{\max_{0\le t\le T}\{\psi'(t)\}}{\min_{0\le t\le T}\{\psi'(t)\}}\right)^{-\frac{1}{p}},$$

we have $\varrho_1\|x\|_{\alpha,\psi,p} \le \|x\|$. □

Lemma 2.4. *A function $x \in E_0^{\alpha,\psi,p}$ is a weak solution of the problem (1.2) if*

$$\begin{aligned}
&\int_0^T \psi'(t)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right) {}^C D_{0^+}^{\alpha,\psi(t)} y(t)dt \\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} |x(t)|^{p-2}x(t)\psi'(t)y(t)dt + \sum_{i=1}^{m} I_i(x(t_i))y(t_i) \\
=&\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} f_i(t,x(t))\psi'(t)y(t)dt
\end{aligned} \tag{2.8}$$

holds for any $y \in E_0^{\alpha,\psi,p}$.

Proof. For any $x,\ y \in E_0^{\alpha,\psi,p}$, by (2.3), we have

$$\begin{aligned}
&\int_0^T \psi'(t)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right) {}^C D_{0^+}^{\alpha,\psi(t)} y(t)dt \\
=&\frac{1}{\Gamma(1-\alpha)}\int_0^T\int_0^t \psi'(t)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right)(\psi(t)-\psi(s))^{-\alpha}y'(s)dsdt \\
=&\frac{1}{\Gamma(1-\alpha)}\int_0^T\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y'(t)dt \\
=&\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y'(t)dt \\
&+\frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{m}\int_{t_i}^{s_i}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y'(t)dt.
\end{aligned} \tag{2.9}$$

Taking into account (2.1), (2.2) and (2.6), one has

$$\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y'(t)dt \\
=&\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y(t)\Big|_{s_i^+}^{t_{i+1}^-} \\
&-\frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\frac{d}{dt}\Bigg(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)} x(s)\right) \\
&\times(\psi(s)-\psi(t))^{-\alpha}ds\Bigg)y(t)dt \\
=&\sum_{i=0}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)} x(t)\right)y(t)\Big|_{s_i^+}^{t_{i+1}^-} + \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\frac{-1}{\Gamma(1-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)
\end{aligned}$$

$$
\begin{aligned}
&\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)}x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)\psi'(t)y(t)dt\\
=&\sum_{i=0}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{s_i^+}^{t_{i+1}^-}\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt\\
=&\sum_{i=0}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{s_i^+}^{t_{i+1}^-}\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} {}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt,
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{m}\int_{t_i}^{s_i}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)}x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y'(t)dt\\
=&\frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{m}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)}x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y(t)\Big|_{t_i^+}^{s_i^-}\\
&-\frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{m}\int_{t_i}^{s_i}\frac{d}{dt}\left(\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)}x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y(t)dt\\
=&\sum_{i=1}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{t_i^+}^{s_i^-}\\
&-\sum_{i=1}^{m}\int_{t_i}^{s_i}\frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_t^T \psi'(s)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(s)}x(s)\right)(\psi(s)-\psi(t))^{-\alpha}ds\right)y(t)dt\\
=&\sum_{i=1}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{t_i^+}^{s_i^-}\\
&-\sum_{i=1}^{m}\int_{t_i}^{s_i}\frac{d}{dt}\left(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)y(t)dt.
\end{aligned}
\tag{2.11}
$$

It follows from (2.9), (2.10) and (2.11) that

$$
\begin{aligned}
&\int_0^T \psi'(t)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right){}^C D_{0^+}^{\alpha,\psi(t)}y(t)dt\\
=&\sum_{i=0}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{s_i^+}^{t_{i+1}^-}\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} {}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt\\
&+\sum_{i=1}^{m} I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)y(t)\Big|_{t_i^+}^{s_i^-}\\
&-\sum_{i=1}^{m}\int_{t_i}^{s_i}\frac{d}{dt}\left(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)y(t)dt
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{i=1}^{m}\Big(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t_i^-)\right)y(t_i^-)\\
&-I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t_i^+)\right)y(t_i^+)\Big)\\
&+\sum_{i=1}^{m}\Big(I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(s_i^-)\right)y(s_i^-)\\
&-I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(s_i^+)\right)y(s_i^+)\Big)\\
&-I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(0)\right)y(0)+I_{T^-}^{1-\alpha,\psi(t)}\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(T)\right)y(T)\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}{}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt\\
=&-\sum_{i=1}^{m}I_i(x(t_i))y(t_i)+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}{}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt.
\end{aligned}
\tag{2.12}
$$

On the other hand,

$$
\begin{aligned}
&\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}{}^C D_{T^-}^{\alpha,\psi(t)}\left(\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right)\right)\psi'(t)y(t)dt\\
=&\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}f_i(t,x(t))\psi'(t)y(t)dt-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}|x(t)|^{p-2}x(t)\psi'(t)y(t)dt.
\end{aligned}
\tag{2.13}
$$

Hence, according to (2.12) and (2.13), we obtain (2.8) holds. □

Define the energy functional $I_\lambda : E_0^{\alpha,\psi,p} \to \mathbb{R}$ by

$$I_\lambda(x) := \Upsilon_1(x) - \lambda\Upsilon_2(x), \tag{2.14}$$

for any $x \in E_0^{\alpha,\psi,p}$, where

$$\Upsilon_1(x) = \frac{1}{p}\|x\|^p + \sum_{i=1}^{m}\int_0^{x(t_i)} I_i(s)ds, \quad \Upsilon_2(x) = \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,x(t))\psi'(t)dt$$

and $F_i(t,x) = \int_0^x f_i(t,s)ds$. Standard arguments show that $I_\lambda \in C^1(E_0^{\alpha,\psi,p},\mathbb{R})$ and

$$
\begin{aligned}
\langle I_\lambda'(x),y\rangle =& \int_0^T \psi'(t)\Phi_p\left({}^C D_{0^+}^{\alpha,\psi(t)}x(t)\right){}^C D_{0^+}^{\alpha,\psi(t)}y(t)dt\\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}|x(t)|^{p-2}x(t)\psi'(t)y(t)dt+\sum_{i=1}^{m}I_i(x(t_i))y(t_i)\\
&-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}f_i(t,x(t))\psi'(t)y(t)dt.
\end{aligned}
\tag{2.15}
$$

Apparently, the weak solutions of the problem (1.2) correspond to the critical points of I_λ.

Lemma 2.5. *If $\frac{1}{p} < \alpha \le 1$, $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\|x\|_\infty \le \ell \|x\|$, where $\ell = \frac{(\psi(T)-\psi(0))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}}$.*

Proof. According to [21, Lemma 4], it is easy to obtain the conclusion. □

Definition 2.2 ([23, Palais-Smale condition])**.** Let $E_0^{\alpha,\psi,p}$ be a real Banach space. If any sequence $\{x_n\} \subset E_0^{\alpha,\psi,p}$ with

$$\{I_\lambda(x_n)\} \text{ being bounded and } \lim_{n\to\infty} I'_\lambda(x_n) = 0$$

possesses a convergent subsequence, then I_λ is called satisfying the Palais-Smale condition.

Lemma 2.6 ([14])**.** *Let $\frac{1}{p} < \alpha \le 1$ and $1 < p < +\infty$. If any sequence $\{x_n\}$ converges weakly to x in $E_0^{\alpha,\psi,p}$, then $x_n \to x$ in $C([0,T])$, i.e., $\|x_n - x\|_\infty \to 0$ as $n \to \infty$.*

Theorem 2.1 ([4, Theorem 3.2])**.** *Suppose that $E_0^{\alpha,\psi,p}$ is a real Banach space. Υ_1, $\Upsilon_2 : E_0^{\alpha,\psi,p} \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals, where Υ_1 is bounded from below and $\Upsilon_1(0) = \Upsilon_2(0) = 0$. Fix $r > 0$ such that $\sup_{x\in\Upsilon_1^{-1}(-\infty,r)} \Upsilon_2(x) < +\infty$ and suppose also that, the functional $I_\lambda = \Upsilon_1 - \lambda\Upsilon_2$ satisfies the Palais-Smale condition and it is unbounded from below for every $\lambda \in \left(0, \frac{r}{\sup_{x\in\Upsilon_1^{-1}(-\infty,r)} \Upsilon_2(x)}\right)$. Then, I_λ possesses two distinct critical points.*

Theorem 2.2 ([5, Theorem 2.1])**.** *Υ_1, $\Upsilon_2 : E_0^{\alpha,\psi,p} \to \mathbb{R}$ are two continuously Gâteaux differentiable functionals, where $E_0^{\alpha,\psi,p}$ is a real Banach space and $\inf_{x\in E_0^{\alpha,\psi,p}} \Upsilon_1(x) = \Upsilon_1(0) = \Upsilon_2(0) = 0$. Suppose that there exist $r \in \mathbb{R}$ and $\zeta \in E_0^{\alpha,\psi,p}$, with $0 < \Upsilon_1(\zeta) < r$, such that*

$$\frac{\sup_{x\in\Upsilon_1^{-1}(-\infty,r)} \Upsilon_2(x)}{r} < \frac{\Upsilon_2(\zeta)}{\Upsilon_1(\zeta)} \tag{2.16}$$

and, for every $\lambda \in \left(\frac{\Upsilon_1(\zeta)}{\Upsilon_2(\zeta)}, \frac{r}{\sup_{x\in\Upsilon_1^{-1}(-\infty,r)} \Upsilon_2(x)}\right)$, the functional $I_\lambda = \Upsilon_1 - \lambda\Upsilon_2$ satisfies Palais-Smale condition and it is unbounded from below. Then, I_λ possesses at least two nontrivial critical points $x_{\lambda,1}, x_{\lambda,2}$ such that $I_\lambda(x_{\lambda,1}) < 0 < I_\lambda(x_{\lambda,2})$.

Theorem 2.3 ([11])**.** *Assume that $E_0^{\alpha,\psi,p}$ is an infinite dimensional real Banach space. $I_\lambda : E_0^{\alpha,\psi,p} \to \mathbb{R}$ is an even functional and continuously differential, satisfying the Palais-Smale condition. Suppose also that*

(A1) $I_\lambda(0) = 0$. There exist $\beta > 0$, $\rho > 0$ such that $\bar{B}_\rho \subset \{x \in E_0^{\alpha,\psi,p} | I_\lambda(x) \ge 0\}$ and $I_\lambda(x) \ge \beta$, $\forall x \in \partial B_\rho$, where $B_\rho = \{x \in E_0^{\alpha,\psi,p} | \|x\| < \rho\}$, $(\rho > 0)$.

(A2) The set $V \bigcap \{x \in E_0^{\alpha,\psi,p} | I_\lambda(x) \ge 0\}$ is bounded for all finite dimensional subspace V in $E_0^{\alpha,\psi,p}$.

Then, I_λ admits infinitely many critical points.

3. Main results

In this section, the multiplicity of solutions for the problem (1.2) are obtained via Theorem 2.1–2.3.

Theorem 3.1. *Assume that* $\frac{1}{p} < \alpha \le 1$, $\gamma > 0$ *and the following conditions hold:*

$(H1)$ *There exist constants* $\Theta \ge 0$, $\varsigma_i > p$, $i = 0, 1, ..., m$, *such that*

$$0 < \varsigma_i F_i(t,x) \le x f_i(t,x), \quad \text{for } t \in (s_i, t_{i+1}],\ |x| > \Theta.$$

$(H2)$ *There exist constants* $p < \vartheta_i < \kappa$, $i = 1, 2, ..., m$, *such that*

$$0 < x I_i(x) \le \vartheta_i \int_0^x I_i(s)ds, \quad \text{for } x \in \mathbb{R}\backslash\{0\},$$

where $\kappa := \min\{\varsigma_0, \varsigma_1, ..., \varsigma_m\}$.

Then, for any $\lambda \in \left(0, \frac{\gamma^p}{p\ell^p \max\limits_{t\in[0,T]}\{\psi'(t)\} \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \max\limits_{|x|\le\gamma} F_i(t,x(t))dt}\right)$, *the problem* (1.2) *has at least two distinct weak solutions.*

Proof. We will show that all the conditions of Theorem 2.1 are satisfies and the proof is completed in three steps.

Step 1. I_λ satisfies the Palais-Smale condition.

Let $\{x_n\} \subset E_0^{\alpha,\psi,p}$ such that $\{I_\lambda(x_n)\}$ is bounded and $\lim\limits_{n\to\infty} I'_\lambda(x_n) = 0$. From (H1), (H2), (2.14) and (2.15), we get

$$\begin{aligned}
&\kappa I_\lambda(x_n) - \langle I'_\lambda(x_n), x_n\rangle \\
=&\frac{\kappa - p}{p}\|x_n\|^p + \kappa\sum_{i=1}^m \int_0^{x_n(t_i)} I_i(s)ds - \kappa\lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t, x_n(t))\psi'(t)dt \\
&- \sum_{i=1}^m I_i(x_n(t_i))x_n(t_i) + \lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i(t, x_n(t))\psi'(t)x_n(t)dt \\
\ge&\frac{\kappa - p}{p}\|x_n\|^p - \lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \psi'(t) \max_{x_n\in[-\Theta,\Theta]} |\kappa F_i(t, x_n(t)) - f_i(t, x_n(t))x_n(t)|\, dt.
\end{aligned}$$

In view of $\kappa > p$, we obtain $\{x_n\}$ is bounded in $E_0^{\alpha,\psi,p}$. Moreover, one has

$$\begin{aligned}
&\langle I'_\lambda(x_n) - I'_\lambda(x), x_n - x\rangle \\
=&\int_0^T \psi'(t)\left(\Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)) - \Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x(t))\right) {}^C D_{0^+}^{\alpha,\psi(t)}(x_n(t) - x(t))dt \\
&+ \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \psi'(t)\left(|x_n(t)|^{p-2}x_n(t) - |x(t)|^{p-2}x(t)\right)(x_n(t) - x(t))dt \\
&+ \sum_{i=1}^m (I_i(x_n(t_i)) - I_i(x(t_i)))(x_n(t_i) - x(t_i)) \\
&- \lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \psi'(t)\left(f_i(t, x_n(t)) - f_i(t, x(t))\right)(x_n(t) - x(t))dt. \qquad (3.1)
\end{aligned}$$

According to the reflexivity of $E_0^{\alpha,\psi,p}$, there exists a subsequence of $\{x_n\}$ which is also designated as $\{x_n\}$ for convenience, such that $x_n \rightharpoonup x$ in $E_0^{\alpha,\psi,p}$. Then, it follows from Lemma 2.6 that $x_n \to x$ in $C([0,T])$. Thus,

$$\sum_{i=1}^{m}(I_i(x_n(t_i)) - I_i(x(t_i)))(x_n(t_i) - x(t_i)) \to 0, \tag{3.2}$$

$$\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \psi'(t)\left(f_i(t,x_n(t)) - f_i(t,x(t))\right)(x_n(t) - x(t))dt \to 0, \text{ as } n \to \infty. \tag{3.3}$$

By $I'_\lambda(x_n) \to 0$ and $x_n \rightharpoonup x$, one has

$$\langle I'_\lambda(x_n) - I'_\lambda(x), x_n - x\rangle \to 0, \text{ as } n \to \infty. \tag{3.4}$$

From [28, Eq (2.2)], there exist δ_1, $\delta_2 > 0$ such that

$$\begin{aligned}
&\int_0^T \psi'(t)\left(\Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)) - \Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x(t))\right) {}^C D_{0^+}^{\alpha,\psi(t)}(x_n(t) - x(t))dt \\
&+\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \psi'(t)\left(|x_n(t)|^{p-2}x_n(t) - |x(t)|^{p-2}x(t)\right)(x_n(t) - x(t))dt \\
&\geq \begin{cases}
\delta_1\left(\displaystyle\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha.\psi(t)}(x_n(t) - x(t))|^p dt \right. \\
\left. +\displaystyle\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \psi'(t)|x_n(t) - x(t)|^p dt\right), \ p \geq 2, \\
\delta_2\left(\displaystyle\int_0^T \frac{\psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^2}{\left(|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|\right)^{2-p}}dt \right. \\
\left. +\displaystyle\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \frac{\psi'(t)|x_n(t) - x(t)|^2}{(|x_n(t)| + |x(t)|)^{2-p}}dt\right), \ 1 < p < 2.
\end{cases}
\end{aligned} \tag{3.5}$$

If $p \geq 2$, then using (3.1)–(3.5), we derive $\|x_n - x\| \to 0$ in $E_0^{\alpha,\psi,p}$.
If $1 < p < 2$, by Hölder's inequality, one has

$$\begin{aligned}
&\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt \\
\leq &\left(\int_0^T \frac{\psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^2}{\left(|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|\right)^{2-p}}dt\right)^{\frac{p}{2}} \\
\times &\left(\int_0^T \psi'(t)(|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|)^p dt\right)^{\frac{2-p}{2}} \\
\leq &\left(\int_0^T \frac{\psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^2}{\left(|{}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |{}^C D_{0^+}^{\alpha,\psi(t)} x(t)|\right)^{2-p}}dt\right)^{\frac{p}{2}}
\end{aligned}$$

$$\times 2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^T \psi'(t)\left(|^C D_{0^+}^{\alpha,\psi(t)} x_n(t)|^p + |^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p\right) dt\right)^{\frac{2-p}{2}}$$

$$\leq 2^{\frac{(p-1)(2-p)}{2}} (\|x_n\|+\|x\|)^{\frac{p(2-p)}{2}} \left(\int_0^T \frac{\psi'(t)|^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^2}{\left(|^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |^C D_{0^+}^{\alpha,\psi(t)} x(t)|\right)^{2-p}} dt\right)^{\frac{p}{2}}. \tag{3.6}$$

Similarly,

$$\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \psi'(t)|x_n(t) - x(t)|^p dt$$

$$\leq \sum_{i=0}^{m} \left(\int_{s_i}^{t_{i+1}} \frac{\psi'(t)|x_n(t) - x(t)|^2}{(|x_n| + |x(t)|)^{2-p}} dt\right)^{\frac{p}{2}} \left(\int_{s_i}^{t_{i+1}} \psi'(t)(|x_n(t)| + |x(t)|)^p dt\right)^{\frac{2-p}{2}}$$

$$\leq \left(\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \frac{\psi'(t)|x_n(t) - x(t)|^2}{(|x_n| + |x(t)|)^{2-p}} dt\right)^{\frac{p}{2}} \left(\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \psi'(t)(|x_n(t)| + |x(t)|)^p dt\right)^{\frac{2-p}{2}}$$

$$\leq 2^{\frac{(p-1)(2-p)}{2}} (\|x_n\| + \|x\|)^{\frac{p(2-p)}{2}} \left(\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \frac{\psi'(t)|x_n(t) - x(t)|^2}{(|x_n| + |x(t)|)^{2-p}} dt\right)^{\frac{p}{2}}. \tag{3.7}$$

Thus, (3.5)–(3.7) yield

$$\int_0^T \psi'(t)\left(\Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x_n(t)) - \Phi_p({}^C D_{0^+}^{\alpha,\psi(t)} x(t))\right) {}^C D_{0^+}^{\alpha,\psi(t)} (x_n(t) - x(t)) dt$$

$$+ \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \psi'(t)\left(|x_n(t)|^{p-2} x_n(t) - |x(t)|^{p-2} x(t)\right)(x_n(t) - x(t)) dt$$

$$\geq \delta_2 \left(\int_0^T \frac{\psi'(t)|^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^2}{\left(|^C D_{0^+}^{\alpha,\psi(t)} x_n(t)| + |^C D_{0^+}^{\alpha,\psi(t)} x(t)|\right)^{2-p}} dt\right.$$

$$\left. + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \frac{\psi'(t)|x_n(t) - x(t)|^2}{(|x_n(t)| + |x(t)|)^{2-p}} dt\right)$$

$$\geq \frac{\delta_2}{2^{\frac{(p-1)(2-p)}{2}} (\|x_n\| + \|x\|)^{2-p}} \left(\left(\int_0^T \psi'(t)|^C D_{0^+}^{\alpha,\psi(t)} x_n(t) - {}^C D_{0^+}^{\alpha,\psi(t)} x(t)|^p dt\right)^{\frac{2}{p}}\right.$$

$$\left. + \left(\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \psi'(t)|x_n(t) - x(t)|^p dt\right)^{\frac{2}{p}}\right)$$

$$\geq \frac{\delta_2}{2^{\frac{(p-1)(2-p)}{2}} \max\{2^{\frac{2}{p}-1}, 1\} (\|x_n\| + \|x\|)^{2-p}} \|x_n - x\|^2. \tag{3.8}$$

From (3.1)–(3.4) and (3.8), it follows that $\|x_n - x\| \to 0$ in $E_0^{\alpha,\psi,p}$. Hence, the Palais-Smale condition is proved.

Step 2. I_λ is unbounded from below.

By (H1), there exist $\epsilon_i,\ L_i > 0$, such that

$$F_i(t, x(t)) \geq \epsilon_i |x|^{\varsigma_i} - L_i,\ \ i = 0, 1, ..., m. \tag{3.9}$$

Similarly to the proof of [37, Theorem 3.1], by (H2), there exist $\iota_i,\ M_i > 0$, such that

$$\int_0^x I_i(s)ds \leq \iota_i |x|^{\vartheta_i} + M_i\ ,\, i = 1, 2, ..., m, \tag{3.10}$$

where

$$\iota_i = \max\left\{\int_0^1 I_i(s)ds, \int_0^{-1} I_i(s)ds\right\}, \quad M_i = \max_{|x|<1}\int_0^x I_i(s)ds. \tag{3.11}$$

Now, using (2.14), (3.9) and (3.10), we have, for any $x \in E_0^{\alpha,\psi,p}$ with $\|x\| = 1$,

$$\begin{aligned}
&I_\lambda(\sigma x)\\
\leq&\frac{1}{p}\|\sigma x\|^p + \sum_{i=1}^m (\iota_i|\sigma x|^{\vartheta_i} + M_i) - \lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \psi'(t)(\epsilon_i|\sigma x|^{\varsigma_i} - L_i)dt\\
\leq&\frac{1}{p}\|\sigma x\|^p + \sum_{i=1}^m \iota_i\|\sigma x\|_\infty^{\vartheta_i} + \sum_{i=1}^m M_i\\
&-\lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m\int_{s_i}^{t_{i+1}} \epsilon_i|\sigma x|^{\varsigma_i}dt + \lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m\int_{s_i}^{t_{i+1}} L_i dt\\
\leq&\frac{\sigma^p}{p}\|x\|^p + \sum_{i=1}^m \iota_i(\sigma\ell)^{\vartheta_i}\|x\|^{\vartheta_i} + \sum_{i=1}^m M_i\\
&-\lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m \epsilon_i\sigma^{\varsigma_i}\int_{s_i}^{t_{i+1}} |x|^{\varsigma_i}dt + \lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m L_i(t_{i+1} - s_i)\\
\leq&\frac{\sigma^p}{p} + \sum_{i=1}^m \iota_i(\sigma\ell)^{\vartheta_i} + \sum_{i=1}^m M_i\\
&-\lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m \epsilon_i\sigma^{\kappa}\int_{s_i}^{t_{i+1}} |x|^{\varsigma_i}dt + \lambda\min_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m L_i(t_{i+1} - s_i).
\end{aligned}$$

Since $p < \vartheta_i < \kappa$ and $\int_{s_i}^{t_{i+1}} |x|^{\varsigma_i}dt > 0$, we can obtain $I_\lambda(\sigma x) \to -\infty$ as $\sigma \to +\infty$. Hence, I_λ is unbounded from below.

Step 3. The problem (1.2) has at least two distinct weak solutions.

Due to $\int_0^x I_i(s)ds > 0$, $x \in \mathbb{R}\backslash\{0\}$, we have

$$\Upsilon_1(x) = \frac{1}{p}\|x\|^p + \sum_{i=1}^m \int_0^{x(t_i)} I_i(s)ds \geq \frac{1}{p}\|x\|^p.$$

Thus, Υ_1 is bounded from below. Obviously, Υ_1 and Υ_2 are continuously Gâteaux differentiable and $\Upsilon_1(0) = \Upsilon_2(0) = 0$.

On the other hand, for each $x \in E_0^{\alpha,\psi,p}$ with $\Upsilon_1(x) \leq r$, taking into account Lemma 2.5, one has $\|x\|_\infty \leq \ell\|x\| \leq \ell\sqrt[p]{pr} := \gamma$. So

$$\sup_{\Upsilon_1(x)\leq r}\Upsilon_2(x) \leq \max_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m\int_{s_i}^{t_{i+1}} \max_{|x|\leq\gamma} F_i(t, x(t))dt < +\infty.$$

Therefore, for every $\lambda \in \left(0, \frac{\gamma^p}{p\ell^p \max\limits_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \max\limits_{|x|\leq\gamma} F_i(t,x(t))dt}\right)$, I_λ has at least two distinct critical points by virtue of Theorem 2.1, that is, the problem (1.2) has at least two distinct weak solutions. □

Before introducing the next theorem, we first define the following notation:

$$
\begin{aligned}
W :=& \frac{(\psi(T)-\psi(0))\,(\psi(t_1)-\psi(0))^p}{p}\left(\frac{1}{(1-\alpha)^p\Gamma^p(1-\alpha)\,(\psi(t_1)-\psi(0))^{p\alpha}}+1\right)\\
&+\sum_{i=1}^m \left(\iota_i(\psi(t_1)-\psi(0))^{\vartheta_i}+M_i\right),
\end{aligned}
$$

where ι_i and M_i are defined in (3.11).

Theorem 3.2. *Assume that (H1), (H2) and the following condition holds:*

$(H3)$ *$F_i(t,x)\geq 0$ for each $(t,x)\in([0,t_1]\cup[s_m,T])\times[0,\psi(t_1)-\psi(0)]$.*

Suppose also that there exists $\gamma>0$ with $W<\frac{\gamma^p}{p\ell^p}$, such that

$$
\begin{aligned}
&\frac{p\ell^p\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\psi'(t)\max_{|x|\leq\gamma}F_i(t,x(t))dt}{\gamma^p}\\
<&\frac{\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\psi'(t)F_i(t,\psi(t_1)-\psi(0))dt}{W}.
\end{aligned}\tag{3.12}
$$

Then, for any

$$
\begin{aligned}
\lambda\in\Bigg(&\frac{W}{\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\psi'(t)F_i(t,\psi(t_1)-\psi(0))dt},\\
&\frac{\gamma^p}{p\ell^p\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\psi'(t)\max_{|x|\leq\gamma}F_i(t,x(t))dt}\Bigg),
\end{aligned}
$$

the problem (1.2) *has at least two nontrivial weak solutions.*

Proof. According to the proof of Theorem 3.1, we can obtain that Υ_1, Υ_2 are two continuously Gâteaux differentiable functionals, $\inf_{x\in E_0^{\alpha,\psi,p}}\Upsilon_1(x)=\Upsilon_1(0)=\Upsilon_2(0)=0$, and I_λ is unbounded from below and satisfies Palais-Smale condition for all $\lambda>0$. Define a function $\zeta:[0,T]\to\mathbb{R}$ by

$$
\zeta(t):=\begin{cases}\psi(t)-\psi(0), & t\in[0,t_1],\\ \psi(t_1)-\psi(0), & t\in[t_1,s_m],\\ \psi(T)-\psi(t), & t\in[s_m,T],\end{cases}
$$

where $s_m=\psi^{-1}\left(\psi(0)+\psi(T)-\psi(t_1)\right)$. Clearly, $\zeta\in E_0^{\alpha,\psi,p}$. Then,

$$
\zeta'(t)=\begin{cases}\psi'(t), & t\in(0,t_1),\\ 0, & t\in(t_1,s_m),\\ -\psi'(t), & t\in(s_m,T),\end{cases}
$$

and

$$
\begin{aligned}
&{}^C D_{0^+}^{\alpha,\psi(t)}\zeta(t)\\
=&\frac{1}{\Gamma(1-\alpha)}\int_0^t (\psi(t)-\psi(s))^{-\alpha}\zeta'(s)ds\\
=&\frac{1}{(1-\alpha)\Gamma(1-\alpha)}\begin{cases}(\psi(t)-\psi(0))^{1-\alpha}, & t\in[0,t_1],\\ (\psi(t_1)-\psi(0))^{1-\alpha}, & t\in[t_1,s_m],\\ (\psi(t_1)-\psi(0))^{1-\alpha}-(\psi(t)-\psi(s_m))^{1-\alpha}, & t\in[s_m,T].\end{cases}
\end{aligned}
$$

Combining with (3.10), we have

$$
\begin{aligned}
\Upsilon_1(\zeta)=&\frac{1}{p}\left(\int_0^T \psi'(t)|{}^C D_{0^+}^{\alpha,\psi(t)}\zeta(t)|^p dt+\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\psi'(t)|\zeta(t)|^p dt\right)\\
&+\sum_{i=1}^m\int_0^{\zeta(t_i)} I_i(s)ds\\
=&\frac{1}{p}\left(\int_0^{t_1}\psi'(t)\frac{|\psi(t)-\psi(0)|^{p-p\alpha}}{(1-\alpha)^p\Gamma^p(1-\alpha)}dt+\int_{t_1}^{s_m}\psi'(t)\frac{|\psi(t_1)-\psi(0)|^{p-p\alpha}}{(1-\alpha)^p\Gamma^p(1-\alpha)}dt\right.\\
&+\int_{s_m}^T\psi'(t)\frac{\left|(\psi(t_1)-\psi(0))^{1-\alpha}-(\psi(t)-\psi(s_m))^{1-\alpha}\right|^p}{(1-\alpha)^p\Gamma^p(1-\alpha)}dt\\
&+\int_0^{t_1}\psi'(t)|\psi(t)-\psi(0)|^p dt+\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\psi'(t)|\psi(t_1)-\psi(0)|^p dt\\
&\left.+\int_{s_m}^T\psi'(t)|\psi(T)-\psi(t)|^p dt\right)+\sum_{i=1}^m\int_0^{\psi(t_1)-\psi(0)} I_i(s)ds\\
\le&\frac{(\psi(T)-\psi(0))\,(\psi(t_1)-\psi(0))^{p-p\alpha}}{p(1-\alpha)^p\Gamma^p(1-\alpha)}+\frac{1}{p}(\psi(T)-\psi(0))\,(\psi(t_1)-\psi(0))^p\\
&+\sum_{i=1}^m\left(\iota_i(\psi(t_1)-\psi(0))^{\vartheta_i}+M_i\right)\\
=&W.
\end{aligned}
$$

Since $W<\frac{\gamma^p}{p\ell^p}=r$, we prove $\Upsilon_1(\zeta)<r$. Additionally, it is obvious that $\Upsilon_1(x)>0$ for any $x\in E_0^{\alpha,\psi,p}\backslash\{0\}$. Therefore, $0<\Upsilon_1(\zeta)<r$.

On the other hand, by (H3), we observe that

$$
\Upsilon_2(\zeta)=\sum_{i=0}^m\int_{s_i}^{t_{i+1}} F_i(t,\zeta(t))\psi'(t)dt\ge\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}} F_i(t,\psi(t_1)-\psi(0))\psi'(t)dt. \quad (3.13)
$$

From (3.12) and (3.13), it follows that

$$
\begin{aligned}
\frac{\sup_{\Upsilon_1(x)\le r}\Upsilon_2(x)}{r}\le&\frac{\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\psi'(t)\max_{|x|\le\gamma}F_i(t,x(t))dt}{r}\\
\le&\frac{p\ell^p\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\psi'(t)\max_{|x|\le\gamma}F_i(t,x(t))dt}{\gamma^p}
\end{aligned}
$$

$$
\begin{aligned}
&<\frac{\sum_{i=1}^{m-1}\int_{s_i}^{t_{i+1}}\psi'(t)F_i(t,\psi(t_1)-\psi(0))dt}{W}\\
&\leq\frac{\Upsilon_2(\zeta)}{\Upsilon_1(\zeta)},
\end{aligned}
$$

which implies that (2.16) holds. Consequently, I_λ possesses at least two nontrivial critical points by means of Theorem 2.2, i.e., the problem (1.2) possesses at least two nontrivial weak solutions. □

Theorem 3.3. *Assume that*

($H4$) $I_i(x), i=1,2,...,m$ *are odd and satisfy* $\int_0^x I_i(s)ds\geq 0$ *for all* $x\in\mathbb{R}$.

($H5$) *There exist constants* σ_i, $\tau_i>0$, *and* $\chi_i\in[0,p-1)$ *such that*

$$|I_i(x)|\leq\sigma_i+\tau_i|x|^{\chi_i},\quad \textit{for } x\in\mathbb{R},\ i=1,2,...,m.$$

($H6$) $0\leq F_i(t,x)=o(|x|^p)$ *as* $|x|\to 0$ *uniformly for* $t\in(s_i,t_{i+1}]$, $i=0,1,...,m$.

($H7$) $\lim\limits_{|x|\to+\infty}\frac{F_i(t,x)}{|x|^p}=+\infty$ *uniformly for* $t\in(s_i,t_{i+1}]$, $i=0,1,...,m$.

Moreover, $f_i(t,x)$, $i=0,1,...,m$ *are odd in* x, *then the problem* (1.2) *has infinitely many weak solutions for* $\lambda\in(0,+\infty)$.

Proof. Obviously, I_λ is continuously differential and $I_\lambda(0)=0$. By (H4) and $f_i(t,x)$, $i=0,1,...,m$ are odd in x, we obtain $I_\lambda(x)$ is even. Next, we complete the proof in two steps.

Step 1. I_λ satisfies the Palais-Smale condition.

We first prove that sequence $\{x_n\}$ is bounded in $E_0^{\alpha,\psi,p}$. If $\{x_n\}$ is unbounded, we set $\|x_n\|\to+\infty$, $n\to\infty$. From (2.14), (H5) and Lemma 2.5, we have

$$
\begin{aligned}
&\frac{I_\lambda(x_n)}{\|x_n\|^p}\\
=&\frac{1}{p}+\frac{\sum_{i=1}^m\int_0^{x_n(t_i)}I_i(s)ds}{\|x_n\|^p}-\frac{\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}F_i(t,x_n(t))\psi'(t)dt}{\|x_n\|^p}\\
\leq&\frac{1}{p}+\frac{\sum_{i=1}^m\sigma_i|x_n(t_i)|+\frac{\tau_i}{\chi_i+1}|x_n(t_i)|^{\chi_i+1}}{\|x_n\|^p}-\frac{\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}F_i(t,x_n(t))\psi'(t)dt}{\|x_n\|^p}\\
\leq&\frac{1}{p}+\frac{\sum_{i=1}^m\sigma_i\ell\|x_n\|+\tau_i\ell^{\chi_i+1}\|x_n\|^{\chi_i+1}}{\|x_n\|^p}-\frac{\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}F_i(t,x_n(t))\psi'(t)dt}{\|x_n\|^p}. \quad (3.14)
\end{aligned}
$$

Due to (H7), we get that $\forall Q>0$, $\exists N-1$, when $n>N-1$, $\frac{F_i(t,\|x_n\|)}{\|x_n\|^p}>Q$, uniformly for $t\in(s_i,t_{i+1}]$, $i=0,1,...,m$. Similarly to the proof of [22, Theorem 1.2], by integral mean value theorem, there exists $\theta_0(N)\in(0,1]$ and $K_0(N)>0$, such that

$$
\begin{aligned}
\int_0^{t_1}F_0(t,x_N(t))dt=&t_1F_0(\theta_0(N)t_1,x_N(\theta_0(N)t_1))\\
=&K_0(N)F_0(\theta_0(N)t_1,\|x_N\|)\\
\geq&\widetilde{K_0}Q\|x_N\|^p,
\end{aligned}
$$

where $\widetilde{K_0} = \inf_{N\leq n\leq N^*}\{K_0(n)\} > 0$, with $n \in \mathbb{N}$, N^* large enough.

Also, there exist $\theta_i(N) \in (0, 1]$, $i = 1, 2, ..., m$, such that

$$\begin{aligned}
&\int_{s_1}^{t_2} F_1(t, x_N(t))dt\\
=&(t_2 - s_1)F_1(s_1 + \theta_1(N)(t_2 - s_1), x_N(s_1 + \theta_1(N)(t_2 - s_1)))\\
=&K_1(N)F_1(s_1 + \theta_1(N)(t_2 - s_1), \|x_N\|)\\
\geq&\widetilde{K_1}Q\|x_N\|^p,\\
&\int_{s_2}^{t_3} F_2(t, x_N(t))dt\\
=&(t_3 - s_2)F_2(s_2 + \theta_2(N)(t_3 - s_2), x_N(s_2 + \theta_2(N)(t_3 - s_2)))\\
=&K_2(N)F_2(s_2 + \theta_2(N)(t_3 - s_2), \|x_N\|)\\
\geq&\widetilde{K_2}Q\|x_N\|^p,\\
&\vdots\\
&\int_{s_m}^{T} F_m(t, x_N(t))dt\\
=&(T - s_m)F_m(s_m + \theta_m(N)(T - s_m), x_N(s_m + \theta_m(N)(T - s_m)))\\
=&K_m(N)F_m(s_m + \theta_m(N)(T - s_m), \|x_N\|)\\
\geq&\widetilde{K_m}Q\|x_N\|^p,
\end{aligned}$$

where $\widetilde{K_i} = \inf_{N\leq n\leq N^*}\{K_i(n)\} > 0$, $i = 1, 2, ..., m$. Let $K = \min\limits_{0\leq i\leq m}\{\widetilde{K_i}\}$, we gain

$$\begin{aligned}
&\frac{\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}} F_i(t, x_N(t))\psi'(t)dt}{\|x_N\|^p}\\
\geq&\frac{\lambda\min\limits_{t\in[0,T]}\{\psi'(t)\}\sum_{i=0}^m\int_{s_i}^{t_{i+1}} F_i(t, x_N(t))dt}{\|x_N\|^p}\\
=&\lambda\min_{t\in[0,T]}\{\psi'(t)\}\Bigg(\frac{t_1F_0(\theta_0(N)t, x_N(\theta_0(N)t)}{\|x_N\|^p}\\
&+\frac{(t_2 - s_1)F_1(s_1 + \theta_1(N)(t_2 - s_1), x_N(s_1 + \theta_1(N)(t_2 - s_1)))}{\|x_N\|^p}\\
&+... + \frac{(T - s_m)F_m(s_m + \theta_m(N)(T - s_m), x_N(s_m + \theta_m(N)(T - s_m)))}{\|x_N\|^p}\Bigg)\\
\geq&\frac{\lambda(m + 1)KQ\min\limits_{t\in[0,T]}\{\psi'(t)\}\|x_N\|^p}{\|x_N\|^p}\\
=&\lambda(m + 1)KQ\min_{t\in[0,T]}\{\psi'(t)\}.
\end{aligned}$$

We can take $Q = \frac{p}{\lambda(m+1)K\min\limits_{t\in[0,T]}\{\psi'(t)\}}$, and choose N such that

$$\frac{\sum_{i=1}^m \sigma_i\ell\|x_n\| + \tau_i\ell^{\chi_i+1}\|x_n\|^{\chi_i+1}}{\|x_n\|^p}$$

becomes small enough. Therefore, it follows from (3.14) that

$$\frac{I_\lambda(x_N)}{\|x_N\|^p} \leq \frac{1}{p} - p.$$

For $\forall Q^* > 0$, $\exists \widehat{N} > 0$ such that $\|x_n\|^p > Q^*$, for $n > \widehat{N}$. Thus, when $N > \widehat{N}$, $I_\lambda(x_n) < -Q^*$ for $n > \widehat{N}$. A contradiction with the boundedness of $I_\lambda(x_n)$. This means that $\{x_n\}$ is bounded. Additionally, since the rest of the proof of Palais-Smale condition is similar to Theorem 3.1, we omit the proving process here.

Step 2. The conditions (A1) and (A2) of Theorem 2.3 are satisfied.

In view of (H6), for $\forall \varepsilon > 0$, $\exists \varpi > 0$ such that

$$F_i(t, x(t)) \leq \varepsilon |x|^p, \quad |x|^p \leq \varpi.$$

Combining with (H4), for each $x \in E_0^{\alpha,\psi,p}$, $|x|^p \leq \varpi$, we acquire

$$\begin{aligned}
I_\lambda(x) =& \frac{1}{p}\|x\|^p + \sum_{i=1}^{m} \int_0^{x(t_i)} I_i(s)ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, x(t))\psi'(t)dt \\
\geq& \frac{1}{p}\|x\|^p - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, x(t))\psi'(t)dt \\
\geq& \frac{1}{p}\|x\|^p - \lambda\varepsilon ||x||_\infty^p \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \psi'(t)dt \\
\geq& \frac{1}{p}\|x\|^p - \lambda\varepsilon\ell^p ||x||^p \int_0^T \psi'(t)dt \\
\geq& \frac{1}{p}\|x\|^p - \lambda\varepsilon\ell^p ||x||^p \left(\psi(T) - \psi(0)\right) \\
\geq& \frac{1}{2p}\|x\|^p,
\end{aligned}$$

where $\varepsilon = \frac{1}{2p\lambda\ell^p(\psi(T)-\psi(0))}$. Let $\rho = \frac{\sqrt[p]{\varpi}}{\ell}$ such that $\bar{B}_\rho \subset \{x \in E_0^{\alpha,\psi,p} | I_\lambda(x) \geq 0\}$ and

$$I_\lambda(x) \geq \frac{1}{2p}\rho^p, \quad \forall x \in \partial B_\rho.$$

This implies that the condition (A1) of Theorem 2.3 is fulfilled. Next, we continue to verify the condition (A2). We also argue by contradiction. Suppose that the set $V \cap \{x \in E_0^{\alpha,\psi,p} | I_\lambda(x) \geq 0\}$ is unbounded, then there exists a sequence $\left\{x_n \in V \cap \{x \in E_0^{\alpha,\psi,p} | I_\lambda(x) \geq 0\}\right\}$ such that $\|x_n\|^p \to +\infty$, $n \to \infty$ and

$$I_\lambda(x_n) = \frac{1}{p}\|x_n\|^p + \sum_{i=1}^{m} \int_0^{x_n(t_i)} I_i(s)ds - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, x_n(t))\psi'(t)dt \geq 0.$$

That is,

$$\frac{1}{p}\|x_n\|^p + \sum_{i=1}^{m} \int_0^{x_n(t_i)} I_i(s)ds \geq \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, x_n(t))\psi'(t)dt.$$

Dividing the two sides of the aforementioned inequality by $\|x_n\|^p$, we have

$$\begin{aligned}&\frac{1}{p}+\frac{\sum_{i=1}^{m}\int_0^{x_n(t_i)} I_i(s)ds}{\|x_n\|^p}\\ \geq&\frac{\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,x_n(t))\psi'(t)dt}{\|x_n\|^p}.\end{aligned}$$

Analogous to the proof of the boundedness of $\{x_n\}$, we derive

$$\frac{1}{p}\geq p,$$

which produces a contradiction. Therefore, the set $V\cap\{x\in E_0^{\alpha,\psi,p}|I_\lambda(x)\geq 0\}$ is bounded, and the condition (A2) holds. By Theorem 2.3, the problem (1.2) admits infinitely many weak solutions. □

4. Examples

Example 4.1. Let $0=s_0<t_1=\frac{1}{4}<s_1=\frac{3}{4}<t_2=1$, $\alpha=0.9$, $\psi(t)=t^3$, $p=6$. Consider the following problem:

$$\begin{cases} \quad {}^C D_{1^-}^{0.9,t^3}\left(\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x(t)\right)\right)+|x(t)|^4x(t)\\ =\lambda f_i(t,x(t)),\quad t\in(s_i,t_{i+1}],\ i=0,1,\\ \Delta\left(I_{1^-}^{0.1,t^3}\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x\right)\right)(t_1)=I_1(x(t_1)),\\ \quad I_{1^-}^{0.1,t^3}\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x\right)(t)\\ =I_{1^-}^{0.1,t^3}\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x\right)(t_1^+),\quad t\in(t_1,s_1],\\ \quad I_{1^-}^{0.1,t^3}\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x\right)(s_1^-)\\ =I_{1^-}^{0.1,t^3}\Phi_6\left({}^C D_{0^+}^{0.9,t^3}x\right)(s_1^+),\\ x(0)=x(1)=0,\end{cases}\tag{4.1}$$

where $I_1(x)=x^3$ and $f_i(t,x)=x^7$. Then, $\int_0^x s^3ds=\frac{1}{4}x^4$ and $F_i(t,x)=\frac{1}{8}x^8$. When $\varsigma_i=7$, $i=0,1$ and $\vartheta_1=\frac{13}{2}$, we obtain

$$\begin{aligned}&0<\frac{7}{8}x^8=\varsigma_i F_i(t,x)\leq xf_i(t,x)=x^8,\quad i=0,1,\\ &0<x^4=xI_1(x)\leq\vartheta_1\int_0^x I_1(s)ds=\frac{13}{8}x^4.\end{aligned}$$

Thus, (H1) and (H2) hold. Choose $\gamma=1$. By Theorem 3.1, we obtain the problem (4.1) has at least two weak solutions for $\lambda\in(0,0.6986)$.

Example 4.2. Let $0=s_0<t_1=\frac{1}{3}<s_1=\frac{2}{5}<t_2=\frac{4}{5}<s_2=\ln(e+1-e^{\frac{1}{3}})<$

$t_3 = 1$, $\alpha = 0.7$, $\psi(t) = e^t$, $p = 3$. Consider the following problem:

$$\begin{cases} \quad {}^C D_{1^-}^{0.7,e^t} \left(\Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x(t) \right) \right) + |x(t)|x(t) \\ = \lambda f_i(t, x(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \\ \Delta \left(I_{1^-}^{0.3,e^t} \Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x \right) \right) (t_i) = I_i(x(t_i)), \quad i = 1, 2, \\ \quad I_{1^-}^{0.3,e^t} \Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x \right) (t) \\ = I_{1^-}^{0.3,e^t} \Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x \right) (t_i^+), \quad t \in (t_i, s_i], \ i = 1, 2, \\ \quad I_{1^-}^{0.3,e^t} \Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x \right) (s_i^-) \\ = I_{1^-}^{0.3,e^t} \Phi_3 \left({}^C D_{0^+}^{0.7,e^t} x \right) (s_i^+), \quad i = 1, 2, \\ x(0) = x(1) = 0, \end{cases} \tag{4.2}$$

where $I_i(x) = x$ and $f_i(t, x) = 6x^5 - 404x^3 + 200x$. Then, $\int_0^x s ds = \frac{1}{2}x^2$ and $F_i(t, x) = x^2(x^2 - 1)(x^2 - 100)$. There exists $0 < |\hat{x}| < 1$ such that $\max\limits_{|x| \le 10} F_i(x) = \max\limits_{|x| \le 1} F_i(x) = F_i(\hat{x})$. By simple calculation, we can obtain $\hat{x} \approx \pm 0.7062$ and $F_i(\hat{x}) \approx 24.8752$. Take $\varsigma_i = 4$, $i = 0, 1, 2$ and $\vartheta_i = \frac{7}{2}$, $i = 1, 2$. Then

$$0 < 4x^2(x^2 - 1)(x^2 - 100) = \varsigma_i F_i(t, x) \le x f_i(t, x) = 6x^6 - 404x^4 + 200x^2, \ |x| > 10,$$

$$0 < x^2 = x I_i(x) \le \vartheta_i \int_0^x I_i(s) ds = \frac{7}{4} x^2.$$

Thus, (H1) and (H2) hold. Choose function

$$\zeta(t) = \begin{cases} e^t - 1, & t \in [0, \frac{1}{3}], \\ e^{\frac{1}{3}} - 1, & t \in [\frac{1}{3}, \ln(e + 1 - e^{\frac{1}{3}})], \\ e - e^t, & t \in [\ln(e + 1 - e^{\frac{1}{3}}), 1]. \end{cases}$$

Obviously, (H3) is also satisfied. Let $\gamma = 10$. By direct computation, we obtain the problem (4.2) has at least two nontrivial weak solutions for each $\lambda \in (0.1608, 5.5078)$ by means of Theorem 3.2.

Example 4.3. Let $\alpha = 0.65$, $\psi(t) = 3t^2$, $p = 8$. Consider the following problem:

$$\begin{cases} \quad {}^C D_{1^-}^{0.65,3t^2} \left(\Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x(t) \right) \right) + |x(t)|^6 x(t) \\ = \lambda f_i(t, x(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, 1, ..., m, \\ \Delta \left(I_{1^-}^{0.35,3t^2} \Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x \right) \right) (t_i) = I_i(x(t_i)), \quad i = 1, 2, ..., m, \\ \quad I_{1^-}^{0.35,3t^2} \Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x \right) (t) \\ = I_{1^-}^{0.35,3t^2} \Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x \right) (t_i^+), \quad t \in (t_i, s_i], \ i = 1, 2, ..., m, \\ \quad I_{1^-}^{0.35,3t^2} \Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x \right) (s_i^-) \\ = I_{1^-}^{0.35,3t^2} \Phi_8 \left({}^C D_{0^+}^{0.65,3t^2} x \right) (s_i^+), \quad i = 1, 2, ..., m, \\ x(0) = x(1) = 0, \end{cases} \tag{4.3}$$

where $I_i(x) = x^5$ and $f_i(t,x) = \frac{1}{10}x^9$. Then, $\int_0^x s^5 ds = \frac{1}{6}x^6$ and $F_i(t,x) = x^{10}$. Evidently, $I_i(x)$ and $f_i(t,x)$ are odd in x, and (H4) holds. Let $\sigma_i = \tau_i = 1$, $\chi_i = 6$, then (H5) is satisfied. Through simple calculation, we obtain (H6) and (H7) are also fulfilled. Thus, by Theorem 3.3, the problem (4.3) admits infinitely many weak solutions.

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