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TWO-DIMENSIONAL SOLITARY WAVE SOLUTIONS FOR WATER WAVES NEAR THE CRITICAL POINT

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Abstract This paper considers that the Bond number is greater than but close to $\frac{1}{3}$ and the Froude number is greater than but close to 1 for the two-dimensional traveling gravity-capillary waves in water of finite depth. The horizontal propagating direction and the stream function are chosen as independent variables. The Euler equations are reduced to a system of ordinary differential equations with dimension 4 by applying a spatial dynamics approach, a center manifold reduction and a normal form analysis. The reduced system has a homoclinic solution near a nonzero equilibrium. The fixed point theorem shows that this homoclinic solution persists for the original system, i.e., the hydrodynamic problem has a solitary wave solution which exponentially approaches a constant function (independent of the horizontal variable but dependent on the stream function) at infinity.

Keywords Solitary wave solutions, center manifold reduction, normal form, homoclinic orbits.

MSC(2010) 35Q35, 37K50, 76B15, 76B25.

1. Introduction

The classical two-dimensional travelling water-wave problem concerns the irrotational flow of a perfect fluid with a constant density ρ subject to the forces of gravity and surface tension. The fluid is bounded above by a free surface and below by a horizontal rigid bottom. The wave is moving with a constant speed c. Assume that the mean depth of the layer is h. It is well-known that the existence of solitary wave solutions is determined by two constants: Bond number $b = T/(\rho h c^2)$ and Froude number $F = c/\sqrt{gh}$ where g is the acceleration of gravity and T is the coefficient of surface tension on the free surface. The distribution of the eigenvalues of the linear operator obtained from the linearized Euler equations is given in Figure 1 where $\lambda = F^{-2}$.

The bifurcation phenomena might happen near the curves C_0^l, C_0^r, E_l and E_r in Figure 1. There are a lot of results near these curves. Amick & Kirchgässner [1] proved a unique solitary wave solution of depression above the curve $E_r \backslash \{(\frac{1}{3}, 1)\}$. Buffoni & Groves [4], Dias & Iooss [10], Iooss & Kirchgässner [19], and Iooss & Pérouème [21] obtained solitary wave solutions near the right side of the curve $C_0^l \backslash \{(\frac{1}{3}, 1)\}$. Beale [2], Iooss & Kirchgässner [20], Lombardi [24] and Sun [27] used

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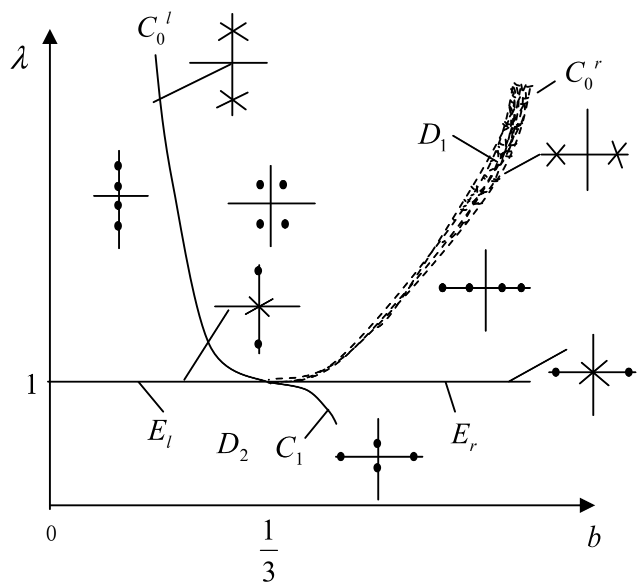

Figure 1. The distribution of the important eigenvalues of the linearized operator. The number of purely imaginary eigenvalues varies near the curves C_0^l, E_l and E_r. Dots and crosses denote simple and double eigenvalues respectively.

different methods and showed the existence of generalized solitary wave solutions which have an oscillatory tail of exponentially small amplitude at infinity below the curve $E_l\backslash\{(\frac{1}{3},1)\}$. Deng and Sun [9] also proved the existence of multi-humbp solutions. At the critical point $(\frac{1}{3},1)$, there is an eigenvalue 0 with multiplicity 4. The dominant system corresponding to this hydrodynamic problem is related to a fourth order ordinary differential equation. It is not easy to find a homoclinic orbit of this fourth order ordinary differential equation. Thus, the problem becomes much more difficult than ones near the curves C_0^l, E_l and E_r. Buffoni, Groves & Toland [5] considered the region D_1 to the left of the curve C_0^r and near the point $(\frac{1}{3},1)$. Using the properties of the Hamiltonian system, they obtained the dominant system

$$A^{(4)} + PA^{(2)} + A - A^2 = 0 \tag{1.1}$$

where P is a constant, which has been studied by many papers (cf. Buffoni, Champneys & Toland [3]). The system (1.1) has a homoclinic solution approaching 0 for some constant P. Using some results in [3], they proved the existence of infinitely many distinct solitary wave solutions. Sun & Shen [28] also investigated this case when (b,λ) approaches $(\frac{1}{3},1)$ along a specific curve in D_1. However, when (b,λ) tends to $(\frac{1}{3},1)$ from the region D_2, the problem of existence of solitary wave solutions is still open. We would like to mention that other properties of the two-dimensional water-wave problem have also been extensively investigated such as the controllability and the stabilization in a basin [7], the propagation of singularities [30], the transverse dynamics [16] and vorticity [11, 12, 14, 26].

Motivated by the above work, in this paper, we consider (b,λ) close to $(\frac{1}{3},1)$

along the curve C_1 defined by

$$\lambda = 1 - \frac{20}{169}\Big(\frac{1}{b} - 3\Big)^2. \tag{1.2}$$

Choosing the horizontal propagating direction x and the stream function as independent variables and adopting a spatial dynamics approach given by Kirchgässner [23], we obtain a different dominant system

$$A^{(4)} + PA^{(2)} - A - A^2 = 0, \tag{1.3}$$

which has a homoclinic solution approaching a nonzero equilibrium. We show that this homoclinic solution will persist when higher order terms are added by applying a perturbation method. This yields that the hydrodynamic problem has a solitary wave solution which exponentially approaches a constant function (independent of x but dependent on the stream function) as $x \to \pm\infty$. Our result differs from ones approaching 0 or a solution periodic in x near the curves C_0^l, E_l, E_r and the region D_1 close to $(\frac{1}{3}, 1)$.

This paper is organized as follows. In Section 2, we change the governing equations of the wave problem into a spatial dynamic system using the horizontal direction x and the stream. The properties of its linear operator are also given. A center manifold reduction theorem obtained by Mielke [25] yields that the spatial dynamic system is reduced to a system of ordinary differential equations with dimension 4. The adjoint operator of this linear operator is also presented. All details in this section can be found in [23] (also see [20]). Section 3 studies the case: (b, λ) near $(\frac{1}{3}, 1)$. There is an eigenvalue zero with multiplicity 4 while other eigenvalues have nonzero real parts. The reduced system consists of four ordinary differential equations. Its normal form is analyzed. Then we concentrate on that (b, λ) approaches $(\frac{1}{3}, 1)$ along the curve C_1 (see Figure 1). The scaling is introduced. Section 4 proves that the reduced system has a nonzero equilibrium. Its dominant system near this equilibrium corresponds to a fourth order ordinary differential equation (1.1), which has a homoclinic solution. Section 5 applies the idea given by Groves & Mielke [15] to prove that this homoclinic solution persists when the higher-order terms are included by using a fixed point theorem and a perturbation method. This gives the existence of a solitary wave solution which exponentially approaches a constant function.

2. Preliminary

Two-dimensional travelling waves of an inviscid, irrotational and incompressible fluid layer are studied subject to gravity and surface tension. The motion is steady in a moving frame. The governing equations are (cf. [20, 23])

$$\begin{aligned}
&\operatorname{div} \underline{v} = \operatorname{curl} \underline{v} = 0, \qquad \text{for } 0 < \eta < z(\xi),\\
&v_2 = 0, \qquad \text{on } \eta = 0,\\
&\left.\begin{aligned}&\frac{1}{2}|\underline{v}|^2 - b\kappa + \lambda z = \text{constant}\\ &v_1\partial_\xi z - v_2 = 0\end{aligned}\right\}, \qquad \text{on } \eta = z(\xi),
\end{aligned} \tag{2.1}$$

where $\kappa = \frac{\partial_\xi^2 z}{(1+(\partial_\xi z)^2)^{3/2}}$ is the curvature of the free surface $\eta = z(\xi)$ and $\underline{v} = (v_1, v_2)$ is the velocity. All quantities are in nondimensional forms. $\lambda = \frac{gh}{c^2}$ is the inverse square of the Froude number and $b = \frac{T}{\rho h c^2}$ is the Bond number. Here g is the acceleration of gravity, h is the mean depth of the layer, c is the wave speed, ρ is the density and T is the coefficient of surface tension.

We look for solutions which are bounded in the flow domain $\Omega = \{(\xi, \eta)|\ \xi \in \mathbf{R}, 0 < \eta < z(\xi)\}$. Due to the moving frame, we have a normalized flux $Q = 1$ through any simple cross section of the layer. The quiescent state of the layer is given by $\underline{v} = (1, 0)$ and $\eta(\xi) = 1$. Let $\Psi(\xi, \eta)$ denote the stream function satisfying

$$\partial_\xi \Psi = -v_2, \qquad \partial_\eta \Psi = v_1, \qquad \Psi|_{\eta=0} = 0, \qquad \Psi(\xi, z(\xi)) = 1 \tag{2.2}$$

since $Q = 1$. Introduce new variables $(x, y) \in \Omega_1 = \mathbf{R} \times (0, 1)$, which are defined by

$$x = \xi, \qquad y = \Psi(\xi, \eta).$$

Note that this is globally invertible from Ω to Ω_1 as long as $|v_1 - 1|$ and v_2 are small. Let

$$W_1 = \frac{1}{2}(v_1^2 + v_2^2 - 1), \qquad W_2 = \frac{v_2}{v_1}, \qquad \beta = W_2(\cdot, 1)$$

and $W = (W_1, W_2)^T$, which change (2.1) into (More details can be seen in [20, 23])

$$\begin{aligned} &\dot{\beta} = \frac{1}{b}(1 + \beta^2)^{3/2}\Big(W_1(\cdot, 1) + \lambda\big([h^{-1}] - 1\big)\Big), \quad \text{on } y = 1, \\ &\dot{W} = K(W)W_y, \quad \text{in } \Omega_1, \end{aligned} \tag{2.3}$$

and

$$W_2 = 0, \qquad \text{on } y = 0 \tag{2.4}$$

where the constant in (2.1) is set to $\lambda + \frac{1}{2}$ and

$$h = \Big(\frac{1 + 2W_1}{1 + W_2^2}\Big)^{1/2}, \quad K(W) = \begin{pmatrix} W_2 h & -h^3 \\ h^{-1} & W_2 h \end{pmatrix}, \quad [f] = \int_0^1 f(y)dy. \tag{2.5}$$

The system (2.3) is reversible with a reverser S defined by

$$S(\beta, W_1, W_2) = (-\beta, W_1, -W_2),$$

that is, $S(\beta, W_1, W_2)(-x)$ is also a solution whenever $(\beta, W_1, W_2)(x)$ is. A solution $(\beta, W_1, W_2)(x)$ is reversible if $S(\beta, W_1, W_2)(-x) = (\beta, W_1, W_2)(x)$.

The system (2.3) is treated as a quasilinear evolution equation in (see [20, 23])

$$\mathcal{X} = \mathbf{R} \times L_2(0, 1) \times L_2(0, 1) \ \text{ with norm } \|\cdot\| \tag{2.6}$$

where

$$u = \begin{pmatrix} \beta \\ W_1 \\ W_2 \end{pmatrix} \in \mathcal{Y} = \mathcal{D}(A) = \mathbf{R} \times H^1(0, 1) \times H^1(0, 1)$$

$$\cap \{W_2(0) = 0, W_2(1) = \beta\} \quad \text{with norm } \|\cdot\|_A. \tag{2.7}$$

A denotes the linearization of the right-hand side of (2.3) at $u = 0$ for fixed $\lambda > 0$ and $b > 0$ and it is given by

$$Au = \begin{pmatrix} \frac{1}{b}(W_1(1) - \lambda[W_1]) \\ -W_{2y} \\ W_{1y} \end{pmatrix}. \tag{2.8}$$

The scalar product in $\mathcal{X}$ is defined by

$$(u, \tilde{u}) = \beta\bar{\tilde{\beta}} + \int_0^1 \big(W_1\bar{\tilde{W}}_1 + W_2\bar{\tilde{W}}_2\big)dy. \tag{2.9}$$

Symbolically, (2.3) can be written as

$$\dot{u} = Au + F(\lambda, b, u), \qquad \text{for } u \in \mathcal{Y} = \mathcal{D}(A) \tag{2.10}$$

where

$$F(\lambda, b, u) = \begin{pmatrix} h_0 - \frac{1}{b}\big(W_1(1) - \lambda[W_1]\big) \\ W_2hW_{1y} - h^3W_{2y} + W_{2y} \\ h^{-1}W_{1y} + W_2hW_{2y} - W_{1y} \end{pmatrix} \tag{2.11}$$

and

$$h_0 = \frac{1}{b}\big(1 + \beta^2\big)^{3/2}\big(W_1(1) + \lambda([h^{-1}] - 1)\big). \tag{2.12}$$

Lemma 2.1. (1) *There exist constants C and $\sigma_0 > 0$ such that each solution $u \in \mathcal{Y} = \mathcal{D}(A)$ of the resolvent equation*

$$(A - i\sigma I)u = u^*, \tag{2.13}$$

where u^* belongs to $\mathcal{X}$ and σ is a real number with $|\sigma| \geq \sigma_0$, satisfies

$$\|u\|_{\mathcal{Y}} \leq C\|u^*\|_{\mathcal{X}}, \tag{2.14}$$

$$\|u\|_{\mathcal{X}} \leq \frac{C}{|\sigma|}\|u^*\|_{\mathcal{X}}. \tag{2.15}$$

(2) *The complex number $\sigma \neq 0$ is an eigenvalue of A if and only if*

$$(\lambda - b\sigma^2) - \sigma\coth\sigma = 0. \tag{2.16}$$

For $\lambda = 1$ and $b \neq \frac{1}{3}$, 0 is a double eigenvalue while it is an eigenvalue with multiplicity 4 for $\lambda = 1$ and $b = \frac{1}{3}$. The spectrum $\tilde{\sigma}(A)$ of A consists of isolated eigenvalues of finite algebraic multiplicity and $\tilde{\sigma}(A) \cap i\mathbf{R}$ is a finite set.

(3) *The adjoint operator* A^* *of* A *is given by*

$$A^*u^* = \Big(- W_1^*(1), \quad -W_{2y}^* + W_2^*(1), \quad W_{1y}^* \Big) \tag{2.17}$$

for $u^* = (\beta^*, W_1^*, W_2^*) \in \mathcal{D}(A^*) = \mathbf{R} \times H^1(0,1) \times H^1(0,1) \cap \{W_2^*(0) = 0, W_2^*(1) = -\frac{1}{b}\beta^*\}$.

This lemma can be found in [20, 23]. The distribution of eigenvalues of A is given in Figure 1. Using a center manifold reduction theorem given by Mielke [25], we have the following lemma [20, 23].

Lemma 2.2. (2.10) *has a finite dimensional center manifold* $M_C^{\tilde{\lambda}}$ *of class* C^k *for any positive integer* k *where* $\tilde{\lambda} = (\lambda, b)$. *The reduced system on* $M_C^{\tilde{\lambda}}$ *preserves reversibility, which* (2.10) *has.*

3. Normal form

In order to use some known results in [23], we set

$$\lambda = 1 + \nu, \qquad \frac{1}{b} = \delta + 3 \tag{3.1}$$

for small ν and δ to be specified later, that is, (b, λ) is near the critical point $(\frac{1}{3}, 1)$. The linear operator A in (2.10) for $b = \frac{1}{3}$ and $\lambda = 1$ has an eigenvalue 0 with multiplicity 4. Its eigenvector and generalized eigenvectors are given by (see [23])

$$U_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -1 \\ 0 \\ -y \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 \\ -\frac{1}{2}y^2 \\ 0 \end{pmatrix}, \quad U_4 = \begin{pmatrix} \frac{1}{6} \\ 0 \\ \frac{1}{6}y^3 \end{pmatrix} \tag{3.2}$$

while the eigenvector and generalized eigenvectors of the eigenvalue 0 corresponding to the adjoint operator A^* are given by

$$U_1^* = 45 \begin{pmatrix} 0 \\ \frac{3}{56} - \frac{5}{42}y^2 + \frac{1}{24}y^4 \\ 0 \end{pmatrix}, \quad U_2^* = 45 \begin{pmatrix} \frac{1}{42} \\ 0 \\ -\frac{5}{21}y + \frac{1}{6}y^3 \end{pmatrix},$$
$$U_3^* = 45 \begin{pmatrix} 0 \\ \frac{1}{6} - \frac{1}{2}y^2 \\ 0 \end{pmatrix}, \quad U_4^* = 45 \begin{pmatrix} \frac{1}{3} \\ 0 \\ -y \end{pmatrix}. \tag{3.3}$$

Moreover,

$$\begin{aligned} &SU_1 = U_1, \qquad SU_2 = -U_2, \qquad SU_3 = U_3, \qquad SU_4 = -U_4, \\ &SU_1^* = U_1^*, \qquad SU_2^* = -U_2^*, \qquad SU_3^* = U_3^*, \qquad SU_3^* = -U_3^*, \\ &(U_i, U_j^*) = 0 \text{ for } i \neq j, \quad (U_i, U_i^*) = 1, \end{aligned}$$

where $i, j = 1, 2, 3, 4$.

Since the spectrum of A consists entirely of isolated eigenvalues of finite algebraic multiplicity, we can write

$$u = AU_1 + BU_2 + CU_3 + DU_4 + v_2$$

where A, B, C and D are real, and v_2 is a linear combination of eigenvectors and generalized eigenvectors corresponding to the rest of eigenvalues. The reverser S (for simplicity, we still use S to denote the reverser since no confusion arises) is now given by

$$S(A, B, C, D) = (A, -B, C, -D).$$

The center manifold reduction Lemma 2.2 shows that the set of all bounded solutions of the system (2.10) is determined solely by the ordinary differential equations about A, B, C and D

$$\dot{X} = LX + F_0(\nu, \delta, A, B, C, D), \tag{3.4}$$

which preserves the reversibility, where $X = (A, B, C, D)^T$,

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$F_0(\nu, \delta, 0, 0, 0, 0) = 0, \quad DF_0(0, 0, 0, 0, 0, 0) = 0, \quad SL = -LS, \quad SF_0 = -F_0 S.$$

For the sake of simplicity, we first let $\nu = \delta = 0$. There exists a change of variables from X to Y which is close to identity, and transforms the system (3.4) into

$$\dot{Y} = LY + \mathcal{P}(Y) + o(|Y|^n), \tag{3.5}$$

where $\mathcal{P}$ is a polynomial of degree $\leq n$ (n is arbitrary but fixed) with $\mathcal{P}(0) = 0$ and $D\mathcal{P}(0) = 0$ (see [13, 18]). For notational simplicity, in the following we still use X to replace Y. Moreover, $\mathcal{P}$ satisfies

$$S\mathcal{P}(X) = -\mathcal{P}(SX) \tag{3.6}$$

and

$$D\mathcal{P}(X)L^*X = L^*\mathcal{P}(X) \tag{3.7}$$

for any X where $L^* = \bar{L}^T$.

Let $\mathcal{P} = (P_1, P_2, P_3, P_3)^T$ and define a differential operator

$$D^* = A\frac{\partial}{\partial B} + B\frac{\partial}{\partial C} + C\frac{\partial}{\partial D}. \tag{3.8}$$

Then, (3.7) is equivalent to

$$D^*\mathcal{P} = L^*\mathcal{P},$$

which gives

$$D^*P_1 = 0, \qquad D^*P_2 = P_1, \qquad D^*P_3 = P_2, \qquad D^*P_4 = P_3. \tag{3.9}$$

It is easy to find that the following are first integrals of $D^* = 0$ (see [17])

$$\begin{aligned} &u_1 = A, \quad u_2 = B^2 - 2AC, \quad u_3 = 3A^2D - 3ABC + B^3, \\ &u_4 = 3B^2C^2 - 6B^3D - 8AC^3 + 18ABCD - 9A^2D^2, \end{aligned} \tag{3.10}$$

where

$$u_3^2 = u_2^3 - u_1^2 u_4.$$

Then we have the following lemma [17].

Lemma 3.1. (1) *Assume that K is a polynomial of X with degree n and $D^*K = 0$. Then*

$$K(X) = K_0(u_1, u_2, u_3, u_4) = K_1(u_1, u_2, u_4) + u_3 K_2(u_1, u_2, u_4),$$

where K_0, K_1 and K_2 are polynomials of their arguments.

(2) *$\mathcal{P}$ in (3.5) can be written as*

$$\begin{aligned} \mathcal{P}(X) =& Q_4(A, u_2, u_4)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + Q_2(A, u_2, u_4)\begin{pmatrix} 0 \\ A \\ B \\ C \end{pmatrix} + Q_5(u_2, u_4)\begin{pmatrix} 0 \\ u_2 \\ u_5 \\ u_6 \end{pmatrix} \\ & + Q_1(A, u_2, u_4)\begin{pmatrix} u_3 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} + Q_3(A, u_2, u_4)\begin{pmatrix} 0 \\ 0 \\ u_3 \\ u_7 \end{pmatrix}, \end{aligned} \tag{3.11}$$

where Q_j are polynomials in their arguments for $j = 1, 2, \cdots, 5$, and

$$\begin{aligned} &u_5 = -3AD + BC, \ u_6 = -3BD + 2C^2, \ u_7 = 3ABD - 2AC^2 + B^2C, \\ &u_8 = -3ACD + 3B^2D - BC^2, \ u_9 = 3BCD - \frac{4}{3}C^3 - 3AD^2, \\ &Q_4(A, u_2, u_4) = Q_{41}A + Q_{42}A^2 + Q_{43}u_2 + O(|(A, B, C, D)|^3), \\ &Q_2(A, u_2, u_4) = Q_{20} + Q_{21}A + O(|(A, B, C, D)|^2), \\ &Q_5(u_2, u_4) = Q_{50} + O(|(A, B, C, D)|^2). \end{aligned}$$

A similar analysis holds if ν and δ are not zero (see [18]). By (3.6), the expression of F in (2.11) and the results in [17], (3.5) becomes

$$\begin{aligned}\dot{A} &= B + P_1(\nu,\delta,A,B,C,D) + f_1(\nu,\delta,A,B,C,D),\\ \dot{B} &= C + P_2(\nu,\delta,A,B,C,D) + f_2(\nu,\delta,A,B,C,D),\\ \dot{C} &= D + P_3(\nu,\delta,A,B,C,D) + f_3(\nu,\delta,A,B,C,D),\\ \dot{D} &= P_4(\nu,\delta,A,B,C,D) + f_4(\nu,\delta,A,B,C,D)\end{aligned} \tag{3.12}$$

where f_k are of order $O(|(A,B,C,D)||(\nu,\delta,A,B,C,D)|^n)$, and polynomials P_k of degree n are given by

$$\begin{aligned}P_1 &= O(|(A,B,C,D)|^3) + |(\nu,\delta)||(A,B,C,D)|^2 + |\nu||\delta||(A,B,C,D)|,\\ P_2 &= p_{24}A^2 + p_{25}(B^2-2AC) + O(|(A,B,C,D)|^3)\\ &\quad + |(\nu,\delta)||(A,B,C,D)|^2 + |\nu||\delta||(A,B,C,D)|,\\ P_3 &= p_{24}AB + p_{25}(BC-3AD) + O(|(A,B,C,D)|^3)\\ &\quad + |(\nu,\delta)||(A,B,C,D)|^2 + |\nu||\delta||(A,B,C,D)|,\\ P_4 &= p_{41}\nu A + p_{42}\nu C + p_{43}\delta C + p_{44}A^2 + p_{45}(B^2-2AC) + p_{25}(2C^2-3BD)\\ &\quad + p_{26}AC + O(|(A,B,C,D)|^3) + |(\nu,\delta)||(A,B,C,D)|^2 + |\nu||\delta||(A,B,C,D)|\end{aligned} \tag{3.13}$$

for $k = 1,2,3,4$. A direct calculation similar to ones in [20,23] yields some important coefficients

$$p_{41} = -45, \quad p_{42} = \frac{30}{7}, \quad p_{43} = -5, \quad p_{44} = \frac{135}{2}. \tag{3.14}$$

Remark 3.1. (1) [23] did not analyze the normal form of (3.5) but calculated all coefficients of terms with degree ≤ 2.

(2) Note that P_2 in (3.13) has no terms with order of $O\big((|\nu|+|\delta|)|(A,B,C,D)|\big)$. Thus, the system (3.12) is not discussed in [17] (see the system (2.19) in [17]).

In the following, we take

$$\nu = -k_1\mu, \qquad \delta = k_2\sqrt{\mu} \tag{3.15}$$

for small $\mu > 0$ where $k_1 > 0$ and k_2 are constants. It means that (b,λ) approaches $(\frac{1}{3},1)$ from the region D_2 (see Figure 1). Then let

$$A = \mu\tilde{A},\ B = \mu^{5/4}\tilde{B},\ C = \mu^{3/2}\tilde{C},\ D = \mu^{7/4}\tilde{D},\ x = \mu^{-1/4}\tilde{x}.$$

By choosing large n and using (3.13)-(3.15), the system (3.12) is changed into (dropping the tilde)

$$\begin{aligned}\dot{A} &= B + R_1(\mu,A,B,C,D),\\ \dot{B} &= C + p_{24}\sqrt{\mu}A^2 + p_{25}\mu(B^2-2AC) + R_2(\mu,A,B,C,D),\\ \dot{C} &= D + p_{24}\sqrt{\mu}AB + p_{25}\mu(BC-3AD) + R_3(\mu,A,B,C,D),\\ \dot{D} &= 45k_1A - 5k_2C - \frac{30}{7}k_1\sqrt{\mu}C + \frac{135}{2}A^2 + p_{45}\sqrt{\mu}(B^2-2AC)\\ &\quad + p_{25}\mu(2C^2-3BD) + p_{26}\sqrt{\mu}AC + R_4(\mu,A,B,C,D),\end{aligned} \tag{3.16}$$

where

$$N_3(\mu, A, B, C, D) \triangleq \begin{pmatrix} R_1(\mu, A, B, C, D) \\ R_2(\mu, A, B, C, D) \\ R_3(\mu, A, B, C, D) \\ R_4(\mu, A, B, C, D) \end{pmatrix}$$

$$= \begin{pmatrix} \mu^{5/4}O(|(A, B, C, D)|) \\ \mu O(|(A, B, C, D)|) \\ \mu^{3/4}O(|(A, B, C, D)|) \\ \sqrt{\mu}O(|(A, B, C, D)|) \end{pmatrix}$$

$$= \sqrt{\mu}O(|(A, B, C, D)|). \tag{3.17}$$

For $\mu = 0$, the system (3.16) corresponds to an ordinary differential equation

$$A^{(4)} + 5k_2A^{(2)} - 45k_1A - \frac{135}{2}A^2 = 0. \tag{3.18}$$

If letting

$$k_2 = -\frac{13}{2\sqrt{5}}\sqrt{k_1}, \quad c = \frac{\sqrt[4]{5}\sqrt[4]{k_1}}{2\sqrt{2}} \tag{3.19}$$

and following the solution form in [28], then (3.18) has a solution given by

$$H_A(x) = \frac{35}{36}k_1 \operatorname{sech}^4(cx) - \frac{2}{3}k_1 \tag{3.20}$$

which exponentially tends to $-\frac{2}{3}k_1$.

Remark 3.2. (1) When (b, λ) approaches $(\frac{1}{3}, 1)$ from the region D_1, [5] and [28] obtained the dominant system

$$A^{(4)} + PA^{(2)} + A - A^2 = 0 \tag{3.21}$$

where P is a constant. We here got a different dominant system from (3.18)

$$A^{(4)} + PA^{(2)} - A - A^2 = 0. \tag{3.22}$$

(2) From (3.1), (3.15) and (3.19), we know that

$$\lambda = 1 - \frac{20}{169}\left(\frac{1}{b} - 3\right)^2, \tag{3.23}$$

which implies that (b, λ) approaches $(\frac{1}{3}, 1)$ along the curve C_1 (see Figure 1) .

In the following, we consider (b, λ) close to $(\frac{1}{3}, 1)$ along the curve C_1.

4. Nonzero equilibrium

Note that the solution in (3.20) approaches $-\frac{2}{3}k_1$, which corresponds to an equilibrium $(-\frac{2}{3}k_1, 0, 0, 0)^T$ of (3.16) for $\mu = 0$. In this section, we will prove that for small $\mu > 0$, (3.16) has a nonzero equilibrium near $(-\frac{2}{3}k_1, 0, 0, 0)^T$.

Let

$$\underline{d} = \underline{\tilde{d}} + (-\frac{2}{3}k_1, 0, 0, 0)^T \tag{4.1}$$

where $\underline{\tilde{d}} = (d_1, d_2, d_3, d_4)^T$ is to be determined. Plugging (4.1) into (3.16) yields

$$\underline{\tilde{d}} = \mathcal{M}_1(\mu, \underline{\tilde{d}}). \tag{4.2}$$

Here $\mathcal{M}_1$ is smooth and

$$\mathcal{M}_1(\mu, \underline{\tilde{d}}) = \begin{pmatrix} \frac{3}{2k_1} d_1^2 + a_1(\mu, d_1, d_2, d_3, d_4) \\ a_2(\mu, d_1, d_2, d_3, d_4) \\ a_3(\mu, d_1, d_2, d_3, d_4) \\ a_4(\mu, d_1, d_2, d_3, d_4) \end{pmatrix} \tag{4.3}$$

where a_k are the rest terms and of order $\sqrt{\mu}O(|(d_1, d_2, d_3, d_4)|)$ for bounded d_k, $k = 1, 2, 3, 4$. Thus, we apply the fixed point theorem to (4.3) and have the following lemma by using the uniqueness and the reversibility.

Lemma 4.1. *For a closed ball $\bar{B}_r(0) \subset \mathbf{R}^4$ with a radius $r = O(\mu^{1/4})$, the map $\mathcal{M}_1$ in (4.3) has a fixed point $\underline{\tilde{d}} \in \bar{B}_r(0) \subset \mathbf{R}^4$ for small $\mu > 0$. Moreover, $S\underline{d} = \underline{d}$ and*

$$|\underline{\tilde{d}}| \leq M\sqrt{\mu} \tag{4.4}$$

where $M > 0$ is a constant.

Let $X = \tilde{X} + \underline{d}$ where $X = (A, B, C, D)^T$ and $\tilde{X} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})^T$ and plug it into (3.16). Note that $\underline{d}$ is an equilibrium of (3.16). We obtain that from (3.17)

$$\dot{\tilde{X}} = \tilde{L}\tilde{X} + \tilde{L}_\mu\tilde{X} + \tilde{N}_2(\tilde{X}) + \hat{N}_2(\mu, \tilde{X}) + \tilde{N}_3(\mu, \tilde{X}) \tag{4.5}$$

where

$$\tilde{L}\tilde{X} = \begin{pmatrix} \tilde{B} \\ \tilde{C} \\ \tilde{D} \\ -5k_2\tilde{C} - 45k_1\tilde{A} \end{pmatrix}, \quad \tilde{L}_\mu\tilde{X} = \begin{pmatrix} 0 \\ l_{42} \\ l_{43} \\ l_{44} \end{pmatrix}, \quad \tilde{N}_2(\tilde{X}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{135}{2}\tilde{A}^2 \end{pmatrix},$$

$$\hat{N}_2(\mu, \tilde{X}) = \begin{pmatrix} 0 \\ p_{24}\sqrt{\mu}\tilde{A}^2 + p_{25}\mu(\tilde{B}^2 - 2\tilde{A}\tilde{C}) \\ p_{24}\sqrt{\mu}\tilde{A}\tilde{B} + p_{25}\mu(\tilde{B}\tilde{C} - 3\tilde{A}\tilde{D}) \\ p_{45}\sqrt{\mu}(\tilde{B}^2 - 2\tilde{A}\tilde{C}) + p_{25}\mu(2\tilde{C}^2 - 3\tilde{B}\tilde{D}) + p_{26}\sqrt{\mu}\tilde{A}\tilde{C} \end{pmatrix},$$

$$\tilde{N}_3(\mu, \tilde{X}) = N_3(\mu, \tilde{X} + \underline{d}) - N_3(\mu, \underline{d}), \tag{4.6}$$

$$l_{42} = 2p_{24}(d_1 - \frac{2}{3}k_1)\sqrt{\mu}\tilde{A} + 2p_{25}\mu\big(- d_3\tilde{A} + d_2\tilde{B} - (d_1 - \frac{2}{3}k_1)\tilde{C}\big),$$

$$l_{43} = p_{24}\sqrt{\mu}\big(d_2\tilde{A} + (d_1 - \frac{2}{3}k_1)\tilde{B}\big) + p_{25}\mu\big(- 3d_4\tilde{A} + d_3\tilde{B} + d_2\tilde{C} - 3(d_1 - \frac{2}{3}k_1)\tilde{D}\big),$$

$$\begin{aligned} l_{44} = &\big(135d_1 - 2p_{45}d_3\sqrt{\mu} + p_{26}d_3\sqrt{\mu}\big)\tilde{A} + 2p_{45}d_2\sqrt{\mu}\tilde{B} \\ &- \big(2p_{45}(d_1 - \frac{2}{3}k_1) + \frac{30}{7}k_1 + p_{26}(d_1 - \frac{2}{3}k_1)\big)\sqrt{\mu}\tilde{C} \\ &+ p_{25}\mu(-3d_4\tilde{B} + 4d_3\tilde{C} - 3d_2\tilde{D}). \end{aligned}$$

Moreover, it is easy to check that for any bounded $\tilde{X}$

$$|\tilde{L}_\mu\tilde{X}| + |\hat{N}_2(\mu, \tilde{X})| + |\tilde{N}_3(\mu, \tilde{X})| = \sqrt{\mu}O(|\tilde{X}|) \tag{4.7}$$

where (3.17) and (4.4) are used. From Lemma 4.1, we know that $S\underline{d} = \underline{d}$ so the system (4.5) is reversible too.

Therefore, the problem of the existence of solitary wave solutions of (3.16) is equivalent to one of (4.5).

Remark 4.1. From the expression of $\tilde{L}$ in (4.6), we got the same system (3.21) after scaling for $P = -\frac{13}{6}$.

Define

$$\mathcal{H}(x) = (H_a, H_b, H_c, H_d)^T(x) = (H_A + \frac{2}{3}k_1, \dot{H}_A, \ddot{H}_A, \dddot{H}_A)^T(x) \tag{4.8}$$

where H_A is given in (3.20). Thus, $\mathcal{H}(x)$ is a homoclinic solution of the system

$$\dot{\tilde{X}} = \tilde{L}\tilde{X} + \tilde{N}_2(\tilde{X}) \tag{4.9}$$

and from (3.19)

$$|\mathcal{H}(x)| \le Me^{-c|x|}, \quad \text{for } x \in \mathbf{R} \tag{4.10}$$

where $M > 0$ is a constant. Moreover,

$$S\mathcal{H}(-x) = \mathcal{H}(x). \tag{4.11}$$

Now we will prove that the homoclinic solution $\mathcal{H}(x)$ will persist when higher order terms are considered, i.e., we will show that there is a homoclinic solution of (4.5).

5. Existence of solitary wave solution

Theorem 5.1. *For* $0 < \mu \le \mu_0$, (4.5) *has a reversible homoclinic solution where* μ_0 *is a positive constant.*

Using the relations among the systems (2.10), (3.16) and (4.5), this theorem shows that (3.16) has a homoclinic solution exponentially approaching a nonzero equilibrium as $x \to \pm\infty$ so (2.10) has a solitary wave solution exponentially approaching a constant function (independent of x but depend on y) as $x \to \pm\infty$.

Remark 5.1. When (b, λ) approaches E_r, E_l, C_0^l or $(\frac{1}{3}, 1)$ in the region D_1, the solitary wave solutions exponentially approach 0 or a periodic solution in x. The solution we here obtained exponentially approaches a constant function.

The proof is divided into two steps. We first show that (4.5) has a solution exponentially approaching 0 as $x \to \infty$. Then we use the reversibility condition to obtain that this solution can be extended to $x \in (-\infty, 0)$. The basic idea comes from [15] (also see [8]).

Step 1. Solution of (4.5) for $x \in [0, \infty)$.

In order to find a solution of (4.5) near $\mathcal{H}(x)$ given in (4.8), we assume that the solution of (4.5) has a form

$$\tilde{S}(x;\mu) = \mathcal{H}(x) + Z(x), \qquad \text{for } x \geq 0 \tag{5.1}$$

which exponentially tends to 0 as $x \to \infty$ where $\tilde{Z}$ is a perturbation term to be determined and also exponentially tends to 0 as $x \to \infty$.

Plugging (5.1) into (4.5) yields the equation about $\tilde{Z}$

$$\dot{Z} = \mathcal{L}(x)(Z) + N_1(x, Z; \mu) \tag{5.2}$$

where $\mathcal{L}(x) = \tilde{L} + d\tilde{N}_2[\mathcal{H}(x)]$, d means taking the Fréchet derivative, and

$$\begin{aligned} N_1(x, Z; \mu) =& \tilde{L}_\mu \mathcal{H} + \tilde{L}_\mu Z - d\tilde{N}_2[\mathcal{H}(x)]Z + \tilde{N}_2(\mathcal{H} + Z) - \tilde{N}_2(\mathcal{H}) \\ &+ \hat{N}_2(\mu, \mathcal{H} + Z) + \tilde{N}_3(\mu, \mathcal{H} + Z). \end{aligned} \tag{5.3}$$

In the following, we let M be a positive constant. Then N_1 satisfies the following estimates by using (4.7) and (4.10).

Lemma 5.1. *If* $|Z| + |Z_1| + |Z_2| \leq M_0$ *for some positive constant* M_0*, then for* $x \geq 0$*,*

$$\begin{aligned} &\big|N_1(x, Z; \mu)\big| \leq M\Big(\sqrt{\mu}(e^{-cx} + |Z|) + |Z|^2\Big), \\ &\big|N_1(x, Z_1; \mu) - N_1(x, Z_2; \mu)\big| \leq M\Big(\sqrt{\mu} + |Z_1| + |Z_2|\Big)|Z_1 - Z_2|. \end{aligned} \tag{5.4}$$

Obviously, the solution of (5.2) exists if x is in a finite interval and an initial condition is given. In order to prove the existence of solutions for $x \geq 0$, we change (5.2) to integral equations and then apply the fixed point theorem to prove the existence of a fixed point of the integral equations.

Now we consider the linear system of (5.2)

$$\dot{Z} = \tilde{L}(Z) + (\mathcal{L}(x) - \tilde{L})(Z) \tag{5.5}$$

where

$$\mathcal{L}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{525}{4}k_1 \mathrm{sech}^4(cx) - 45k_1 & 0 & -5k_2 & 0 \end{pmatrix} \tag{5.6}$$

and $\tilde{L}$ is given in (4.6). The linear system (5.5) yields the following differential equation

$$A^{(4)} + 5k_2 A^{(2)} - \big(\frac{525}{4} k_1 \mathrm{sech}^4(cx) - 45k_1\big) A = 0. \tag{5.7}$$

Lemma 5.2. *The equation* (5.7) *with initial conditions* $A'(0) = A^{(3)}(0) = 0$ *has a trivial solution only in* $L_2(0,\infty)$ (see Lemma 1 in [28]).

Obviously, $\mathcal{L}(x)$ exponentially tends to $\tilde{L}$ as $x \to \pm\infty$ and $\tilde{L}$ has four eigenvalues $\pm\lambda_1, \pm\lambda_2$ where

$$\lambda_1 = \frac{3\sqrt[4]{5}}{\sqrt{2}} \sqrt[4]{k_1}, \qquad \lambda_2 = \sqrt{2}\sqrt[4]{5}\sqrt[4]{k_1}.$$

Now we look for a fundamental matrix of the system (5.5). Since $\mathcal{H}(x)$ is a solution of the system (4.9), we obtain that

$$s_1(x) = \frac{d}{dx}\mathcal{H}(x) \tag{5.8}$$

is a solutions of the system (5.5) that satisfies

$$s_1(0) = \big(0, -\frac{35\sqrt{5}}{72} k_1^{3/2}, 0, \frac{1225}{288} k_1^2\big)^T \tag{5.9}$$

and by (4.10)

$$|s_1(x)| \le M\mathrm{e}^{-cx} \text{ for } x \in [0,\infty). \tag{5.10}$$

Using the relationship between $\mathcal{L}(x)$ and $\tilde{L}$ (see Problem 29 in Chapter 3 in the book by Coddington & Levinson [6]), let $s_1(x)$, $s_2(x)$, $u_1(x)$ and $u_2(x)$ be linearly independent solutions of (5.5) such that $u_1(x), u_2(x) > 0$ for large x. By Lemma 5.2, we may choose

$$u_1[2](0) = u_1[4](0) = u_2[2](0) = u_2[4](0) = 0 \tag{5.11}$$

and for $x \in [0,\infty)$

$$|u_1(x)| + |u_2(x)| \le M\mathrm{e}^{cx}, \qquad |s_2(x)| \le M\mathrm{e}^{-cx}. \tag{5.12}$$

Here $f[i]$ denotes the i-th component of f.

Define

$$\mathcal{B}(x) = \big(s_1(x), s_2(x), u_1(x), u_2(x)\big)$$

which is a fundamental matrix of (5.5). Note that the trace of $\mathcal{L}(x)$ in (5.6) is equal to 0, which gives that $\mathrm{Det}\mathcal{B}(x)$ is independent of x and equal to $\mathrm{Det}\mathcal{B}(0) = M$.

Assume that the fundamental set of solutions for the adjoint equation of (5.5) is

$$\big\{s_1^*(x), s_2^*(x), u_1^*(x), u_2^*(x)\big\}$$

which is the dual of $\{s_1(x), s_2(x), u_1(x), u_2(x)\}$ in the sense of the Euclidean inner product on $\mathbf{R}^4$ for each fixed x. It follows from (5.10) and (5.12) that for $x \in [0,\infty)$

$$|s_1^*(x)| + |s_2^*(x)| \le M\mathrm{e}^{cx}, \qquad |u_1^*(x)| + |u_2^*(x)| \le M\mathrm{e}^{-cx}. \tag{5.13}$$

The solution of (5.2) that decays to zero at infinity can be found as

$$Z = \mathcal{G}(Z;\mu) \tag{5.14}$$

where

$$\mathcal{G}(Z;\mu) = \sum_{k=1}^{2}\int_0^x \langle N_1(t,Z;\mu), s_k^*(t)\rangle dt\ s_k(x) - \sum_{k=1}^{2}\int_x^\infty \langle N_1(t,Z;\mu), u_k^*(t)\rangle dt\ u_k(x)$$

and $\langle\cdot\rangle$ denotes the Euclidean inner product on $\mathbf{R}^4$.

Take any positive constant τ such that $\tau/c \in (\frac{1}{2},1)$. Consider (5.14) as a fixed point problem in a Banach space

$$E_\tau = \left\{Z \in C(0,\infty) \mid \sup_{x\in[0,\infty)} \{|Z(x)|\mathrm{e}^{\tau x}\} < \infty\right\}$$

with the norm

$$\|Z\|_\tau = \sup\left\{|Z(x)|\mathrm{e}^{\tau x}|\ x \in [0,\infty)\right\},$$

which implies that Z exponentially tends to zero as $x \to \infty$. It is easy to obtain the following lemma using Lemma 5.1, (5.10), and (5.12) and (5.13).

Lemma 5.3. *For $x \geq 0$, the function $\mathcal{G}$ satisfies*

$$\begin{aligned}
&\big\|\mathcal{G}(Z;\mu)\big\|_\tau \leq M\Big(\|Z\|_\tau^2 + \sqrt{\mu}(1+\|Z\|_\tau)\Big),\\
&\big\|\mathcal{G}(Z_1;\mu) - \mathcal{G}(Z_2;\mu)\big\|_\tau \leq M\Big(\sqrt{\mu} + \|Z_1\|_\tau + \|Z_2\|_\tau\Big)\|Z_1 - Z_2\|_\tau
\end{aligned} \tag{5.15}$$

for $Z, Z_1, Z_2 \in E_\tau$.

Let $\bar{B}_{\tilde{r}}(0) \subset E_\tau$ be a closed ball in E_τ with a radius $\tilde{r} = O(\mu^{1/4})$. Thus, the contraction mapping theorem yields that $\mathcal{G}$ has a fixed point Z in $\bar{B}_{\tilde{r}}(0)$ for small $\mu > 0$, i.e., (5.14) has a unique solution $Z(x;\mu)$ satisfying

$$\|Z(x;\mu)\|_\tau \leq M\sqrt{\mu}. \tag{5.16}$$

Using the same argument as that for (5.16) and an extension of a contraction mapping principle in [29], we can show that $Z(x;\mu)$ is smooth in its arguments.

Step 2. Solution of (4.5) for $x \in (-\infty, 0]$.

Using (4.11), (5.1), (5.11) and (5.14), it is easy to verify that the following equation

$$(I - S)\tilde{\mathcal{S}}_1(0;\mu) = 0 \tag{5.17}$$

is true.

To construct the solution for $x < 0$, we know from the reversibility of the system (4.5) that both $\tilde{\mathcal{S}}_1(x;\mu)$ and $S\left(\tilde{\mathcal{S}}_1(-x;\mu)\right)$ are solutions of (4.5) and at $x = 0$ from (5.17)

$$S\left(\tilde{\mathcal{S}}_1(0;\mu)\right) = \tilde{\mathcal{S}}_1(0;\mu).$$

Thus, by the uniqueness of the solution for an initial value problem, we can define a solution of (4.5) as

$$\tilde{\mathcal{S}}_2(x) = \begin{cases} \tilde{\mathcal{S}}_1(x;\mu), & \text{for } x \geq 0,\\ S\left(\tilde{\mathcal{S}}_1(-x;\mu)\right), & \text{for } x \leq 0. \end{cases}$$

Then $S\tilde{\mathcal{S}}_2(-x) = \tilde{\mathcal{S}}_2(x)$. Thus, the solution $\tilde{\mathcal{S}}_2(x)$ of (4.5) is a reversible homoclinic solution. This completes the proof of Theorem 5.1.

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