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ANALYSIS OF A STOCHASTIC TREE-GRASS MODEL WITH MEAN-REVERTING ORNSTEIN-UHLENBECK PROCESS

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Abstract Considering the survival regulation mechanisms of trees and grasses in savannas ecosystems, the stochastic variability of ecosystems and the effects of fire, a stochastic tree-grass model with mean-reverting Ornstein-Uhlenbeck process is developed and investigated in this paper. Firstly, the biological and environmental components of the tree-grass model and the biological significance of each parameter are described, while the mean-reverting Ornstein-Uhlenbeck process is introduced and its biological significance is explained. Then we list some dynamical properties of the model and give proofs. The existence and moment estimates of the global solution of the stochastic model and sufficient conditions for the existence of a stationary distribution are given. In addition, we give sufficient conditions for extinction of species. Finally, we verify that the theories are valid by numerical simulation.

Keywords Tree-grass model, Ornstein-Unlenbeck process, global solution, stationary distribution, extinction.

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1. Introduction

While grasslands and forests are both extremely important ecosystems in nature, trees and grasses mix together to create a landscape that is neither grassland nor forest, which is called savannas ecosystems [14]. Disturbance is central to the coexistence of trees and grasses in humid climates, most typically in the form of fire [1]. This shows that the tree-grass system affected by fire is very complex, mathematical model should be used to deal with this problem.

Several scholars have now developed systems of ordinary differential equations to describe the asymmetric competition between trees and grasses. Tamen [15] presented a simple model of tree-grass dynamics that treats fire as a continuous event with positive initial conditions $G(0) = G_0$ and $T(0) = T_0$ as follow

$$\begin{cases} \dfrac{\mathrm{d}G}{\mathrm{d}t} = (\gamma_G - \delta_{G0})G - b_1 G^2 - cTG - \lambda_1 f G, \\ \dfrac{\mathrm{d}T}{\mathrm{d}t} = (\gamma_T - \delta_T)G - b_2 T^2 - \lambda_2 f \dfrac{G^2}{G^2 + \alpha^2} T, \end{cases}$$

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where T and G are tree and grass biomasses, γ_T and γ_G are tree and grass biomass productivity, δ_T and δ_{G0} are natural mortality rate, b_1 and b_2 are mortality associated with intraspecific competition, f is the fire frequency, λ_1 and λ_2 are the mortality rate caused by fire, c is mortality due to competition between trees and grasses, $\frac{G^2}{G^2+\alpha^2}$ describes the interrelationship between grass biomass and fire intensity. To facilitate the study of the dynamical properties of the model, we consider to simplify the parameters of the model by combining parameters that describe the natural variation in organisms

$$\begin{cases} \dfrac{\mathrm{d}G}{\mathrm{d}t} = G(a_1 - b_1 G - cT) - \lambda_1 f G, \\ \dfrac{\mathrm{d}T}{\mathrm{d}t} = T(a_2 - b_2 T) - \lambda_2 f \dfrac{G^2}{G^2+\alpha^2} T, \end{cases}$$

where $a_1 = \gamma_G - \delta_{G0}$ and $a_2 = \gamma_T - \delta_T$ denotes the natural growth rate of grass and trees respectively. However, populations in nature are constantly subject to environmental change and are in a state of dynamic flux. Most of the parameters involved in ecological dynamics, such as growth rates and mortality rates, therefore fluctuate stochastically around some mean value [5, 9]. Mao et al. [13] also demonstrated that stochastic noise can have an unstable effect on biological populations, implies that the introduction of stochastic noise would bring biological models closer to realistic populations. We assume the natural growth rate of the grass a_1 and the tree a_2 to be stochastic variables $a_1(t)$ and $a_2(t)$. In general, there are two common approaches to model real-world stochastic effects, namely linear perturbations [4, 7, 10] and the mean-reverting Ornstein-Uhlenbeck process. According to Zhou [18], we know that if a linear perturbation is used as the parameter of a stochastic process, its stochastic fluctuations will become large in a small time interval, which is an unreasonable result. Therefore we let $a_1(t)$ and $a_2(t)$ be fluctuated by the mean-reverting Ornstein-Uhlenbeck process [12, 16, 17]

$$\begin{cases} \mathrm{d}a_1(t) = \beta_1[\bar{a}_1 - a_1(t)]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}a_2(t) = \beta_2[\bar{a}_2 - a_2(t)]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \end{cases}$$

where $\bar{a}_1$ and $\bar{a}_2$ denote the average of $a_1(t)$ and $a_2(t)$ over time, i.e. $\mathbb{E}[a_1(t)] = \bar{a}_1$, $\mathbb{E}[a_2(t)] = \bar{a}_2$. $B_i(t)(i = 1, 2)$ denotes independent standard Brownian motion, $\beta_i(i = 1, 2)$ indicates the speed of reversion, σ_i indicates intensity of fluctuations. Thus we can construct the following tree-grass model with the mean-reverting Ornstein-Uhlenbeck process

$$\begin{cases} \mathrm{d}G(t) = \big\{G(t)[a_1(t) - b_1 G(t) - cT(t)] - \lambda_1 f G(t)\big\}\mathrm{d}t, \\ \mathrm{d}T(t) = \big\{T(t)[a_2(t) - b_2 T] - \lambda_2 f \dfrac{G(t)^2}{G(t)^2+\alpha^2} T(t)\big\}\mathrm{d}t, \\ \mathrm{d}a_1(t) = \beta_1[\bar{a}_1 - a_1(t)]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}a_2(t) = \beta_2[\bar{a}_2 - a_2(t)]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t). \end{cases} \tag{1.1}$$

Currently, there are fewer papers dedicated to stochastic models with the mean-reverting Ornstein-Uhlenbeck process. Therefore, the study of tree-grass models affected by fire in combination with the mean-reverting Ornstein-Uhlenbeck process will be a new breakthrough in the study of stochastic populations and tree-grass models.

We will next investigate the dynamical properties of the system (1.1), including the existence of global solutions, boundedness of the θth moments, the existence of the stationary distribution, and the extinction of the trees and grasses.

2. Materials and methods

Assume that $X(t)$ is a homogeneous Markov process defined in k-dimensional space and characterized by the following k-dimensional stochastic differential equation

$$\mathrm{d}X(t) = h(t, X(t))\mathrm{d}t + \sum_{j=1}^{k} \sigma_j(t, X(t))\mathrm{d}B_j(t).$$

Then we have the Khasminskii Theorem [6].

Lemma 2.1. (Khasminskii) *There exists a constant Q that makes*

$$|h(s,x) - h(s,y)| + \sum_{j=1}^{k} |\sigma_j(s,x) - \sigma_j(s,y)| \leqslant Q|x-y|,$$

$$|h(s,x)| + \sum_{j=1}^{k} |\sigma_j(s,x)| \leqslant Q(1+|x|).$$

Furthermore, there exists a non-negative function $U(x)$ for any $x \in \mathbb{R}^k \backslash \mathbb{H}$ such that

$$\mathcal{L}U(x) \leqslant -1,$$

where $\mathbb{H}$ is a compact subset defined on $\mathbb{R}^k$.

Then, the solution $X(t)$ is a stationary Markov process, i.e. $X(t)$ has a stationary distribution.

3. The format of theorem, lemma, proof, etc.

Theorem 3.1. *For any initial value $(G(0), T(0), a_1(0), a_2(0) \in \mathbb{R}_+^2 \times \mathbb{R}^2$, there exists a unique solution $(G(t), T(t), a_1(t), a_2(t))$ of system (1) on $t \geqslant 0$, and it will remain in $\mathbb{R}_+^2 \times \mathbb{R}^2$ with probability one (a.s.).*

Proof. For convenience, we define that $\mathbb{S}_{n_0} = (-n_0, n_0) \times (-n_0, n_0) \times (-n_0, n_0) \times (-n_0, n_0)$. It is clearly that the coefficients of system (1.1) are all Lipschitz continuous, according to Mao et al., we know that there exists a local unique solution $(G(t), T(t), a_1(t), a_2(t)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ of the system (1.1) on $t \in [0, \tau_e]$, where τ_e is an explosion time. A sufficiently large $n_0 > 0$ can be found to make $(\ln G(0), \ln T(0), a_1(0), a_2(0)) \in \mathbb{S}_{n_0}$. For any integer $n > n_0$, the stopping time of the solution can be defined as

$$\begin{aligned}\tau_n = &\inf\{t \in (0, \tau_e) | \ln G(t) \notin (-n, n), \text{ or } \ln T(t) \notin (-n, n), \\ &\text{or } a_1(t) \notin (-n, n), \text{ or } a_2(t) \notin (-n, n)\}.\end{aligned}$$

It is obviously that τ_n is monotonically increasing to infinity as n increases. We let $\tau_\infty = \lim_{n\to\infty} \tau_n$, it can be found that $\tau_\infty \leqslant \tau_e (a.s.)$. To prove Theorem 3.1, we only need to verify $\tau_\infty = \infty$. Using the contradiction method, i.e. $\tau_\infty < \infty$ a.s.. Then there exists constants $(\varepsilon, T_0) \in ((0,1), \mathbb{R}_+)$ such that $\mathbb{P}(\tau_\infty \leqslant T_0) \geqslant \varepsilon$. Therefore there exists an integer $n_1 \geqslant n_0$ such that $\mathbb{P}\{\tau_n \leqslant T_0\} \geqslant \varepsilon,\ n \geqslant n_1$.

For any $t \leqslant \tau_n$, a non-negative C^2-fuction $V(G(t), T(t), a_1(t), a_2(t))$ is defined as

$$V(G,T,a_1,a_2) = G - 1 - \ln G + T - 1 - \ln T + \frac{a_1^4}{4} + \frac{a_2^4}{4}.$$

Applying the Itô formula, which yields

$$\begin{aligned}
\mathcal{L}V =& [G(a_1 - b_1G - cT) - \lambda_1 fG] - (a_1 - b_1 G - cT - \lambda_1 f)\\
&+ [T(a_2 - b_2T) - \lambda_2 f\frac{G^2}{\alpha^2 + G^2}T] - (a_2 - b_2T - \lambda_2 f\frac{G^2}{\alpha^2+G^2})\\
&+ \frac{3\sigma_1^2 a_1}{2} + \beta_1 a_1^3(\bar{a_1} - a_1) + \frac{3\sigma_2^2 a_2}{2} + \beta_2 a_2^3(\bar{a_2} - a_2)\\
\leqslant& \lambda_1 f + \lambda_2 f + G(|a_1| + b_1) + T(|a_2| + b_2 + c) - b_1G^2 - b_2T^2\\
&+ \frac{3\sigma_1^2 a_1}{2} + \beta_1 a_1^3\bar{a_1} - \beta_1 a_1^4 + \frac{3\sigma_2^2 a_2}{2} + \beta_2 a_2^3 \bar{a_2} - \beta_2 a_2^4\\
\leqslant& K_0\\
\leqslant& \infty,
\end{aligned} \tag{3.1}$$

where $K_0 = \sup\{\lambda_1 f + \lambda_2 f + G(|a_1| + b_1) + T(|a_2| + b_2 + c) - b_1G^2 - b_2T^2 + \frac{3\sigma_1^2 a_1}{2} + \beta_1 a_1^3\bar{a_1} - \beta_1 a_1^4 + \frac{3\sigma_2^2 a_2}{2} + \beta_2 a_2^3\bar{a_2} - \beta_2 a_2^4\}$ is a positive number. According to the inequality (3.1), we have

$$\mathrm{d}V(G,T,a_1,a_2) \leqslant K_0\mathrm{d}t + \sigma_1 a_1^3\mathrm{d}B_1(t) + \sigma_2 a_2^3 \mathrm{d}B_2(t). \tag{3.2}$$

Integrating from 0 to $\tau_n \wedge T_0$ and taking expectation on both sides of the inequality (3.2), we can get that

$$\begin{aligned}
&\mathbb{E}V(G(\tau_n\wedge T_0), T(\tau_n \wedge T_0), a_1(\tau_n\wedge T_0), a_2(\tau_n \wedge T_0))\\
\leqslant& V(G(0), T(0), a_1(0), a_2(0)) + K_0T_0.
\end{aligned} \tag{3.3}$$

Notice that for any $\omega \in \{\tau_n \leqslant T_0\}$, at least one of G and T is equal to n or $1/n$, we have

$$\begin{aligned}
&V(G(\tau_n,\omega), T(\tau_n,\omega), a_1(\tau_n,\omega), a_2(\tau_n,\omega))\\
\geqslant& \min\{(\sqrt{n} - 1 - \ln\sqrt{n}), (\sqrt{\frac{1}{n}} - 1 - \ln\sqrt{\frac{1}{n}})\}.
\end{aligned}$$

According to the inequality (3.3), it can be derived that

$$\begin{aligned}
&V(G(0), T(0), a_1(0), a_2(0)) + K_0T_0\\
\geqslant& \mathbb{E}[1_{\{\tau_n \leqslant T_0\}}(\omega) V(G(\tau_n), T(\tau_n), a_1(\tau_n), a_2(\tau_n))]\\
\geqslant& \varepsilon \min\{(\sqrt{n} - 1 - \ln\sqrt{n}), (\sqrt{\frac{1}{n}} - 1 - \ln\sqrt{\frac{1}{n}})\},
\end{aligned}$$

where $1_{\{\tau_n \leqslant T_0\}}$ is an index function of $\{\tau_n \leqslant T_0\}$. Let $n \to \infty$, we have

$$\infty > V(G(0), T(0), a_1(0), a_2(0)) + K_0T_0 = \infty,$$

which is contradictory to the assumptions. □

Theorem 3.2. *For any initial value* $(G(0), T(0), a_1(0), a_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$*, the solution* (G, T, a_1, a_2) *of system* (1.1) *satisfy*

$$\mathbb{E}|(G,T)|^\theta \leqslant K(\theta) \tag{3.4}$$

for any $\theta > 0$*, where* $K(\theta)$ *is a continuous function with respect to* θ*. That is to say, the* θ*th moment of the solution of system* (1.1) *is bounded.*

Proof. For any $\theta > 2$, a non-nagetive C^2-funtion $V_1(G,T,a_1,a_2) : \mathbb{R}_+^2 \times \mathbb{R}^2$ can be defined by

$$V_1(G,T,a_1,a_2) = \frac{G^\theta}{\theta} + \frac{T^\theta}{\theta} + \frac{a_1^{2\theta}}{2\theta} + \frac{a_2^{2\theta}}{2\theta}. \tag{3.5}$$

Applying differential operator $\mathcal{L}$ to $e^{\gamma t}V_1(G,T,a_1,a_2)$, taking expections on the both sides of the formula, we have

$$\begin{aligned}\mathbb{E}[e^{\gamma t}V_1(G,T,a_1,a_2)] =& \mathbb{E}[V_1(G(0),T(0),a_1(0),a_2(0))] \\ &+ \int_0^t \mathbb{E}\{\mathcal{L}[e^{\gamma s}V_1(G(s),T(s),a_1(s),a_2(s))]\}\mathrm{d}s,\end{aligned} \tag{3.6}$$

where $\gamma = \theta \min\{\beta_1, \beta_2\}$. According to Itô formula, it can be gotten that

$$\begin{aligned}\mathcal{L}(e^{\gamma t}V_1) =& \gamma e^{\gamma t}V_1 + e^{\gamma t}\mathcal{L}V_1 \\ =& \gamma e^{\gamma t}(\frac{G^\theta}{\theta} + \frac{T^\theta}{\theta} + \frac{a_1^{2\theta}}{2\theta} + \frac{a_2^{2\theta}}{2\theta}) + e^{\gamma t}G^\theta(a_1 - b_1G - cT - \lambda_1 f) \\ &+ e^{\gamma t}[a_1^{2\theta-1}\beta_1(\bar{a_1} - a_1) + (2\theta-1)a_1^{2\theta-2}\sigma_1^2] \\ &+ e^{\gamma t}[a_2^{2\theta-1}\beta_2(\bar{a_2} - a_2) + (2\theta-1)a_2^{2\theta-2}\sigma_2^2] \\ &+ e^{\gamma t}T^\theta(a_2 - b_2T - \lambda_2 f\frac{G^2}{\alpha^2+G^2}) \\ \leqslant& e^{\gamma t}(\frac{\gamma G^\theta}{\theta} + a_1G^\theta - b_1G^{\theta+1} - cG^\theta T - \lambda_1 fG^\theta + \frac{\gamma T^\theta}{\theta} + a_2T^\theta \\ &- b_2T^{\theta+1} + \lambda_2 fT^\theta - \frac{\beta_1 a_1^{2\theta}}{2} + \beta_1\bar{a_1}a_1^{2\theta-1} + (2\theta-1)a_1^{2\theta-2}\sigma_1^2 \\ &- \frac{\beta_2 a_2^{2\theta}}{2} + \beta_2\bar{a_2}a_2^{2\theta-1} + (2\theta-1)a_2^{2\theta-2}\sigma_2^2) \\ \leqslant& K_1(\theta)e^{\gamma t},\end{aligned} \tag{3.7}$$

where $K_1(\theta) = \sup_{(G,T,a_1,a_2)\in R_+^2\times R^2}\{e^{\gamma t}(\frac{\gamma G^\theta}{\theta} + a_1G^\theta - b_1G^{\theta+1} - cG^\theta T - \lambda_1 fG^\theta + \frac{\gamma T^\theta}{\theta} + a_2T^\theta - b_2T^{\theta+1} + \lambda_2 fT^\theta - \frac{\beta_1 a_1^{2\theta}}{2} + \beta_1\bar{a_1}a_1^{2\theta-1} + (2\theta-1)a_1^{2\theta-2}\sigma_1^2 - \frac{\beta_2 a_2^{2\theta}}{2} + \beta_2\bar{a_2}a_2^{2\theta-1} + (2\theta-1)a_2^{2\theta-2}\sigma_2^2)\} < \infty$. Combining formula (3.6) and (3.7), we obtain

$$\mathbb{E}[e^{\gamma t}V_1(G,T,a_1,a_2)] \leqslant \mathbb{E}[V_1(G(0),T(0),a_1(0),a_2(0))] + \frac{K_1(\theta)(e^{\gamma t}-1)}{\gamma},$$

which is equivalent to the following inequality

$$\mathbb{E}[V_1(G,T,a_1,a_2)] \leqslant e^{-\gamma t}\mathbb{E}[V_1(G(0),T(0),a_1(0),a_2(0))] + \frac{K_1(\theta)(e^{\gamma t}-1)}{\gamma e^{\gamma t}},$$

then we obtain that

$$\limsup_{t\to\infty} \mathbb{E}V_1(G,T,a_1,a_2) \leqslant \frac{K_1(\theta)}{\gamma}. \tag{3.8}$$

Besides, it can be easily gotten that

$$|(G,T)|^\theta \leqslant 2^{\frac{\theta}{2}}\theta \max\{\frac{G^\theta}{\theta}, \frac{T^\theta}{\theta}\} \leqslant 2^{\frac{\theta}{2}}\theta V_1(G,T,a_1,a_2). \tag{3.9}$$

Then combining the inequalities (3.8) and (3.9), it can be derived that

$$\limsup_{t\to\infty} \mathbb{E}|(G,T)|^\theta \leqslant 2^{\frac{\theta}{2}}\theta \limsup_{t\to\infty} \mathbb{E}V_1(G,T,a_1,a_2) \leqslant 2^{\frac{\theta}{2}}\theta\frac{K_1(\theta)}{\gamma} = K(\theta). \tag{3.10}$$

For any $\theta \in (0,2)$, according to Hölder's inequality and the inequality (3.10), we have

$$\limsup_{t\to\infty} \mathbb{E}|(G,T)|^\theta \leqslant \limsup_{t\to\infty} \mathbb{E}[|(G,T)|^2]^{\frac{\theta}{2}} \leqslant (K(2))^{\frac{\theta}{2}}. \tag{3.11}$$

The proof of the theorem is finished. $\square$

Theorem 3.3. *For any initial value* $(G(0),T(0),a_1(0),a_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$, *if* $\bar{a}_1 + \bar{a}_2 > \lambda_1 f + \lambda_2 f$ *and* $P_0^2 < \min\{\frac{2}{b_1}, \frac{2b_2}{(b_2+c)^2}\}$, *where* P_0 *can satisfy* $-P_0(\bar{a}_1+\bar{a}_2-\lambda_1 f-\lambda_2 f)+K_2=-2$ *with the constant*

$$\begin{aligned}K_2 := \sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2} &\{|a_1|G - cGT - \lambda f_1 G + |a_2|T + \frac{3a_1^2\sigma_1^2}{2} + \frac{3a_2^2\sigma_2^2}{2} + \beta_1|a_1|^3\bar{a}_1 \\ &+ \beta_2|a_2|^3\bar{a}_2 - \frac{b_1G^2}{2} - \frac{b_2^2T^2}{2} - \frac{\beta_1a_1^4}{2} - \frac{\beta_2a_2^4}{2}\}.\end{aligned}$$

Then system (1.1) *has a stationary distribution* $l(\cdot)$ *on* $\mathbb{R}_+^2\times\mathbb{R}^2$.

Proof. We define $V_0(G,T,a_1,a_2): \mathbb{R}_+^2\times\mathbb{R}^2$ by

$$V_0(G,T,a_1,a_2) = P_0[-\ln G(t) - \ln T(t) - \frac{a_1(t)}{\beta_1} - \frac{a_2(t)}{\beta_2}] + G(t) + T(t) + \frac{a_1^4(t)}{4} + \frac{a_2^4(t)}{4}.$$

According to Itô formula, we have

$$\begin{aligned}\mathcal{L}V_0 =& P_0[-(a_1 - b_1G - cT - \lambda_1 f) - (a_2 - b_2T - \lambda_2 f\frac{G^2}{\alpha^2+G^2}) - (\bar{a}_1 - a_1) \\ &- (\bar{a}_2 - a_2)] + G(a_1 - b_1G - cT - \lambda_1 f) + T(a_2 - b_2T - \lambda_2 f\frac{G^2}{\alpha^2+G^2}T) \\ &+ \beta_1a_1^3(\bar{a}_1 - a_1) + \frac{3a_1^2\sigma_1^2}{2} + \beta_2a_2^3(\bar{a}_2 - a_2) + \frac{3a_2^2\sigma_2^2}{2} \\ \leqslant& -P_0(\bar{a}_1+\bar{a}_2-\lambda_1 f-\lambda_2 f) + \sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\{|a_1|G - cGT - \lambda f_1 G + |a_2|T \\ &+ \frac{3a_1^2\sigma_1^2}{2} + \frac{3a_2^2\sigma_2^2}{2} + \beta_1|a_1|^3\bar{a}_1 + \beta_2|a_2|^3\bar{a}_2 - \frac{b_1G^2}{2} - \frac{b_2T^2}{2} - \frac{\beta_1a_1^4}{2} - \frac{\beta_2a_2^4}{2}\} \\ &+ P_0b_1G + P_0cT + P_0b_2T - \frac{b_1G^2}{2} - \frac{b_2T^2}{2} - \frac{\beta_1a_1^4}{2} - \frac{\beta_2a_2^4}{2}\end{aligned}$$

$$= -2 + P_0 b_1 G + P_0 b_2 T + P_0 cT - \frac{b_1 G^2}{2} - \frac{b_2 T^2}{2} - \frac{\beta_1 a_1^4}{2} - \frac{\beta_2 a_2^4}{2}. \tag{3.12}$$

It is easy to obtain that as $|(G,T,a_1,a_2)| \to \infty$, the function $V_0(G,T,a_1.a_2)$ tends to ∞ or (G,T) approaches the boundary of $\mathbb{R}^2_+$. Therefore the function has a minimum value $V_0(G^0,T^0,a_1^0,a_2^0)$ in $\mathbb{R}^2_+ \times \mathbb{R}^2$. Thus a non-negative function can be constructed as follows

$$V(G,T,a_1,a_2) = V_0(G,T,a_1.a_2) - V_0(G^0,T^0,a_1^0,a_2^0).$$

According to Itô formula and the inequality (3.12), we obtain

$$\begin{aligned} LV(G,T,a_1,a_2) \leqslant& -2 + P_0 b_1 G + P_0 b_2 T + P_0 cT - \frac{b_1 G^2}{2} - \frac{b_2 T^2}{2} \\ &- \frac{\beta_1 a_1^4}{2} - \frac{\beta_2 a_2^4}{2}. \end{aligned} \tag{3.13}$$

Then we consider a closed set $\mathbb{H}_\varepsilon$ as

$$\mathbb{H}_\varepsilon = \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | G \in [\varepsilon, \frac{1}{\varepsilon}], T \in [\varepsilon, \frac{1}{\varepsilon}], a_1 \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}], a_2 \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]\},$$

where $\varepsilon \in (0,1)$ is a sufficiently small number which can satisfy the following inequalities

$$-2 + K_3 - \frac{\min\{b_1,b_2\}}{4}(\frac{1}{\varepsilon})^2 \leqslant -1, \tag{3.14}$$

$$-2 + K_3 - \frac{\min\{\beta_1,\beta_2\}}{4}(\frac{1}{\varepsilon})^4 \leqslant -1, \tag{3.15}$$

$$-2 + P_0 b_2 T + P_0 cT - \frac{b_2 T^2}{2} + P_0 b_1 \varepsilon \leqslant -1, \tag{3.16}$$

$$-2 + P_0 b_1 G - \frac{b_1 G^2}{2} + (P_0 b_2 + P_0 c)\varepsilon \leqslant -1, \tag{3.17}$$

where $K_3 = \sup_{(G,T,a_1,a_2)\in\mathbb{R}^2_+\times\mathbb{R}^2}\{P_0 b_1 G + P_0 b_2 T + P_0 cT - \frac{b_1 G^2}{4} - \frac{b_2 T^2}{4} - \frac{\beta_1 a_1^4}{4} - \frac{\beta_2 a_2^4}{4}\} < \infty$. □

Note that $(\mathbb{R}_+ \times \mathbb{R}^2)\backslash\mathbb{H}_\varepsilon = \bigcup_{k=1}^6 \mathbb{H}^c_{k,\varepsilon}$, where

$$\begin{aligned} \mathbb{H}^c_{1,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | G \in (\frac{1}{\varepsilon}, \infty)\}, \\ \mathbb{H}^c_{2,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | T \in (\frac{1}{\varepsilon}, \infty)\}, \\ \mathbb{H}^c_{3,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | |a_1| \in (\frac{1}{\varepsilon}, \infty)\}, \\ \mathbb{H}^c_{4,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | |a_2| \in (\frac{1}{\varepsilon}, \infty)\}, \\ \mathbb{H}^c_{5,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | G \in (0,\varepsilon)\}, \\ \mathbb{H}^c_{6,\varepsilon} &= \{(G,T,a_1,a_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 | T \in (0,\varepsilon)\}. \end{aligned}$$

Next we will verify $LV(G,T,a_1,a_2)\leqslant -1$ for any $(G,T,a_1,a_2)\in(\mathbb{R}_+^2\times\mathbb{R}^2)\backslash\mathbb{H}_\varepsilon$. This part of the proof can be divided into six parts.

Case 1. If $(G,T,a_1,a_2)\in\mathbb{H}_{1,\varepsilon}^c$, according to the inequality (3.14), we have

$$\begin{aligned}\mathcal{L}V(G,T,a_1,a_2)\leqslant&-2+\sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\{P_0b_1G+P_0b_2T+P_0cT-\frac{b_1G^2}{4}-\frac{b_2T^2}{4}\\&-\frac{\beta_1a_1^4}{4}-\frac{\beta_2a_2^4}{4}\}-\frac{b_1G^2}{4}\\\leqslant&-2+K_3-\frac{\min\{b_1,b_2\}}{4}(\frac{1}{\varepsilon})^2\\\leqslant&-1.\end{aligned}$$

Case 2. If $(G,T,a_1,a_2)\in\mathbb{H}_{2,\varepsilon}^c$, according to the inequality (3.14), we have

$$\begin{aligned}\mathcal{L}V(G,T,a_1,a_2)\leqslant&-2+\sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\{P_0b_1G+P_0b_2T+P_0cT-\frac{b_1G^2}{4}-\frac{b_2T^2}{4}\\&-\frac{\beta_1a_1^4}{4}-\frac{\beta_2a_2^4}{4}\}-\frac{b_2T^2}{4}\\\leqslant&-2+K_3-\frac{\min\{b_1,b_2\}}{4}(\frac{1}{\varepsilon})^2\\\leqslant&-1.\end{aligned}$$

Case 3. If $(G,T,a_1,a_2)\in\mathbb{H}_{3,\varepsilon}^c$, combined with the inequality (3.15), we obtain

$$\begin{aligned}\mathcal{L}V(G,T,a_1,a_2)\leqslant&-2+\sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\{P_0b_1G+P_0b_2T+P_0cT-\frac{b_1G^2}{4}-\frac{b_2T^2}{4}\\&-\frac{\beta_1a_1^4}{4}-\frac{\beta_2a_2^4}{4}\}-\frac{\beta_1a_1^4}{4}\\\leqslant&-2+K_3-\frac{\min\{\beta_1,\beta_2\}}{4}(\frac{1}{\varepsilon})^4\\\leqslant&-1.\end{aligned}$$

Case 4. If $(G,T,a_1,a_2)\in\mathbb{H}_{4,\varepsilon}^c$, combined with the inequality (3.15), it can be derived that

$$\begin{aligned}\mathcal{L}V(G,T,a_1,a_2)\leqslant&-2+\sup_{(G,T,a_1,a_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\{P_0b_1G+P_0b_2T+P_0cT-\frac{b_1G^2}{4}-\frac{b_2T^2}{4}\\&-\frac{\beta_1a_1^4}{4}-\frac{\beta_2a_2^4}{4}\}-\frac{\beta_2a_2^4}{4}\\\leqslant&-2+K_3-\frac{\min\{\beta_1,\beta_2\}}{4}(\frac{1}{\varepsilon})^4\\\leqslant&-1.\end{aligned}$$

Case 5. If $(G,T,a_1,a_2)\in\mathbb{H}_{5,\varepsilon}^c$ and $P_0^2<\frac{2b_2}{(b_2+c)^2}$, combined with the inequality (3.16), it can be derived that

$$-2+P_0b_2T+P_0cT-\frac{b_2T^2}{2}+P_0b_1\varepsilon\leqslant-1.$$

Case 6. If $(G,T,a_1,a_2)\in\mathbb{H}^c_{6,\varepsilon}$ and $P_0^2<\frac{2}{b_1}$, combined with the inequality (3.17), it can be derived that

$$-2+P_0b_1G-\frac{b_1G^2}{2}+(P_0b_2+P_0c)\varepsilon\leqslant-1.$$

Based on the six cases above, we can derive that there is a sufficiently small constant $\varepsilon>0$ which can satisfy $\mathcal{L}V(G,T,a_1,a_2)\leqslant-1$ for any $(G,T,a_1,a_2)\in(\mathbb{R}^2_+\times\mathbb{R}^2)\backslash\mathbb{H}_\varepsilon$, where ε satisfies $\varepsilon\leqslant\min\{1,\frac{1-P_0b_2T-P_0cT+\frac{b_2T^2}{2}}{P_0b_1},\frac{1-P_0b_1G+\frac{b_1G^2}{2}}{P_0b_2+P_0c}\}$ when $K_3\leqslant1$, and if $K_3>1$, we have

$$\varepsilon\leqslant\min\{1,\sqrt[2]{\frac{\min\{b_1,b_2\}}{4(K_3-1)}},\sqrt[4]{\frac{\min\{\beta_1,\beta_2\}}{4(K_3-1)}}\}.$$

According to Lemma 2.1, system (1.1) has a stationary distribution $l(\cdot)$ on $\mathbb{R}^2_+\times\mathbb{R}^2$ when $\bar{a}_1+\bar{a}_2>\lambda_1f+\lambda_2f$.

Theorem 3.4. *For any initial value $(G(0),T(0),a_1(0),a_2(0))\in\mathbb{R}^2_+\times\mathbb{R}^2$, the solution $(G(t),T(t),a_1(t),a_2(t))$ of system (1.1) satisfy*

$$\limsup_{t\to\infty}\frac{\ln G(t)}{t}\leqslant\bar{a}_1,\limsup_{t\to\infty}\frac{\ln T(t)}{t}\leqslant\bar{a}_2.$$

Particularly, if $\bar{a}_1<0,\bar{a}_2<0$, then $G(t)$ and $T(t)$ are extinct.

Proof. Applying to *Itô* formula to $\ln G(t)$ and $\ln T(t)$, it can be gotten that

$$\begin{cases}\mathrm{d}\ln G=(a_1-b_1G-cT-\lambda_1f)\mathrm{d}t,\\ \mathrm{d}\ln T=(a_2-b_2T-\lambda_2f\dfrac{G^2}{G^2+\alpha^2})\mathrm{d}t.\end{cases}\tag{3.18}$$

Integrating both sides of the equalities (3.18) from 0 to t, we have

$$\begin{cases}\ln G(t)=\ln G(0)+\displaystyle\int_0^t(a_1(t)-b_1G(t)-cT(t)-\lambda_1f)\mathrm{d}s,\\ \ln T(t)=\ln T(0)+\displaystyle\int_0^t(a_2(t)-b_2T(t)-\lambda_2f\frac{G(t)^2}{G(t)^2+\alpha^2})\mathrm{d}s.\end{cases}\tag{3.19}$$

According to the equality (3.19), it can be obtained that

$$\ln G(t)\leqslant\ln G(0)+\int_0^ta_1(s)\mathrm{d}s,\ln T(t)\leqslant\ln T(0)+\int_0^ta_2(s)ds.\tag{3.20}$$

According to the definition of Ornstein-Uhlenbeck process and the strong law of large numbers [8], it can be gotten that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^ta_1(s)\mathrm{d}s=\bar{a}_1,\lim_{t\to\infty}\frac{1}{t}\int_0^ta_2(s)\mathrm{d}s=\bar{a}_2.\tag{3.21}$$

Combining (3.20) and (3.21), we have

$$\begin{cases}\displaystyle\limsup_{t\to\infty}\frac{\ln G(t)}{t}\leqslant\limsup_{t\to\infty}\frac{\int_0^ta_1(s)\mathrm{d}s}{t}=\bar{a}_1,\\ \displaystyle\limsup_{t\to\infty}\frac{\ln T(t)}{t}\leqslant\limsup_{t\to\infty}\frac{\int_0^ta_2(s)\mathrm{d}s}{t}=\bar{a}_2.\end{cases}\tag{3.22}$$

Therefore, when $\bar{a}_1 < 0, \bar{a}_2 < 0$, it means that $\lim_{n\to\infty} G(t) = 0, \lim_{n\to\infty} T(t) = 0$, in other words, $G(t)$ and $T(t)$ are extinct. Then Theorem 3.4 has proved. $\square$

4. Example

In this section, we will introduce Milstein's higher-order method [3] to our numerical tests in order to verify our conclusions. The corresponding discretization equation of system (1.1) can be gotten

$$\begin{cases} G(k+1) = G(k) + G(k)[a_1(k) - b_1G(k) - cT(k) - \lambda_1 f]\Delta t, \\ T(k+1) = T(k) + T(k)[a_2(k) - b_2T(k) - \lambda_2 f\dfrac{G^2}{\alpha^2 + G^2}]\Delta t, \\ a_1(k+1) = a_1(k) + \beta_1[\bar{a_1} - a_1(k)]\Delta t + \sigma_1\xi_k\sqrt{\Delta t}, \\ a_2(k+1) = a_2(k) + \beta_2[\bar{a_2} - a_2(k)]\Delta t + \sigma_2\eta_k\sqrt{\Delta t}, \end{cases} \tag{4.1}$$

where $\Delta t > 0$ denotes time increment, ξ_k and η_k are independent Gaussian random variables with the Gaussian distribution $\mathbb{N}(0,1)$.

According to Tamen [15], we can obtain a reasonable range of values for the parameters in system (1.1). For the convenience of readers, it will be shown in Table 1. And we provides several combinations of parameters in Table 2 which will be used in the following examples.

Table 1. Parameter values of system (1.1).

Parameters	Values	Units
a_1	$0.3 - 4.5$	yr^{-1}
a_2	$0.156 - 7.17$	yr^{-1}
b_1	0.1	$ha.t^{-1}.yr^{-1}$
b_2	0.3	$ha.t^{-1}.yr^{-1}$
c	0.19	$ha.t^{-1}.yr^{-1}$
λ_1	$0.1 - 1$	yr^{-1}
λ_2	$0.005 - 1$	yr^{-1}
α	$0.54 - 1.73$	$t.ha^{-1}$
f	$0 - 1$	yr^{-1}

Table 2. Several combinations of biological parameters of system (1.1).

Combination	Values
$(\mathcal{A}_1)$	$a_{10} = 2.4, a_{20} = 3, b_1 = 0.1, b_2 = 0.3, c = 0.19, \lambda_1 = 0.2, \lambda_2 = 0.2,$
	$f = 0.5, \alpha = 0.8, \sigma_1 = 0.08, \sigma_2 = 0.08, \beta_1 = 0.4, \beta_2 = 0.5$
$(\mathcal{A}_2)$	$a_{10} = 3, a_{20} = 4, b_1 = 0.1, b_2 = 0.3, c = 0.19, \lambda_1 = 0.3, \lambda_2 = 0.3,$
	$f = 0.5, \alpha = 0.8, \sigma_1 = 0.05, \sigma_2 = 0.05, \beta_1 = 0.4, \beta_2 = 0.5, q = 2$
$(\mathcal{A}_3)$	$a_{10} = -0.4, a_{20} = -0.5, b_1 = 0.1, b_2 = 0.3, c = 0.19, \lambda_1 = 0.2, \lambda_2 = 0.2,$
	$f = 0.5, \alpha = 0.8, \sigma_1 = 0.08, \sigma_2 = 0.08, \beta_1 = 0.4, \beta_2 = 0.5$

Example 4.1. At first, we will verify the conclusion of Theorem 3.1. By Table 2, we choose the combination $\mathcal{A}_1$ as the value of the parameters of the system (1.1), and draw with R language. Then the results of computer simulation is shown in Figure 1. It means that the savannas ecosystem model is reasonably well constructed, and the existence of the global solution of the system (1.1) is verified.

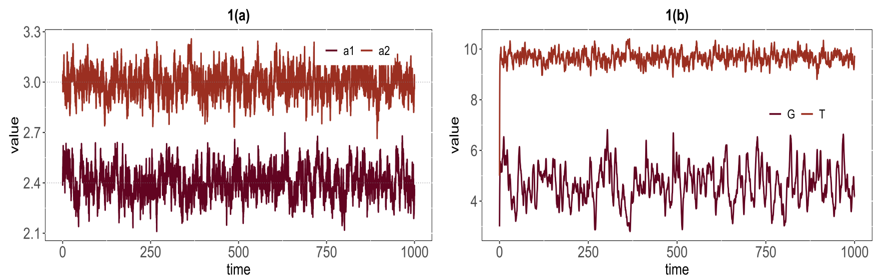

Figure 1. The existence of global solution.

Example 4.2. Then we will verify the conclusion of Theorem 3.2. We will use numerical simulations to find an upper bound on the θth moments. By Table 2, we choose the combination $\mathcal{A}_2$ as the value of the biological parameters of the system (1.1). The results of computer simulation in Figure 2 express that the θth moments of the solution of the system (1.1) have an upper bound. It means that there is an upper limit to the growth of the population so it is not infinite.

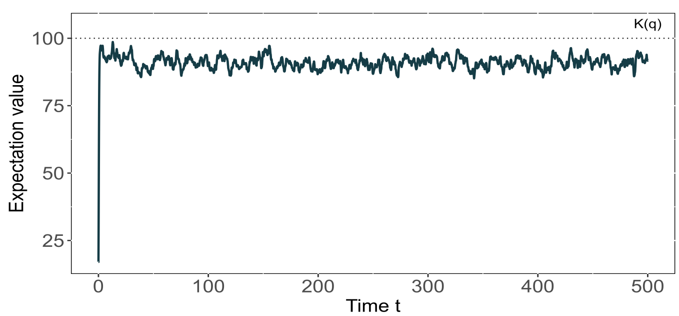

Figure 2. Boundness of θth moment.

Example 4.3. Based on the equations (4.1) we can count the frequency of the solutions. In order to verify the conclusion of Theorem 3.3, We will plot histograms to model the distribution of the solution values. By Table 2, we choose the combination $\mathcal{A}_1$ as the value of the parameters of the system (1.1), and draw with R language. Then the results of computer simulation is shown in Figure 3. Based on the results of the frequency histogram, it can be found that G, T, a_1, a_2 all tend to a stationary distribution when $\bar{a}_1 + \bar{a}_2 > \lambda_1 f + \lambda_2 f$ and $P_0^2 < \min\{\frac{2}{b_1}, \frac{2b_2}{(b_2+c)^2}\}$. It means that the savannas ecosystem will remain stable in the long term, when subject to a number of conditions. And the existence of the stationary distribution of the system (1.1) is verified.

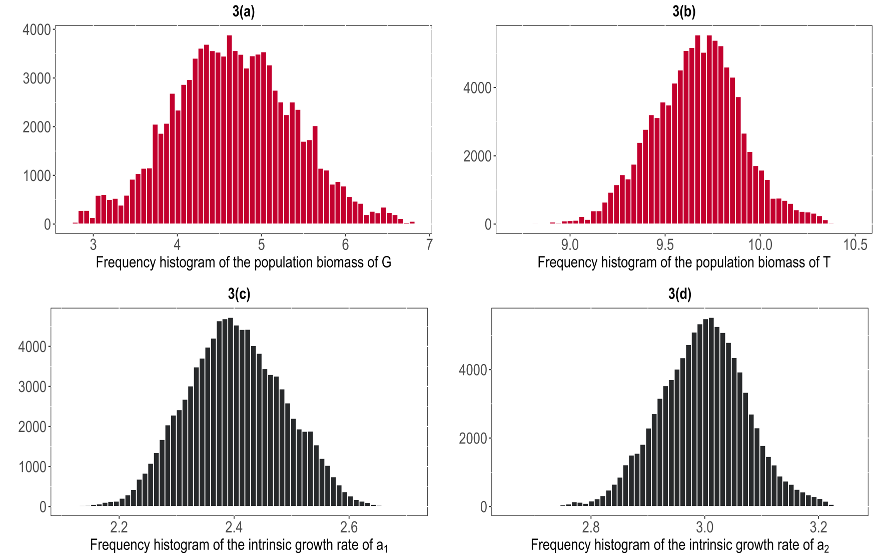

Figure 3. Frequency histogram of solutions.

Example 4.4. In order to verify the conclusion in Theorem 3.4 about the extinction of the system (1.1), we choose to perform numerical simulations in conjunction with $\mathcal{A}_3$ to determine whether both G and T become extinct under the conditions $\mathcal{A}_3$. The results of the numerical simulation are presented in Figure 4. The results show that both G and T become extinct at the same time if $\bar{a}_1 < 0, \bar{a}_2 < 0$.

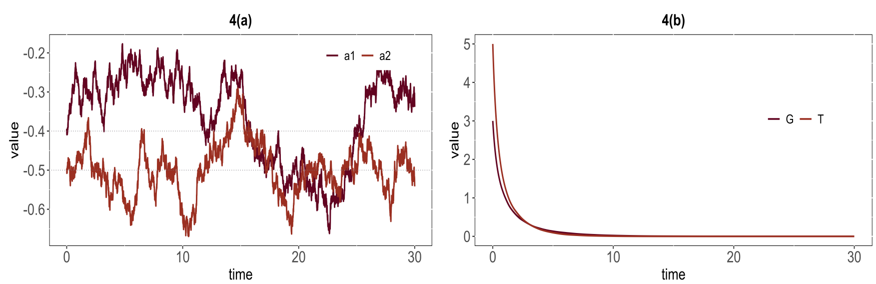

Figure 4. Extinction of solutions.

5. Conclusions

In this paper, we first introduce the mixed tree-grass ecosystems commonly affected by fire in nature, and then combine this with related studies to construct differential equation models that attempt to describe the asymmetric interactions between trees and grasses. Secondly, to more realistically reflect the role of environmental noise in nature, i.e. the effect of environmental noise on the variability of biomass, we introduced stochastic variables to simulate the stochastic effects in the real world. Comparing the linear perturbation and the mean-reverting Ornstein-Uhlenbeck process, we find that the mean-reverting Ornstein-Uhlenbeck process can avoid explosive random fluctuations in the growth rate and thus better simulate environmental noise [2, 11]. Accordingly, we developed a tree-grass model with the mean-reverting Ornstein-Uhlenbeck process affected by fire. Next we prove some of the dynamical properties of the tree-grass model. First we show that the solution $(G(t), T(t), a_1(t), a_2(t))$ is global and unique, which is not the same as the linear perturbation stochastic model with a global and unique positive solution. Secondly we prove that for any $\theta > 0$, the θth moments of the solution of the model are bounded. We next prove the existence of a stationary distribution for system (1.1) when $\bar{a}_1 + \bar{a}_2 > \lambda_1 f + \lambda_2 f$, and satisfy $P_0^2 < \min\{\frac{2}{b_1}, \frac{2b_2}{(b_2+c)^2}\}$, which implies that the population in system (1.1) is weakly persistent. Finally we prove that when the solution $(G(t), T(t), a_1(t), a_2(t))$ of system (1.1) satisfy $\bar{a}_1 < 0, \bar{a}_2 < 0$, Trees and Grasses are extinct.

Therefore we can give the conclusion that our model is able to describe the variation of tree and grass in savannas ecosystem, so the model design is reasonable. Meanwhile our model illustrates that both trees and grasses have an upper bound on their growth without exploding. And the growth of trees and grasses constrain each other. Under some conditions, trees and grasses can coexist stably for a long period of time. While the growth rate is negative, both trees and grasses will tend to become extinct. Our model is also still partially underconsidered. The effect of time lag is not considered in this paper. Whether it is fire or ecological changes, the impacts they bring are often time-lagged. We will introduce a time lag term to represent this in future studies. In savannas ecosystems, the water content of the soil also needs to be taken into account, and we did not consider the water content of the soil in our model. Therefore, we will add equation for soil water content ratio to the model in future studies. Also, we did not consider the coefficients about fire well enough. The intensity of fire actually changes with the number of trees and grasses. We will consider these factors in our future studies, which will be a big challenge.

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Conflict of interest. The authors declare there is no conflicts of interest.

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