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EXISTENCE AND EXPONENTIAL STABILITY FOR A NEW CLASS OF FIRST-ORDER IMPULSIVE EVOLUTION EQUATIONS*

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Abstract The aim of this paper is to present systematic methods for analyzing the existence and the exponential stability of a new class of first-order impulsive evolution equations. We initially provide two existence results of mild solutions for the equations using two kinds of methods. Subsequently, we also explore the exponential stability of the equation. Lastly, we present some applications in differential hemivariational inequalities to demonstrate our main results.

Keywords Impulsive delay evolution equation, mild solution, existence, exponential stability.

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1. Introduction

Impulsive evolution equations constitute an intriguing and vibrant domain of mathematical inquiry, attracting considerable attention in recent times. These equations are distinguished by their capacity to simulate systems that experience sudden alterations or impulses at specific instances of time.This characteristic distinguishes them from traditional evolution equations, which usually depict systems evolving smoothly over time. The importance of impulsive evolution equations lies in their capability to capture and analyze intricate phenomena present in various real-world systems, such as biological populations, financial markets, and control systems.

Additionally, impulsive evolution equations are utilized in various interdisciplinary fields. In biology, they are applied to simulate the dynamics of infectious diseases using impulsive vaccination strategies or the effects of harvesting on prey-predator systems. In engineering, impulsive equations can represent the behavior of

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systems that undergo abrupt changes in operating conditions, such as power grids experiencing sudden load changes. In the field of economics, impulsive evolution equations are capable of capturing the repercussions of policy alterations or market interventions on the dynamics of the economy. The wide-ranging applicability of these equations renders them an invaluable resource for researchers from diverse disciplines, who aim to comprehend and model systems exhibiting impulsive characteristics.

Studying impulsive evolution equations necessitates the creation of novel mathematical tools and methods. One example is the extension of the monotone iterative technique, commonly utilized in the examination of ordinary and partial differential equations, to impulsive fractional evolution equations. This provides fresh perspectives on the dynamics of intricate systems. By including impulsive impacts, these equations are able to represent the abrupt shifts and interruptions that frequently occur in real-world systems.

This enables a more accurate and realistic modeling of such systems, leading to a deeper understanding of their behavior and performance. This technique allows for the construction of approximate solutions and the proof of existence and uniqueness results under certain conditions.This facilitates the development of more precise and practical models for these systems, fostering a profounder comprehension of their operational patterns and effectiveness. Furthermore, this methodology facilitates the derivation of approximate solutions and the verification of the existence and uniqueness of outcomes under specified circumstances. Additionally, Lipschitz conditions, Gronwall's inequality, compact semigroup, and noncompactness measure have been utilized in the analysis of these equations, offering robust instruments for evaluating solutions and constraining their behaviors. Regarding the fundamental principles and recent advancements in impulsive differential equations, we recommend consulting [1, 19, 20, 25, 26] for further insights.

The importance of exponential stability in differential equations is deep and varied. Exponential stability is an essential idea in the examination of dynamical systems, especially those characterized by differential equations. It denotes the feature of a system's solutions to converge towards an equilibrium point at an exponential rate as time progresses.

This trait is greatly appreciated in different scientific and engineering fields because of its implications for system performance, predictability, and robustness (see [3, 4, 6, 12, 27]). An important aspect of exponential stability is its ability to guarantee the quick convergence of system states.In numerous scenarios, it is crucial for the system to quickly reach a stable state, whether it is a control system adjusting to a desired setup or a physical process stabilizing after a disruption. Additionally, exponential stability is closely connected to the resilience of dynamical systems. Moreover, the idea of exponential stability is relevant to the study of chaotic dynamics and nonlinear systems. Though typically associated with linear systems, the application of exponential stability to nonlinear systems offers valuable understanding of the dynamics of intricate systems.

In particular, the exponential convergence of trajectories in nonlinear systems can help identify regions of stability and predict the long-term behavior of the system, even in the presence of chaotic dynamics. The significance of exponential stability is also reflected in its role in system identification and model reduction. Specifically, the rapid convergence of paths in nonlinear systems can aid in determining stable areas and forecasting the system's future behavior, even when chaotic dynamics are present. The importance of exponential stability is also evident in its contribution to system recognition and simplification. In numerous situations, it is vital to approximate intricate systems with simpler models that are easier to analyze and control. Exponential stability offers a measure for evaluating the accuracy of these approximations. If the simplified model preserves the exponential stability of the initial system, it is more probable to offer dependable forecasts and uphold the desired system performance.

Consider an interval $J=[0,b](b>0)$ and a finite set of points

$$D=\{t_i\in(0,b), i=1,2,\cdots,m\},\quad 0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=b.$$

In this paper, we consider the following form:

$$\begin{cases} x'(t)=Ax(t)+f(t,x(t))+B(t,x(t))g(t,x(t)), & t\in(0,b]-D,\\ \Delta x(t_k)=G(t_k,x(t_k)), \qquad t_k\in D, & \\ x(t)=\theta(t), \qquad t\in[-\tau,0]. & \end{cases} \tag{1.1}$$

$A:D(A)\subseteq X\to X$, is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a reflexive Banach space X. $\theta:[-\tau,0]\to X(\tau>0)$ is a continuous function. Let V be a separable Banach space. $f:J\times X\to X, g:J\times X\to V, B:J\times X\to\mathcal{L}(V,X)$ $G:D\times X\to X$ are given functions to be specified later. Here, $x(t_k)=x(t_k^-)$ and $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t=t_k\in D$, respectively.

The objective of this paper is to furnish methodical methodologies for studying the existence and exponential stability of the first-order impulsive evolution equation (1.1), which incorporates Caputo first-order derivatives in separable reflexive Banach spaces. Firstly, we establish two existence result for mild solutions to the equations, employing two kinds of methods. In addressing the existence results under varying hypotheses, we invoke several fixed point theorems. Additionally, we consider the exponential stability of the equation. Lastly, to exemplify our key findings, we present several applications pertaining to differential hemivariational inequalities.

The main novelties of the paper are following. First, the existence and the exponential stability of the first-order impulsive evolution equation considered in this work has not been studied. Such equations are important and useful in the applications of many practical problems. Second, the method with almost history-dependent operators is quite useful in the study of mild solutions to evolution equations, and has applied to deal frictional contact problems (see [17, 18]). Moreover, the method involving feedback control is new to prove existence for impulsive evolution equations, which can be applied to several problems, for instance, evolutionary hemivariational inequalities, differential variational inequalities. Third, we provide a new idea to study the differential hemivariational inequalities as a application of the abstract existence result.

We now discuss the difference between the two methods. The first method is restricted by the Lipschitz conditions for the data by using the fixed point theorem for almost history-dependent operator, which can derives the existence and uniqueness of solution. For the second method, it is restricted by the compactness conditions for the semigroup and is involved in feedback control system, which only derives the existence of solution. They are both new and useful to study impulsive evolution equations. It is the first step to use them to investigate such equations, which can be generalized and improved to deal with more complex evolution equations, for instance, fractional evolution equations and functional differential equations.

The rest of this paper is organized as follows. In Section 2, we will present some preliminaries which will be used to prove our main results. In Section 3 and Section 4, some sufficient conditions are established to guarantee some existence of mild solutions of problem (1.1). The subsequent structure of this paper is as follows. Section 2 introduces foundational concepts that will be instrumental in demonstrating our primary results. Sections 3 and 4 establish sufficient conditions to ensure the existence of mild solutions for problem (1.1). Theorem 3.2 and Theorem 4.1 are the main results. Section 5 delves into the exponential stability of problem (1.1), and the final section shows some applications.

2. Preliminaries

The norm of a Banach space X will be denoted by $\|\cdot\|_X$. Let $\bar{J} = J \cup [-\tau, 0]$. Let $C(J, X)$ denote the Banach space of continuous functions from J into X with the norm $\|x\|_C = \sup_{t\in J}\|x(t)\|_X$ and $L^2(J, X)$ denote the Banach space of twice integrable functions from J into X with the norm $\|x\|_{L^2} = \left(\int_{t\in J} \|x(t)\|_X^2\right)^{\frac{1}{2}}$. In order to define the mild solutions of problem (1.1), we also consider the Banach space $PC(J, X) = \{x \in C((t_k, t_{k+1}], X) : x(t_k^-), x(t_k^+) \text{ exist}, k = 0, 1, 2, \cdots, m \text{ and } x(t_k) = x(t_k^-)\}$ with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|_X$. Let $\mathcal{C} := C([-\tau, 0], X) \cap PC(J, X)$ with the norm $\|x\|_{\mathcal{C}} = \sup_{t\in[-\tau,b]} \|x(t)\|_X$.

As is well-known, the concept of history-dependent operator was initially presented in [17], while the definition of almost history-dependent operator was explored in [18], both of which have garnered numerous practical applications. Here, we define the class of almost history-dependent operators over bounded intervals I as detailed below:

Definition 2.1 ([18]). An operator $F : C(I, X) \to C(I, X)$ is called an almost history-dependent operator if for any compact set $K \subset I$, there exist constants $\ell_K \in [0, 1), L_K \geq 0$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K\|x(t) - y(t)\|_X + L_K \int_0^t \|x(s) - y(s)\|_X ds, \qquad (2.1)$$

for all $x, y \in C(I, X),\ t \in K$.

There are three special cases of general history-dependent operators.

(i) If $L_K = 0$, then (2.1) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K\|x(t) - y(t)\|_X,$$

which implies that

$$\|Fx - Fy\|_{C(I,X)} \leq \ell_K\|x - y\|_{C(I,X)},$$

i.e., F is a Lipschitz continuous operator on $C(I, X)$ with constant $\ell_K \in [0, 1)$.

(ii) If $\ell_K = 0$, then (2.1) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq L_K \int_0^t \|x(s) - y(s)\|_X ds,$$

i.e., F is a history-dependent operator on $C(I, X)$ with constant $L_K \geq 0$ (cf. [17]).

Theorem 2.1 ([18]). *If $F : C(I,X) \to C(I,X)$ is a almost history-dependent operator, then F has a fixed point in $C(I,X)$.*

Definition 2.2 ([9,15,25]). A function $x \in \mathcal{C}$ is called a mild solution to problem (1.1) if it satisfies

$$x(t) = \begin{cases} \theta(t), \qquad t \in [-\tau, 0], \\ T(t)\theta(0) + \displaystyle\int_0^t T(t-s)[f(s,x(s)) + B(s,x(s))g(s,x(s))]ds, \qquad t \in (0, t_1], \\ T(t)\theta(0) + \displaystyle\sum_{i=1}^{k} T(t-t_i)G(t_i, x(t_i)) + \int_0^t T(t-s)[f(s,x(s)) \\ +B(s,x(s))g(s,x(s))]ds, \qquad t \in (t_k, t_{k+1}],\ k = 1, \cdots, m. \end{cases}$$

3. Existence result I

In the sequel, we will establish the following assumptions regarding the data of our problem.
(H_f) : $f : J \times X \to X$ is continuous on $J \times X$, and there exist a function $\phi \in L^2(J, \mathbb{R}_+)$ and a constant $L_f > 0$ such that

$$\begin{aligned} &\|f(t,0)\|_X \leq \phi(t), \\ &\|f(t,x_1) - f(t,x_2)\|_X \leq L_f \|x_1 - x_2\|_X \end{aligned}$$

for all $x_1,\ x_2 \in X$ and a.e. $t \in J$.
(H_B) : $B : J \times X \to L(V, X)$ is continuous on $J \times X$, and there exist a function $\varphi \in L^2(J, \mathbb{R}_+)$ and a constant $L_B > 0$ such that

$$\begin{aligned} &\|B(t,0)\|_{L(V,X)} \leq \varphi(t), \\ &\|B(t,x_1) - B(t,x_2)\|_{L(V,X)} \leq L_B \|x_1 - x_2\|_X \end{aligned}$$

for all $x_1,\ x_2 \in X$ and a.e. $t \in J$.
(H_g) : $g : J \times X \to V$ is continuous on $J \times X$, and there exist a function $\psi \in L^\infty(J, \mathbb{R}_+)$ and a constant $L_g > 0$ such that

$$\begin{aligned} &\|g(t,x)\|_X \leq \psi(t), \\ &\|g(t,x_1) - g(t,x_2)\|_X \leq L_g \|x_1 - x_2\|_X \end{aligned}$$

for all $x_1,\ x_2 \in X$ and a.e. $t \in J$.
(H_G) : There exist constants $L_k \geq 0$ with $M \sum_{i=1}^{k} L_i < 1$ ($k = 1, 2, \cdots, m$) such that

$$\|G(t_k, x_1) - G(t_k, x_2)\|_X \leq L_k \|x_1 - x_2\|_X$$

for all $t_k \in D, x, y \in X$.

For fixed $\eta \in C(J, V)$, consider the following problem:

$$x(t) = \begin{cases} \theta(t), \qquad t \in [-\tau, 0], \\ T(t)\theta(0) + \displaystyle\int_0^t T(t-s)[f(s, x(s)) + B(s, x(s))\eta(s))]ds, \qquad t \in (0, t_1], \\ T(t)\theta(0) + \displaystyle\sum_{i=1}^{k} T(t-t_i)G(t_i, x(t_i)) + \int_0^t T(t-s)[f(s, x(s)) \\ +B(s, x(s))\eta(s)]ds, \qquad t \in (t_k, t_{k+1}],\ k = 1, \cdots, m. \end{cases} \tag{3.1}$$

Theorem 3.1. *Assume that the hypotheses* $(H_f), (H_B), (H_g), (H_G)$ *are satisfied. Then for each* $\eta \in C(J, V)$*, problem (3.1) has a unique mild solution on* $x_\eta \in \mathcal{C}$*. Moreover, for two solutions* $x_{\eta_1}, x_{\eta_2} \in \mathcal{C}$ *of problem (3.1) corresponding to* $\eta_1, \eta_2 \in C(J, V)$*, respectively, one has*

$$\|x_{\eta_1}(t) - x_{\eta_2}(t)\|_X \le \overline{M} \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds, \qquad \forall t \in J \tag{3.2}$$

for some $\overline{M} > 0$.

Proof. The mild solution to problem (3.1) exists and is unique, as demonstrated in Theorem 3.1 in [24]. We now show the boundedness of x_η. If $t \in (0, t_1]$, then from the hypotheses $(H_f), (H_B), (H_g)$ and the Hölder's inequality, we obtain

$$\begin{aligned} &\|x_\eta(t)\|_X \\ \le\ & \|T(t)\theta(0)\|_X + \int_0^t \|T(t-s)[f(s, x_\eta(s)) + B(s, x_\eta(s))\eta(s)]\|_X ds \\ \le\ & M\|\theta(0)\|_X + M\int_0^t [\|f(s, x_\eta(s)) - f(s, 0)\| + \|f(s, 0)\| \\ & +(\|B(s, x_\eta(s)) - B(s, 0)\| + \|B(s, 0)\|)\|\eta(s)\|]ds \\ \le\ & M\|\theta(0)\|_X + M\int_0^t (L_f\|x_\eta(s)\|_X + \|\phi(s)\|_X + L_B\|\eta\|_C\|x_\eta(s)\|_X)ds \\ \le\ & M\|\theta(0)\|_X + M\sqrt{b}\|\phi\|_{L^2(J,\mathbb{R}_+)} + M(L_f + L_B\|\eta\|_C)\int_0^t \|x_\eta(s)\|_X ds. \end{aligned}$$

Therefore, by using the standard Gronwall's inequality, we obtain

$$\|x_\eta(t)\|_X \le M_1 e^{M_2} \qquad \text{for some } M_1, M_2 > 0.$$

If $t \in (t_k, t_{k+1}] (k = 1, \cdots, m)$, from the hypotheses $(H_f), (H_B), (H_g), (H_G)$ and the Hölder's inequality, we have

$$\begin{aligned} &\|x_\eta(t)\|_X \\ \le\ & \|T(t)\theta(0)\|_X + \|\sum_{i=1}^{k} T(t-t_i)G(t_i, x_\eta(t_i))\|_X \\ & + \int_0^t \|T(t-s)[f(s, x_\eta(s)) + B(s, x_\eta(s))\eta(s)]\|_X ds \end{aligned}$$

$$
\begin{aligned}
&\le M\|\theta(0)\|_X + M\|\sum_{i=1}^{k}(\|G(t_i,x_\eta(t_i)) - G(t_i,0)\|_X + \|G(t_i,0)\|_X) \\
&\quad + M\int_0^t [\|f(s,x_\eta(s)) - f(s,0)\| + \|f(s,0)\| \\
&\quad + (\|B(s,x_\eta(s)) - B(s,0)\| + \|B(s,0)\|)\|\eta(s)\|]ds \\
&\le M\|\theta(0)\|_X + M\sum_{i=1}^{k} L_i\|x_\eta(t_i)\|_X + M\|G(t_i,0)\|_X \\
&\quad + M\int_0^t (L_f\|x_\eta(s)\|_X + \|\phi(s)\|_X + L_B\|\eta\|_C\|x_\eta(s)\|_X)ds \\
&\le M\|\theta(0)\|_X + M\sum_{i=1}^{k} L_i\|x_\eta(t_i)\|_X + M\|G(t_i,0)\|_X \\
&\quad + M\sqrt{b}\|\phi\|_{L^2(J,\mathbb{R}_+)} + M(L_f + L_B\|\eta\|_C)\int_0^t \|x_\eta(s)\|_X ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sup_{s\in[0,t]}\|x_\eta(s)\|_X \\
&\le M\|\theta(0)\|_X + M\sum_{i=1}^{k} L_i \sup_{s\in[0,t]}\|x_\eta(s)\|_X + M\|G(t_i,0)\|_X \\
&\quad + M\sqrt{b}\|\phi\|_{L^2(J,\mathbb{R}_+)} + M(L_f + L_B\|\eta\|_C)\int_0^t \sup_{r\in[0,s]}\|x_\eta(r)\|_X ds,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&(1 - M\sum_{i=1}^{k} L_i)\sup_{s\in[0,t]}\|x_\eta(s)\|_X \\
&\le M\|\theta(0)\|_X + M\sum_{i=1}^{k} L_i \sup_{s\in[0,t]}\|x_\eta(s)\|_X + M\|G(t_i,0)\|_X \\
&\quad + M\sqrt{b}\|\phi\|_{L^2(J,\mathbb{R}_+)} + M(L_f + L_B\|\eta\|_C)\int_0^t \sup_{r\in[0,s]}\|x_\eta(r)\|_X ds.
\end{aligned}
$$

Since $M\sum_{i=1}^{k} L_i < 1$, by utilizing the standard Gronwall's inequality once more, we obtain

$$
\sup_{s\in[0,t]}\|x_\eta(s)\|_X \le M_3 e^{M_4} \qquad \text{for some } M_3, M_4 > 0.
$$

Afterwards, we demonstrate (3.2). Let $\eta_1, \eta_2 \in C(J,V)$ and $t \in (0,t_1]$. Then

$$
\begin{aligned}
&\|x_{\eta_1}(t) - x_{\eta_2}(t)\|_X \\
&\le M\int_0^t [\|f(s,x_{\eta_1}(s)) - f(s,x_{\eta_2}(s))\|_X \\
&\quad + \|B(s,x_{\eta_1}(s))\eta_1(s) - B(s,x_{\eta_2}(s))\eta_2(s)\|_X]ds.
\end{aligned}
$$

Because of

$$\begin{aligned}
&\|B(s,x_{\eta_1}(s))\eta_1(s)) - B(s,x_{\eta_2}(s))\eta_2(s))\| \\
\le\ &\|B(s,x_{\eta_1}(s))\eta_1(s) - B(s,x_{\eta_2}(s))\eta_1(s)\| \\
&+\|B(s,x_{\eta_2}(s))\eta_1(s) - B(s,x_{\eta_2}(s))\eta_2(s))\| \\
\le\ &\|B(s,x_{\eta_1}(s)) - B(s,x_{\eta_2}(s))\|\|\eta_1(s)\| + \|B(s,x_{\eta_2}(s))\|\|\eta_1(s)-\eta_2(s))\| \\
\le\ &L_B\|\eta_1\|_C\|x_{\eta_1}(s) - x_{\eta_2}(s)\| \\
&+(\|B(s,x_{\eta_2}(s)) - B(s,0_X)\| + \|B(s,0_X)\|)\|\eta_1(s)-\eta_2(s))\| \\
\le\ &L_B\|\eta_1\|_C\|x_{\eta_1}(s) - x_{\eta_2}(s)\| + (L_B\|x_{\eta_2}(s)\| + \varphi(s)\|)\|\eta_1(s)-\eta_2(s))\| \\
\le\ &L_B\|\eta_1\|_C\|x_{\eta_1}(s) - x_{\eta_2}(s)\| + (L_B M_1 e^{M_2} + \varphi(s))\|\eta_1(s)-\eta_2(s))\|,
\end{aligned}$$

one has

$$\begin{aligned}
&\|x_{\eta_1}(t) - x_{\eta_2}(t)\|_X \\
\le\ &M(L_f + L_B\|\eta_1\|_C)\int_0^t [\|x_{\eta_1}(s) - x_{\eta_2}(s)\|_X ds \\
&+M(L_B M_1 e^{M_2} + \|\varphi\|_{L^2})\int_0^t \|\eta_1(s)-\eta_2(s))\| ds.
\end{aligned}$$

The result (3.2) is derived from the standard Gronwall's inequality. When $t \in (t_k, t_{k+1}](k = 1, \cdots, m)$, the equation (3.2) can be derived using the same reasoning as before. The proof is finished. □

We are now capable of demonstrating the existence of mild solutions for problem (1.1) as follows.

Theorem 3.2. *Assume that the hypotheses Theorem 3.1 are satisfied. Then problem (1.1) has a unique mild solution on $\mathcal{C}$.*

Proof. Define the operator $\digamma : C(J,V) \to C(J,V)$ by

$$(\digamma\eta)(t) = g(t, x_\eta(t)), \qquad \forall \eta \in C(J,V).$$

Then the problem of finding mild solutions for problem (1.1) is reduced to find fixed points of $\digamma$. Let $\eta_1, \eta_2 \in C(J,V)$ and $t \in J$. Then

$$\begin{aligned}
&\|g(t,x_{\eta_1}(t)) - g(t,x_{\eta_2}(t))\|_V \le L_g\|x_{\eta_1}(t) - x_{\eta_2}(t)\|_X, \\
&\|x_{\eta_1}(t) - x_{\eta_2}(t)\|_X \le \overline{M}\int_0^t \|\eta_1(s)-\eta_2(s)\|_V ds, \\
&\|\digamma\eta_1(t) - \digamma\eta_2(t)\|_V \le L_g\overline{M}\int_0^t \|\eta_1(s)-\eta_2(s)\|_V ds.
\end{aligned}$$

This inequality illustrates that $\digamma$ is an almost history-dependent operator, and hence by applying Theorem 2.1, we conclude that there exists unique fixed point $\eta^* \in C(J,V)$. Let $x^* = x_{\eta^*}$, i.e. x^* solves problem (3.1) with $\eta = \eta^* = g(t, x^*(t))$. Then x^* is the mild solution of problem (3.1). In addition x^* is the unique mild solution of problem (3.1) due to the uniqueness of the fixed point of $\digamma$. □

4. Existence result II

In this section, we will consider alternative hypotheses that without Lipschitz conditions as discussed in Section 3.

(H_A) : $T(t)$ is compact for every $t > 0$.
(H_{f_1}) : $f : J \times X \to X$ is Borel measurable on $J \times X$ and continuous on X, and there exist a function $\phi_1 \in L^2(J, \mathbb{R}_+)$ and a constant $L_{f1} > 0$ such that

$$\|f(t,x)\|_X \le \phi_1(t) + L_{f1}\|x\|_X$$

for all $x \in X$ and a.e. $t \in J$.
(H_{B_1}) : There exist a function $\varphi_1 \in L^2(J, \mathbb{R}_+)$ and a constant $L_{B1} > 0$ such that

$$\|B(t,x)\|_{\mathcal{L}(V,X)} \le \varphi_1(t) + L_{B1}\|x\|_X$$

for all $x \in X$ and a.e. $t \in J$.
(H_{g_1}) : There exist a function $\psi_1 \in L^2(J, \mathbb{R}_+)$ and a constant $L_{g1} > 0$ such that

$$\|g(t,x)\|_V \le \psi_1(t) + L_{g1}\|x\|_X$$

for all $x \in X$ and a.e. $t \in J$.
(H_{G_1}) : $G(t_k, \cdot) : X \to X$ is continuous and there exist constants $a_k, b_k \ge 0$ with $M\sum_{i=1}^{k} a_i < 1$ $(k = 1, 2, \cdots, m)$ such that

$$\|G(t_k, x)\|_X \le a_k\|x\|_X + b_k$$

for all $t_k \in D, x, y \in X$.

Lemma 4.1 ([7]). *If (H_A) holds, then the operator $\Psi : L^2(J,X) \to C(J,X)$, given by*

$$(\Psi h)(\cdot) = \int_0^{\cdot} T(\cdot - s)h(s)ds, \qquad \forall h \in L^2(J,X), \tag{4.1}$$

is compact.

We are now ready to present the primary finding of this section.

Theorem 4.1. *Assume that the hypotheses $(H_A), (H_{f1}), (H_{B1}), (H_{g1}), (H_{G1})$ are satisfied. Then problem (1.1) has a mild solution on J.*

Proof. **Case 1.** Let $t \in [0, t_1]$. For any $n > 0$, let $\tau_{0,j} = \frac{j}{n}t_1,\ 0 \le j \le n-1$. We set

$$u_{0,n}(t) = \sum_{j=0}^{n-1} u^{0,j}\chi_{[\tau_{0,j},\tau_{0,j+1})}(t), \qquad t \in [0, t_1],$$

where $\chi_{[\tau_{0,j},\tau_{0,j+1})}$ represents the character function of the interval $[\tau_{0,j}, \tau_{0,j+1})$. The sequence $\{u^{0,j}\}$ is constructed as follows. To begin with, we set $u^{0,0} = g(0, \theta(0))$. According to Theorem 3.3 in [24], there exists $x_{0,n}(\cdot)$ such that $x_{0,n}(t) = \theta(t)$ for $t \in [-\tau, 0]$ and

$$x_{0,n}(t) = T(t)\theta(0) + \int_0^t T(t-s)[f(s, x_{0,n}(s)) + B(s, x_{0,n}(s))u^{0,0}]ds, \quad t \in (0, \tau_{0,1}].$$

Then take $u^{0,1} = g(\tau_{0,1}, x_{0,n}(\tau_{0,1}))$. By following the same steps, we can obtain $x_{0,n}$ on $[\tau_{0,1}, \tau_{0,2}]$ etc. By induction, we arrive at the following:

$$\begin{cases} \quad x_{0,n}(t) \\ = T(t)\theta(0) + \int_0^t T(t-s)[f(s, x_{0,n}(s)) + B(s, x_{0,n}(s))u_{0,n}(s)]ds, \qquad t \in (0, t_1], \\ u_{0,n}(t) = g(\tau_{0,j}, x_{0,n}(\tau_{0,j})), \qquad t \in [\tau_{0,j}, \tau_{0,j+1}),\ 0 \le j \le n-1. \end{cases}$$

By $(H_{f1}), (H_{B1}), (H_{g1})$ and the proof of Theorem 3.1 there exist $r_{0,0}, r_{0,1}, r_{0,2}, r_{0,3} > 0$ such that

$$\|x_{0,n}\|_{C([0,t_1],X)} \leq r_{0,0}$$

and

$$\|f(\cdot, x_{0,n}(\cdot))\|_{L^2([0,t_1],X)} \leq r_{0,1},$$
$$\|g(\cdot, x_{0,n}(\cdot))\|_{L^2([0,t_1],X)} \leq r_{0,2},$$
$$\|u_{0,n}(\cdot)\|_{L^2([0,t_1],V)} \leq r_{0,3}.$$

Then, in the case of subsequences,

$$f(\cdot, x_{0,n}(\cdot)) \rightharpoonup \overline{f}_0(\cdot) \quad \text{in } L^2([0,t_1], X), \tag{4.2}$$
$$B(\cdot, x_{0,n}(\cdot)) \rightharpoonup \overline{B}_0(\cdot) \quad \text{in } L^2([0,t_1], \mathcal{L}(V, X)), \tag{4.3}$$
$$u_{0,n}(\cdot) \rightharpoonup \overline{u}_0(\cdot) \quad \text{in } L^2([0,t_1], V). \tag{4.4}$$

By (H_A) and Lemma 4.1, we can conclude that

$$\begin{aligned}&\int_0^t T(t-s)[f(s, x_{0,n}(s)) + B(s, x_{0,n}(s))u_{0,n}(s)]ds\\ \to &\int_0^t T(t-s)[\overline{f}_0(s) + \overline{B}_0(s)\overline{u}_0(s)]ds, \quad t \in (0, t_1]\end{aligned}$$

and the sequence $\{x_{0,n}\}$ is relatively compact in $C((0,t_1], X)$.

$$x_{0,n}(\cdot) \to \overline{x}_0(\cdot) \quad \text{in } C((0,t_1], X) \tag{4.5}$$

and

$$\overline{x}_0(t) = T(t)\theta(0) + \int_0^t T(t-s)[\overline{f}_0(s) + \overline{B}_0(s)\overline{u}_0(s)]ds, \qquad t \in (0, t_1].$$

Furthermore, from the definition of $u_{0,n}(\cdot)$, for sufficiently large n we obtain

$$u_{0,n}(t) = g(\tau_{0,j}, x_{0,n}(\tau_{0,j})) \in g(O_\delta(t, \overline{x}_0(t))), \tag{4.6}$$

for all $t \in [\tau_{0,j}, \tau_{0,j+1})$, $0 \leq j \leq n-1$, where $O_\delta(y)$ is the δ-neighborhood of y.

Secondly, by (4.2) and the Mazur's theorem (Chapter 2, Corollary 2.8, [7]), let $a^0_{il}, b^0_{il}, c^0_{il} \geq 0$ and $\sum_{i\geq 1} a^0_{il} = \sum_{i\geq 1} b^0_{il} = \sum_{i\geq 1} c^0_{il} = 1$ such that

$$\overline{f}_{0,l}(\cdot) = \sum_{i\geq 1} a^0_{il} f(\cdot, x_{0,i+l}(\cdot)) \to \overline{f}_0(\cdot), \quad \text{in } L^2([0,t_1], X),$$
$$\overline{B}_{0,l}(\cdot) = \sum_{i\geq 1} b^0_{il} B(\cdot, x_{0,i+l}(\cdot)) \to \overline{B}_0(\cdot) \quad \text{in } L^2([0,t_1], \mathcal{L}(V, X)),$$
$$\overline{u}_{0,l}(\cdot) = \sum_{i\geq 1} c^0_{il} u_{0,i+l} \to \overline{u}_0(\cdot) \quad \text{in } L^2([0,t_1], V).$$

Then, in the case of subsequences,

$$\overline{f}_{0,l}(t) \to \overline{f}_0(t) \quad \text{in } X, \ \text{ a.e. } t \in [0, t_1],$$

$$\overline{B}_{0,l}(t) \to \overline{B}_0(t), \quad \text{in } \mathcal{L}(V,X), \quad \text{a.e. } t \in [0,t_1],$$
$$\overline{u}_{0,l}(t) \to \overline{u}_0(t), \quad \text{in } X, \quad \text{a.e. } t \in [0,t_1].$$

Therefore, based on (4.5) and (4.6), for sufficiently large l, we have

$$\begin{aligned}
&\overline{f}_{0,l}(t) \in cof(t, O_\delta(\overline{x}_0(t))) \quad \text{a.e. } t \in [0,t_1],\\
&\overline{B}_{0,l}(t) \in coB(t, O_\delta(\overline{x}_0(t))) \quad \text{a.e. } t \in [0,t_1],\\
&\overline{u}_{0,l}(t) \in cog(t, O_\delta(\overline{x}_0(t))) \quad \text{a.e. } t \in [0,t_1].
\end{aligned}$$

Thus, for any $\delta > 0$,

$$\begin{aligned}
&\overline{f}_0(t) \in \overline{co}f(t, O_\delta(\overline{x}_0(t))), \quad \text{a.e. } t \in [0,t_1],\\
&\overline{u}_0(t) \in \overline{co}g(t, O_\delta(\overline{x}_0(t))), \quad \text{a.e. } t \in [0,t_1].
\end{aligned}$$

As a result,

$$\overline{f}_0(t) = f(t,\overline{x}_0(t)),\ \overline{B}_0(t) = B(t,\overline{x}_0(t)),\ \overline{u}_0(t) = g(t,\overline{x}_0(t)) \quad \text{a.e. } t \in [0,t_1].$$

Case 2. Let $t \in (t_1, t_2]$. For any $n > 0$, let $\tau_{1,j} = t_1 + \frac{j}{n}(t_2 - t_1),\ 0 \le j \le n-1$. We set

$$u_{1,n}(t) = \begin{cases} \overline{u}_0(t), & t \in [0,t_1],\\ \displaystyle\sum_{j=0}^{n-1} u^{1,j}\chi_{[\tau_{1,j},\tau_{1,j+1})}(t), & t \in (t_1,t_2]. \end{cases}$$

The sequence $\{u^{1,j}\}$ is created in the following manner. Take $u^{1,0} = g(t_1, x(t_1))$. According to Theorem 3.2, there exists $x_{1,n}(\cdot)$ which is given by

$$x_{1,n}(t) = \begin{cases} \theta(t), & t \in [-\tau,0],\\ \overline{x}_0(t), & t \in (0,t_1],\\ T(t)\theta(0) + T(t-t_1)G(t_1, x_{1,n}(t_1)) & \\ + \displaystyle\int_0^t T(t-s)[f(s,x_{1,n}(s)) + B(s,x_{1,n}(s))u^{1,0}]ds, & t \in (t_1, \tau_{1,1}]. \end{cases}$$

Then take $u^{1,1} = g(\tau_{1,1}, x_{1,n}(\tau_{1,1}))$. By applying the same process repeatedly, we can calculate $x_{1,n}$ on $(\tau_{1,1}, \tau_{1,2}]$. By induction, we eventually derive the following result:

$$x_{1,n}(t) = \begin{cases} \theta(t), & t \in [-\tau,0],\\ \overline{x}_0(t), & t \in (0,t_1],\\ T(t)\theta(0) + T(t-t_i)G(t_1, x_{1,n}(t_1)) & \\ + \displaystyle\int_0^t T(t-s)[f(s,x_{1,n}(s)) + B(s,x_{1,n}(s)u_{1,n}(s)ds, & t \in (t_1,t_2], \end{cases}$$
$$u_{1,n}(t) = g(\tau_{1,j}, x_{1,n}(\tau_{1,j})),\ t \in (\tau_{1,j}, \tau_{1,j+1}],\ 0 \le j \le n-1.$$

By $(H_{f1}),(H_{B1}),(H_{g1}),(H_{G1})$ and the proof of Theorem 3.2, there exist $r_{1,0}, r_{1,1}$, $r_{2,1}, r_{3,1} > 0$ such that

$$\|x_{1,n}\|_{PC((0,t_2],X)} \le r_{1,0}$$

and

$$\begin{aligned}&\|f(\cdot, x_{1,n}(\cdot))\|_{L^2([0,t_2],X)} \le r_{1,1},\\ &\|B(\cdot, x_{1,n}(\cdot))\|_{L^2([0,t_2],\mathcal{L}(V,X))} \le r_{1,2}, *\\ &\|u_{1,n}(\cdot)\|_{L^2([0,t_2],V)} \le r_{1,3}.\end{aligned}$$

Then, there exists a subsequence $\{f(\cdot, x_{1,n}(\cdot))\}$ such that

$$f(\cdot, x_{1,n}(\cdot)) \rightharpoonup \overline{f}_1(\cdot) \qquad \text{in } L^2([0,t_2],X), \tag{4.7}$$

$$B(\cdot, x_{1,n}(\cdot)) \rightharpoonup \overline{B}_1(\cdot) \qquad \text{in } L^2([0,t_2],\mathcal{L}(V,X)), \tag{4.8}$$

$$u_{1,n}(\cdot) \rightharpoonup \overline{u}_0(\cdot) \quad \text{in } L^2([0,t_1],V). \tag{4.9}$$

It is clear that $\overline{f}_1|_{[0,t_1]} = \overline{f}_0, \overline{u}_1|_{[0,t_1]} = \overline{u}_0$.

Employing (H_A) and Lemma 4.1 again, we derive that

$$\begin{aligned}&\int_0^t T(t-s)f(s,x_{1,n}(s))ds \to \int_0^t T(t-s)\overline{f}_1(s)ds, \qquad t\in(t_1,t_2],\\ &x_{1,n}(\cdot)\to \overline{x}_1(\cdot), \qquad \text{in } PC((0,t_2],X)\end{aligned} \tag{4.10}$$

and

$$\overline{x}_1(t) = \begin{cases}\theta(t), \qquad t\in[-\tau,0],\\ \overline{x}_0(t), \qquad t\in(0,t_1],\\ T(t)\theta(0)+T(t-t_1)G(t_1,\overline{x}_1(t_1))\\ +\displaystyle\int_0^t T(t-s)[\overline{f}_1(s)+\overline{B}_1(s)\overline{u}_1(s))ds, \qquad t\in(t_1,t_2].\end{cases}$$

Furthermore, as n becomes sufficiently large, according to the definition of $u_{1,n}(\cdot)$, we have

$$u_{1,n}(t) = g(\tau_{1,j}, x_{1,n}(\tau_{1,j})) \in g(O_\delta(t,\overline{x}_1(t))), \tag{4.11}$$

for all $t\in(\tau_{1,j},\tau_{1,j+1}]$, $0\le j\le n-1$.

Secondly, utilizing (4.7) and the Mazur's theorem once more, let $a^1_{il}, b^1_{il}, c^1_{il} \ge 0$ and $\sum_{i\ge1} a^1_{il} = \sum_{i\ge1} b^1_{il} = \sum_{i\ge1} c^1_{il} = 1$ such that

$$\begin{aligned}&\overline{f}_{1,l}(\cdot) = \sum_{i\ge1} a^1_{il} f(\cdot, x_{1,i+l}(\cdot)) \to \overline{f}_1(\cdot) \qquad \text{in } L^2([0,t_2],X),\\ &\overline{B}_{1,l}(\cdot) = \sum_{i\ge1} a^1_{il} B(\cdot, x_{1,i+l}(\cdot)) \to \overline{B}_1(\cdot) \qquad \text{in } L^2([0,t_2],\mathcal{L}(V,X)),\\ &\overline{u}_{1,l}(\cdot) = \sum_{i\ge1} a^1_{il} u_{1,i+l}(\cdot) \to \overline{u}_1(\cdot) \qquad \text{in } L^2([0,t_2],V).\end{aligned}$$

Subsequently, in the case of subsequences,

$$\begin{aligned}
&\overline{f}_{1,l}(t) \to \overline{f}_1(t) \qquad \text{in } X, \qquad \text{a.e. } t \in [0,t_2],\\
&\overline{B}_{1,l}(t) \to \overline{B}_1(t) \qquad \text{in } \mathcal{L}(V,X), \qquad \text{a.e. } t \in [0,t_2],\\
&u_{1,l}(t) \to \overline{u}_1(t) \qquad \text{in } V, \qquad \text{a.e. } t \in [0,t_2].
\end{aligned}$$

Thus, for sufficiently large l, based on (4.10) and (4.11),

$$\begin{aligned}
&\overline{f}_{1,l}(t) \in cof(t, O_\delta(\overline{x}_1(t))),\ \overline{B}_{1,l}(t) \in coB(t, O_\delta(\overline{x}_1(t))),\\
&\overline{u}_{1,l}(t) \in cog(t, O_\delta(\overline{x}_1(t))),\ \ \text{a.e. } t \in [0,t_2].
\end{aligned}$$

Thus, for any $\delta > 0$,

$$\begin{aligned}
&\overline{f}_1(t) \in \overline{co}f(t, O_\delta(\overline{x}_1(t))), \qquad \text{a.e. } t \in [0,t_2],\\
&\overline{B}_1(t) \in \overline{co}B(t, O_\delta(\overline{x}_1(t))), \qquad \text{a.e. } t \in [0,t_2],\\
&\overline{u}_1(t) \in \overline{co}g(t, O_\delta(\overline{x}_1(t))), \qquad \text{a.e. } t \in [0,t_2].
\end{aligned}$$

As a result,

$$\overline{f}_1(t) = f(t, \overline{x}_1(t)),\ \overline{B}_1(t) = B(t, \overline{x}_1(t)),\ \overline{u}_1(t) = g(t, \overline{x}_1(t)), \qquad \text{a.e. } t \in [0,t_2].$$

Case 3. Let $t \in (t_k, t_{k+1}](k = 2, \cdots, m)$. For any $n > 0$, let $\tau_{k,j} = t_k + \frac{j}{n}(t_{k+1} - t_k),\ 0 \le j \le n-1$. We set

$$u_{k,n}(t) = \begin{cases} \overline{u}_{k-1}(t), \qquad t \in [0, t_k],\\ \displaystyle\sum_{j=0}^{n-1} u^{k,j}\chi_{(\tau_{k,j},\tau_{k,j+1}]}(t), \qquad t \in (t_k, t_{k+1}]. \end{cases}$$

By induction, we arrive at the following conclusion:

$$x_{k,n}(t) = \begin{cases} \theta(t), \qquad t \in [-\tau, 0],\\ \overline{x}_{k-1}(t), \qquad t \in (0, t_k],\\ T(t)\theta(0) + \displaystyle\sum_{i=1}^{k} T(t-t_i)G(t_i, x_{i,n}(t_i))\\ + \displaystyle\int_0^t T(t-s)[f(s, x_{k,n}(s)) + B(s, x_{k,n}(s)u_{k,n}(s)]ds, \qquad t \in (t_k, t_{k+1}], \end{cases}$$

$$u_{k,n}(t) = g(\tau_{k,j}, x_{k,n}(\tau_{k,j})), \qquad t \in (\tau_{k,j}, \tau_{k,j+1}],\ 0 \le j \le n-1.$$

There exists $r_{k,0}, r_{k,1}, r_{k,2}, r_{k,3} > 0$ such that

$$\begin{aligned}
&\|x_{k,n}\|_{PC((0,t_{k+1}],X)} \le r_{k,0},\\
&\|f(\cdot, x_{k,n}(\cdot))\|_{L^2([0,t_{k+1}],X)} \le r_{k,1},\\
&\|B(\cdot, x_{k,n}(\cdot), u_{k,n}(\cdot))\|_{L^2([0,t_{k+1}],\mathcal{L}(V,X))} \le r_{k,2},\\
&\|u_{k,n}(\cdot)\|_{L^2([0,t_{k+1}],V)} \le r_{k,3}.
\end{aligned}$$

Next, for subsequences.

$$f(\cdot, x_{k,n}(\cdot)) \rightharpoonup \overline{f}_k(\cdot) \quad \text{in } L^2([0,t_{k+1}],X), \tag{4.12}$$

$$B(\cdot, x_{k,n}(\cdot)) \rightharpoonup \overline{B}_k(\cdot) \quad \text{in } L^2([0, t_{k+1}], \mathcal{L}(V, X)), \tag{4.13}$$
$$u_{k,n}(\cdot) \rightharpoonup \overline{u}_k(\cdot) \quad \text{in } L^2([0, t_{k+1}], V). \tag{4.14}$$

It is clear that $\overline{f}_k|_{[0,t_k]} = \overline{f}_{k-1}, \overline{B}_k|_{[0,t_k]} = \overline{B}_{k-1}, \overline{u}_k|_{[0,t_k]} = \overline{u}_{k-1}$ and

$$\begin{aligned} &\int_0^t T(t-s)[f(s, x_{k,n}(s)) + B(s, x_{k,n}(s))u_{k,n}(s)]ds \\ \to &\int_0^t T(t-s)[\overline{f}_k(s) + \overline{B}_k(s)\overline{u}_k(s)]ds, \qquad t \in (t_k, t_{k+1}]. \end{aligned}$$

We obtain similar results as in the previous proof

$$x_{k,n}(\cdot) \to \overline{x}_k(\cdot) \qquad \text{in } PC((0, t_{k+1}], X), \tag{4.15}$$

and

$$\overline{x}_k(t) = \begin{cases} \theta(t), \qquad t \in [-\tau, 0], \\ \overline{x}_{k-1}(t), \qquad t \in (0, t_k], \\ T(t)\theta(0) + \displaystyle\sum_{i=1}^{k} T(t-t_i)G(t_i, \overline{x}_i(t_i)) \\ + \displaystyle\int_0^t T(t-s)[\overline{f}_k(s) + \overline{B}_k(s)\overline{u}_k(s)]ds, \qquad t \in (t_k, t_{k+1}]. \end{cases}$$

Furthermore, from the definition of $u_{k,n}(\cdot)$ for sufficiently large n, we obtain

$$u_{k,n}(t) = g(\tau_{k,j}, x_{k,n}(\tau_{k,j})) \in g(O_\delta(t, \overline{x}_k(t))), \tag{4.16}$$

for all $t \in (\tau_{k,j}, \tau_{k,j+1}],\ 0 \le j \le n-1$.

Furthermore, by equation (4.14) and the Mazur's theorem once more let a_{il}^k, b_{il}^k, $b_{il}^k \ge 0$ and $\sum_{i\ge1} a_{il}^k = \sum_{i\ge1} b_{il}^k = \sum_{i\ge1} c_{il}^k = 1$ such that

$$\begin{aligned} \overline{f}_{k,l}(\cdot) &= \sum_{i\ge1} a_{il}^k f(\cdot, x_{k,i+l}(\cdot)) \to \overline{f}_k(\cdot) \qquad \text{in } L^2([0, t_{k+1}], X), \\ \overline{B}_{k,l}(\cdot) &= \sum_{i\ge1} a_{il}^k B(\cdot, x_{k,i+l}(\cdot)) \to \overline{B}_k(\cdot) \qquad \text{in } L^2([0, t_{k+1}], \mathcal{L}(V, X)), \\ \overline{u}_{k,l}(\cdot) &= \sum_{i\ge1} c_{il}^k u_{k,i+l}(\cdot) \to \overline{u}_k(\cdot) \qquad \text{in } L^2([0, t_{k+1}], V). \end{aligned}$$

Subsequently, in the case of subsequences,

$$\begin{aligned} &\overline{f}_{k,l}(t) \to \overline{f}_k(t) \qquad \text{in } X, \qquad \text{a.e. } t \in [0, t_{k+1}], \\ &\overline{B}_{k,l}(t) \to \overline{B}_k(t) \qquad \text{in } \mathcal{L}(V, X), \qquad \text{a.e. } t \in [0, t_{k+1}], \\ &\overline{u}_{k,l}(t) \to \overline{u}_k(t) \qquad \text{in } V, \qquad \text{a.e. } t \in [0, t_{k+1}]. \end{aligned}$$

Therefore, based on (4.15) and (4.16), for sufficiently large l,

$$\begin{aligned} &\overline{f}_{k,l}(t) \in cof(t, O_\delta(\overline{x}_k(t))),\ \overline{B}_{k,l}(t) \in coB(t, O_\delta(\overline{x}_k(t))), \\ &\overline{u}_{k,l}(t) \in cog(t, O_\delta(\overline{x}_k(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}]. \end{aligned}$$

Therefore, for any $\delta > 0$,

$$\begin{aligned}
&\overline{f}_k(t) \in \overline{co} f(t, O_\delta(\overline{x}_k(t))),\ \overline{B}_k(t) \in \overline{co} B(t, O_\delta(\overline{x}_k(t))),\\
&\overline{u}_k(t) \in \overline{co} g(t, O_\delta(\overline{x}_k(t))), \quad \text{a.e. } t \in (t_k, t_{k+1}].
\end{aligned}$$

As a result,

$$\overline{f}_k(t) = f(t, \overline{x}_k(t)),\ \overline{B}_k(t) = B(t, \overline{x}_k(t)),\ \overline{u}_k(t) = g(t, \overline{x}_k(t)), \qquad \text{a.e. } t \in (t_k, t_{k+1}].$$

Let

$$\begin{aligned}
&\overline{x}(t) = \begin{cases} \theta(t), & t \in [-\tau, 0],\\ \overline{x}_0(t), & t \in (0, t_1],\\ \overline{x}_k(t), & t \in (t_k, t_{k+1}],\ k = 1, \cdots, m, \end{cases}\\
&\overline{u}(t) = \overline{u}_{m+1}(t).
\end{aligned}$$

Hence, $\overline{x} \in \mathcal{C}$ is a mild solution to problem (1.1). The proof is finished. □

5. Exponential stability

In this section, we delve into the exponential stability of the problem (1.1).

Definition 5.1. A mild solution x of (1.1) is called globally exponentially stable if there exists two constants $L > 0$ and $\omega > 0$ such that

$$\|x(t)\|_X \le L e^{-\omega t}.$$

For this purpose, we must impose the subsequent hypothesis.
(H_T) : there exist two constants $M, w > 0$ such that

$$\|T(t)\| \le M e^{-wt}, \qquad \text{for every } t \ge 0.$$

Theorem 5.1. *Under the hypotheses of Theorem 3.2 and (H_T), the mild solution of (1.1) is globally exponentially stable if $M(L_f + L_B l_g) < w$.*

Proof. Initially, Theorem 3.2 ensures that (1.1) possesses a unique mild solution. If $t \in (0, t_1]$, we can deduce from the assumption (H_T) and the Hölder's inequality that

$$\begin{aligned}
&\|x(t)\|_X\\
&\le \|T(t)\theta(0)\|_X + \int_0^t \|T(t-s)[f(s, x(s)) + B(s, x(s))g(s, x(s))]\|_X ds\\
&\le M e^{-\omega t}\|\theta(0)\|_X + M\int_0^t e^{-\omega(t-s)}[\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|\\
&\quad + (\|B(s, x(s)) - B(s, 0)\| + \|B(s, 0)\|)\|g(s, x(s))\|]ds\\
&\le M e^{-\omega t}\|\theta(0)\|_X + M\int_0^t e^{-\omega(t-s)}(L_f\|x(s)\|_X + |\phi(s)| + L_B\|\varphi\|_{L^\infty}\|x(s)\|_X)ds\\
&\le M e^{-\omega t}\|\theta(0)\|_X + \int_0^t e^{-\omega(t-s)}|\phi(s)|ds
\end{aligned}$$

$$+M(L_f+L_B\|\varphi\|_{L^\infty})\int_0^t e^{-\omega(t-s)}\|x(s)\|_X ds.$$

Then

$$e^{\omega t}\|x(t)\|_X\leq M\|\theta(0)\|_X+\int_0^t e^{\omega s}|\phi(s)|ds+M(L_f+L_B\|\varphi\|_{L^\infty})\int_0^t e^{\omega s}\|x(s)\|_X ds.$$

Therefore, upon employing the standard Gronwall's inequality once more, we derive

$$e^{\omega t}\|x(t)\|_X\leq M_5e^{M_6t},$$

where

$$M_5=M\|\theta(0)\|_X+\sup_{t\in J}\int_0^t e^{\omega s}|\phi(s)|ds,\ M_6=M(L_f+L_B\|\varphi\|_{L^\infty}),$$

implying that

$$\|x(t)\|_X\leq Le^{-\omega t}$$

with

$$L=M_5,\omega=w-M_6.$$

If $t\in(t_k,t_{k+1}](k=1,\cdots,m)$, from the hypotheses $(H_f),(H_B),(H_g),(H_G)$ and the Hölder's inequality, we have

$$\begin{aligned}
&\|x(t)\|_X\\
&\leq\|T(t)\theta(0)\|_X+\|\sum_{i=1}^k T(t-t_i)G(t_i,x(t_i))\|_X\\
&\quad+\int_0^t\|T(t-s)[f(s,x(s))+B(s,x(s))G((s,x(s))]\|_X ds\\
&\leq Me^{-\omega t}\|\theta(0)\|_X+Me^{-\omega t}\|\sum_{i=1}^k(\|G(t_i,x(t_i))-G(t_i,0)\|_X+\|G(t_i,0)\|_X)\\
&\quad+M\int_0^t e^{-\omega(t-s)}[\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\\
&\quad+(\|B(s,x_\eta(s))-B(s,0)\|+\|B(s,0)\|)\|\eta(s)\|]ds\\
&\leq Me^{-\omega t}\|\theta(0)\|_X+Me^{-\omega t}\sum_{i=1}^k L_i\|x(t_i)\|_X+Me^{-\omega t}\|G(t_i,0)\|_X\\
&\quad+Me^{-\omega t}\|\theta(0)\|_X+\int_0^t e^{-\omega(t-s)}|\phi(s)|ds\\
&\quad+M(L_f+L_B\|\varphi\|_{L^\infty})\int_0^t e^{-\omega(t-s)}\|x(s)\|_X ds\\
&\leq Me^{-\omega t}\|\theta(0)\|_X+M\sum_{i=1}^k L_i\|x(t_i)\|_X+Me^{-\omega t}\|G(t_i,0)\|_X\\
&\quad+\int_0^t e^{\omega s}|\phi(s)|ds+M(L_f+L_B\|\varphi\|_{L^\infty})\int_0^t e^{\omega s}\|x(s)\|_X ds.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\sup_{s\in[0,t]}\|x(s)\|_X\\
&\le Me^{-\omega t}\|\theta(0)\|_X + M\sum_{i=1}^{k} L_i \sup_{s\in[0,t]}\|x(s)\|_X + Me^{-\omega t}\|G(t_i,0)\|_X\\
&\quad+\int_0^t e^{-\omega(t-s)}|\phi(s)|ds + M(L_f + L_B\|\varphi\|_{L^\infty})\int_0^t e^{-\omega(t-s)}\|x(s)\|_X ds,
\end{aligned}
$$

thereby

$$
\begin{aligned}
&(1-M\sum_{i=1}^{k} L_i)e^{\omega t}\sup_{s\in[0,t]}\|x(s)\|_X\\
&\le M\|\theta(0)\|_X + M\|G(t_i,0)\|_X\\
&\quad+\int_0^t e^{\omega s}|\phi(s)|ds + M(L_f + L_B\|\varphi\|_{L^\infty})\int_0^t e^{\omega s}\sup_{r\in[0,s]}\|x(r)\|_X ds.
\end{aligned}
$$

Since $M\sum_{i=1}^{k} L_i < 1$, by applying standard Gronwall's inequality again, we deduce that

$$
e^{\omega t}\sup_{s\in[0,t]}\|x(s)\|_X \le M_7 e^{M_8 t},
$$

where

$$
M_7 = M\|\theta(0)\|_X + M\|G(t_i,0)\|_X + \sup_{t\in J}\int_0^t e^{\omega s}|\phi(s)|ds,\ M_8 = M(L_f + L_B\|\varphi\|_{L^\infty}).
$$

This entails that

$$
\|x(t)\|_X \le Le^{-\omega t}
$$

with

$$
L = M_7,\ \omega = w - M_8.
$$

The proof is complete. □

6. Applications

6.1. Differential hemivariational inequalities

Differential variational inequalities (DVIs, abbreviated) are intricate systems that intertwine differential or partial differential equations with a time-dependent variational inequality (refer to [10, 11]). They constitute a formidable mathematical instrument utilized in dissecting a vast array of pragmatic challenges encountered in contact and impact mechanics, electrical circuits incorporating ideal diodes, economic dynamics, and transportation networks. Differential hemivariational inequalities (DHVIs, for short) are a significant extension of DVIs that combine differential or partial differential equations with a hemivariational inequality.

Let us revisit the definition of Clarke's subdifferential for a locally Lipschitz function. $j : K \subset V \to \mathbb{R}$, where K is a nonempty subset of a Banach space V (one can see [2,13]). We denote by $j^0(u; v)$ the Clarke's generalized directional derivative of j at the point $u \in K$ in the direction $v \in X$, that is

$$j^0(u; v) := \limsup_{\lambda \to 0^+,\ \zeta \to u} \frac{j(\zeta + \lambda v) - j(\zeta)}{\lambda}.$$

Recall also that the Clarke's subdifferential or generalized gradient of j at $u \in K$, denoted by $\partial j(u)$, is a subset of V^* given by

$$\partial j(u) := \{x^* \in X^* : j^0(u; v) \geq \langle u^*, v\rangle,\ \forall v \in V\}.$$

In this subsection, we consider the impulsive differential hemivariational inequality:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) + B(t, x(t))u(t), \qquad t \in (0, b] - D, \\ u(t) \in SOL(K; C, h, \Phi, j), \qquad \text{a.e. } t \in (0, b] - D, \\ \Delta x(t_k) = G(t_k, x(t_k)), \qquad t_k \in D, \\ x(t) = \theta(t), \qquad t \in [-\tau, 0]. \end{cases} \tag{6.1}$$

Here $SOL(K; C, h, \Phi, j)$ represents the set of solutions to the hemivariational inequality in V: find $u : J \to K \subset V$ such that

$$\begin{aligned} &\langle Cu(t) + h(t, x(t)), v - u(t)\rangle_V + \Phi(u(t), v) - \Phi(u(t), u(t)) \\ &\quad + j^\circ(u(t); v - u(t)) \geq 0, \quad \forall v \in K,\ \text{a.e. } t \in J. \end{aligned} \tag{6.2}$$

Consider the following assumptions regarding the data in problem (6.2).
(H_K) : K is a closed convex subset of V such that $0_V \in K$.
(H_C) : $C : V \to V^*$ is pseudomonotone and there exists a constant $\alpha_C > 0$ such that

$$\langle Cv_1 - Cv_2, v_1 - v_2\rangle_V \geq \alpha_C \|v_1 - v_2\|_V^2$$

for all $v_1,\ v_2 \in V$.
(H_h) : $h : J \times X \to V^*$ is continuous on $J \times X$, and there exist constants $l_h, L_h > 0$ such that

$$\begin{aligned} &\|h(t, x)\|_{V^*} \leq l_h, \\ &\|h(t, x_1) - \tau(t, x_2)\|_X \leq L_h \|x_1 - x_2\|_X \end{aligned}$$

for all $x_1,\ x_2 \in X$ and a.e. $t \in J$.
(H_Φ) : $\Phi(u, \cdot)$: $V \to \mathbb{R}$ is convex and lower semicontinuous on V for all $u \in V$, $\Phi(u, \lambda v) = \lambda \Phi(u, v)$, $\Phi(v, v) \geq 0$ for all $u, v \in V, \lambda > 0$, and there exists $\alpha_\Phi > 0$ such that

$$\Phi(u_1, v_2) - \Phi(u_1, v_1) + \Phi(u_2, v_1) - \Phi(u_2, v_2) \leq \alpha_\Phi \|u_1 - u_2\|_V \|v_1 - v_2\|_V$$

for all $u_1, u_2, v_1, v_2 \in V$.
(H_j) : j: $V \to \mathbb{R}$ is locally Lipschitz, and there exist $c_0, c_1, \alpha_j > 0$ such that

$$\|\partial j(u)\|_{V^*} \leq c_0 + c_1 \|u\|_V,$$

$$j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \le \alpha_j \|u_1 - u_2\|_V^2$$

for all $u, u_1, u_2 \in V$.

Theorem 6.1 ([8], Lemma 3.3)**.** *Assume that the hypotheses* $(H_K), (H_C), (H_h), (H_\Phi), (H_j)$ *are satisfied and*

$$\alpha_\Phi + \alpha_j < \alpha_C.$$

Then problem (6.2) possesses a unique solution $u \in C(J, V)$*. Moreover,* u *satisfies the following estimate*

$$\|u(t)\|_V \le \frac{1}{m_C - \alpha_j}(\|C0_V\|_{V^*} + \|l_h\|_{V^*} + c_0). \tag{6.3}$$

Furthermore, if $u_1, u_2 \in C(J, V)$ *denote the solution of problem (6.2) for* $x_1, x_2 \in C(J, X)$*, then it holds that*

$$\|u_1(t) - u_2(t)\|_V \le \frac{L_h}{\alpha_C - \alpha_\Phi - \alpha_j}\|x_1(t) - x_2(t)\|_X.$$

The subsequent finding regarding the existence and uniqueness of the solution stems directly from Theorem 3.1.

Theorem 6.2. *Assume that* $(H_f), (H_B), (H_g), (H_G)$ *and the hypotheses of Theorem 6.1 are hold. Then problem (6.1) has a unique mild solution on* $\mathcal{C}$*.*

6.2. A boundary value problem

For $a, b > 0$, we consider the following boundary value problem.

$$\begin{cases} \dfrac{\partial}{\partial t}x(r,t) \\ = \delta \dfrac{\partial^2}{\partial r^2}x(r,t) - cx(r,t) + \mu(r,t)\sin x(r,t) \\ +\nu(r,t)\alpha(r,t)|x(r,t)|\dfrac{x^2(r,t)}{1+x^2(r,t)}, \quad 0 \le r \le a, \quad t \in (0,b] - D, \\ \Delta x(r,t_k) = -\beta(r,t_k)\dfrac{x(r,t_k)}{1+|x(r,t_k)|}, \quad 0 \le r \le a, t_k \in D, \\ x(0,t) = x(a,t) = 0, \quad 0 \le t \le b, \\ x(r,\sigma) = \theta(r,\sigma), \quad 0 \le r \le a, \quad \sigma \in [-\tau, 0], \end{cases} \tag{6.4}$$

where $\delta, c > 0, \mu, \nu, \alpha, \beta \in C([0,a] \times [0,b]) \to \mathbb{R}_+$. Here, $x(r,t)$ represents the temperature of the point w at time t.

To represent problem (6.4) as the form of problem (1.1), we take $X = V = L^2([0,a])$ and $x(t)(\cdot) := x(t, \cdot)$. Let A: $D(A) \to X$ be the operator given by

$$A\xi = \delta\xi'' - c\xi,$$

with the domain

$$D(A) = H^2([0,a]) \cap H_0^1([0,a]) = \Big\{\xi \in X : \ \xi', \ \xi'' \in X, \ \xi(0) = \xi(a) = 0\Big\}.$$

From [15], it is easy to verify that A generates a C_0 semigroup $\{T(t) = e^{At}\}_{t\geq 0}$ satisfying

$$\|T(t)\| \leq e^{-(c+\delta\pi^2/4)t}, \qquad \text{for every } t \geq 0.$$

This implies that (H_T) holds with $M = 1, w = c + \delta\pi^2/4$.

Now define the map $f : J \times X \to X, g : J \times X \to V, B : J \times X \to \mathcal{L}(V, X)$ $G : D \times X \to X$, respectively, as

$$\begin{aligned}
&f(t,x)(r) = f(r,t,x(r)) = \mu(r,t)\sin x(r),\\
&B(t,x)(r) = B(r,t,x(r)) = \nu(r,t)|x(r)|,\\
&g(t,x)(r) = g(r,t,x(r)) = \alpha(r,t)\frac{x^2(r)}{1+x^2(r)},\\
&G(t,x)(r) = G(r,t,x(r)) = -\beta(r,t)\frac{x(r)}{1+|x(r)|},
\end{aligned}$$

for any $t \in [0, b]$ and $x \in X$. Then under these notations problem (6.4) is rewritten into the form of (1.1). One can verify easily that $(H_f), (H_B), (H_g), (H_G)$ holds with $L_f = \|\mu\|_C, L_B = \|\nu\|_C, L_g = \|\alpha\|_C, L_k = \|\beta\|_C$.

Consequently, when $2k\|\beta\|_C < 1$ and $\|\mu\|_C + \|\alpha\|_C\|\nu\|_C < c + \delta\pi^2/4$, from Theorem 3.2 and Theorem 5.1, problem (6.4) admits one unique mild solution which is globally exponentially stable.

7. Conclusions

We investigate in this work the existence and the exponential stability of a new class of first-order impulsive evolution equations, in which two kinds of methods are provided for the existence of mild solutions. Moreover, based on the existence result, the exponential stability of the solution is also present. Thereafter, we show an interesting application to prove the existence of solutions to a class of differential hemivariational inequalities.

It is valuable to study such results for the equations by using similar method or finding more methods. Furthermore, it is still interesting to weaken the hypotheses of Lipschitz conditions and the compactness conditions.

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