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BACKWARDS DYNAMICS OF TIME-DEPENDENT ATTRACTOR FOR PLATE EQUATION ON THE WHOLE SPACE*

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Abstract In this paper, we give the definition and criterion for the existence of backwards compact time-dependent attractor which is the minimal one among the backwards compact and pullback attracting sets in time-dependent whole space. Combining with the method of C_t-limit compact and backwards asymptotic estimates outside a ball, the existence of backwards compact time-dependent attractor for stretchable plate equation with localized weak damping on the whole space is obtained.

Keywords Backwards compact, time-dependent attractor, stretchable plate equation, unbounded domain.

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1. Introduction

In this paper, we consider the existence of the backwards compact attractor for the non-autonomous stretchable plate with local damping

$$\begin{cases} \varepsilon(t)u_{tt} + \alpha(x)u_t + \Delta^2 u + \lambda u + (p - \|\nabla u\|^2)\Delta u + f(x,u) = g(x,t), \ t > \tau, \\ u(x,\tau) = u_0(x), u_t(x,\tau) = u_1(x), \ x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

on time-dependent unbounded domain, where $x \in \mathbb{R}^n (n \geq 5), \ \lambda > 0, \ p \in \mathbb{R}, \ \varepsilon = \varepsilon(t)$ is a decreasing bounded function along with

$$\lim_{t \to +\infty} \varepsilon(t) = 0, \tag{1.2}$$

and there exists a constant $L > 0$ such that

$$\sup_{t \in \mathbb{R}} [|\varepsilon(t)| + |\varepsilon'(t)|] \leq L. \tag{1.3}$$

Moreover, we assume that the function in (1.1) satisfies the following conditions:

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Function $\alpha(x) \in L^\infty(\mathbb{R}^n)$, and

$$\alpha'(x) \in L^\infty(\mathbb{R}^n),\ \alpha(\cdot) \geq \alpha_0 > 0 \ \ a.e.\ x \in \mathbb{R}^n. \tag{1.4}$$

Nonlinear function $f(x,s) \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ satisfies $\forall\ s \in \mathbb{R}, x \in \mathbb{R}^n$, there is

$$|f(x,u)| \leq c_0|u|^q + \phi_1(x), \qquad \phi_1(x) \in L^2(\mathbb{R}^n), \tag{1.5}$$

$$f(x,u)u - c_1F(x,u) \geq \phi_2(x), \qquad \phi_2(x) \in L^1(\mathbb{R}^n), \tag{1.6}$$

$$F(x,u) \geq c_2|u|^{q+1} - \phi_3(x), \qquad \phi_3(x) \in L^1(\mathbb{R}^n), \tag{1.7}$$

$$|\frac{\partial f}{\partial u}(x,u)| \leq \beta; \quad |\frac{\partial f}{\partial x}(x,s)| \leq \phi_4(x), \qquad \phi_4(x) \in L^2(\mathbb{R}^n), \tag{1.8}$$

where $c_i > 0, i = 0,1,2,\ \beta > 0,\ F(x,u) = \int_0^u f(x,s)ds$, and $0 < q \leq \frac{n}{n-4}$ for $n \geq 5$. The condition for g is given below.

The problem (1.1) stems from the elastic equation established by Woinowsky-Krieger, Ball [4, 15] etc. It is clear that (1.1) becomes an autonomous plate equation when $\varepsilon(t)$ is a positive constant. In this case, we can characterize the long-time behavior of solutions by virtue of the concept of global attractors under the framework of semigroup. Some authors have extensively studied the existence of global attractors for the autonomous plate equation on unbounded domain, one can see [3, 12, 13, 23] and references therein. On the other hand, when g is dependent on t, the attractors are studied for the non-autonomous system on bounded and unbounded domain, respectively, one can refer to [1, 10, 11, 24, 26, 27] and so on.

Just for our problem (1.1), since the presence of $\varepsilon(t)$ vanishing at infinity leads to time-dependent terms at functional level, the dynamical system associated with such problem is still understood under the non-autonomous framework even the forcing term g in the equation is independent of time t. In order to describe the long-term behavior of solution for these problems, the concept of time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}}$ is introduced [8, 22] and the methods for verifying the compactness are extended to time-dependent bounded domain, such as contractive functions and C_t-condition [20, 21]. However, when the domain is unbounded, in [18, 19] we have given the method of asymptotic contractive function to verify the compactness of the solution process family and the method is used for plate equation, which obtain the existence of time-dependent global attractor in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. But when the force term g depends on t simultaneously, we want to know how the backwards behavior of the solution is going under the influence of the non-autonomous external force term and the degenerate coefficient, where the conditions for damping coefficient are taken from [3].

For the backwards dynamics of pullback attractors, the backwards asymptotic behavior of some kinds of equation have been discussed. In [5, 6], when $\alpha(x) \equiv \beta(t)$ the authors proved that the evolution process for wave equation on bounded domain have a pullback attractor $\mathcal{A} = \{A(t)\}$ that is backwards bounded, i.e. $\bigcup_{s\leq t} A_s$ is bounded for each t, for more results about the backwards compactness for pullback attractor about evolution equation with time-dependent coefficient, one can refer to [2, 9, 16, 17, 25, 29] and so on.

So in this section, combing with the pullback attraction of time-dependent attractor and the results of backwards dynamics for non-autonomous wave equations, under some suitable assumptions of the external force term [9, 28], we discuss the existence of the backwards compact attractor of equation (1.1) in time-dependent whole space. From the discussion that follows, the pullback-bounded family depends on the conditions of the external force term, which is consistent with respect to the parameter $\varepsilon(t)$.

As we know, the existence of attractor depends on some kind compactness for a dynamical system. For our system, it is different from the result of wave equation [28], because there are some novelties in showing the main results:

(i) For the existence of degenerate coefficient, the goal of this paper is to define a suitable object family $\mathfrak{A} = \{A_t\}_{t\in\mathbb{R}} \subset X_t$ and for each $t \in \mathbb{R}$, every $s \leq t, A_s$ and $\bigcup_{s\leq t} A_s$ are able to attract(under the topology of X_t) all solution of the system originating sufficiently far in the past.

(ii) To overcome the difficulty of lacking the compactness of Sobolev embedding in $\mathbb{R}^n$, we take advantage of the idea of ω_t-limit compact in time-dependent space along with tail estimate to overcome the difficulties caused by nonlinear term $\|\nabla u\|^2\Delta u$ and critical nonlinear function.

(iii) Since the presence of localized weak damping, which makes the estimation more complicated, so we can't directly use the method [18] to settle the problem. Moreover, we think the paper extend the already result in [19].

For brevity, we denote C be a family of positive constant, which will change in the different line, even in the same line.

The rest of this article consists of two sections. In Section 2, we define some functions sets and give some useful lemmas. In Section 3, existence of solution and the time-dependent absorbing set as well as tail estimate and the existence of the time-dependent global attractor for (1.1) based on the asymptotic contractive process are obtained.

2. Preliminaries

2.1. Abstract results

In the following, according to the definition of the pullback attractors and backwards compact attractors proposed in [5,9,14,28], as well as the definition and topological structure of the time-dependent phase space X_t [8, 22], we give the definition of backwards compact time-dependent attractors and relevant conclusions in X_t .

Definition 2.1. The non-autonomous set $\mathcal{D} = \{\mathcal{D}_t\}_{t\in\mathbb{R}} \subset X_t$ is increasing if for any $s \leq t$, there have $\mathcal{D}_s \subset \mathcal{D}_t$.

Definition 2.2. Non-autonomous set $\mathcal{D} = \{\mathcal{D}_t\}_{t\in\mathbb{R}} \subset X_t$ is called backwards compact if for each $t \in \mathbb{R}$, D_t and $\overline{\bigcup_{s\leq t} D_s}$ is compact in X_t; if for each $t \in \mathbb{R}$, $\bigcup_{s\leq t} D_s$ is bounded in X_t, D_t is said to be backwards bounded.

Next, we give the definition of the backwards compact time-dependent attractors in X_t:

Definition 2.3. A family of compact subsets $\mathcal{A} = \{\mathcal{A}_t\}_{t\in\mathbb{R}} \subset X_t$ is called time-dependent attractor if it fulfills the following properties:

(i) (invariance) $U(t, s)\mathcal{A}(s) = \mathcal{A}(t)$, for every $s \leq t$;

(ii) (pullback attraction) for every pullback-bounded family $\mathcal{B}$ and every $t \in \mathbb{R}$,

$$\lim_{s\to-\infty} \mathrm{dist}_{\mathrm{X_t}}(U(t, s)\mathcal{A}(s), \mathcal{A}(t)) = 0.$$

If property (ii) holds uniformly with regard to $t \in \mathbb{R}$, then $\mathcal{A}$ is a uniform time-dependent attractor.

Definition 2.4. Non-autonomous set $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}} \subset X_t$ is called a backwards compact time-dependent attractor for the process $U(t, \tau)$ if

(i) For each $t \in \mathbb{R}$, A_t is backwards compact;
(ii) For each $t \in \mathbb{R}, A_t$ pullback attracts every bounded subset $B_t \subset X_t$;
(iii) A_t is the smallest set satisfying properties (i), (ii).

The results of smallest property for $\mathcal{A}$, we can see [8], and the authors pointed out when the process satisfies some continuity, then it's invariant.

Theorem 2.1. *A backwards compact time-dependent attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}} \subset X_t$ must be a time-dependent attractor and backwards compact, vice versa.*

Proof. From the definition of the attractor and the inclusion relation between them, the result can be obtained. □

According to the definition of the time-dependent absorbing set in X_t, there is:

Definition 2.5. A time-dependent absorbing set for the process $\{U(t, \tau)\}_{t\geq\tau}$ is a pullback bounded family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following property: for every $R > 0$ there exists a $t_0 = t_0(t) \leq t$ such that

$$\tau \leq t - t_0 \Rightarrow U(t, \tau)\mathbb{B}_\tau(R) \subset B_t.$$

The non-autonomous set $\mathcal{B}$ is a backwards time-dependent absorbing set for the process U means: for any $t \in \mathbb{R},\ s \leq t$ and any $R \geq 0$, there exists $s_0 = s_0(R) \geq 0$ such that when $\tau \leq s - s_0$, there is

$$U(s, \tau)\mathbb{B}_\tau(R) \subset B_t,$$

where $\mathbb{B}_t(R)$ denotes the R-ball of X_t.

Definition 2.6. The process U is called pullback asymptotically compact in X_t: for any $t \in \mathbb{R}$, with the bounded sequence $\{x_n\} \subset X_{\tau_n}$ and $\{\tau_n\}_{n=1}^{\infty} \subset \mathbb{R}^{-t}$ satisfies that when $n \to \infty$, $\tau_n \to -\infty$, there is sequence

$$\{U(t, \tau_n)x_n\}_{n\in\mathbb{N}} \text{ in } X_t \text{ has convergent subsequence,}$$

where $\mathbb{R}^{-t} = \{\tau : \tau \in \mathbb{R}, \tau \leq t\}$.

Further, the process U is called backwards pullback asymptotically compact: for any $t \in \mathbb{R},\ s_n \leq t$, with the bounded sequence $\{x_n\} \subset X_{\tau_n}$ and $\{\tau_n\}_{n=1}^{\infty} \subset \mathbb{R}^{-t}$ satisfies that when $n \to \infty$, $\tau_n \to -\infty$, the sequence

$$\{U(s_n, \tau_n)x_n\}_{n\in\mathbb{N}} \text{ in } X_t \text{ has convergent subsequence.}$$

Theorem 2.2. *Let U be a backwards pullback asymptotically compact process on Banach space X_t, then the following propositions are equivalent:*

(i) U has a backwards compact time-dependent attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$;

(ii) U has an increasing, bounded and time-dependent absorbing set $\mathcal{K}^1 = \{K_t^1\}_{t\in\mathbb{R}}$;

(iii) U has a bounded and backwards time-dependent absorbing set $\mathcal{K}^2 = \{K_t^2\}_{t\in\mathbb{R}}$;

and the attractor coincides with the time-dependent ω-limit of any pullback absorbing set, i.e.

$$A_t := \omega(K_t^1, t) = \omega(K_t^2, t), \ \forall \ t \in \mathbb{R},$$

where $\omega(K_t, t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) K_\tau}$.

Proof. $(i) \Rightarrow (ii)$. For any $t \in \mathbb{R}$, define $K_t^1 = \bigcup_{s \le t} \mathcal{O}_s^1(A_s) = \mathcal{O}_t^1(\bigcup_{s \le t} A_s)$, where $\mathcal{O}_s^1$ is defined as the 1-neighborhood in X_s. Obviously, K_t^1 is increasing. Since A_t is backwards compact, we know that K_t^1 is bounded. Combined with the pullback attracting of A_t, which implies $\mathcal{O}_s^1(A_s)$ is the time-dependent pullback absorbing set, therefore K_t^1 is the time-dependent absorbing set.

$(ii) \Rightarrow (iii)$. If $\mathcal{K}^1 = \{K_t^1\}_{t \in \mathbb{R}}$ is an increasing time-dependent absorbing set, then for any $t \in \mathbb{R}, s \le t$ and for the bounded subset $B_\tau \subset X_\tau$, there exists $s_0 := s_0(s,t) < s$ such that for any $\tau < s - s_0$, there is

$$U(s,\tau) B_\tau \subset K_s^1 \subset K_t^1.$$

Obviously, K_t^1 is a backwards time-dependent absorbing set.

$(iii) \Rightarrow (i)$. Since $\mathcal{K}^2 = \{K_t^2\}_{t \in \mathbb{R}}$ is a backwards time-dependent absorbing set, we know that for any $s \in \mathbb{R}, \ s \le t$ and the bounded subset $B_\tau \subset X_\tau$, there is

$$\lim_{\tau \to -\infty} \operatorname{dist}_{X_t}(U(s,\tau) B_\tau, K_t^2) = 0,$$

namely, K_t^2 pullback attracts every bounded set at any past time, so the process $U(t,\tau)$ is strongly pullback bounded dissipation (see [6]). And then combined the pullback asymptotic compactness of U in X_t [6, 8], it is known that there exist a family of time-dependent attractors $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$, and

$$A_t = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) B_\tau}.$$

Then we prove A_t is backwards bounded, that is, for any $t \in \mathbb{R}$, $\bigcup_{s \le t} \mathcal{A}_s$ is bounded in X_t. From the backward pullback absorbing of K_t^2, for any $s \le t, \ \tau \in \mathbb{R}$, there is

$$U(s,\tau) K_\tau^2 \subset K_t^2.$$

Using the definition of $\mathcal{A}_t$,

$$\mathcal{A}_s \subset \overline{K_t^2}.$$

So we know that $\bigcup_{s \le t} \mathcal{A}_s$ is bounded.

To prove the backward compactness of $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ in X_t, we need prove that for any sequence $\{x_n\} \in \bigcup_{s \le t} A_s$, which exists convergent subsequence in X_t.

From the backwards boundedness of $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$, we know $\bigcup_{s \le t} A_s$ is bounded, let $x_n \in A_{s_n}, \ s_n \le t$, that is

$$y_n \in A_{s_n} = U(s_n, \tau_n) A_{\tau_n} \subset U(s_n, \tau_n)(\bigcup_{\tau_n \le t} A_{\tau_n}),$$

where we use the minimalism of the attractor $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$, it is invariant. Combined with the backwards compactness of the process $\{U(t,\tau)\}_{t \ge \tau}$, we know that the sequence $\{y_n\}$ exists convergent subsequence.

By Theorem 2.1, we infer $\mathcal{A} = \{A_t\}_{t \in \mathbb{R}}$ is a backwards compact time-dependent attractor. $\square$

2.2. Notations

For problem (1.1), we first introduce the following abbreviations:

$$L^p = L^p(\mathbb{R}^n), \quad \mathrm{H} = L^2(\mathbb{R}^n), \quad \mathrm{H}^s = W^{s,2}(\mathbb{R}^n), \quad \|\cdot\| = \|\cdot\|_{L^2},$$

the notation $\langle\cdot,\cdot\rangle$ represents the L^2-inner product and can also be used for the notation of dual pairs of conjugate spaces. And we use the symbol $\|\cdot\|_s^2$ to define the norms in Hilbert space H^s.

For $t \in \mathbb{R}$, consider the time-dependent phase space

$$\mathcal{H}_t = \mathrm{H}^2 \times \mathrm{H}_t, \quad \mathcal{H}_t^1 = \mathrm{H}^3 \times \mathrm{H}_t^1$$

endowed with the following norms

$$\|z\|_{\mathcal{H}_t}^2 = \|(u, u_t)\|_{\mathcal{H}_t}^2 = \|u\|_2^2 + \varepsilon(t)\|u_t\|^2, \tag{2.1}$$

and

$$\|z\|_{\mathcal{H}_t^1}^2 = \|(u, u_t)\|_{\mathcal{H}_t^1}^2 = \|u\|_3^2 + \varepsilon(t)\|u_t\|_1^2.$$

For convenience, for $z \in \mathcal{H}_t$, we define the following norm

$$\|z\|_{\mathcal{H}_t}^2 = \|(u, u_t)\|_{\mathcal{H}_t}^2 = \|\Delta u\|^2 + \lambda\|u\|^2 + \varepsilon(t)\|u_t\|^2, \tag{2.2}$$

obviously, it is equivalent to the norm in (2.1).

Furthermore, in order to prove the result, for non-autonomous external force term $g \in L^2_{loc}(\mathbb{R}, \mathrm{H}^1)$, we assume the following backwards tempered conditions [9, 28]:

(1) g is backwards tempered, i.e

$$\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}[\|g(\cdot,r)\|^2 + \|g(\cdot,r)\|_1^2]dr < \infty, \forall\ t \in \mathbb{R},\ \gamma > 0; \tag{2.3}$$

(2) g is backwards tempered tail-small, i.e

$$\lim_{k\to\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}\int_{|x|\ge k}|g(x,s)|^2 dxdr = 0,\ \forall\ t \in \mathbb{R},\ \gamma > 0; \tag{2.4}$$

(3) g is backwards tempered complement-small, i.e

$$\lim_{i\to\infty}\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}\|(I - P_i)g(x,s)\|dr = 0,\ \forall\ t \in \mathbb{R},\ \gamma > 0. \tag{2.5}$$

We give the following lemma, which is indispensable for our proof.

Lemma 2.1. ([7]) *Let $\Omega \subset \mathbb{R}^n$ having the extension property, or $\Omega = \mathbb{R}^n$, $1 < p,\ p_0,\ p_1 < +\infty,\ 0 \le s_0 < s_1 < +\infty, 0 < \theta < 1$,*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then for any $u \in W^{s_0,p_0}(\Omega) \cap W^{s_1,p_1}(\Omega)$ satisfies

$$\|u\|_{W^{s^-,p}(\Omega)} \le C(\theta)\|u\|_{W^{s_1,p_1}(\Omega)}^{\theta}\|u\|_{W^{s_0,p_0}(\Omega)}^{1-\theta},$$

where

$$s^- \begin{cases} = s,\ s_0,\ s,\ s_1 \in \mathbb{N}\ or\ s_0,\ s,\ s_1 \in \mathbb{R}\setminus\mathbb{N}, \\ < s,\ other\ \ case. \end{cases}$$

3. Existence of time-dependent pullback attractor

By the Faedo-Galerkin procedure, under the conditions of (1.2)-(1.8) and (2.3)-(2.5), we can obtain the existence of the solution for equation (1.1), as well as there exists a unique weak solution $z(t) = (u, u_t)$ under the partial sense:

$$u \in C([\tau, T]; \mathrm{H}^2), \quad u_t \in C([\tau, T]; \mathrm{H}),$$

furthermore, the mapping $(u_0, u_1) \to (u(t, \tau, u_0), u_t(t, \tau, u_0))$ is continuous in $\mathcal{H}_t$, that is: given $R > 0$, for every pair of initial data $z_i(\tau) = \{u_0^i, u_1^i\} \in \mathcal{H}_\tau, i = 1, 2$, and the solution $\{z_i(t), g_i\} = \{u^i, u_t^i\}$, if $\|z_i(\tau)\|_{\mathcal{H}_\tau} \le R,\ i = 1, 2$, then the difference of the corresponding solutions satisfies

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_t} \le e^{C(t-\tau)} \left[\|z_1(\tau) - z_2(\tau)\|_{\mathcal{H}_\tau} + \|g_1 - g_2\|^2_{L^2_b(\mathbb{R}, L^2)}\right], \ \forall\ t \ge \tau, \tag{3.1}$$

where $C = C(R) \ge 0$.

The well-posedness of weak solutions for the problem can be shown by Galerkin approximation and compactness method as in [14, 19, 29] on unbounded domain, so the proof we omitted here.

Global existence of (weak) solutions u to (1.1) is classical, and based on the continuous dependence of the solution on the initial data, problem (1.1) generates a strongly continuous process, i.e.

$$U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t,\ t \ge \tau \in \mathbb{R} \ \text{ acting as } U(t, \tau)z(\tau) = \{u(t), u_t(t)\},$$

where u is the unique solution of problem (1.1) with initial time τ and initial condition $z = z(\tau) = \{u_0, u_1\} \in \mathcal{H}_\tau$.

3.1. Dissipation estimates

In order to prove the existence of backwards compact time-dependent attractors, firstly, we make some backwards asymptotic estimates of the solution, such that the process $\{U(t, \tau)\}_{t\ge\tau}$ exists an increasing, bounded and time-dependent pullback absorbing set:

Lemma 3.1. *Assume that (1.2)-(1.8) and (2.3) are hold, for any $t \in \mathbb{R},\ s \le t$ and $R_0 > 0$, there exists $s_0 := s_0(s, t) > 0$ such that for any $s - \tau \ge s_0$, $z(\tau) \in \mathbb{B}_\tau(R_0) \subset \mathcal{H}_\tau,\ \xi \in [\tau - s, 0]$, there have*

$$\begin{aligned}
&\sup_{s\le t}\left[\|u(s+\xi, \tau, u_0)\|^2_{\mathrm{H}^2} + \varepsilon(s+\xi)\|v(s+\xi, \tau, v_0)\|^2\right] \le Ce^{-\frac{\delta_0}{2}\xi}(1 + G(t)),\\
&\sup_{s\le t}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}(\|u(r, \tau, u_0)\|^2_{\mathrm{H}^2} + \|v(r, \tau, u_0)\|^2)dr \le C(1 + G(t)),
\end{aligned} \tag{3.2}$$

where $G(t)$ is given by

$$G(t) := \sup_{s\le t}\int_{-\infty}^s e^{\frac{\delta_0}{2}(r-s)}\|g(\cdot, r)\|^2 dr,\ \forall\ t \in \mathbb{R}.$$

Proof. Fixed $t \in \mathbb{R}$, for any $s \le t$. Multiplying (1.1) by $v(s) := v(s, \tau, v_0) = u_t(s) + \delta u(s)$ and integrating on $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{d}{ds}[\|\Delta u(s)\|^2 + \lambda\|u(s)\|^2 + \varepsilon(s)\|v(s)\|^2 + 2\langle F(x, u(s)), 1\rangle] \\ &+ 2\delta\|\Delta u(s)\|^2 + 2\delta\lambda\|u(s)\|^2 \\ &- \varepsilon'(s)\|v(s)\|^2 + 2\int_{\mathbb{R}^n}(\alpha(x) - \delta\varepsilon(s))|v(s)|^2dx - 2\delta\int_{\mathbb{R}^n}\alpha(x)u(s)\cdot v(s)dx \\ &+ 2\delta^2\varepsilon(s)\langle u(s), v(s)\rangle + 2\langle(p - \|\nabla u\|^2)\Delta u, v\rangle + 2\delta\langle f(x, u(s)), u\rangle \\ =&2\langle g(x,s), v\rangle. \end{aligned}$$

By Hölder, Young inequalities, we deduce

$$\begin{aligned} &-\langle(\|\nabla u\|^2 - p)\Delta u, v\rangle \ge \frac{1}{4}\frac{d}{ds}(\|\nabla u\|^2 - p)^2 + \frac{\delta}{2}(\|\nabla u\|^2 - p)^2 - \frac{\delta}{2}p^2, \\ &2\langle g(x,s), v\rangle \le \frac{\alpha_0}{2}\|v(s)\|^2 + \frac{2}{\alpha_0}\|g(x,s)\|^2. \end{aligned}$$

From (1.4), there exists $L_1 > 0$, such that $\|\alpha(x)\|_{L^\infty} < L_1$. Choosing $0 < \delta \le \min\{\frac{\alpha_0}{8L}, \frac{\lambda\alpha_0}{2L_1^2 + L\alpha_0}\}$, such that

$$\frac{1}{2}\alpha_0 - 3\delta\varepsilon(s) \ge \delta\varepsilon(s); \quad 2\delta\lambda - \frac{2\delta^2L_1^2}{\alpha_0} - \delta^3L \ge \delta\lambda. \tag{3.3}$$

Then combined with Hölder's and Young's inequalities, we get

$$\begin{aligned} &2\delta\lambda\|u(s)\|^2 - \varepsilon'(s)\|v(s)\|^2 + 2\int_{\mathbb{R}^n}(\alpha(x) - \delta\varepsilon(s))|v(s)|^2dx \\ &- 2\delta\int_{\mathbb{R}^n}\alpha(x)u(s)\cdot v(s)dx + 2\delta^2\varepsilon(s)\langle u(s), v(s)\rangle \\ \ge&(2\alpha_0 - 3\delta\varepsilon(s))\|v\|^2 + 2\delta\lambda\|u(s)\|^2 - 2\delta L_1\|u(s)\|\|v(s)\| - \delta^3L\|u\|^2 \\ \ge&\delta\varepsilon(s)\|v(s)\|^2 + (2\delta\lambda - \frac{2\delta^2L_1^2}{\alpha_0} - \delta^3L)\|u(s)\|^2 + \frac{\alpha_0}{2}\|v(s)\|^2. \end{aligned} \tag{3.4}$$

Therefore, from (1.6), Hölder's and Young's inequalities, taking $\delta_0 = \min\{\delta, \delta c_1\}$, there is

$$\frac{d}{ds}E(s) + \delta_0E(s) + \frac{\alpha_0}{2}\|v(s)\|^2 \le M + M\|g(s)\|^2, \tag{3.5}$$

where

$$E(s) = \|\Delta u(s)\|^2 + \lambda\|u(s)\|^2 + \varepsilon(s)\|v(s)\|^2 + \frac{1}{2}(\|\nabla u\|^2 - p)^2 + 2\langle F(x, u(s)), 1\rangle,$$

and positive constant $M = \max\{\delta p^2 + 2\delta\|\phi_2(x)\|, \frac{2}{\alpha_0}\}$.

Applying the Gronwall lemma to (3.5) on $[\tau, s+\xi]$, $\xi \in [\tau - s, 0]$, it yields

$$\begin{aligned} &E(s+\xi) + \frac{\delta_0}{2}e^{-\frac{\delta_0}{2}\xi}\int_\tau^{s+\xi}e^{\frac{\delta_0}{2}(r-s)}E(r)dr + \frac{\alpha_0}{2}e^{-\frac{\delta_0}{2}\xi}\int_\tau^{s+\xi}e^{\frac{\delta_0}{2}(r-s)}\|v(r)\|^2dr \\ \le&e^{-\frac{\delta_0}{2}\xi}e^{-\frac{\delta_0}{2}(s-\tau)}E(\tau) + Me^{-\frac{\delta_0}{2}\xi}\int_\tau^{s+\xi}e^{\frac{\delta_0}{2}(r-s)}(1 + \|g(\cdot, r)\|^2)dr. \end{aligned} \tag{3.6}$$

Exploiting (1.5) as well as continuous embedding $\mathrm{H}^2 \subset L^{q+1}$, we know that for any $x \in \mathbb{R}^n,\ u \in \mathbb{R}$ there is

$$\int_{\mathbb{R}^n} F(x,u)dx \leq c_3(1 + \|u_0\|_{\mathrm{H}^2}^{q+1} + \|u_0\|^2). \tag{3.7}$$

Thus, there exists $s_0 := s_0(R_0, s, t) > 0$, such that when $s - \tau \geq s_0$ it yields

$$e^{-\frac{\delta_0}{2}\xi} e^{-\frac{\delta_0}{2}(s-\tau)} E(\tau) \leq c_4 e^{-\frac{\delta_0}{2}\xi}(1 + R_0^{q+1} + R_0^4 + p^2) \leq C(p, L, R_0)e^{-\frac{\delta_0}{2}\xi}. \tag{3.8}$$

From the above estimates

$$\begin{aligned} &E(s+\xi) + \frac{\delta_0}{2} e^{-\frac{\delta_0}{2}\xi} \int_\tau^{s+\xi} e^{\frac{\delta_0}{2}(r-s)} E(r)dr + \frac{\alpha_0}{2} e^{-\frac{\delta_0}{2}\xi} \int_\tau^{s+\xi} e^{\frac{\delta_0}{2}(r-s)} \|v(r)\|^2 dr \\ \leq &Ce^{-\frac{\delta_0}{2}\xi}(1 + \int_{-\infty}^{s} e^{\frac{\delta_0}{2}(r-s)} \|g(\cdot, r)\|^2 dr). \end{aligned} \tag{3.9}$$

Recall (1.7), and $\xi \leq 0$, we have

$$\begin{aligned} &-2\int_{\mathbb{R}^n} F(x, u(s+\xi))dx - \delta_0 \int_\tau^{s+\xi} e^{\frac{\delta_0}{2}(r-s-\xi)} \int_{\mathbb{R}^n} F(x, u(r))dxdr \\ \leq &2\|\phi_3(x)\| \\ \leq &2e^{-\frac{\delta_0}{2}\xi}\|\phi_3(x)\|. \end{aligned} \tag{3.10}$$

Using (3.9)-(3.10), by considering the supremum over the past time $s \leq t$, we get that

$$\begin{aligned} &\sup_{s \leq t} \|U(s+\xi, \tau)z_\tau\|_{\mathcal{H}_{s+\xi}}^2 + \frac{\delta_0}{2} e^{-\frac{\delta_0}{2}\xi} \sup_{s \leq t} \int_\tau^{s+\xi} e^{\frac{\delta_0}{2}(r-s)} \|u(r, \tau, u_0)\|_{\mathrm{H}^2}^2 dr \\ &+ \frac{\alpha_0}{2} e^{-\frac{\delta_0}{2}\xi} \sup_{s \leq t} \int_\tau^{s+\xi} e^{\frac{\delta_0}{2}(r-s)} \|v(r)\|^2 dr \\ \leq &Ce^{-\frac{\delta_0}{2}\xi} \sup_{s \leq t}(1 + \int_{-\infty}^{s} e^{\frac{\delta_0}{2}(r-s)} \|g(\cdot, r)\|^2 dr) \\ \leq &Ce^{-\frac{\delta_0}{2}\xi}(1 + G(t)), \end{aligned}$$

then the results is proved. □

Lemma 3.2. *Under the assumptions (1.2)-(1.8) and (2.3), for any initial data $z(\tau) \in \mathbb{B}_\tau(R_1) \subset \mathcal{H}_\tau^1$, there exists $\tau_1 := \tau_1(R_1)$, such that when $s \leq t$ there is*

$$\begin{aligned} &\sup_{s \leq t} \sup_{z(\tau) \in \mathbb{B}_\tau(R_1)} \{\varepsilon(s)\|A^{\frac{1}{4}} v(s)\|^2 + \|A^{\frac{3}{4}} u(s)\|^2 + \lambda \|A^{\frac{1}{4}} u(s)\|^2\} \\ &+ \int_\tau^s e^{-\frac{\delta_0}{2}(s-r)} \|A^{\frac{1}{4}} v(r)\|^2 dr \\ \leq &C(1 + G(t) + G_1(t) + \frac{2G^3(t)}{\delta_0}),\ \forall\ \tau \leq t, \end{aligned} \tag{3.11}$$

where

$$G_1(t) := \sup_{s \leq t} \int_\tau^s e^{\frac{\delta_0}{2}(r-s)} \|g(\cdot, r)\|_{\mathrm{H}^1}^2 dr,\ \forall\ t \in \mathbb{R}.$$

Proof. For any $t \in \mathbb{R},\ s \le t$, multiplying (1.1) by $A^{\frac{1}{2}}v(s) = A^{\frac{1}{2}}u_s(s) + \delta A^{\frac{1}{2}}u(s)$ and integrating on $\mathbb{R}^n$, we obtain

$$\begin{aligned}
&\frac{d}{ds}[\varepsilon(s)\|A^{\frac{1}{4}}v(s)\|^2 + \|A^{\frac{3}{4}}u(s)\|^2 + \lambda\|A^{\frac{1}{4}}u(s)\|^2] + 2\langle\alpha(x)(v-\delta u(s)), A^{\frac{1}{2}}v(s)\rangle\\
&- \varepsilon'(s)\|A^{\frac{1}{4}}v(s)\|^2 - 2\delta\varepsilon(s)\|A^{\frac{1}{4}}v(s)\|^2 + 2\delta^2\varepsilon(s)\langle u, A^{\frac{1}{2}}v\rangle\\
&+ 2\delta\|A^{\frac{3}{4}}u\|^2 + 2\delta\lambda\|A^{\frac{1}{4}}u(s)\|^2\\
&+ 2\langle(p-\|\nabla u\|^2)\Delta u, A^{\frac{1}{2}}v\rangle + 2\langle f(x,u(s)), A^{\frac{1}{2}}v(s)\rangle\\
=&2\langle g(x,s), A^{\frac{1}{2}}v(s)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
2\langle(p-\|\nabla u\|^2)\Delta u, A^{\frac{1}{2}}v\rangle =&\frac{d}{dt}(\|\nabla u\|^2\|A^{\frac{1}{2}}u\|^2 - p\|A^{\frac{1}{2}}u\|^2)\\
&+ 2\delta(\|\nabla u\|^2\|A^{\frac{1}{2}}u\|_2^2 - p\|A^{\frac{1}{2}}u\|^2) - \|A^{\frac{1}{2}}u\|^2\langle\nabla u, \nabla u_t\rangle.
\end{aligned}$$

Combing with Hölder, Young inequalities

$$\|A^{\frac{1}{2}}u\|^2|\langle\nabla u, \nabla u_t\rangle| \le C(\|A^{\frac{1}{2}}u\|^6 + \delta^2\|u\|^2 + \|v\|^2), \tag{3.12}$$

$$2|\langle g(\cdot,s), A^{\frac{1}{2}}v\rangle| \le \frac{\alpha_0}{4}\|A^{\frac{1}{4}}v\|^2 + \frac{4}{\alpha_0}\|g(\cdot,s)\|^2_{\mathrm{H}^1}, \tag{3.13}$$

$$\begin{aligned}
2|\langle f(x,u), A^{\frac{1}{2}}v\rangle| &= 2\int_{\mathbb{R}^n}|\frac{\partial f}{\partial u}(x,u)\cdot A^{\frac{1}{4}}u\cdot A^{\frac{1}{4}}v + \frac{\partial f}{\partial x}(x,u)\cdot A^{\frac{1}{4}}v|dx\\
&\le 2\beta\|A^{\frac{1}{4}}u\|\cdot\|A^{\frac{1}{4}}v\| + 2\|\phi_4(x)\|\|A^{\frac{1}{4}}v\|\\
&\le \frac{\alpha_0}{2}\|A^{\frac{1}{4}}v\|^2 + \frac{4\beta^2}{\alpha_0}\|A^{\frac{1}{4}}u\|^2 + \frac{4}{\alpha_0}\|\phi_4(x)\|^2.
\end{aligned}\tag{3.14}$$

From the above estimates

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{ds}}E_1(s) + \delta_0 E_1(s) + \frac{\alpha_0}{4}\|A^{\frac{1}{4}}v(s)\|^2\\
\le&C(\|g(\cdot,s)\|^2_{\mathrm{H}^1} + \|A^{\frac{1}{4}}u(s)\|^2 + \|A^{\frac{1}{2}}u\|^6 + \|v\|^2),
\end{aligned}\tag{3.15}$$

where

$$E_1(s) = \varepsilon(s)\|A^{\frac{1}{4}}v(s)\|^2 + \|A^{\frac{3}{4}}u(s)\|^2 + \lambda\|A^{\frac{1}{4}}u(s)\|^2 + \|\nabla u\|^2\|A^{\frac{1}{2}}u\|^2 - p\|A^{\frac{1}{2}}u\|^2.$$

By virtue of Lemma 2.1,

$$\|\nabla u\| \le \eta\|u\| + C_\eta\|\Delta u\|, \quad \forall\eta > 0,$$

and applying the Gronwall lemma to (3.15) over $[\tau, s]$, from (3.2) we get

$$\begin{aligned}
&E_1(s) + \frac{\delta_0}{2}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}E_1(r)dr + \frac{\alpha_0}{4}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}\|A^{\frac{1}{4}}v(r)\|^2dr\\
\le&e^{-\frac{\delta_0}{2}(s-\tau)}E_1(\tau) + C\int_\tau^s e^{-\frac{\delta_0}{2}(s-r)}(1 + \|g(\cdot,r)\|^2_{\mathrm{H}^1}\\
&+ \|A^{\frac{1}{4}}u(s)\|^2 + \|A^{\frac{1}{2}}u\|^6 + \|v\|^2)dr.
\end{aligned}\tag{3.16}$$

Then, by considering the supremum over the past time $s \le t$, we infer

$$\begin{aligned}&\sup_{s\le t}\Big[E_1(s)+\frac{\delta_0}{2}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}E_1(r)dr+\frac{\alpha_0}{4}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}\|A^{\frac14}v(r)\|^2dr\Big]\\ \le & C(1+G(t)+G_1(t))+G^3(t)\sup_{s\le t}\int_\tau^s e^{\frac{\delta_0}{2}(r-s)}dr)\\ \le & C(1+G(t)+G_1(t)+\frac{2G^3(t)}{\delta_0}),\end{aligned}$$

the result is proved. □

3.2. Backwards asymptotic estimates outside a ball

Lemma 3.3. *Assume that (1.2)-(1.8) and (2.3)-(2.5) are hold, for any $\eta > 0,\ t \in \mathbb{R}$ there exist $T_1 = T_1(\eta, R)$ and $K = K(\eta, t) > 1,$ such that when $\tau \ge T_1,\ k \ge K$ and $z(\tau) \in \mathbb{B}_\tau(R_1)$,*

$$\sup_{s\le t}\int_{\Omega_k^c}\big(\varepsilon(s)|v(s,\tau,v_0)|^2+|\Delta u(s,\tau,u_0)|^2+\lambda|u(s,\tau,u_0)|^2\big)dx\le C\eta,$$

where $\Omega_k^c = \{x \in \mathbb{R}^n : |x| \ge k\},\ C$ is a positive constant.

Proof. Choosing smooth function θ satisfies: for any $s \in \mathbb{R},\ 0 \le \theta \le 1$, and

$$\theta(s)=\begin{cases}0, & \text{when }\ 0\le|s|\le1,\\ 1, & \text{when }\ |s|\ge2,\end{cases}\tag{3.17}$$

so, there exists a positive contant $\tilde{C_0}$, such that for any $s \in \mathbb{R}^+,\ \max\{|\theta'(s)|, |\theta''(s)|\} \le \tilde{C_0}$.

Let $s \le t$ with $t \in \mathbb{R}$ fixed. Taking the inner product of (1.1) with $\theta_k v(s) := \theta_k v(s, \tau, v_0) = \theta(\frac{|x|^2}{k^2})v(s, \tau, v_0)$ in $L^2(\mathbb{R}^n)$, we infer

$$\begin{aligned}&\frac{d}{ds}\Big[\int_{\mathbb{R}^n}\theta_k\cdot\big(\varepsilon(s)|v(s)|^2+\lambda|u(s)|^2\big)\,dx\Big]+2\int_{\mathbb{R}^n}\theta_k\alpha(x)(v(s)-\delta u(s))v(s)dr\\ &+(-2\delta\varepsilon(t)-\varepsilon'(s))\int_{\mathbb{R}^n}\theta_k\cdot|v|^2dx+2\delta^2\varepsilon(t)\int_{\mathbb{R}^n}\theta_k\cdot u\cdot vdx\\ &+2\int_{\mathbb{R}^n}\theta_k\Delta^2u(s)\cdot v(s)dx+2\delta\lambda\int_{\mathbb{R}^n}\theta_k|u(s)|^2dx\\ =&2\int_{\mathbb{R}^n}\theta_k(p-\|\nabla u\|^2)\cdot v\cdot\Delta udx-2\int_{\mathbb{R}^n}\theta_kf(x,u(s))v(s)dx\\ &+2\int_{\mathbb{R}^n}\theta_kg(x,s)v(s)dx.\end{aligned}\tag{3.18}$$

Next, we deal with each term in the equation term by term:

Firstly,

$$\begin{aligned}2\int_{\mathbb{R}^n}\theta_k\Delta^2u\cdot vdx\ge&\frac{d}{dt}\int_{\mathbb{R}^n}\theta_k|\Delta u|^2dx+2\delta\int_{\mathbb{R}^n}\theta_k|\Delta u|^2dx\\ &-(\frac{4\sqrt2\tilde{C_0}}{k}+\frac{8\tilde{C_0}}{k^2})\|\Delta u\|^2-\frac{4\sqrt2\tilde{C_0}}{k}\|\nabla v\|^2-\frac{8\tilde{C_0}}{k^2}\|v\|^2.\end{aligned}\tag{3.19}$$

Similar to (3.4), we get

$$\begin{aligned}
&2\int_{\mathbb{R}^n}\theta_k\alpha(x)(v-\delta u)vdx-(2\delta\varepsilon(t))\varepsilon'(t))\int_{\mathbb{R}^n}\theta_k\cdot|v|^2dx\\
&+2\delta^2\varepsilon(t)\int_{\mathbb{R}^n}\theta_k\cdot u\cdot vdx\\
\geq&(\alpha_0-3\delta\varepsilon(t))\|v\|^2-(\frac{\delta^2L_1}{\alpha_0}+\delta^3L)\int_{\mathbb{R}^n}\theta_k\cdot|u|^2dx, \qquad (3.20)\\
2|&\int_{\mathbb{R}^n}\theta_kg(x,t)vdx|\leq\frac{\alpha_0}{2}\int_{\mathbb{R}^n}\theta_k|v|^2dx+\frac{2}{\alpha_0}\int_{\mathbb{R}^n}\theta_k|g(x,t)|^2dx, \qquad (3.21)\\
&|2\int_{\mathbb{R}^n}\theta_k(p-\|\nabla u\|^2)\cdot v\cdot\Delta udx|\\
\leq&\frac{\alpha_0}{2}\int_{\mathbb{R}^n}\theta_k|v|^2dx+\frac{4}{\alpha_0}(p^2+\|\nabla u\|^4)\int_{\mathbb{R}^n}\theta_k|\Delta u|^2dx. \qquad (3.22)
\end{aligned}$$

From the above estimates

$$\begin{aligned}
&\frac{d}{ds}E_2(s)+\delta_0E_2(s)\\
\leq&(\frac{4\sqrt{2}\tilde{C}_0}{k}+\frac{8\tilde{C}_0}{k^2})\|\Delta u(s)\|^2+\frac{4\sqrt{2}\tilde{C}_0}{k}\|\nabla v(s)\|^2+\frac{8\tilde{C}_0}{k^2}\|v(s)\|^2\\
&+\frac{4}{\alpha_0}(p^2+\|\nabla u\|^4)\int_{\mathbb{R}^n}\theta_k|\Delta u|^2dx+(\frac{\delta^2L_1}{\alpha_0}+\delta^3L)\int_{\mathbb{R}^n}\theta_k\cdot|u|^2dx\\
&+\frac{2}{\alpha_0}\int_{\mathbb{R}^n}\theta_k|g(x,s)|^2dx+c\int_{\mathbb{R}^n}\theta_k[|\phi_2(x)|^2+|\phi_3(x)|^2]dx,
\end{aligned}$$

where

$$E_2(s)=\int_{\mathbb{R}^n}\theta_k(|\Delta u(s)|^2+\lambda|u(s)|^2+\varepsilon(s)|v(s)|^2+2F(x,u(s)))dx.$$

Take $k_1(\eta)>0$, and for any $0<\eta<1$, such that when $k\geq k_1(\eta)$, there is

$$\begin{aligned}
&(\frac{4\sqrt{2}\tilde{C}_0}{k}+\frac{8\tilde{C}_0}{k^2})\|\Delta u\|^2+\frac{4\sqrt{2}\tilde{C}_0}{k}\|\nabla v\|^2+\frac{8\tilde{C}_0}{k^2}\|v\|^2\\
&+\frac{4}{\alpha_0}(p^2+\|\nabla u\|^4)\int_{\mathbb{R}^n}\theta_k|\Delta u|^2dx+(\frac{\delta^2L_1}{\alpha_0}+\delta^3L)\int_{\mathbb{R}^n}\theta_k\cdot|u|^2dx\\
<&\eta((p^2+1)\|\Delta u\|^2+\|\nabla v\|^2+\|v\|^2+\|\nabla u\|^4\|\Delta u\|^2),
\end{aligned}$$

and there exists $k_2(\eta)>0$, such that when $k\geq k_2(\eta)$,

$$c\int_{\mathbb{R}^n}\theta_k[|\phi_2(x)|^2+|\phi_3(x)|^2]dx<\eta.$$

Thus, we get the following inequality

$$\begin{aligned}
&\frac{d}{ds}E_2(s)+\delta_0E_2(s)\\
\leq&c\eta(1+\|\Delta u\|^2+\|\nabla v(s)\|^2+\|v(s)\|^2+\|\nabla u\|^4\|\Delta u\|^2)+\int_{\mathbb{R}^n}\theta_k|g(x,s)|^2dx.
\end{aligned}$$

Applying the Gronwall lemma on $[\tau, s]$, from Lemma 3.1-3.2, taking $K_1 = \max\{k_1, k_2\}$, for any $s \le t$, we get

$$\begin{aligned} E_2(s) \le & e^{-\delta_0(s-\tau)} E_2(\tau) + c\eta \int_\tau^s e^{-\delta_0(s-r)} [\|\Delta u(r)\|^2 + \|\nabla v(r)\|^2 + \|v(r)\|^2 \\ & + \|\nabla u\|^4 \|\Delta u\|^2] ds + \int_\tau^s e^{-\delta_0(s-r)} \int_{\mathbb{R}^n} \theta_k |g(x,r)|^2 dx dr \\ \le & C(R_0) e^{-\delta_0 \tau} + C\eta(1 + G(t) + G_1(t) + (1 + \frac{1}{\delta_0}) G^3(t)) \\ & + C \int_\tau^s e^{-\delta_0(s-r)} \int_{|x| \ge k} |g(\cdot, r)|^2 dx dr. \end{aligned}$$

In line with (3.8), there is

$$e^{-\delta_0(s-\tau)} E_2(\tau) \le e^{-\delta_0(s-\tau)} C(R_1^{q+1} + R_1^4 + p^2) < C\eta, \ \forall \ \tau < s.$$

Then from (1.7), there exists $K \ge K_1, \ \tau \ge T_1$, such that

$$-2 \int_{\mathbb{R}^n} \theta_k F(x, u(s)) dx \le -2c_2 \int_{\mathbb{R}^n} \theta_k |u|^{q+1} + 2 \int_{\mathbb{R}^n} \theta_k \phi_3(x) dx \le C\eta.$$

So, for all $k \ge K, \ \tau \ge T_1$, we get

$$E_2(s) \le C\eta(1 + G(t) + G_1(t) + G^3(t)) + C \int_\tau^s e^{-\delta_0(s-r)} \int_{|x| \ge k} |g(\cdot, r)|^2 dx dr.$$

Since $G(\cdot)$ is increasing and bounded, by taking the supremum with respect to the past time and for K large enough, we infer

$$\begin{aligned} & \sup_{s \le t} \int_{\Omega_k^c} (|\Delta u(s)|^2 + |u(s)|^2 + \varepsilon(s) |v(s)|^2) dx \\ \le & C\eta + \sup_{s \le t} \int_\tau^s e^{-\delta_0(s-r)} \int_{|x| \ge k} |g(\cdot, r)|^2 dx dr. \end{aligned}$$

Combing with (2.4), the result is proved. □

3.3. Backwards asymptotic estimates inside a ball

Next, we prove the process $U(t, \tau)$ is backwards pullback asymptotically compact inside a ball under the topology of $\mathcal{H}_t$. For $Q_{\sqrt{2}k} = \{x \in \mathbb{R}^n : |x| < \sqrt{2}k\}, \ k \ge 1$. Let $q_k(x) = 1 - \theta_k(x)$, it is obvious that

$$\begin{cases} \widehat{u}(t) = \widehat{u}(t, \tau, \widehat{u_0}) = q(\frac{|x|^2}{k^2}) u(t, \tau, u_0), \\ \widehat{v}(t) = \widehat{v}(t, \tau, \widehat{v_0}) = q(\frac{|x|^2}{k^2}) v(t, \tau, v_0), \end{cases} \tag{3.23}$$

and for any $t \in \mathbb{R}$, it satisfies the following equation

$$\begin{cases} \widehat{v} = \dfrac{d\widehat{u}}{dt} + \delta\widehat{u},\ x \in Q_{\sqrt{2}k}, \\ \varepsilon(t)\dfrac{d\widehat{v}}{dt} + (\alpha(x) - \delta\varepsilon(t))\widehat{v} + \Delta^2\widehat{u} - \delta(\alpha(x) - \delta\varepsilon(t))\widehat{u} + \lambda\widehat{u} + q_k f(x,u) \\ = q_k g(x,t) + 4\Delta\nabla q_k \nabla u + 6\Delta q_k \Delta u + 4\nabla q_k \Delta\nabla u + u\Delta^2 q_k \\ \quad +(p - \|\nabla u\|^2)\cdot q_k \cdot \Delta u,\ x \in Q_{\sqrt{2}k}, \\ \widehat{u}(x,\tau) = \widehat{u_0} = q_k u_0,\ \widehat{v}(x,\tau) = \widehat{v_0} = q_k v_0,\ x \in Q_{\sqrt{2}k}, \\ \widehat{u} = \widehat{v} = 0, \quad |x| = \sqrt{2}k. \end{cases} \tag{3.24}$$

Consider characteristic function

$$A\widehat{u} = \lambda\widehat{u},\ x \in Q_{\sqrt{2}k};\ \widehat{u} = \frac{\partial\widehat{u}}{\partial\mathbf{n}} = 0,\ x \in \partial Q_{\sqrt{2}k}, \tag{3.25}$$

and there exists a family of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ corresponding to the eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$, which satisfies

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots,\ \lambda_i \to +\infty\ (i \to +\infty),$$

and $\{e_i\}_{i\in\mathbb{N}}$ is orthogonal in $H_0^2(Q_{\sqrt{2}k})$. For any n, let $X_n = \operatorname{span}\{e_1,\cdots,e_n\}$ and $P_n : L^2(Q_{\sqrt{2}k}) \to X_n$ be the projection operator. Then for any $(\widehat{u},\widehat{v}) \in H_0^2(Q_{\sqrt{2}k}) \times L^2(Q_{\sqrt{2}k})$, there exists a unique orthogonal decomposition

$$(\widehat{u},\widehat{v}) = P_n(\widehat{u},\widehat{v}) \oplus (I - P_n)(\widehat{u},\widehat{v}) := (\widehat{u_1},\widehat{v_1}) + (\widehat{u_2},\widehat{v_2}).$$

Lemma 3.4. *Assume that (1.2)-(1.8) and (2.3)-(2.5) are hold, for $\eta > 0, \tau \in \mathbb{R}, R > 0$, there exists $T_0 := T_0(\eta, R_1) > 0, K = K(\eta) \geq 1,\ N = N(\eta,t) \geq 1$ and $\tau \geq T_0,\ k \geq K,\ n \geq N, z(\tau) \in \mathbb{B}_\tau(R_1)$, when $(\widehat{u}_0,\widehat{v}_0) \in B_\tau$, the solution $(\widehat{u},\widehat{v})$ satisfies*

$$\sup_{s\leq t}\|(I - P_n)U(s, s-\tau, z_0)(\widehat{u}_0,\widehat{v}_0)\|^2_{\mathcal{H}_s(\sqrt{2}k)} \leq \eta.$$

Proof. Let $t \in \mathbb{R}, n \in \mathbb{N}$, for each $s \leq t$, taking the inner product of $\widehat{v_2} := \widehat{v_2}(s,\tau,q_k v_0)$ with the second equation of (3.24) on $L^2(Q_{\sqrt{2}k})$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}E_3(s) + \delta_0 E_3(s) \leq &\langle q_k f_u(x,u)u_s, \widehat{u}_2\rangle + \langle q_k g(x,t) + 4\Delta\nabla q_k\nabla u + 6\Delta q_k\Delta u \\ &+ 4\nabla q_k\Delta\nabla u + u\Delta^2 q_k, \widehat{v}_2\rangle + \langle (p - \|\nabla u\|^2)\cdot q_k\cdot\Delta u, \widehat{v}_2\rangle, \end{aligned}$$

where

$$E_3(s) = \varepsilon(s)\|\widehat{v}_2(s)\|^2 + \|\Delta\widehat{u}_2(s)\|^2 + \lambda\|\widehat{u}_2(s)\|^2 + 2\langle q_k f(x,u), \widehat{u}_2(s)\rangle.$$

For any $\eta > 0$, there exists $K := K(\eta) \geq 1$ such that for any $k \geq K_1$,

$$\begin{aligned} &2\langle q_k g(x,s), \widehat{v}_2\rangle \leq \frac{\alpha_0}{4}\|\widehat{v}_2\|^2 + C\|(I - P_n)g(x,s)\|^2, \\ &\quad\langle 4\Delta\nabla q_k\nabla u + 6\Delta q_k\Delta u + 4\nabla q_k\Delta\nabla u + u\Delta^2 q_k, \widehat{v}_2\rangle \\ &\leq \frac{\alpha_0}{4}\|\widehat{v}_2\|^2 + C\eta(\|u\|^2_{\mathrm{H}^2} + \|\nabla\Delta u\|^2), \end{aligned}$$

$$|\langle (p-\|\nabla u\|^2)\Delta u\cdot q_k\cdot \Delta u, \widehat{v}_2\rangle| \leq \frac{\alpha_0}{4}\|\widehat{v}_2\|^2 + C\eta(p^2\|\Delta u\|^2 + \|\nabla u\|^6 + \|\Delta u\|^6).$$

For nonlinearities, we can obtain

$$\begin{aligned}
&2\langle q_k f_u(x,u)u_s, \widehat{u}_2\rangle\\
\leq&2\int_{Q_{\sqrt{2}k}} (c_0|u|^{q-1} + \phi_1(x))\cdot q_k\cdot u_s\cdot \widehat{u}_2 dx\\
\leq&2c_0\|u\|^{q-1}_{L^{10}}\cdot\|u_s\|\cdot\|\widehat{u}_2\|_{L^{\frac{10}{6-q}}} + 2\|\phi_1(x)\|_{L^\infty}\|u_s\|\cdot\|\widehat{u}_2\|\\
\leq&C\|u\|_2^{q-1}\cdot\|u_s\|\cdot\|\widehat{u}_2\|^{\frac{6-q}{5}}\|\widehat{u}_2\|^{\frac{q-1}{5}}_{L^{10}} + C\|u_s\|\cdot\|\widehat{u}_2\|\\
\leq&C\lambda_{i+1}^{\frac{q-6}{10}}\|u\|_2^{q-1}\cdot\|u_s\|\cdot\|\widehat{u}_2\|_2 + C\lambda_{i+1}^{-\frac{1}{2}}\|u_s\|\cdot\|\widehat{u}_2\|_2\\
\leq&\frac{\delta_0}{4}\|\widehat{u}_2\|^2 + C\lambda_{i+1}^{\frac{q-6}{5}}(\|u\|_2^{\frac{10(q-1)}{4}} + \|u\|_2^{10} + \|v\|^{10}) + C\lambda_{i+1}^{-1}(\|u\|^2+\|v\|^2)\\
\leq&\frac{\delta_0}{4}\|\widehat{u}_2\|^2 + C\lambda_{i+1}^{\frac{q-6}{5}}(1 + \|u\|_2^{10} + \|v\|^{10}) + C\lambda_{i+1}^{-1}(\|u\|^2+\|v\|^2).
\end{aligned} \tag{3.26}$$

So for any $k\geq K$, we infer

$$\begin{aligned}
&\frac{d}{ds}E_3(s) + \delta_0 E_3(s)\\
\leq&C(\lambda_{i+1}^{\frac{q-6}{5}} + \lambda_{i+1}^{-1})(1+\|u\|_2^{10} + \|v\|^{10})\\
&+ C\eta(\|u\|^2_{\mathrm{H}^2} + \|\nabla\Delta u\|^2 + \|\nabla u\|^6 + \|\Delta u\|^6) + C\|(I-P_n)g(\cdot,s)\|^2.
\end{aligned} \tag{3.27}$$

Applying the Gronwall lemma on $[\tau,s]$, for all $k\geq K$,

$$\begin{aligned}
E_3(s)\leq&e^{-\delta_0(s-\tau)}E_3(\tau) + c(\lambda_{i+1}^{\frac{q-6}{5}} + \lambda_{i+1}^{-1})\int_\tau^s e^{-\delta_0(s-r)}(1+\|u(r)\|_2^{10} + \|v(r)\|^{10})dr\\
&+ C\eta\int_\tau^s e^{-\delta_0(s-r)}(\|u(r)\|^2_{\mathrm{H}^2} + \|\nabla u\|^6 + \|\Delta u\|^6 + \|\nabla\Delta u(r)\|^2)dr\\
&+ C\int_\tau^s e^{-\delta_0(s-r)}\|(I-P_n)g(\cdot,r)\|^2 dr.
\end{aligned} \tag{3.28}$$

We now consider the supremum of each term in (3.28) over the past time $s\leq t$, and then prove all terms on the right side of the inequality tend to zero.

By (3.7), for $\tau < T_0 < t$, we have

$$e^{-\delta_0(s-\tau)}E_3(\tau) \leq e^{-\delta_0\tau}(1+(L+1)R_1^2 + R_1^{q+1}) < C\eta. \tag{3.29}$$

Since $1\leq q\leq \frac{n}{n-4}$ and when $i\to\infty$, there is $\lambda_i\to\infty$, combined with the increasing property of (3.2) and $G(\cdot)$, there exists $N := N(\eta)\geq 1$, such that for all $i\geq N$

$$\begin{aligned}
&C(\lambda_{i+1}^{\frac{p-6}{5}} + \lambda_{i+1}^{-1})\sup_{s\leq t}\int_\tau^s e^{-\delta_0(s-r)}(1+\|u(r)\|_2^{10} + \|v(r)\|^{10})dr\\
\leq&C\eta G^5(t)\sup_{s\leq t}\int_{\tau-s}^0 e^{\delta_0 r}dr\\
\leq&\frac{C\eta}{\delta_0}(1+G^5(t)).
\end{aligned} \tag{3.30}$$

Similarly, from Lemma 3.1-3.2, for any $\tau \geq T_0$ we get

$$\begin{aligned} &C\eta \int_{-\tau}^{0} e^{\delta_0 r}(\|u(r+s)\|_{\mathrm{H}^2}^2 + \|\nabla\Delta u(r+s)\|^2 + \|\nabla u(r+s)\|^6 + \|\Delta u(r+s)\|^6)dr \\ \leq &C\eta(1+G^3(t)+G_1(t)). \end{aligned} \tag{3.31}$$

By (2.5), we know

$$c \lim_{n\to\infty} \sup_{s\leq t} \int_{\tau}^{s} e^{-\delta_0(r-\tau)}\|(I-P_n)g(\cdot,r)\|^2 dr = 0.$$

Combined with the above estimates, for all $k \geq K,\ \tau \geq T_0$ and $n \geq N$, there is

$$\sup_{s\leq t} E_3(s) \leq C\eta(1+G^5(t)+G^3(t)+G_1(t)).$$

Reusing (1.5) and (3.2), for all $\tau \geq \tau_1$, it follows

$$\begin{aligned} 2|\sup_{s\leq t}\langle q_k f(x,u), \widehat{u}_2(s)\rangle| \leq &2\sup_{s\leq t}(c_0\|u(s)\|_{2q}^q\|\widehat{u}_2(s)\| + 2\|\phi_1(x)\|\|\widehat{u}_2(s)\|) \\ \leq &C\lambda_{i+1}^{-\frac{1}{2}}\sup_{s\leq t}(\|u(s)\|_2^q\|\Delta\widehat{u}_2(s)\| + \|\Delta\widehat{u}_2(s)\|\cdot\|u\|) \\ \leq &\frac{1}{2}\|\Delta\widehat{u}_2(s)\|^2 + C\lambda_{i+1}^{-1}(1+G(t)+G^q(t)). \end{aligned} \tag{3.32}$$

Then for any $k \geq K,\ n \geq N,\ \tau \geq T_0$, we obtain

$$\begin{aligned} &\sup_{s\leq t}(\|(I-P_n)\widehat{u}(s,\tau,q_k u_0))\|_{H_0^2(Q(\sqrt{2}k))}^2 + \varepsilon(s)\|(I-P_n)\widehat{v}(s,\tau,\widehat{v}_0)\|_{L^2(Q_{\sqrt{2}k})}^2) \\ \leq &c\eta(1+G^5(t)+G^3(t)+G_1(t)). \end{aligned}$$

Combining with the boundedness of $G(t),\ G_1(t)$, the result is proved. □

3.4. Backwards compact attractors in $\mathcal{H}_t$

Theorem 3.1. *Assume that (1.2)-(1.8) and (2.3)-(2.5) are hold, then the problem (1.1) generated the process $U(t,\tau)$ possesses a backwards compact time-dependent attractor $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}_t$, which is the minimal one among the backwards compact and pullback attracting sets in $\mathcal{H}_t$.*

Proof. For each $t \in \mathbb{R}$, let

$$K(t) = \{(u,v) \in \mathcal{H}_t : \|(u,u_t)\|_{\mathcal{H}_t} \leq C(1+G(t))\}. \tag{3.33}$$

Since $G(\cdot)$ is increasing and finite, we can conclude that the time-dependent set K is increasing and bounded. Combining with (3.2), we infer that K is a pullback absorbing set.

In the following we will show that K is backwards pullback asymptotic compactness. For any $t \in \mathbb{R}$, we need only prove whenever $s_n \leq t, \tau_n \to -\infty$, for any bounded set $\{u_{0,n}, v_{0,n}\}$ in the topological sense of $\mathcal{H}_t$,

$$\{(u_n,v_n)\} := \{(u(s_n,\tau_n,u_{0n}), v(s_n,\tau_n,v_{0n}))\} \text{ is precompact in } \mathcal{H}_t.$$

Combined with Lemma 3.3, we know that there exists $n_1 \in \mathbb{N},\ k_1 \geq 1$, such that

$$\int_{|x|\geq k_1} (|u_n|^2 + |\Delta u_n|^2 + \varepsilon(t)|v_n|^2)dx < \eta. \tag{3.34}$$

By Lemma 3.4, for $n \in \mathbb{N},\ r \geq k_1$, there exists $n_2 \geq n_1$ such that

$$\|(I-P_n)\widehat{u}(s,\tau,q_k u_0))\|^2_{H_0^2(Q_{\sqrt{2}r})} + \varepsilon(s)\|(I-P_n)\widehat{v}(s,\tau,\widehat{v}_0)\|^2_{L^2(Q_{\sqrt{2}r})} < \eta. \tag{3.35}$$

On the other hand, from (3.2), there exists $n_3 \geq n_2$ such that for all $n \geq n_3$,

$$\|u_n\|^2_{\mathrm{H}^2} + \varepsilon(t)\|v_n\|^2 \leq C(1+G(t)). \tag{3.36}$$

For any fixed $T,\ \zeta \in [T,t]$, from (1.3), we know $\varepsilon(\zeta)$ is bounded, which implies that the sequence $\{(u_n,v_n)\}$ is bounded in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and thus $\{(\widehat{u}_n,\widehat{v}_n)\}$ is also bounded in $H_0^2(Q_{\sqrt{2}r})\times L^2(Q_{\sqrt{2}r})$. Therefore, the sequence $\{(P_i u_n, P_i v_n)\}$ is bounded in the finite-dimensional subspace $P_i(H_0^2(Q_{\sqrt{2}r}))\times P_i(L^2(Q_{\sqrt{2}r}))$, which further implies that the sequence $\{(P_i u_n, P_i v_n)\}$ has a convergent subsequence in $H_0^2(Q_{\sqrt{2}r})\times L^2(Q_{\sqrt{2}r})$. It is a Cauchy sequence. Subsequently, there exists $n_4 \geq n_3$ such that for all $n,m \geq n_4$, there is

$$\|(P_i u_n, P_i v_n) - (P_i u_m, P_i v_m)\|^2_{\mathcal{H}_t(Q_{\sqrt{2}r})} < \eta. \tag{3.37}$$

Since $(\widehat{u}_n,\widehat{v}_n) = (q_k u_n, q_k v_n) = (u_n, v_n)$ on Q_r, it follows from (3.36) and (3.37) that for all $n,m \geq n_4$,

$$\begin{aligned}
&\|(u_n-u_m, v_n-v_m)\|^2_{\mathcal{H}_t(Q_r)}\\
=&\|(\widehat{u}_n-\widehat{u}_m,\widehat{v}_n-\widehat{v}_m)\|^2_{\mathcal{H}_t(Q_r)}\\
\leq&\|(\widehat{u}_n-\widehat{u}_m,\widehat{v}_n-\widehat{v}_m)\|^2_{\mathcal{H}_t(Q_{\sqrt{2}r})}\\
\leq&2\|P_i(\widehat{u}_n-\widehat{u}_m)\|^2_{H_0^2(Q_{\sqrt{2}r})} + 2\varepsilon(t)\|P_i(\widehat{v}_n-\widehat{v}_m)\|^2_{L^2(Q_{\sqrt{2}r})}\\
&+2\|(I-P_i)\widehat{u}_n\|^2_{H_0^2(Q_{\sqrt{2}r})} + 2\varepsilon(t)\|(I-P_i)\widehat{v}_n\|^2_{L^2(Q_{\sqrt{2}r})}\\
&+2\|(I-P_i)\widehat{u}_m\|^2_{H_0^2(Q_{\sqrt{2}r})} + 2\varepsilon(t)\|(I-P_i)\widehat{v}_m\|^2_{L^2(Q_{\sqrt{2}r})}\\
<&6\eta.
\end{aligned} \tag{3.38}$$

Note that, (3.35) holds true if k_1 is replaced by the larger r. It follows from (3.35) and (3.38) that for all $n,m \geq n_4$,

$$\begin{aligned}
\|(u_n-u_m, v_n-v_m)\|^2_{\mathcal{H}_t} \leq&\|(u_n-u_m, v_n-v_m)\|^2_{H_0^2(Q_r)\times L^2(Q_r)} + 2\|u_n\|^2_{\mathrm{H}^2(Q_r^c)}\\
&+2\varepsilon(t)\|v_n\|^2_{L^2(Q_r^c)} + 2\|u_m\|^2_{\mathrm{H}^2(Q_r^c)} + 2\varepsilon(t)\|v_m\|^2_{L^2(Q_r^c)}\\
<&10\eta.
\end{aligned}$$

Therefore, the subsequence $\{(u_n,v_n)\}$ is a Cauchy sequence and converges in $\mathcal{H}_t$, which implies the theorem is proved. □

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