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EXACT SOLUTIONS OF A GENERALIZED TIME-FRACTIONAL KDV EQUATION UNDER RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL DIFFERENTIAL OPERATORS*

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Abstract It is well known that searching exact solutions of nonlinear fractional partial differential equations (PDEs) is a very difficult work. In this paper, based on a modified separation method of variables and the dynamic system method, a combinational method is proposed in order to develop new methods for solving nonlinear time-fractional PDEs. Compared with the traditional separation method of variables, the modified separation method of variables has some advantages in reducing nonlinear time-fractional PDEs. As an example for the application of this combinational method, a generalized nonlinear time-fractional KdV equation is studied under the Riemann-Liouville and Caputo fractional differential operators, respectively. In different parametric regions, different kinds of phase portraits of the dynamic systems derived from the generalized time-fractional KdV equation are presented. Existence and dynamic properties of solutions of the generalized time-fractional KdV equation are investigated. In some special parametric conditions, many exact solutions are obtained, some of them are parametric form. Such solutions of parametric form are usually unable to be obtained by other methods, which also shows an advantage of dynamic system method.

Keywords Separation method of semi-fixed variables, dynamic system method, nonlinear time-fractional PDEs, generalized time-fractional KdV equation.

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1. Introduction

It is well known that the basic concept of fractional derivative was born in the Leibniz era. At first, the study of fractional calculus was only limited to pure math-

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ematical theory because it lacked the application background for a long time. For a long period of history, there had no idea what mathematical models could be modeled by fractional differential calculus. This situation did not change until the middle of the last century. Since 1960s, more and more researchers have found that the fractional-order differential models can be more accurately described complex problems in various fields such as mathematical mechanics, control theory, signal processing, aerodynamics, chemistry, biology and so forth. However, compared with the number of integer-order differential equations, the number of existing fractional differential models is very small. In order to make up for this deficiency, many researchers directly changed some classical integer-order PDEs into fractional PDEs to study, so as to develop new solution methods for more complex nonlinear fractional PDEs. From a mathematical point of view, this is meaningful and very necessary to do so in the current shortage of mathematical models.

In terms of solving fractional differential equations, the current works mainly focuses on the investigations of exact solutions, approximate solutions and numerical solutions. On the methods for searching exact solutions and approximate analytic solutions of fractional PDEs, many effective methods were proposed in recent decades. These methods include Adomian decomposition method [1, 9], homotopy analysis method [2, 3, 22], invariant analysis method [4, 40], fractional variational iteration method [17, 30, 31, 47], invariant subspace method [13, 41, 43], method of fractional complex transformation [10, 26, 27] and the method of separating variables [7, 18, 29], etc. Of cause, some methods for investigate numerical solutions of fractional differential equations also deserve attention [5, 12, 48].

Although exact solutions of some fractional differential equations can be obtained by above methods, these methods are not general and have their own limitations in their application. And we found that the method of fractional complex transformation appeared in Refs. [10, 26, 27] is based on the fractional chain rule given by Jumarie in Refs. [19–21]. Unfortunately, Jumarie's fractional chain rule has been verified that it is not valid in Refs. [14, 35, 44]. This shows that existing methods are not sufficient to meet people's demand on solving complex fractional PDEs. This means peoples need to develop more new methods to solve those very complex nonlinear fractional PDEs. For this purpose, based on the separation method of semi-fixed variables and combined with other methods, we introduced several new combinational methods [36, 37, 46] for solving nonlinear time-fractional PDEs, recently. Expressly, we noted that the dynamic system method [15, 23–25] based on the method of bifurcation theory [8] is very effective in searching travelling wave solutions of nonlinear integer-order PDEs and nonlinear fractional PDEs defined by the conformable fractional derivative [28]. However, the pure dynamical system method can not directly be used to solve nonlinear time-fractional PDEs defined by Riemann-Liouville fractional derivative or Caputo fractional derivative due to the fractional chain rule does not hold under these two definitions of fractional derivative, so the application of the dynamical system method needs to combine with separation method of variables. It is precisely for this reason, by using a combinational method based on the modified separation method of variables and dynamical system method, we successfully investigated exact solutions and dynamic properties of the time-fractional biology model of $(2 + 1)$ dimensions in [38]. Obviously, the several methods in [36–38, 46] mentioned above unlike the traditional separation method of variables. Specifically speaking, the function $T(t)$ of the part of time t

in the traditional separation method of variables

$$u(x,t) = v(x)T(t)$$

is fixed into some specific special functions such as Mittag-Leffler functions or power function. So, we call this modified separation method of variables as separation method of semi-fixed variables. Although these combinational methods are successfully used to solve some nonlinear time-fractional PDEs in the above references, but most of these nonlinear time-fractional PDEs belong to second order PDEs, so we will naturally ask whether these methods can be solved those nonlinear time-fractional PDEs with higher-order terms? In the next, we answer this question by solving a nonlinear time-fractional PDEs with higher-order terms (such as u_{xxx}).

In this paper, by using the separation method semi-fixed variables together with the dynamic system method, we will investigate exact solutions, existence and dynamic properties of solutions to the following generalized time-fractional KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u_x + \rho\frac{\partial^\alpha u_{xx}}{\partial t^\alpha} + \beta u_{xxx} + \kappa u u_x + \frac{1}{3}\kappa\rho(u u_{xxx} + 2u_x u_{xx}) = 0, \tag{1.1}$$

where the sign $\frac{\partial^\alpha}{\partial t^\alpha}$ defines fractional differential operator of Riemann-Liouville type or Caputo type and $0 < \alpha < 1,\ u = u(x,t),\ t > 0,\ x \in \mathbb{R}$, the parameters $\rho,\ \beta,\ \kappa$ are nonzero constants. In particular, when $\alpha \to 1$, the equation (1.1) becomes a generalized KdV equation

$$u_t + u_x + \rho u_{xxt} + \beta u_{xxx} + \kappa u u_x + \frac{1}{3}\kappa\rho(u u_{xxx} + 2u_x u_{xx}) = 0, \tag{1.2}$$

which was first derived by Fokas [11]. Compared with the classical KdV equation, the generalized KdV equation (1.2) has very strong nonlinear structure. So, equation (1.2) exhibits a much richer phenomenology than the classical KdV equation. In [39], some new soliton-like solutions and periodic wave solutions with loop of Eq. (1.2) were studied.

Although the generalized time-fractional KdV equation (1.1) is not a practical physical model, only converted from an integer-order soliton equation (1.2), it is very meaningful for the purpose of developing a solution method for nonlinear fractional PDEs. Moreover, when an integer-order soliton equation is converted into a nonlinear time-fractional PDE, how the types and dynamic properties of the solutions will change is also a very interesting question, which deserves our in-depth exploration and research.

It is well known that when an integer-order nonlinear PDE is changed into a fractional nonlinear PDE, it becomes very difficult to solve, because many classical methods in the field of integer-order differential equations will completely lose their efficacy in the field of fractional differential equations. In other words, the classical methods in the integer-order field cannot be directly applied to the fractional field, because there is no succinct chain rule and Leibniz rule as in the integer-order calculus. Therefore, how to obtain the exact solutions of the generalized time-fractional KdV equation (1.1) is a very interesting and expectant question. We will introduce a new combinational method and use it to solve this difficult problem in this paper.

The organization of this paper is as follows: In Sec. 2, we will summary traditional and modified separation methods of variables for fractional PDEs. In Sec.

3, we will investigate exact solutions and dynamic properties of the generalized time-fractional KdV equation under the definition of Caputo fractional derivative. In Sec. 4, we will investigate its exact solutions and dynamic properties under the definition of Riemann-Liouville fractional derivative.

2. Summary of traditional and modified separation methods of variables for fractional PDEs

2.1. Traditional separation method of variables for fractional PDEs

We all known that a linear fractional PDE is always able to separate it into two independent differential equations via traditional separation method of variables. For example, a linear fractional PDE formed as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_0 u + a_1 \frac{\partial^{\sigma_1} u}{\partial x^{\sigma_1}} + a_2 \frac{\partial^{\sigma_2} u}{\partial x^{\sigma_2}} + \cdots + a_n \frac{\partial^{\sigma_n} u}{\partial x^{\sigma_n}} \tag{2.1}$$

is always able to separate it into two independent differential equations via the following traditional separation method of variables

$$u(x,t) = v(x)T(t), \tag{2.2}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ and $\frac{\partial^{\sigma_i}}{\partial x^{\sigma_i}}$ are Riemann-Liouville or Caputo differential operator and $0 < \alpha < 1$, $\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_n$ are arbitrary positive constants. Indeed, substituting (2.2) into (2.1), it yields

$$v\ \frac{d^\alpha T}{dt^\alpha} = T\ \left(a_0 v + a_1 \frac{d^{\sigma_1} v}{dx^{\sigma_1}} + a_2 \frac{d^{\sigma_2} v}{dx^{\sigma_2}} + \cdots + a_n \frac{d^{\sigma_n} v}{dx^{\sigma_n}}\right), \tag{2.3}$$

where $\frac{d^\alpha}{dt^\alpha}$ and $\frac{d^{\sigma_i}}{dx^{\sigma_i}}$ are Riemann-Liouville or Caputo differential operator. Separating variables, Eq. (2.3) can be rewritten as

$$\frac{\frac{d^\alpha T}{dt^\alpha}}{T} = \frac{a_0 v + a_1 \frac{d^{\sigma_1} v}{dx^{\sigma_1}} + a_2 \frac{d^{\sigma_2} v}{dx^{\sigma_2}} + \cdots + a_n \frac{d^{\sigma_n} v}{dx^{\sigma_n}}}{v} = \lambda. \tag{2.4}$$

Obviously, Eq. (2.4) can be separated into two independent differential equations as follows:

$$\begin{cases} \dfrac{d^\alpha T}{dt^\alpha} - \lambda T = 0, \\ (a_0 - \lambda) v + a_1 \dfrac{d^{\sigma_1} v}{dx^{\sigma_1}} + a_2 \dfrac{d^{\sigma_2} v}{dx^{\sigma_2}} + \cdots + a_n \dfrac{d^{\sigma_n} v}{dx^{\sigma_n}} = 0. \end{cases} \tag{2.5}$$

If $\frac{d^\alpha}{dt^\alpha} = {}_0^{RL}D_t^\alpha$ is Riemann-Liouville differential operator, then the fractional differential equation $\frac{d^\alpha T}{dt^\alpha} - \lambda T = 0$ in (2.5) has general solution formed as

$$T = \delta\ t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^\alpha\right), \tag{2.6}$$

where $E_{\alpha,\alpha}\left(\lambda t^\alpha\right)$ is a special case of the two-parameter Mittag-Leffler function, its definition can be seen *Appendix* at the end of article, the parameter δ is an arbitrary constant.

If $\frac{d^\alpha}{dt^\alpha} = {}_0^C D_t^\alpha$ is Caputo differential operator, then the fractional differential equation $\frac{d^\alpha T}{dt^\alpha} - \lambda T = 0$ in (2.5) has general solution formed as

$$T = \delta\ E_\alpha\left(\lambda t^\alpha\right). \tag{2.7}$$

Obviously, solving the first equation of (2.5) is much simpler. But, the second equation of (2.5) is complex linear fractional ODE, it is hard to obtain its exact solutions except some special cases. Especially, when $\sigma_1 = 1,\ \sigma_2 = 2,\ \cdots,\ \sigma_n = n$ and $n \in \mathbb{N}^+$, the second equation of (2.5) is a n-oder linear ODE and very easy to solve it.

However, for the nonlinear fractional PDEs, the separation method of variables has no efficiency for them at all. Even for a relatively simple nonlinear time-fractional PDE, the traditional separation method of variables is not necessarily able to separate it into two independent differential systems. For example, a non-linear time-fractional PDE formed as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \eta u + \beta u^2 + \kappa\left(\frac{\partial u}{\partial x}\right)^2 + \rho u \frac{\partial^2 u}{\partial x^2} \tag{2.8}$$

cannot be separated into two independent differential systems by use of (2.2). Substituting (2.2) into (2.8), it yields

$$v\ \frac{d^\alpha T}{dt^\alpha} = \eta v T + \beta T^2 v^2 + \kappa T^2\left(\frac{dv}{dx}\right)^2 + \rho T^2 v \frac{d^2 v}{dx^2}. \tag{2.9}$$

It is easy to find that the equation (2.9) cannot be separated into two independent differential equations as in (2.3).

Specially, when $\eta = 0$, Eq. (2.8) can be reduced to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \beta u^2 + \kappa\left(\frac{\partial u}{\partial x}\right)^2 + \rho u \frac{\partial^2 u}{\partial x^2}. \tag{2.10}$$

Substituting (2.2) into (2.10), we obtain

$$v \frac{d^\alpha T}{dt^\alpha} = T^2\left[\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v \frac{d^2 v}{dx^2}\right]. \tag{2.11}$$

Separating variables, Eq. (2.11) can be rewritten as

$$\frac{\frac{d^\alpha T}{dt^\alpha}}{T^2} = \frac{\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v \frac{d^2 v}{dx^2}}{v} = \lambda. \tag{2.12}$$

Obviously, Eq. (2.12) can be separated into two independent differential systems as follows:

$$\begin{cases} \dfrac{d^\alpha T}{dt^\alpha} - \lambda T^2 = 0, \\ -\lambda v + \beta v^2 + \kappa\left(\dfrac{dv}{dx}\right)^2 + \rho v \dfrac{d^2 v}{dx^2} = 0. \end{cases} \tag{2.13}$$

If $\frac{d^\alpha}{dt^\alpha} = {}_0^{RL} D_t^\alpha$ is Riemann-Liouville differential operator, then the nonlinear fractional differential equation $\frac{d^\alpha T}{dt^\alpha} - \lambda T^2 = 0$ in (2.13) has general solution formed as

$$T = \delta\ \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\ t^{-\alpha}, \tag{2.14}$$

where δ is arbitrary constant and $\alpha \neq \frac{1}{2}$. The second equation of (2.13) is a nonlinear ODE, its solutions can be obtained by dynamic system method or other methods.

2.2. Modified separation method of variables for nonlinear time-fractional PDEs

Inspired by the above discussions, by using the formulas of fractional derivatives of Mittag-Leffler functions and power function given in *Appendix* at the end of article, we modify the expression (2.2) of the traditional separation method of variables as follows:

(i) If $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{RL}D_t^\alpha$ is Riemann-Liouville differential operator and $0 < \alpha < 1$, then the function $T(t)$ in (2.2) can be fixed to $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ or t^γ, that is

$$u(x,t) = v(x)\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right] \tag{2.15}$$

or

$$u(x,t) = v(x)\ t^\gamma, \quad \gamma > -1. \tag{2.16}$$

(ii) If $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{C}D_t^\alpha$ is Caputo differential operatorand $0 < \alpha < 1$, then the function $T(t)$ in (2.2) can be fixed to $E_\alpha(\lambda t^\alpha)$, that is

$$u(x,t) = v(x)E_\alpha(\lambda t^\alpha). \tag{2.17}$$

We call above modified separation methods as separation methods of semi-fixed variables.

In order to test effect of the above separation methods of semi-fixed variables, we will make some discussions on the applications of (2.15), (2.16) and (2.17) in the next.

As example, when $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{RL}D_t^\alpha$ is Riemann-Liouville differential operator, substituting (2.15) into the nonlinear time-fractional PDE (2.8), it yields

$$(\lambda-\eta)v[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)] = \left[\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v\frac{d^2v}{dx^2}\right][t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)]^2. \tag{2.18}$$

In Eq. (2.18), respectively letting the coefficients of the Mittag-Leffler functions $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ and $\left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right]^2$ as zero, it yields

$$\begin{cases} (\lambda-\eta)v = 0, \\ \beta v^2 + \kappa\left(\dfrac{dv}{dx}\right)^2 + \rho v\dfrac{d^2v}{dx^2} = 0. \end{cases} \tag{2.19}$$

Taking $\lambda = \eta$, Eq. (2.19) can be reduced to

$$\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v\frac{d^2v}{dx^2} = 0. \tag{2.20}$$

Similarly, when $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{C}D_t^\alpha$ is Caputo differential operator, substituting (2.17) into the nonlinear time-fractional PDE (2.8), it yields

$$(\lambda-\eta)vE_\alpha(\lambda t^\alpha) = \left[\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v\frac{d^2v}{dx^2}\right][E_\alpha(\lambda t^\alpha)]^2. \tag{2.21}$$

Similarly, respectively making the coefficients of the Mittag-Leffler functions $E_\alpha(\lambda t^\alpha)$ and $[E_\alpha(\lambda t^\alpha)]^2$ as zero in above equation, Eq. (2.21) also can be reduced to the ODE (2.20).

From above discussions, it is found that the nonlinear time-fractional PDE (2.8) cannot be solved by (2.2), but it can be easily solved by (2.15) and (2.17). This shows that the separation method of semi-fixed variables have some advantages in reducing nonlinear time-fractional PDEs.

When $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{RL}D_t^\alpha$ is Riemann-Liouville differential operator, substitute (2.16) to the nonlinear time-fractional PDE (2.13), it yields

$$\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} v t^{\gamma-\alpha} = \left[\beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v\frac{d^2v}{dx^2}\right] t^{2\gamma}. \tag{2.22}$$

In (2.22), letting numbers of power of the functions $t^{\gamma-\alpha}$ and $t^{2\gamma}$ equal, it yields

$$\gamma - \alpha = 2\gamma.$$

Thus, we obtain

$$\gamma = -\alpha > -1, \quad \Omega_0 \equiv \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}, \quad \alpha \neq \frac{1}{2}. \tag{2.23}$$

Substituting the condition (2.23) into (2.22) and then dividing both sides of the equation by $t^{-2\alpha}$, it yields

$$-\Omega_0 v + \beta v^2 + \kappa\left(\frac{dv}{dx}\right)^2 + \rho v\frac{d^2v}{dx^2} = 0. \tag{2.24}$$

Further, using dynamic system method, exact solutions of the nonlinear ODE (2.24) always can be obtained. Obviously, this method is also very convenient.

To test the efficiency of the modified separation method of variables introduced above, we next discuss exact solutions and dynamic properties of Eq. (1.1).

3. Exact solutions and dynamic properties of Eq. (1.1) under Caputo fractional differential operator

If $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{C}D_t^\alpha$ is Caputo differential operator, then Eq. (1.1) can be rewritten as

$${}_0^{C}D_t^\alpha u + u_x + \rho\, {}_0^{C}D_t^\alpha u_{xx} + \beta u_{xxx} + \kappa u u_x + \frac{1}{3}\kappa\rho(u u_{xxx} + 2u_x u_{xx}) = 0, \tag{3.1}$$

where $0 < \alpha < 1,\ u = u(x,t),\ t > 0,\ x \in \mathbb{R}$, the parameters $\rho,\ \beta,\ \kappa$ are nonzero constants. We suppose that Eq. (3.1) has solution formed as follows:

$$u = \delta + v(x)E_\alpha(\lambda t^\alpha), \tag{3.2}$$

where $v = v(x)$ and $\delta,\ \lambda$ are nonzero constants which can be determined in a later discussions.

Substituting (3.2) into (3.1), the equation can be reduced to

$$\left[\lambda v + (1+\kappa\delta)v_x + \lambda\rho v_{xx} + \left(\beta + \frac{1}{3}\kappa\rho\delta\right) v_{xxx}\right] E_\alpha\left(\lambda t^\alpha\right) + \left[\kappa v v_x + \frac{1}{3}\kappa\rho v v_{xxx} + \frac{2}{3}\kappa\rho v_x v_{xx}\right]\left[E_\alpha\left(\lambda t^\alpha\right)\right]^2 = 0. \tag{3.3}$$

In Eq. (3.3), letting the coefficients of the Mittag-Leffler functions $E_\alpha\left(\lambda t^\alpha\right)$ and $\left[E_\alpha\left(\lambda t^\alpha\right)\right]^2$ equal zero, it yields

$$\begin{cases} \lambda v + (1+\kappa\delta)v_x + \lambda\rho v_{xx} + \left(\beta + \frac{1}{3}\kappa\rho\delta\right) v_{xxx} = 0, \\ v v_x + \frac{2}{3}\rho v_x v_{xx} + \frac{1}{3}\rho v v_{xxx} = 0. \end{cases} \tag{3.4}$$

The first equation of (3.4) is linear ODE, but the second equation of (3.4) is nonlinear ODE. From the theory of ordinary differential equations, we know that the solution to the first equation in (3.4) is not necessarily the solution of the second equation in (3.4), but the solution of the second equation may be the solution of the first equation. Therefore, we plan to solve the second nonlinear ODE in (3.4) firstly, and then test the obtained results into the first linear ODE in (3.4).

Integrating the second nonlinear ODE in (3.4), we obtain

$$\frac{1}{2}v^2 + \frac{1}{3}\rho v v_{xx} + \frac{1}{6}\rho v_x^2 = g_1, \tag{3.5}$$

where g_1 is an integral constant and $v_x = \frac{dv}{dx}$. Eq. (3.5) can be rewritten as

$$3v^2 + 2\rho v v_{xx} + \rho v_x^2 = g, \tag{3.6}$$

where g is an arbitrary constant and $g = 6g_1$. Letting $\frac{dv}{dx} = y$, Eq. (3.6) can be reduced to

$$\begin{cases} \dfrac{dv}{dx} = y, \\ \dfrac{dy}{dx} = \dfrac{g - 3v^2 - \rho y^2}{2\rho v}. \end{cases} \tag{3.7}$$

Obviously, the $\frac{dy}{dx}$ cannot be defined when $v = 0$, so the system (3.7) is not equivalent to the equation (3.6) at $v = 0$. However, $v = 0$ is a trivial solution of equation (3.6). In order to obtain a completely equivalent system to the equation (3.6) no mater how the function v vary, we make a scalar transformation as follows:

$$dx = 2\rho v d\tau, \tag{3.8}$$

where τ is a parameter. Under the transformation (3.8), the singular system (3.7) is reduced to a regular system as follows:

$$\begin{cases} \dfrac{dv}{d\tau} = 2\rho v y, \\ \dfrac{dy}{d\tau} = g - 3v^2 - \rho y^2. \end{cases} \tag{3.9}$$

Obviously, both systems (3.7) and (3.9) have a same first integral as follows:

$$y^2 = \frac{h + gv - v^3}{\rho v}, \tag{3.10}$$

where h is an integral constant. We rewrite Eq. (3.10) as

$$H(v, y) \equiv -gv + v^3 + \rho v y^2 = h. \tag{3.11}$$

When $g > 0$ and $\rho > 0$, the system (3.9) has four equilibrium points $A_{1,2}\left(\pm\sqrt{\frac{g}{3}},\ 0\right)$ and $B_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$. When $g > 0$ and $\rho < 0$, the system (3.9) has two equilibrium points $A_{1,2}\left(\pm\sqrt{\frac{g}{3}},\ 0\right)$. When $g < 0$ and $\rho < 0$, the system (3.9) has two equilibrium points $B_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$. When $g = 0$ and $\rho \neq 0$, the system (3.9) has only one equilibrium point $O(0,0)$. When $g < 0$ and $\rho > 0$, the system (3.9) has not any equilibrium point.

Respectively substituting these equilibrium points into (3.11), we get

$$\begin{cases} h_0 = H(0,0) = 0, \\ h_{1,2} = H\left(\pm\sqrt{\dfrac{g}{3}}, 0\right) = \mp\dfrac{2g}{3}\sqrt{\dfrac{g}{3}}, \\ h_{3,4} = H\left(0, \pm\sqrt{\dfrac{g}{\rho}}\right) = 0. \end{cases} \tag{3.12}$$

Writing $P(v, y) = 2\rho v y,\ Q(v, y) = g - 3v^2 - \rho y^2$, we get the Jacobian matrix and Jacobian determinant of system (3.9) as follows:

$$M(v, y) \equiv \begin{bmatrix} P_v & P_y \\ Q_v & Q_y \end{bmatrix} = \begin{bmatrix} 2\rho y & 2\rho v \\ -6v & -2\rho y \end{bmatrix}, \tag{3.13}$$

$$J(v, y) \equiv \det M(v, y) = 12\rho v^2 - 4\rho^2 y^2, \quad \mathrm{Trace}M(v, y) = 0. \tag{3.14}$$

Respectively substituting above equilibrium points into (3.14), we get

$$J_0 \equiv J(0,0) = 0 \ \text{ and } \ \mathrm{Trace}M(0,0) = 0, \tag{3.15}$$

$$J_{1,2} \equiv J\left(\pm\sqrt{\frac{g}{3}},\ 0\right) = 4\rho g, \qquad J_{3,4} \equiv J\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right) = -4\rho g. \tag{3.16}$$

According to the classification method of equilibrium points in the bifurcation theory [8, 15, 23–25] and using the equations (3.15) and (3.16), we can classify the equilibrium points of the system (3.9). Obviously, the origin point $O(0,0)$ is a higher-order equilibrium point due to its equilibrium-point index is grater than 0. When $g = 0$ and $\rho < 0$, the origin point $O(0,0)$ is a saddle-cusp point, this complex equilibrium point can be regarded as a combination of degenerative saddle point and cusp point. When $g = 0$ and $\rho > 0$, the origin point $O(0,0)$ is a center-cusp point, this complex equilibrium point can be regarded as a combination of degenerative center point and cusp point.

When $g > 0$ and $\rho > 0$, the equilibrium points $A_{1,2}\left(\pm\sqrt{\frac{g}{3}},\ 0\right)$ are two center points and the equilibrium points $B_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$ are two saddle points. When

$g > 0$ and $\rho < 0$, the equilibrium points $A_{1,2}\left(\pm\sqrt{\frac{g}{3}},\ 0\right)$ are two saddle points. When $g < 0$ and $\rho < 0$, the equilibrium points $B_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$ are two saddle points.

According to the above information, under different parametric conditions, we draw the phase portraits of system (3.9), which are shown in the Figure 1 of the below. In the below graphs, the orbits defined by $h = 0$ are marked in black, the orbits defined by $h > 0$ are marked by red, the orbits defined by $h < 0$ are marked by blue, the singular line $v = 0$ is marked by green.

From the theory of the dynamic system, we know that the various orbits in the phase portraits of the system (3.9) correspond to the various solutions of the nonlinear ODE (3.6). Therefore, in different parametric conditions we can obtain different kinds of exact solutions of the nonlinear ODE (3.6) by the orbits of the phase portraits in the Figure 1. Further, we can obtain different kinds of exact solutions of the equation (3.4). Next, we will investigate exact solutions of (3.4).

Case 1. When $g = 0,\ \rho < 0$ and $h = h_0 = 0$, there are two straight-line orbits passing through the origin $O(0, 0)$ which are marked by black in Figure 1(**a**). Substituting these parametric conditions into (3.10), we obtain expression of the two line orbits as follows:

$$y = \pm\frac{v}{\sqrt{-\rho}}. \tag{3.17}$$

Plugging (3.17) into the first equation $\frac{dv}{dx} = y$ of (3.7) and then integrating it, we get

$$v = C_1 e^{\frac{x}{\sqrt{-\rho}}}, \tag{3.18}$$

$$v = C_2 e^{-\frac{x}{\sqrt{-\rho}}}, \tag{3.19}$$

where $C_1,\ C_2$ are arbitrary constants. Respectively, substituting (3.18) and (3.19) into the first equation of (3.4), it yields

$$\left[\frac{1+\kappa\delta}{\sqrt{-\rho}} - \frac{\kappa\rho\delta + 3\beta}{3\rho\sqrt{-\rho}}\right] C_1 e^{\frac{x}{\sqrt{-\rho}}} = 0, \tag{3.20}$$

$$-\left[\frac{1+\kappa\delta}{\sqrt{-\rho}} - \frac{\kappa\rho\delta + 3\beta}{3\rho\sqrt{-\rho}}\right] C_2 e^{-\frac{x}{\sqrt{-\rho}}} = 0. \tag{3.21}$$

Solving Eqs. (3.20) and (3.21), we can always obtain

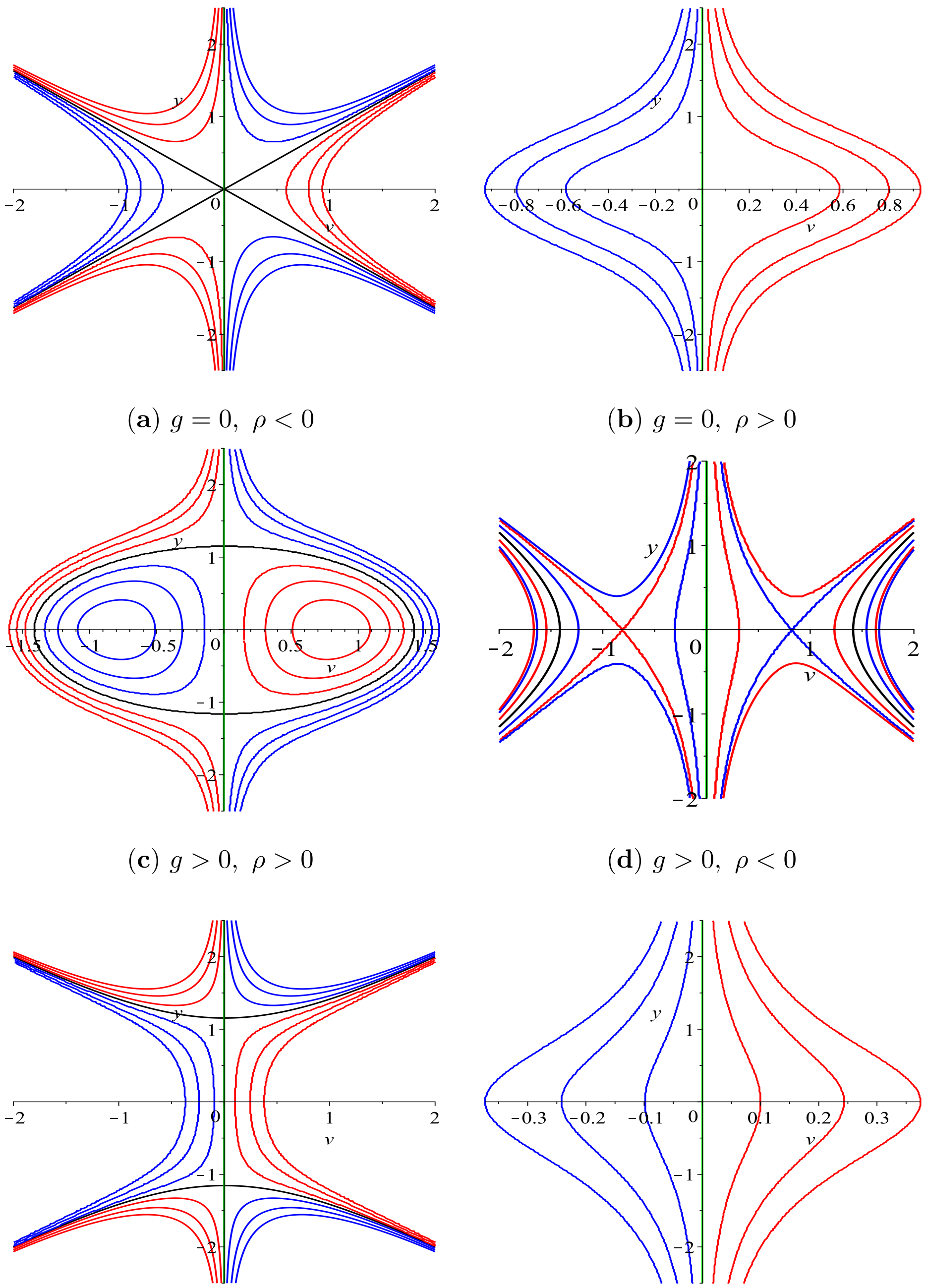
$$\delta = \frac{3(\beta - \rho)}{2\kappa\rho}. \tag{3.22}$$

Respectively plugging (3.18), (3.19) and the parametric condition (3.22) into (3.2), we obtain two exact solutions of the generalized time-fractional KdV equation defined by Caputo differential operator as follows:

$$u(x,t) = \frac{3(\beta - \rho)}{2\kappa\rho} + C_1 e^{\frac{x}{\sqrt{-\rho}}} E_\alpha\left(\lambda t^\alpha\right), \tag{3.23}$$

$$u(x,t) = \frac{3(\beta - \rho)}{2\kappa\rho} + C_2 e^{-\frac{x}{\sqrt{-\rho}}} E_\alpha\left(\lambda t^\alpha\right). \tag{3.24}$$

The above two solutions are general solutions under free conditions. Once, the boundary conditions and initial value conditions are given, we can determine the



(**a**) $g = 0,\ \rho < 0$ (**b**) $g = 0,\ \rho > 0$

(**c**) $g > 0,\ \rho > 0$ (**d**) $g > 0,\ \rho < 0$

(**e**) $g < 0,\ \rho < 0$ (**f**) $g < 0,\ \rho > 0$

Figure 1. Bifurcation graphs of phase portraits of system (3.9).

special solution of equation (3.1). For example, in the boundary condition $u(0,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + E_\alpha(-2t^\alpha)$ and the initial value condition $u(x,0) = \frac{3(\beta-\rho)}{2\kappa\rho} + e^{\frac{x}{\sqrt{-\rho}}}$ or $u(x,0) = \frac{3(\beta-\rho)}{2\kappa\rho} + e^{-\frac{x}{\sqrt{-\rho}}}$, by using (3.23) and (3.24) we obtain two special solutions of equation (3.1) as follows:

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + e^{\frac{x}{\sqrt{-\rho}}} E_\alpha(-2t^\alpha), \tag{3.25}$$

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + e^{-\frac{x}{\sqrt{-\rho}}} E_\alpha(-2t^\alpha). \tag{3.26}$$

Case 2. When $g > 0,\ \rho > 0$ and $h = h_{3,4} = 0$, there are two closed orbits shaped as half-moon passing through the saddle points $B_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$ which are marked by black in Figure 1(**c**). Substituting these parametric conditions into (3.10), we obtain expression of the two line orbits as follows:

$$y = \pm\frac{\sqrt{g-v^2}}{\sqrt{\rho}}. \tag{3.27}$$

Plugging (3.27) into the first equation $\frac{dv}{dx} = y$ of (3.7) and then integrating it, we get

$$v = \sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_3\right), \tag{3.28}$$

$$v = -\sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_4\right), \tag{3.29}$$

where $C_3,\ C_4$ are arbitrary constants. Respectively, substituting (3.28) and (3.29) into the first equation of (3.4), it yields

$$\left[\frac{1+\kappa\delta}{\sqrt{-\rho}} - \frac{\kappa\rho\delta+3\beta}{3\rho\sqrt{-\rho}}\right]\sqrt{g}\cos\left(\frac{x}{\sqrt{\rho}} + C_3\right) = 0, \tag{3.30}$$

$$-\left[\frac{1+\kappa\delta}{\sqrt{-\rho}} - \frac{\kappa\rho\delta+3\beta}{3\rho\sqrt{-\rho}}\right]\sqrt{g}\cos\left(\frac{x}{\sqrt{\rho}} + C_4\right) = 0. \tag{3.31}$$

Solving Eqs. (3.30) and (3.31), we also obtain

$$\delta = \frac{3(\beta-\rho)}{2\kappa\rho}$$

which is as same as the condition (3.22). This is not a coincidence because the two differential equations in (3.4) have same solutions in the parametric condition (3.22). Therefore, in the below discussions, the parametric condition (3.22) need not to be repeatedly verified, we can apply it directly. Respectively plugging (3.28), (3.29) and the parametric condition (3.22) into (3.2), we obtain two exact solutions of the generalized time-fractional KdV equation defined by Caputo differential operator as follows:

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + \sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_3\right) E_\alpha(\lambda t^\alpha), \tag{3.32}$$

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} - \sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_4\right)E_\alpha\left(\lambda t^\alpha\right). \tag{3.33}$$

These two solutions are general solutions under free conditions. Once, the boundary conditions and initial value conditions are determined as in Case 1, the special solution of equation (3.1) can be always determined, here we omit the parts of discussions. In the below process we will not discuss the problem on special solution of equation (3.1) and only discuss general solutions under free conditions.

Case 3. When $g > 0,\ \rho < 0$ and $h = 0$, there are two hyperbola orbits which are marked by black in Figure 1(**d**). Substituting these parametric conditions into (3.10), we obtain expression of the two line orbits as follows:

$$y = \pm\frac{\sqrt{v^2-g}}{\sqrt{-\rho}}. \tag{3.34}$$

Plugging (3.34) into the first equation $\frac{dv}{dx} = y$ of (3.7) and then integrating it, we get

$$v = \sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_5\right), \tag{3.35}$$

$$v = -\sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_6\right), \tag{3.36}$$

where $C_5,\ C_6$ are arbitrary constants. As in the Case 1 and Case 2, we obtain two general exact solutions of the generalized time-fractional KdV equation defined by Caputo differential operator as follows:

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + \sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_5\right)E_\alpha\left(\lambda t^\alpha\right), \tag{3.37}$$

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} - \sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_6\right)E_\alpha\left(\lambda t^\alpha\right). \tag{3.38}$$

Case 4. When $g > 0,\ \rho < 0$ and $h = h_{1,2} = \mp\frac{2g}{3}\sqrt{\frac{g}{3}}$, there are four orbits passing though the two saddle points $A_{1,2}\left(0,\ \pm\sqrt{\frac{g}{\rho}}\right)$ which are marked by blue and red in Figure 1(**d**). Substituting above conditions into (3.10), we obtain two expressions of the four orbits as follows:

$$y = \pm\frac{v-\sqrt{\frac{g}{3}}}{\sqrt{-\rho}}\sqrt{\frac{v+2\sqrt{\frac{g}{3}}}{v}}, \quad (v > 0) \tag{3.39}$$

and

$$y = \pm\frac{v+\sqrt{\frac{g}{3}}}{\sqrt{-\rho}}\sqrt{\frac{v-2\sqrt{\frac{g}{3}}}{v}}, \quad (v < 0). \tag{3.40}$$

Respectively plugging (3.39) and (3.40) into the first equation $\frac{dv}{dx} = y$ of (3.7) to integrate, we get

$$\int \frac{1}{v-\sqrt{\frac{g}{3}}}\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}dv = \pm\int\frac{1}{\sqrt{-\rho}}dx \tag{3.41}$$

and

$$\int \frac{1}{v+\sqrt{\frac{g}{3}}}\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}dv = \pm\int \frac{1}{\sqrt{-\rho}}dx. \tag{3.42}$$

Completing above integrals in (3.41) and (3.42), we obtain four solutions of implicit function form of Eq. (3.6) as follows:

$$\ln\left|\frac{\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}+1}{\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}-1}\right| + \frac{\sqrt{3}}{3}\ln\left|\frac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right| = \pm\frac{x}{\sqrt{-\rho}} + \tilde{C}_{7,8}, \tag{3.43}$$

$$\ln\left|\frac{\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}+1}{\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}-1}\right| + \frac{\sqrt{3}}{3}\ln\left|\frac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right| = \pm\frac{x}{\sqrt{-\rho}} + \tilde{C}_{9,10}, \tag{3.44}$$

where $\tilde{C}_{7,8,9,10}$ are arbitrary constants.

Note. Letting $\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}} = \phi$, the first integral in (3.41) can be reduced to

$$\int \frac{1}{v-\sqrt{\frac{g}{3}}}\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}dv = \int\left(\frac{1}{\phi+1} - \frac{1}{\phi-1} + \frac{1}{\sqrt{3}\phi-1} - \frac{1}{\sqrt{3}\phi+1}\right)d\phi.$$

By using the solutions (3.43), (3.44) and (3.2), (3.22), we obtain eight exact solution of parametric form of Eq. (3.1) as follow:

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \quad \left(0<v<\sqrt{\dfrac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}\right)\right] + C_7, \end{cases} \tag{3.45}$$

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \quad \left(0<v<\sqrt{\dfrac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}\right)\right] + C_8, \end{cases} \tag{3.46}$$

where v is a parameter and $0<v<\sqrt{\frac{g}{3}}$, the $C_{7,8}$ are arbitrary constants. The (3.45) and (3.46) are two bounded solutions

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \quad \left(v>\sqrt{\dfrac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_7, \end{cases} \tag{3.47}$$

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \quad \left(v>\sqrt{\dfrac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_8, \end{cases} \tag{3.48}$$

where v is a parameter and $v > \sqrt{\frac{g}{3}}$, the $C_{7,8}$ are arbitrary constants. The (3.47) and (3.48) are two unbounded solutions

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \qquad \left(-\sqrt{\dfrac{g}{3}} < v < 0\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}\right)\right] + C_9, \end{cases} \tag{3.49}$$

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \qquad \left(-\sqrt{\dfrac{g}{3}} < v < 0\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}\right)\right] + C_{10}, \end{cases} \tag{3.50}$$

where v is a parameter and $-\sqrt{\frac{g}{3}} < v < 0$, the $C_{9,10}$ are arbitrary constants. The (3.49) and (3.50) are two bounded solutions

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \qquad \left(v < -\sqrt{\dfrac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_9, \end{cases} \tag{3.51}$$

$$\begin{cases} u = \dfrac{3(\beta-\rho)}{2\kappa\rho} + vE_\alpha\left(\lambda t^\alpha\right), \qquad \left(v < -\sqrt{\dfrac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_{10}, \end{cases} \tag{3.52}$$

where v is a parameter and $v < -\sqrt{\frac{g}{3}}$, the $C_{9,10}$ are arbitrary constants. The (3.51) and (3.52) are two unbounded solutions.

Case 5. When $g < 0$, $\rho < 0$ and $h = 0$, there are two parabolic orbits which are marked by black in Figure 1(**e**). Substituting these parametric conditions into (3.10), we obtain expression of the two line orbits as follows:

$$y = \pm\frac{\sqrt{v^2 + (\sqrt{-g})^2}}{\sqrt{-\rho}}. \tag{3.53}$$

Plugging (3.53) into the first equation $\frac{dv}{dx} = y$ of (3.7) and then integrating it, we get

$$v = \sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{11}\right), \tag{3.54}$$

$$v = -\sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{12}\right), \tag{3.55}$$

where $C_{11},\ C_{12}$ are arbitrary constants. As in the Cases 1 and Case 2, we obtain two general exact solutions of the generalized time-fractional KdV equation defined by Caputo differential operator as follows:

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} + \sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{11}\right) E_\alpha\left(\lambda t^\alpha\right), \tag{3.56}$$

$$u(x,t) = \frac{3(\beta-\rho)}{2\kappa\rho} - \sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{12}\right) E_\alpha\left(\lambda t^\alpha\right). \tag{3.57}$$

In the other parametric conditions for $h \neq 0$, we have no way to obtain the exact solution of Eq. (3.1) by integral because the expressions of those orbits are too complex. But in practice applications, we can find the numerical solutions of Eq. (3.1) in those complex parametric conditions, and the types of the numerical solutions can be determined by the orbit types in the phase portraits.

In order to intuitively show the dynamic property of above solutions, as examples, under the integral constants $C_i = 0,\ (i = 3, 5, 8, 11)$, the 3D-graphs of the solutions (3.32), (3.37), (3.46) and (3.56) are illustrated, which are shown in Figure 2(**a**), (**b**), (**c**) and (**d**) respectively.

4. Exact solutions of Eq. (1.1) under Riemann-Liouville fractional differential operator

If $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^{RL}D_t^\alpha$ is Riemann-Liouville differential operator, then Eq. (1.1) can be rewritten as

$${}_0^{RL}D_t^\alpha u + u_x + \rho\, {}_0^{RL}D_t^\alpha u_{xx} + \beta u_{xxx} + \kappa u u_x + \frac{1}{3}\kappa\rho(u u_{xxx} + 2u_x u_{xx}) = 0, \tag{4.1}$$

where $0 < \alpha < 1,\ u = u(x,t),\ t > 0,\ x \in \mathbb{R}$, the parameters $\rho,\ \beta,\ \kappa$ are nonzero constants. We suppose that Eq. (3.1) has solution formed as follows:

$$u = v(x)[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right)], \tag{4.2}$$

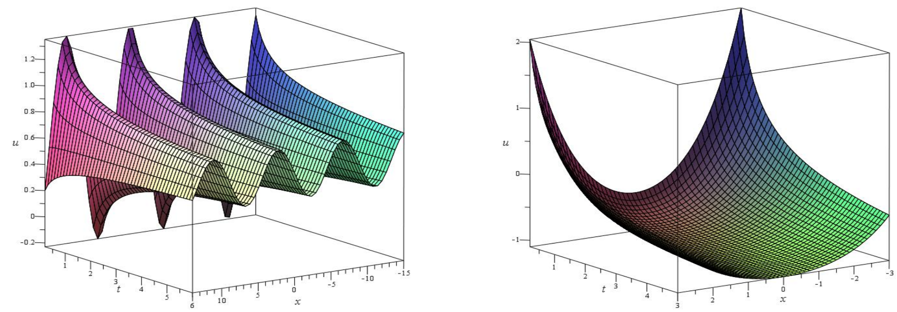
where $v = v(x)$ is function to be determined and λ is nonzero constant. Substituting (4.2) into (4.1), the equation can be reduced to

$$\begin{aligned} &\left[\lambda v + v_x + \lambda\rho v_{xx} + \beta v_{xxx}\right] t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right) \\ &+ \left[\kappa v v_x + \frac{1}{3}\kappa\rho v v_{xxx} + \frac{2}{3}\kappa\rho v_x v_{xx}\right]\left[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right)\right]^2 = 0. \end{aligned} \tag{4.3}$$

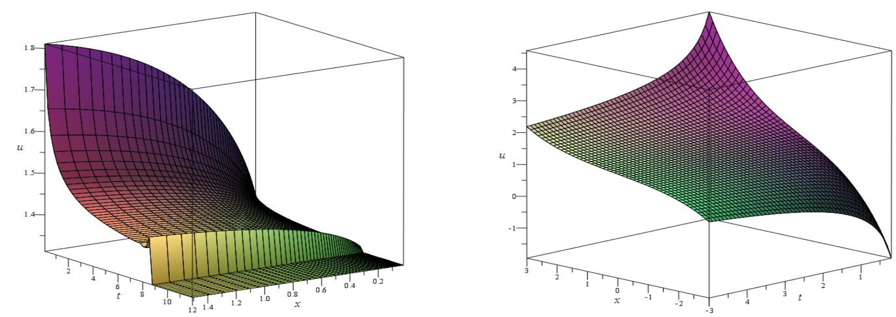
Letting the coefficients of Mittag-Leffler functions $t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right)$, $\left[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^\alpha\right)\right]^2$ equal to zero, it yields

$$\begin{cases} \lambda v + v_x + \lambda\rho v_{xx} + \beta v_{xxx} = 0, \\ v v_x + \frac{2}{3}\rho v_x v_{xx} + \frac{1}{3}\rho v v_{xxx} = 0. \end{cases} \tag{4.4}$$

Compared with the systems (4.4) and (3.4), it is easily find that the second equation in (4.4) is as same as the second equation in (3.4), just that their first equations are different.



(**a**) Solution (3.32): $g = 3, \rho = 2, \kappa = 4$ (**b**) Solution (3.37): $g = 3, \rho = -2, \kappa = 4$



(**c**) Solution (3.46): $g = 3, \rho = -2, \kappa = -4$ (**d**) Solution (3.56): $g = -3, \rho = -2, \kappa = -4$

Figure 2. The 3D-graphs of the profiles of four solutions: $\beta = 5, \lambda = -2, \alpha = 0.5$.

Integrating the second nonlinear ODE in (4.4), we obtain

$$\frac{1}{2}v^2 + \frac{1}{3}\rho v v_{xx} + \frac{1}{6}\rho v_x^2 = g_1, \tag{4.5}$$

where g_1 is an integral constant. Eq. (4.5) can be rewritten as

$$3v^2 + 2\rho v v_{xx} + \rho v_x^2 = g, \tag{4.6}$$

where g is an arbitrary constant. Obviously, Eq. (4.6) is identical to Eq. (3.6), so their solutions are also same. Thus, we can directly obtain exact solutions of Eq. (4.1) by using the solutions of Eq. (3.6). The concrete approach is that we substitute solutions of Eq. (3.6) into the first equation of (4.4) one by one so that we can find corresponding parametric condition as in Sec. 3. And then, substituting the obtained parametric condition and the corresponding solutions into (4.2), so that we can directly obtain the solutions of Eq. (4.1) without having to solve the Eq. (4.4) again.

(i) When $g = 0,\ \rho < 0$ and $h = 0$, Eq. (4.6) has the following two exact solutions as same to those of Eq. (3.6)

$$v = C_1 e^{\frac{x}{\sqrt{-\rho}}}, \tag{4.7}$$

$$v = C_2 e^{-\frac{x}{\sqrt{-\rho}}}. \tag{4.8}$$

Respectively, substituting (4.7) and (4.8) into the first equation $\lambda v + v_x + \lambda \rho v_{xx} + \beta v_{xxx} = 0$ in (4.4), it yields

$$\left(\frac{1}{\sqrt{-\rho}} - \frac{\beta}{\rho\sqrt{-\rho}}\right) C_1 e^{\frac{x}{\sqrt{-\rho}}} = 0, \tag{4.9}$$

$$-\left(\frac{1}{\sqrt{-\rho}} - \frac{\beta}{\rho\sqrt{-\rho}}\right) C_2 e^{-\frac{x}{\sqrt{-\rho}}} = 0. \tag{4.10}$$

Solving Eqs. (4.9) and (4.10), we can always obtain

$$\beta = \rho. \tag{4.11}$$

Thus, when $\beta = \rho < 0$, Eq. (4.1) has two exact solutions as follows:

$$u(x,t) = C_1 e^{\frac{x}{\sqrt{-\rho}}} [t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})], \tag{4.12}$$

$$u(x,t) = C_2 e^{-\frac{x}{\sqrt{-\rho}}} [t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})]. \tag{4.13}$$

(ii) When $g > 0,\ \rho > 0$ and $h = 0$, Eq. (4.6) has the following two periodic solutions as same to those of Eq. (3.6)

$$v = \sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_3\right), \tag{4.14}$$

$$v = -\sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_4\right). \tag{4.15}$$

Respectively, substituting (4.14) and (4.15) into the first equation $\lambda v + v_x + \lambda \rho v_{xx} + \beta v_{xxx} = 0$ in (4.4), it yields

$$\left(\frac{1}{\sqrt{\rho}} - \frac{\beta}{\rho\sqrt{\rho}}\right)\sqrt{g}\cos\left(\frac{x}{\sqrt{\rho}} + C_3\right) = 0, \tag{4.16}$$

$$-\left(\frac{1}{\sqrt{\rho}} - \frac{\beta}{\rho\sqrt{\rho}}\right)\sqrt{g}\cos\left(\frac{x}{\sqrt{\rho}} + C_4\right) = 0. \tag{4.17}$$

Solving Eqs. (4.16) and (4.17), we obtain the parametric condition of $\beta = \rho$ again. Also, this is not a coincidence, because the two differential equations in (4.4) have same solutions in the parametric condition (4.11). Therefore, in the below discussions, the parametric condition (4.11) need not to be repeatedly verified, we can apply it directly. Thus, when $\beta = \rho < 0$, Eq. (4.1) has two exact solutions as follows:

$$u(x,t) = \sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_3\right) [t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})], \tag{4.18}$$

$$u(x,t) = -\sqrt{g}\sin\left(\frac{x}{\sqrt{\rho}} + C_4\right) [t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})]. \tag{4.19}$$

(iii) When $g > 0,\ \rho < 0$ and $h = 0$, Eq. (4.6) has the following two exact solutions as same to those of Eq. (3.6)

$$v = \sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_5\right), \tag{4.20}$$

$$v = -\sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_6\right). \tag{4.21}$$

Thus, when $g > 0,\ \beta = \rho < 0$ and $h = 0$, Eq. (4.1) has two exact solutions as follows:

$$u(x,t) = \sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_5\right)[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)], \tag{4.22}$$

$$u(x,t) = -\sqrt{g}\cosh\left(\frac{x}{\sqrt{-\rho}} + C_6\right)[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)]. \tag{4.23}$$

(iv) When $g > 0,\ \rho < 0$ and $h = \mp\frac{2g}{3}\sqrt{\frac{g}{2}}$, Eq. (4.6) has the following four exact solutions of parametric form as same to those of Eq. (3.6)

$$\ln\left|\frac{\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}+1}{\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}-1}\right| + \frac{\sqrt{3}}{3}\ln\left|\frac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right| = \pm\frac{x}{\sqrt{-\rho}} + \tilde{C}_{7,8}, \tag{4.24}$$

$$\ln\left|\frac{\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}+1}{\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}-1}\right| + \frac{\sqrt{3}}{3}\ln\left|\frac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right| = \pm\frac{x}{\sqrt{-\rho}} + \tilde{C}_{9,10}, \tag{4.25}$$

where $\tilde{C}_{7,8,9,10}$ are arbitrary constants. Thus, when $g > 0,\ \rho < 0$ and $h = \mp\frac{2g}{3}\sqrt{\frac{g}{2}}$, Eq. (4.1) has four eight solutions of parametric form as follows:

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha), \quad \left(0 < v < \sqrt{\frac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\frac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \frac{\sqrt{3}}{3}\ln\left(\frac{1-\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}\right)\right] + C_7, \end{cases} \tag{4.26}$$

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha), \quad \left(0 < v < \sqrt{\frac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\frac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \frac{\sqrt{3}}{3}\ln\left(\frac{1-\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}}\right)\right] + C_8, \end{cases} \tag{4.27}$$

where v is a parameter and $0 < v < \sqrt{\frac{g}{3}}$, the $C_{7,8}$ are arbitrary constants. The (4.26) and (4.27) are two bounded solutions

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha), \quad \left(v > \sqrt{\frac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\frac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \frac{\sqrt{3}}{3}\ln\left(\frac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_7, \end{cases} \tag{4.28}$$

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \quad \left(v > \sqrt{\frac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v+2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v+2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_8, \end{cases} \tag{4.29}$$

where v is a parameter and $v > \sqrt{\frac{g}{3}}$, the $C_{7,8}$ are arbitrary constants. The (4.28) and (4.29) are two unbounded solutions

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \quad \left(-\sqrt{\frac{g}{3}} < v < 0\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}\right)\right] + C_9, \end{cases} \tag{4.30}$$

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \quad \left(-\sqrt{\frac{g}{3}} < v < 0\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{1-\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}{1+\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}}\right)\right] + C_{10}, \end{cases} \tag{4.31}$$

where v is a parameter and $-\sqrt{\frac{g}{3}} < v < 0$, the $C_{9,10}$ are arbitrary constants. The (4.30) and (4.31) are two bounded solutions

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \quad \left(v < -\sqrt{\frac{g}{3}}\right), \\ x = \sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_9, \end{cases} \tag{4.32}$$

$$\begin{cases} u = vt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \quad \left(v < -\sqrt{\frac{g}{3}}\right), \\ x = -\sqrt{-\rho}\left[\ln\left(\dfrac{1+\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}{1-\sqrt{\frac{v}{v-2\sqrt{\frac{g}{3}}}}}\right) + \dfrac{\sqrt{3}}{3}\ln\left(\dfrac{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}-1}{\sqrt{\frac{3v}{v-2\sqrt{\frac{g}{3}}}}+1}\right)\right] + C_{10}, \end{cases} \tag{4.33}$$

where v is a parameter and $v < -\sqrt{\frac{g}{3}}$, the $C_{9,10}$ are arbitrary constants. The (4.32) and (4.33) are two unbounded solutions.

(v) When $g < 0$, $\rho < 0$ and $h = 0$, Eq. (4.6) has the following two exact solutions as same to those of Eq. (3.6)

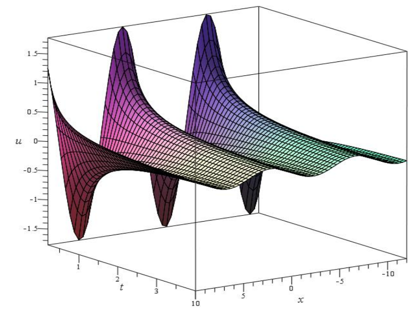
$$v = \sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{11}\right), \tag{4.34}$$

$$v = -\sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{12}\right). \tag{4.35}$$

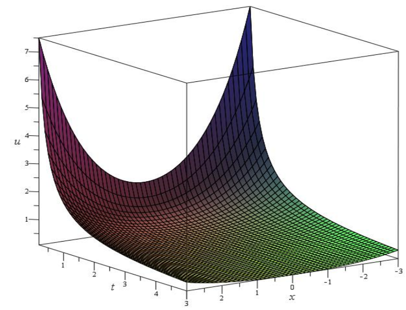
Thus, when $g < 0$, $\rho < 0$ and $h = 0$, Eq. (4.1) has two exact solutions as follows:

$$u(x,t) = \sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{11}\right)[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)], \tag{4.36}$$

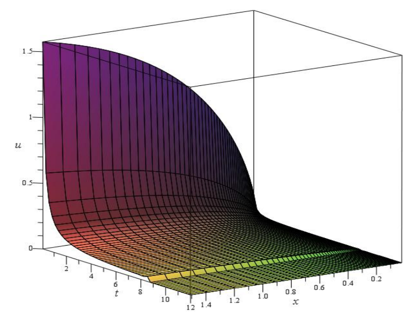
$$u(x,t) = -\sqrt{-g}\sinh\left(\frac{x}{\sqrt{-\rho}} + C_{12}\right)[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})]. \tag{4.37}$$



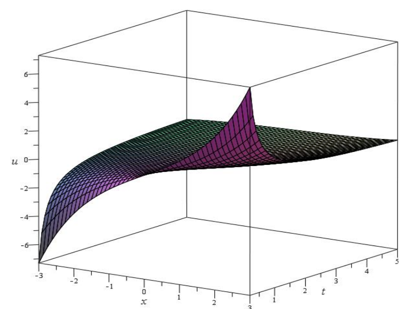
(**a**) Solution (4.18): $g = 3, \rho = 2$



(**b**) Solution (4.22): $g = 3, \rho = -2$



(**c**) Solution (4.27): $g = 3, \rho = -2$



(**d**) Solution (4.36): $g = -3, \rho = -2$

Figure 3. The 3D-graphs of the profiles of four solutions: $\lambda = -2, \alpha = 0.5$.

In order to intuitively show the dynamic property of above solutions, as examples, under the integral constants $C_i = 0$, $(i = 3, 5, 8, 11)$, the 3D-graphs of the solutions (4.18), (4.22), (4.27) and (4.36) are illustrated, which are shown in Figure 3(**a**), (**b**), (**c**) and (**d**) respectively.

Comparing every graph in Figure 2 and Figure 3, it is easy to find that the reduced speeds of amplitudes of these two kinds of solutions are different although their dynamic properties are quite similar. Obviously, according to the time t increase, the solution of the time-fractional KdV equation (1.1) under the definition of Riemann-Liouville derivative is decayed faster than the solution under the definition of Caputo derivative. This is also because the two-parameter Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$ in solutions converges faster than the one-parameter Mittag-Leffler function $E_{\alpha}(\lambda t^{\alpha})$ when $\lambda < 0$.

5. Conclusion

In this work, based on a modified separation method of variables and the dynamic system method, a combinational method is proposed. Investigation has shown that the modified separation method of variables has some advantages in reducing nonlinear time-fractional PDEs. As an example for the application of this combinational method, a generalized nonlinear time-fractional KdV equation is studied under the Riemann-Liouville and Caputo fractional differential operators, respectively.

Under the definition of Caputo fractional differential operator, taking the integral constant $h = 0$, we obtained eight explicit solution such as (3.23), (3.24), (3.32), (3.33), (3.37), (3.38), (3.56) and (3.57) of the generalized nonlinear time-fractional KdV equation. As examples, we obtain two special solutions such as (3.25) and (3.26) in two kinds of boundary conditions and initial value conditions. When $g > 0$, taking the integral constant $h = \mp\frac{2g}{3}\sqrt{\frac{g}{3}}$, we obtained eight exact solutions of parametric form such as (3.45), (3.46), (3.47), (3.48), (3.49), (3.50), (3.51) and (3.52) of the generalized nonlinear time-fractional KdV equation. It is not difficult to find that these solutions of parametric form are usually unable to be obtained by other methods such as invariant subspace method, which is an advantage of the dynamical system method. Of course, the invariant subspace method also has its own advantages. In fact, whether it is the modified separation method of variables or invariant subspace method, their ideas both originate from the concept of separating variables. The difference lies in that the modified separation method of variables assumes the time variables of solutions assumed structure as a certain fixed function, such as a Mittag-Leffler function or a power function, while the invariant subspace method assumes the space variables of solutions assumed structure as certain elementary functions, which are obtained through invariant subspaces. It's worth mentioning that some derivative methods of the invariant subspace method have also been produced, such as the combinations of the invariant subspace method with Lie symmetry analysis and other techniques. Through these approaches, some authors have investigated exact solutions of some nonlinear time-fractional PDEs including coupled time fractional PDEs and time-fractional differential-difference equations [33,34,42,45]. Similarly, there are new ideas for the development of the separation method of semi-fixed variables. Recently, combining with the extended separation method of semi-fixed variables and the mapping method of Riccati equation, a new approach for searching exact solutions on time-fractional PDEs is introduced in [16].

Under the definition of Riemann-Liouville fractional differential operator, when $\beta = \rho$ and the integral constant $h = 0$, we obtained eight explicit solution such as (4.12), (4.13), (4.18), (4.19), (4.22), (4.23), (4.36) and (4.37) of the generalized nonlinear time-fractional KdV equation. When $\beta = \rho$, $g > 0$ and the integral constant $h = \mp\frac{2g}{3}\sqrt{\frac{g}{3}}$, we obtained eight exact solutions of parametric form such as (4.26), (4.27), (4.28), (4.29), (4.30), (4.31), (4.32) and (4.33) of the generalized nonlinear time-fractional KdV equation. Obviously, solutions of the generalized nonlinear time-fractional KdV equation (1.1) is not limited by the condition $\beta = \rho$ under the definition of Caputo fractional differential operator. This also fully shows that the difficulty on solving a nonlinear time-fractional PDE of Riemann-Liouville type is much greater than that for a nonlinear time-fractional PDE of Caputo type.

In the results mentioned above, we did not obtain any soliton solutions of fractional generalized KdV equation (1.1) from first to last. However, when $\alpha = 1$, there are many soliton solutions for the integer-order generalized KdV equation (1.2). Does this mean that there are no soliton solutions when an integer-order soliton equation is changed into a nonlinear time-fractional PDE? This is a fascinating question, but in the present way, we cannot answer it with certainty, we have to leave it to those readers who are interested.

A. Appendix

A.1. The definitions of Riemann-Liouville fractional derivative and Caputo fractional derivative

Definition A.1. ([32]) If $f(t)$ is a continuous function at the interval $[a,t)$, then its arbitrary α-order (fractional) derivative of Riemann-Liouville type is defined by

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n-1 \le \alpha < n,\ t > 0. $$

Definition A.2. ([6]) If $f(t)$ is a n-order smooth function at the interval $[a,t)$, then its arbitrary α-order (fractional) derivative of Caputo type is defined by

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha \le n,\ t > 0. $$

Obviously, the above two definitions are only different in their operation order, the Riemann-Liouville fractional differential operator is the integral operator at first and the differential operator in the later, but the Caputo fractional differential operator is just the opposite.

A.2. The formulas of fractional derivatives of Mittag-Leffler functions and power function under the Riemann-Liouville and Caputo differential operators

$$ {}_0^{RL}D_t^{\alpha} \left[t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)\right] = \lambda t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \tag{A.1} $$

$$ {}_0^{RL}D_t^{\alpha}\ t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > -1, \tag{A.2} $$

$$ {}_0^{C}D_t^{\alpha}\ E_{\alpha}\left(\lambda t^{\alpha}\right) = \lambda E_{\alpha}\left(\lambda t^{\alpha}\right), \tag{A.3} $$

$$ {}_0^{C}D_t^{\alpha}\ t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \tag{A.4} $$

where $0 < \alpha < 1,\ t > 0$, the $E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)$ is called two-parameter Mittag-Leffler function, the $E_{\alpha}\left(\lambda t^{\alpha}\right)$ is called one-parameter Mittag-Leffler function, which are defined by

$$ E_{\alpha,\beta}\left(\lambda t^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(\alpha k+\beta)}, \qquad E_{\alpha}\left(\lambda t^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(\alpha k+1)}. $$

Especially, when $\beta = \alpha$, the $E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)$ becomes $E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)$.

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