Journal of Applied Analysis and Computation
Volume 15, Number 6, December 2025, 3603–3628
Website:http://www.jaac-online.com
DOI:10.11948/20240557

A STUDY ON FRACTIONAL LANE–EMDEN EQUATION OF ASTROPHYSICS AS THERMAL EXPLOSIONS USING CHEBYSHEV WAVELET METHOD

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Abstract Thermal explosions in astrophysical systems are crucial for understanding stellar evolution and dynamics. The fractional Lane–Emden equation is a key mathematical tool for modeling these explosions, providing insight into the thermodynamic processes within stellar interiors. Knowledge of such equations is significant because they quantify the temperature field and transport of energy within self-gravitating systems. A notable challenge in solving these equations arises from the singularity at $x = 0$, which requires careful numerical handling. Standard analytical methods may not give exact solutions to fractional-order models, necessitating effective numerical solutions. In this paper, we use the second-kind Chebyshev wavelet approximation to solve the fractional Lane–Emden equation effectively. This method utilizes orthogonality and the wavelet operational matrices to convert the original problem into algebraic equations, significantly reducing the computational burden. Numerical experiments confirm that the presented technique is not only efficient and precise but also has the least computational cost compared to traditional numerical methods. Therefore, it makes it highly suitable for solving other complex fractional models in astrophysics.

Keywords Fractional derivative, fractional integral, operational matrix, Lane−Emden differential equations, fractional Lane−Emden differential equations, second kind Chebyshev wavelets.

MSC(2010) 34A08, 34A45, 65T60.

1. Introduction

Differential equations are widely recognized as highly effective mathematical problems in the physical science and engineering sectors. The Lane–Emden differential equation belongs to a part of the dimensionless Poisson equation, which is commonly employed to determine the gravitational potential of stars. Jonathan Homer Lane first attempted to investigate the Lane–Emden differential equation, and then Robert Emden continued to expand his work. Their solutions provided a detail in

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the thermal behavior of a spherical gas cloud, which depends on the mutual gravitational attraction between the molecules within the cloud and follows the classical thermodynamic principles [59]. It is an important differential equation with use in many arenas of mathematics; flexible expressions are used to model a wide variety of dynamics and phenomena. The Lane–Emden equation versatility allows one to accurately solve the important number of astrophysical and cosmological problems. An equation is widely applied in self-gravitating polytropic sphere models and isothermal gas distributions; many authors concentrated their efforts on the equation itself [2, 3, 6, 26, 38, 39, 47, 49, 50, 54, 55, 62–64, 68, 71, 76, 79–81, 83, 85]. The Lane–Emden equation is both of linear and non-linear kinds, and in this form it just defines the problem regarding the equilibrium density distribution within such systems. The Lane–Emden problem poses numerical challenges due to the singularity at the origin.

Fractional calculus theory allows the differentiation and integration of functions to arbitrary fractional values, whereby the area of study extends beyond integer orders. Fractional differential equations garnered significant attention in the research community due to their ability to accurately represent phenomena in a wide range of scientific and engineering disciplines [4, 6, 11, 16, 19, 30, 36, 40, 43, 46, 53, 72, 79, 80]. Many researchers studied various methods to attain appropriate solutions to fractional-order differential equations [16, 28, 29, 31, 72, 76]. The author developed a fractional differential equation model of population growth in [32]. In [8] studied Ralston's cubic convergence in population models, but their subsequent work [7] provided numerical methods generalized towards population models of nuclear decay. P. K. Shaw et.al [66] studied a curative and preventive treatment fractional plant disease model. Futher they [67] investigated two quadratic schemes on world population growth models of fractional differentiation. The fractional Lane–Emden differential equations incorporate fractional derivatives into the classical Lane–Emden equation. They are used usually in the study of polytropic gas spheres. Owing to their far-reaching applications in modelling the thermal behavior of a spherical gas cloud under the mutual attraction influence of its molecules, they are governed by classical thermodynamic laws. These provide some insight into how pressure and density are connected in gravitationally self-bound, symmetrically spherical clouds of gas within hydrostatic equilibrium. The inclusion of fractional derivatives makes the equation modeling behavior of such gas clouds significantly more flexible, rendering deeper insight into the structure and evolution of astrophysical systems. However, finding an analytical solution to the fractional Lane–Emden type differential equation proves quite challenging.

In recent decades, researchers have proposed various strategies to approximate solutions for equations of these types. These methods include the algorithm of the operational matrix based on Chebyshev wavelets [17], the way of Adomian decomposition [12, 58, 77–79], the modified Legendre–Spectral approach [1], Jacobi–Gauss collocation technique [10], Hermite functions collocation method [55], reproducing kernel methodology [5], the method of ultra-spherical wavelets [85], the modified differential transform scheme [38], the technique of Legendre multi-wavelets [63], the Legendre pseudo spectral method [56], Legendre wavelet and quasilinearization technique [24] the strategy of homotopy pertubation [35, 45, 81], Orthonormal Bernoulli's polynomials [62], Numerical simulation [20] and the Variational Iteration Method (VIM) [21, 44, 82].

Numerous researchers have conducted an extensive investigation on Chebyshev wavelets and other numerical techniques for solving differential equations. In [9] developed a Chebyshev wavelet operational matrix method for solving differential equations by wavelet basis functions efficiently. Chebyshev finite difference method for integro-differential equations have been studied in [15]. M. A. Iqbal et.al [25] modified the Chebyshev wavelet to deal with fractional delay differential equations. The authors [34, 74] employed wavelet-based techniques to solve Ricatti and non-linear Volterra integro-differential equations. In [41, 51] they considered Lane–Emden–type problems for self-gravitating gas spheres, analytically. The Haar wavelet collocation approaches for singular differential and Lane–Emden–type equations developed in [69, 70]. In [86, 87] the authors solved convection-diffusion and fractional integro-differential equations using second-kind Chebyshev wavelets. The one-dimensional adaptive grid generation technique to improve numerical solutions by enhancing grid point distribution have been studied in [27].

This paper discusses the applications of second-kind Chebyshev wavelets to find numerical solutions for fractional-order Lane–Emden–type equations. Researchers make use of orthogonal wavelets, known as Chebyshev wavelets in numerical techniques and approximation theory because of their exceptional approximation properties. Wavelet based methods have played a significant role in solving fractional order differential equations over the recent past (see [17,23,26,33,36,48,63,84,85]). An application of wavelet approximation in solving the fractional order Lane–Emden type equations is established in [26]. In [65], they used the Chebyshev wavelet operational matrix to figure out a solution for the Lane–Emden differential equations. Furthermore, in [49, 50], the authors proposed the Chebyshev wavelet finite difference method for solving these equations. In [37] authors solve the fractional Lane–Emden differential equations using the one-layer Chebyshev wavelet Neural network method. The fractional Lane−Emden differential equations are solved using the one−layer Chebyshev wavelet Neural network method in [37]. The proposed method involves the transformation of the fractional order Lane−Emden type differential equation into a system of algebraic equations using second kind Chebyshev wavelets. This transformed system can then be solved using standard numerical techniques.

This paper examines the fractional order Lane–Emden type differential equation of the form [37, 52, 57, 60, 61]. The differential equation is given by:

$$D_t^{\delta\delta} x(t) + \frac{K}{t^\delta} D_t^{\delta} x(t) + g(t, x) = f(t), \quad t > 0, \quad 0 < \delta \le 1 \tag{1.1}$$

with initial conditions: $x(0) = x_0$, and $x'(0) = x_1$, where x_0, x_1 and K are constants, $g(t, x)$ is a continuous real-valued function and $f(t) \in C[0, 1)$. For $K = 2$ and $\delta = 1$ the equation reduces to the classical Lane–Emden type equations. The present study examines a specific operational matrix for the Riemann−Liouville fractional order integral operator, thereby providing the right approach to precisely determine its components. Our solution will provide an accurate and enhanced efficiency of computing pertaining to such wavelet-based approaches.

In [42] the method of solution starts with transforming fractional differential equations to linear equations by applying the quasi-linearization technique. The process simplifies the nonlinear terms such that a linear equation can approximate the problem. Subsequent to the transformation, the authors apply the Chebyshev wavelet collocation method to discretize the problem and calculate numerical solutions.

On the other hand, our suggested technique takes a different route by converting the given fractional differential equation into an algebraic system directly, which could either be linear or nonlinear in nature based on the nature of the original problem. Rather than using quasi-linearization, we take a collocation approach at given points to discretize the problem. After transforming the system into algebraic form, researchers apply numerical solutions–e.g., Newton's method for nonlinear and direct solvers for linear to achieve precise and effective numerical solutions.There is no need for quasi-linearization as our method has greater freedom in treating nonlinearity. Therefore, it is a more effective process for solving difficult fractional differential equations with high degrees of accuracy.

The structure of the paper involves the following sections: Section 2 discusses fundamentals of fractional calculus and the Sumudu transform. Section 3 contains an approximation of the functions employing the second kind Chebyshev wavelets. Section 4 deals with the creation of an operational matrix for the Riemann−Liouville fractional integral operator. The solutions to fractional Lane–Emden differential equation are present in Section 5. Section 6 focuses on the convergence method. Section 7 includes numerical examples and discussions. At last, Section 8 deals with the outcomes of the study.

2. Preliminaries

The Caputo-Fabrizio and Atangana-Baleanu derivatives with non-singular kernels and exponential decay are usable for complex systems but less manageable in conventional astrophysical applications. The Hadamard derivative, while dominant in logarithmic scale problems, is not standardly used for power-law behavior problems, which is the nature of Lane–Emden equations. The Lane–Emden equation models astrophysical phenomena with singularities at $x = 0$ but the symmetric nature of the Riesz derivative does not naturally accommodate such singular initial conditions. Similarly, though the Fractal-Fractional derivative has a commendable recent history, it is computationally demanding and does not have exhaustive theoretical testing in astrophysical scenarios. Therefore, the application of Caputo and Riemann-Liouville derivatives presents a balance of mathematical tractability and physical relevance and is the most desirable choice for the solution of the fractional Lane–Emden equation in thermal explosion theory.

Here, we cover the essential definitions pertaining to fractional derivatives and integrals.

Definition 2.1. The Riemann−Liouville (R−L) fractional integral operator of fractional order $\delta > 0$ with respect to t of a function $f(t)$ is defined by [57]

$$I^{\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} f(\tau)\, d\tau, \tag{2.1}$$

where $i - 1 < \delta < i$, $i \in \mathbb{N}$.

Definition 2.2. The Riemann−Liouville fractional derivative of fractional order δ of a function $f(t)$ with respect to t is of the form [57]

$$\frac{d^{\delta}}{dt^{\delta}} f(t) = \frac{1}{\Gamma(i-\delta)} \frac{d^i}{dt^i} \left(\int_0^t (t-\tau)^{i-\delta-1} f(\tau)\, d\tau \right), \tag{2.2}$$

where $i-1 < \delta < i$, $i \in \mathbb{N}$ and the function $f(t)$ has absolutely continuous derivatives upto order $i-1$.

Definition 2.3. The Caputo fractional derivative of fractional order δ with respect to t of a function $f(t)$ is defined as [57]

$$\frac{d^\delta}{dt^\delta} f(t) = \frac{1}{\Gamma(i-\delta)} \left(\int_0^t (t-\tau)^{i-\delta-1} \frac{d^i}{d\tau^i} f(\tau)\, d\tau \right), \tag{2.3}$$

where $i-1 < \delta < i$, $i \in \mathbb{N}$ and the function $f(t)$ has absolutely continuous derivatives upto order $i-1$.

Some properties of fractional Caputo derivative and fractional R-L integral are:

$$D^\delta I^\delta f(t) = f(t), \tag{2.4}$$

$$I^\delta D^\delta f(t) = f(t) - \sum_{k=0}^{i-1} f^{(k)}(0^+) t^k, \quad \delta > 0, \quad i-1 < \delta < i, \tag{2.5}$$

$$I^\delta t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\delta)} t^{\delta+\gamma}, \tag{2.6}$$

$$\frac{d^\delta}{dt^\delta} t^\gamma = \begin{cases} \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\delta)} t^{\gamma-\delta}, & \text{if } n-1 < \delta < n,\ \gamma > n-1,\ \gamma \in \mathbb{R}, \\ 0, & \text{if } n-1 < \delta < n,\ \gamma \leq n-1,\ \gamma \in \mathbb{R}. \end{cases} \tag{2.7}$$

Sumudu Transform. The Sumudu transform simplifies the process of solving linear differential equations by converting them into algebraic equations in the transformed (u) domain, similar to the Laplace transform. However, it retains the physical significance of variables, making it easier to interpret solutions. Unlike the Laplace transform, the transformed variables in the u -domain are treated as replicas of the original function. This method retains the physical significance of variables, unlike the Laplace transform that treats variables as dummies. The Sumudu transform simplifies transfer function calculations. Unlike the Laplace transform, the unit step function transforms to unity, making system response analysis more intuitive. Useful in analyzing block diagrams while maintaining unit consistency. The Sumudu transform is useful for applied mathematics and control engineering. It offers an alternative to the Laplace transform with better visualization and unit consistency.

If $f(t)$ is of exponential order then the Sumudu transform of the function $f(t)$ is given by [75]

$$S\{f(t)\} = T(u) = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t)\, dt, \quad t > 0 \tag{2.8}$$

and if $T(u) = S\{f(t)\}$ is the Sumudu transform of $f(t)$, then the inverse Sumudu transform of $S\{f(t)\}$ is given by: [75]

$$\begin{aligned} S^{-1}\{T(u)\} &= f(t) \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{tu} \frac{T\left(\frac{1}{u}\right)}{u}\, du \\ &= \sum \operatorname{Res}\left[\frac{e^{tu} T(u)}{u}\right] \quad \text{where } \operatorname{Re}(u) < a. \end{aligned}$$

The Sumudu transform of the convolution of two functions $f(t)$ and $g(t)$ is given by

$$S(f(t) \star g(t)) = u\, S[f(t)]\, S[g(t)].$$

The relation between Sumudu and Laplace transforms is given by

$$F\left(\frac{1}{u}\right) = uT(u),$$

where $F(.)$ is the Laplace transformation of $f(t)$ and $T(.)$ is Sumudu transformation of $f(t)$.

3. The second kind Chebyshev wavelet

Researchers formulate wavelets by means of dilating and translating a single function $\phi(t)$, frequently mentioned as the mother wavelet. This results in a family of continuous wavelets that change with the translation and dilation parameters.

The family of continuous wavelets that follows functions as

$$\phi_{ab}(t) = |a|^{-\frac{1}{2}} \phi\left(\frac{t-b}{a}\right) \quad \text{where } a, b \in \mathbb{R},\ a \neq 0. \tag{3.1}$$

If we restrict the parameters a and b to only discreet values, as

$$a = a_0^{-k}, \quad b = nb_0 a_0^{-k}, \quad a_0 > 1, \quad b_0 > 0.$$

The set of discrete wavelets that we possess is presented below

$$\phi_{kn}(t) = |a_0|^{\frac{k}{2}} \phi(a_0^k x - nb_0) \quad \text{where } k, n \in \mathbb{Z}. \tag{3.2}$$

The wavelet basis for $L^2(\mathbb{R})$ is denoted as $\phi(t)$. When $a_0 = 2$ and $b_0 = 1$, the function $\phi_{kn}(x)$ forms an orthonormal basis.

For the interval [0,1) the second kind Chebyshev wavelets can be characterized as follows [37,48]

$$\phi_{pq}(t) = \begin{cases} \sqrt{\frac{2}{\pi}}\, 2^{\frac{k}{2}}\, U_q\left(2^k t - 2p + 1\right), & t \in \left[\frac{p-1}{2^{k-1}}, \frac{p}{2^{k-1}}\right), \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

where $p = 1, 2, \ldots, 2^{k-1}$ where $q = 0, 1, 2, \ldots, M-1$, and k and M are fixed positive integers, $U_q(t)$ signify the second kind Chebyshev polynomials of degree q, which are defined on the interval $[-1, 1]$ as follows:

$$U_q(t) = \sum_{l=0}^{q} \sum_{n=0}^{l} \frac{(-1)^{l+n} 2^l (q+l+1)!l!}{n!(l-n)!(q-l)!(2l+1)!} t^n. \tag{3.4}$$

According to the weight function $w_p(t)$, The second kind of Chebyshev wavelets functions are orthogonal, where the weight function $w_p(t)$ is defined by:

$$w_p(t) = \begin{cases} \sqrt{1 - (2^k t - 2p + 1)^2}, & \text{if } t \in \left[\frac{p-1}{2^{k-1}}, \frac{p}{2^{k-1}}\right), \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Further more, the Hilbert space $L^2_\omega[0,1)$ is defined as

$$L^2_\omega[0,1) = \left\{ f(t) \;\middle|\; \int_0^1 |f(t)|^2 \omega(t)\, dt < \infty \right\}.$$

Chebyshev wavelet method. The second kind Chebyshev wavelets can be used to expand a function $x(t) \in L^2_\omega[0,1)$ as illustrated below:

$$x(t) = \sum_{p=1}^{\infty} \sum_{q=0}^{\infty} x_{pq} \phi_{nm}(t), \tag{3.6}$$

where

$$x_{pq} = \langle x(t), \phi_{pq}(t) \rangle_{w_p} = \int_0^1 x(t) \phi_{pq}(t) \omega_p(t)\, dt. \tag{3.7}$$

We approximate by truncating the infinite series as shown below.

$$x(t) \approx \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} x_{pq} \phi_{pq}(t) = \; X^T \Phi(t). \tag{3.8}$$

The column vectors X and $\phi(t)$ are $2^{k-1}M \times 1$ are given by

$$\mathbf{X} = \begin{bmatrix} x_{10}, x_{11}, \ldots, x_{1(M-1)}, x_{20}, x_{21}, \ldots, x_{2(M-1)}, \cdots \\ x_{2^{k-1}0}, x_{2^{k-1}1}, \ldots, x_{2^{k-1}(M-1)} \end{bmatrix}^T, \tag{3.9}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} \phi_{10}, \phi_{11}, \ldots, \phi_{1(M-1)}, \phi_{20}, \phi_{21}, \ldots, \phi_{2(M-1)}, \cdots \\ \phi_{2^{k-1}0}, \phi_{2^{k-1}1}, \ldots, \phi_{2^{k-1}(M-1)} \end{bmatrix}^T. \tag{3.10}$$

4. The fractional integral of the second kind Chebyshev wavelet

Here, the researchers use the second kind Chebyshev polynomials to develop a fractional integral operational matrix for second kind Chebyshev wavelets in the Riemann-Liouville sense. As a vital factor, this operational matrix deals with the time fractional differential equations.

In [39] they finding fractional integral operational matrix using Laplace transform, but here we used Sumudu transform. Many authors utilize the Laplace transform to obtain a fractional integral operational matrix. In contrast to the Laplace transform, where function multiplication is equivalent to convolution, the Sumudu transform provides more straightforward manipulation for multiplication operations with less computational complexity. The Sumudu transform preserves the differentiation and integration properties for easy calculation, which is beneficial in the derivation of an integral operational matrix for wavelet-based approaches.

Theorem 4.1. *Let $\phi_{pq}(t)$ be the second kind Chebyshev wavelet defined in the interval $[0,1)$ with the compact support $\left[\frac{p-1}{2^k}, \frac{p}{2^k}\right)$. Then the fractional integral of* *the second kind Chebyshev wavelet is given by:*

$$I^{\delta}\phi_{pq}(t)=\begin{cases}0, \quad t<\dfrac{p-1}{2^{k-1}},\\ \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left[\displaystyle\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}\frac{2^l l!(l+q+1)!(-1)^{l+m+n}2^{k(n-m)}}{(2l+1)!m!(q-l)!(l-n)!\Gamma(\delta-m+n+1)}\right.\\ \left.\times\left(t-\dfrac{2p-2}{2^k}\right)^{\delta-m+n}\right], \quad \dfrac{p-1}{2^{k-1}}\leq t<\dfrac{p}{2^{k-1}},\\ \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left[\displaystyle\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}\frac{2^l l!(-1)^{l+n}(l+q+1)!2^{k(n-m)}}{(2l+1)!m!(q-l)!(l-n)!\Gamma(\delta-m+n+1)}\right.\\ \left.\times\left((-1)^m\left(t-\dfrac{2p-2}{2^k}\right)^{\delta-m+n}-\left(t-\dfrac{2p}{2^k}\right)^{\delta-m+n}\right)\right], \quad t\geq\dfrac{p}{2^{k-1}}.\end{cases}$$

Proof. The second kind Chebyshev polynomials is

$$U_q(t)=\sum_{l=0}^{q}\sum_{n=0}^{l}\frac{(-1)^{l+n}2^l(q+l+1)!l!}{n!(l-n)!(q-l)!(2l+1)!}t^n.$$

By using the definition of second kind Chebyshev wavelets and unit step function $H_d(t)$

$$\phi_{pq}(t)=\sqrt{\frac{2}{\pi}}\,2^{k/2}\left(H_{\frac{2p-2}{2^k}}(t)\,U_q(2^kt-2p+1)-H_{\frac{2p}{2^k}}(t)\,U_q(2^kt-2p+1)\right),$$

where $p=1,2,\ldots,2^{k-1}$, $q=0,1,2,\ldots,M-1$ and $H_d(t)$ is defined as:

$$H_d(t)=\begin{cases}1, & t\geq d,\\ 0, & t<d.\end{cases}$$

By taking Sumudu transform on both sides the following is derived

$$\begin{aligned}S(\phi_{pq}(t))=&\sqrt{\frac{2}{\pi}}\,2^{k/2}\,S\left\{H_{\frac{2p-2}{2^k}}(t)\,U_q\left(2^kt-2p+1\right)-H_{\frac{2p}{2^k}}(t)\,U_q\left(2^kt-2p+1\right)\right\}\\ =&\sqrt{\frac{2}{\pi}}\,2^{k/2}\,S\left\{H_{\frac{2p-2}{2^k}}(t)\,U_q\left(2^k\left(t-\frac{2p-2}{2^k}\right)-1\right)\right.\\ &\left.-H_{\frac{2p}{2^k}}(t)\,U_q\left(2^k\left(t-\frac{2p}{2^k}\right)+1\right)\right\}\\ =&\sqrt{\frac{2}{\pi}}\,2^{k/2}\left[e^{-\frac{2p-2}{2^ku}}S\left(U_q\left(2^kt-1\right)\right)-e^{-\frac{2p}{2^ku}}S\left(U_q\left(2^kt+1\right)\right)\right].\end{aligned}$$

Using the general form of second kind Chebyshev Polynomial

$$\begin{aligned}=&\sqrt{\frac{2}{\pi}}2^{\frac{k}{2}}\left[e^{-\frac{2p-2}{2^ku}}S\left(\sum_{l=0}^{q}\sum_{n=0}^{l}\frac{(-1)^{l+n}2^l(q+l+1)!l!}{n!(l-n)!(q-l)!(2l+1)!}(2^kt-1)^n\right)\right.\\ &\left.-\,e^{-\frac{2p}{2^ku}}S\left(\sum_{l=0}^{q}\sum_{n=0}^{l}\frac{(-1)^{l+n}2^l(q+l+1)!l!}{n!(l-n)!(q-l)!(2l+1)!}(2^kt+1)^n\right)\right]\end{aligned}$$

where

$$
\begin{aligned}
T_{q,l,n} =&\frac{(-1)^{l+n}\, 2^{l}\, (q+l+1)!\, l!}{n!\, (l-n)!\, (q-l)!\, (2l+1)!} \\
=&\sqrt{\frac{2}{\pi}}\, 2^{k/2} \left[e^{-\frac{2p-2}{2^k u}} S\left(\sum_{l=0}^{q} \sum_{n=0}^{l} T_{q,l,n} (2^k t-1)^n \right) \right. \\
&\left. -e^{-\frac{2p}{2^k u}} S\left(\sum_{i=0}^{q} \sum_{n=0}^{l} T_{q,l,n} (2^k t+1)^n \right) \right] \\
=&\sqrt{\frac{2}{\pi}}\, 2^{k/2} \left[e^{-\frac{2p-2}{2^k u}} S\left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{r} T_{q,l,n} \frac{n!}{m!(n-m)!} (-1)^m 2^{k(n-m)} t^{n-m} \right) \right. \\
&\left. -e^{-\frac{2p}{2^k u}} S\left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!(n-m)!} 2^{k(n-m)} t^{n-m} \right) \right] \\
=&\sqrt{\frac{2}{\pi}}\, 2^{k/2} \left[e^{-\frac{2p-2}{2^k u}} \left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!} (-1)^m 2^{k(n-m)} u^{n-m} \right) \right. \\
&\left. -e^{-\frac{2p}{2^k u}} \left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!} 2^{k(n-m)} u^{n-m} \right) \right].
\end{aligned}
$$

With the help of Riemann–Liouville fractional integral order δ

$$
I^{\delta} f(t) = \frac{1}{\Gamma(\delta)} t^{\delta-1} \star f(t).
$$

Here $\star$ represents convolution between two function, and we know that

$$
\begin{aligned}
&S\left[\frac{t^{\delta-1}}{\Gamma(\delta)} \right] = u^{\delta-1}, \\
&\quad S\left[I^{\delta} \phi_{nm}(t) \right] \\
&= u\, S\left[\frac{t^{\delta-1}}{\Gamma(\delta)} \right] S\left[\phi_{nm}(t) \right] \\
&= \sqrt{\frac{2}{\pi}}\, 2^{k/2} \left[e^{-\frac{2p-2}{2^k u}} \left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!} (-1)^m 2^{k(n-m)} u^{n-m+\delta} \right) \right. \\
&\quad \left. -e^{-\frac{2p}{2^k u}} \left(\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!} 2^{k(n-m)} u^{n-m+\delta} \right) \right].
\end{aligned}
$$

By taking the inverse Sumudu transform,

$$
\begin{aligned}
&I^{\delta}(\phi_{pq}(t)) \\
=&\sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} \left[\sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{l} T_{q,l,n} \frac{n!}{m!} (-1)^m 2^{k(n-m)} S^{-1} \left(e^{-\frac{2p-2}{2^k u}} u^{n-m+\delta} \right) \right. \\
&\left. - \sum_{l=0}^{q} \sum_{n=0}^{l} \sum_{m=0}^{n} T_{q,l,n} \frac{n!}{m!} 2^{k(n-m)} S^{-1} \left(e^{-\frac{2p}{2^k u}} u^{n-m+\delta} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&=\sqrt{\frac{2}{\pi}}2^{\frac{k}{2}}\left[\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}T_{q,l,n}\frac{n!}{m!}(-1)^{m}2^{k(n-m)}\frac{H_{(\frac{2p-2}{2^k})}(t)(t-\frac{2p-2}{2^k})^{n-m+\delta}}{\Gamma(n-m+\delta+1)}\right.\\
&\quad\left.-\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}T_{q,l,n}\frac{n!}{m!}2^{k(n-m)}\frac{H_{(\frac{2p}{2^k})}(t)(t-\frac{2p}{2^k})^{n-m+\delta}}{\Gamma(n-m+\delta+1)}\right]\\
&=\begin{cases}
0, & t<\dfrac{p-1}{2^{k-1}},\\[2ex]
\dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left[\displaystyle\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}T_{q,l,n}\frac{n!}{m!}(-1)^{m}2^{k(n-m)}\right.\\
\quad\left.\times\dfrac{(t-\frac{2p-2}{2^k})^{n-m+\delta}}{\Gamma(n-m+\delta+1)}\right], & \dfrac{p-1}{2^{k-1}}\le t<\dfrac{p}{2^{k-1}},\\[2ex]
\dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left[\displaystyle\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}T_{q,l,n}\frac{n!}{m!}(-1)^{m}2^{k(n-m)}\frac{(t-\frac{2p-2}{2^k})^{n-m+\delta}}{\Gamma(n-m+\delta+1)}\right.\\
\left.-\displaystyle\sum_{l=0}^{q}\sum_{n=0}^{l}\sum_{m=0}^{n}T_{q,l,n}\frac{n!}{m!}2^{k(n-m)}\frac{(t-\frac{2p}{2^k})^{n-m+\delta}}{\Gamma(n-m+\delta+1)}\right], & t\ge\dfrac{p}{2^{k-1}},
\end{cases}
\end{aligned}
$$

where

$$T_{q,l,n}=\frac{(-1)^{l+n}\,2^{l}\,(q+l+1)!\,l!}{n!\,(l-n)!\,(q-l)!\,(2l+1)!}.$$

□

5. Solution of the fractional Lane–Emden differential equations

The second kind Chebyshev wavelets are utilized to discover an approximate solution to a class of fractional Lane–Emden differential equations (1.1).

To solve this problem, we assume:

$$D_t^{\delta\delta}x(t)\approx\sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1}x_{pq}\phi_{pq}(t)=C^{T}\Phi(t). \tag{5.1}$$

Equations (3.9) and (3.10) define C and $\Phi(t)$ as $2^{k-1}M\times 1$ column vectors, respectively.

Integrating Eq. (5.1) with the Riemann-Liouville fractional integral operator yields

$$x(t)=x(0)+tx'(0)+\sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1}x_{pq}I^{\delta\delta}\phi_{pq}(t), \tag{5.2}$$

$$D_t^{\delta}x(t)=\sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1}x_{pq}\left(I^{\delta}\phi_{pq}(t)\right)+\frac{x'(0)\,t^{1-\delta}}{\Gamma(2-\delta)}. \tag{5.3}$$

The Equation (1.1) implies

$$\sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} x_{pq}\phi_{pq}(t) + \frac{K}{t^\delta}\sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} x_{pq} I^\delta(\phi_{pq}(t))$$
$$+ x'(0)\frac{t^{1-\delta}}{\Gamma(2-\delta)} + g\left(t, x(0) + tx'(0) + \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} x_{pq} I^{\delta\delta}(\phi_{pq}(t))\right) = f(t).$$

Taking the points

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \dots, 2^{k-1}M.$$

After that, we obtain an algebraic linear or non-linear system of equations containing $2^{k-1}M$ equations. After transforming the system into its algebraic representation, we apply fundamental numerical techniques. We used Newton's method for nonlinear systems and direct solvers for linear systems to achieve efficient and accurate numerical solutions.

6. Convergence analysis

Convergence analysis is critical in ensuring the quality and reliability of numerical solutions derived from a method. It determines whether the numerical solutions converge towards the exact solution of the fractional Lane–Emden equation. They assess this by refining computational parameters, such as grid size or time step, etc. Convergence analysis allows us to quantify the errors in numerical approximations. By examining how these errors evolve with changes in computational parameters, we can gauge the accuracy of the numerical method and make informed parameter selection decisions to achieve the desired levels of precision.

The convergence analysis in [42] is based on the Cauchy sequence and Bessel's inequality. The Cauchy sequence guarantees that the approximations are arbitrarily close, and Bessel's inequality gives a bound on the series expansion in Hilbert space. In this paper, we use Chebyshev polynomials and the Chebyshev wavelet bound to examine convergence. The Chebyshev wavelet boundedness property is used to control the error in the approximation, while the norm in Hilbert space guarantees stability and accuracy of the numerical approximation.

Theorem 6.1. *The function $D_t^{\delta\delta}x(t) \in L^2_\omega[0,1)$ is developed into an infinite sum of the second kind Chebyshev wavelet with $|D_t^{\delta\delta+2}x(t)| \leq L$.*

$$D_t^{\delta\delta}x(t) = \sum_{p=1}^{2^{k-1}}\sum_{q=0}^{M-1} x_{pq}\phi_{pq}(t). \tag{6.1}$$

Then the coefficients x_{pq} satisfy the following inequality,

$$|x_{pq}| \leq \frac{\sqrt{\pi}\,L}{4\,p^{5/2}(q+1)^2}. \tag{6.2}$$

Proof. This theorem's proof resembles Theorem 3 in [18]. □

Theorem 6.2. *Let $k, M \to \infty$. Then the series solution*

$$D_t^{\delta\delta}\overline{x(t)} \approx \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} x_{pq}\phi_{pq}(t),$$

converges to

$$D_t^{\delta\delta}x(t) \approx \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} x_{pq}\phi_{pq}(x).$$

Proof. Let $L^2_\omega[0,1)$ be the Hilbert space and $\phi_{nm}(x)$ form a basis for $L^2_\omega[0,1)$. Let us consider,

$$\begin{aligned}
D_t^{\delta\delta}\overline{x(t)} &\approx \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} x_{pq}\phi_{pq}(t),\\
x_{pq} &= \langle D_t^{\delta\delta}x(t), \phi_{pq}(t)\rangle_{w_p} = \int_0^1 D_t^{\delta\delta}x(t)\phi_{pq}(t)w_p(t)\,dt.
\end{aligned}$$

Now consider the sequence,

$$\begin{aligned}
S_{k,M} &= D_t^{\delta\delta}x(t) - D_t^{\delta\delta}\overline{x(t)},\\
\|S_{k,M}\| &= \|D_t^{\delta\delta}x(t) - D_t^{\delta\delta}\overline{x(t)}\|\\
&= \left\|\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} x_{pq}\phi_{pq}(t) - \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} x_{pq}\phi_{pq}(t)\right\|\\
&= \left\|\sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} x_{pq}\phi_{pq}(t)\right\|\\
&\leq \sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} \|x_{pq}\phi_{pq}(t)\|\\
&= \sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} |x_{pq}|\|\phi_{pq}(t)\|.
\end{aligned}$$

Since $\|\phi_{pq}(t)\| = 1$,

$$\left\|D_t^{\delta\delta}x(t) - D_t^{\delta\delta}\overline{x(t)}\right\| \leq \sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} |x_{pq}|.$$

From Theorem 6.1,

$$|x_{pq}| < \frac{\sqrt{\pi}L}{4p^{5/2}(q+1)^2}.$$

Hence, we get

$$\left\|D_t^{\delta\delta}x(t) - D_t^{\delta\delta}\overline{x(t)}\right\| \leq \sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} \frac{\sqrt{\pi}L}{4p^{5/2}(q+1)^2}.$$

Taking the limit as $k \to \infty$ and as $M \to \infty$,

$$\lim_{k,M\to\infty} \left\| D_t^{\delta\delta} x(t) - D_t^{\delta\delta}\overline{x(t)} \right\| \leq \lim_{k,M\to\infty} \left(\sum_{p=2^{k-1}}^{\infty} \sum_{q=M-1}^{\infty} \frac{\sqrt{\pi}L}{4p^{5/2}(q+1)^2} \right),$$

$$\left\| D_t^{\delta\delta} x(t) - \lim_{k,M\to\infty} D_t^{\delta\delta}\overline{x(t)} \right\| \leq 0,$$

$$\left\| D_t^{\delta\delta} x(t) - \lim_{k,M\to\infty} D_t^{\delta\delta}\overline{x(t)} \right\| = 0,$$

$$D_t^{\delta\delta} x(t) - \lim_{k,M\to\infty} D_t^{\delta\delta}\overline{x(t)} = 0,$$

$$\lim_{k,M\to\infty} D_t^{\delta\delta}\overline{x(t)} = D_t^{\delta\delta} x(t).$$

Hence, $D_t^{\delta\delta}\overline{x(t)}$ converges to $D_t^{\delta\delta} x(t)$. □

7. Illustrative examples

Example 7.1. The fractional nonlinear Lane−Emden type differential equation for order $\delta = 0.75$,

$$D_t^{\delta\delta} x(t) + \frac{2}{t^\delta} D_t^{\delta} x(t) - e^{-x(t)} = 0$$

with the initial conditions $x(0) = 1$, $x'(0) = 0$.

Researchers obtained solution to this problem using both the Chebyshev wavelet finite difference (CWFD) method [23] and the Chebyshev neural network (CHNN) method [37]. Table 1 shows the results achieved using CWFD [23], CHNN [37], and the second kind Chebyshev wavelet method (SKCWD) in the specified domain [0.1, 0.9]. The graph shows a comparison of numerical results produced using the CWFD, CHNN, and SKCWD methods. It is worth noting that the results from SKCWD are consistent with the numerical results of CWFD and CHNN.

Table 1a. Comparison of numerical methods and Chebyshev wavelets results (Example 7.1).

Inputs Values	CWFD [23]	CHNN [37]	Chebyshev wavelet method $k = 2, M = 4$
0.1	0.00746419	0.00737216	0.00746434775119274383287189323161 9
0.2	0.02098214	0.02089361	0.020978100949719264522379638733257
0.3	0.03824551	0.03824741	0.038241318629259294351653642582255
0.4	0.05834999	0.05879991	0.058338777592145446360782417417029
0.5	0.08073004	0.08120049	0.080669335457035979535595821775624
0.6	0.10497857	0.11087687	0.10484525160664135062439704968769
0.7	0.13078223	0.12992310	0.13054031496253050748879076321275
0.8	0.15789189	0.16988731	0.15748066790040949855862318107268
0.9	0.18610806	0.182806	0.18544370729583394862485108783737

The graphical outputs demonstrate a comparative study of three numerical techniques: The Chebyshev Wavelet Finite Difference Method (CWFD), the Chebyshev Neural Network Method (CHNN), and the Chebyshev wavelet method. All three techniques have a comparable rising pattern in $x(t)$ as t moves from 0.1 to 0.9, ascertaining their consistency in solving the solution of the underlying differential equation. Of these, the Chebyshev wavelet method is closest to the CWFD method with high numerical accuracy. However, the CHNN method deviates minutely, especially for higher values of t, like $t = 0.8$ and $t = 0.9$. The Chebyshev wavelet method provides more precise solutions, as seen in its more decimal-based data points, while the CWFD and CHNN methods depict minute differences within a reasonable error range. The smoothness of each curve signals numerical stability, with the Chebyshev wavelet method showing robustness owing to its spectral approximation properties. This implies that the Chebyshev wavelet method is an efficient and reliable method for solving fractional differential equations.

Table 1b. Comparison of numerical solutions and Chebyshev wavelet method for integer order (Example 7.1).

Inputs values	Exact solution	Haar wavelet solution	Chebyshev wavelet solution $k = 2$ $M = 4$	Mirza [41]	Nouh [51]	Hunter [22]
0.1	0.001666	0.0016658	0.001666	0.0016	0.0166	0.0016
0.2	0.006653	0.0066534	0.006653	0.0066	0.0333	0.0065
0.3	0.014933	0.0149329	0.014933	0.0149	0.0500	0.0145
0.4	0.026455	0.0264555	0.026456	0.0266	0.0666	0.0253
0.5	0.041154	0.0411540	0.041154	0.0416	0.0833	0.0385
0.6	0.058944	0.0589441	0.058944	0.0598	0.1000	0.0536
0.7	0.079726	0.0797260	0.079727	0.0813	0.1166	0.0700
0.8	0.103386	0.1033861	0.103386	0.1060	0.1333	0.0870
0.9	0.129799	0.1297985	0.129799	0.1338	0.1500	0.1038

Computation of L_∞ norm

From the above table, it is evident that the Chebyshev wavelet method provides the best accuracy with $L_\infty = 0.000001$, indicating the smallest maximum error. The Haar method also performs well with very low error, while the Nouh method exhibits the highest error, making it the least accurate among the compared methods.

Table 1c. Computation of L_∞ Norm for Different Methods (Example 7.1).

Method	$\mathbf{L_\infty}$ **Norm**
Haar wavelet	5×10^{-6}
Chebyshev Wavelet	1×10^{-6} (Best Accuracy)
Mirza	4×10^{-3}
Nouh	4.2×10^{-2}
Hunter	2.5×10^{-2}

Example 7.2. Consider the linear fractional order differential Lane–Emden equa-

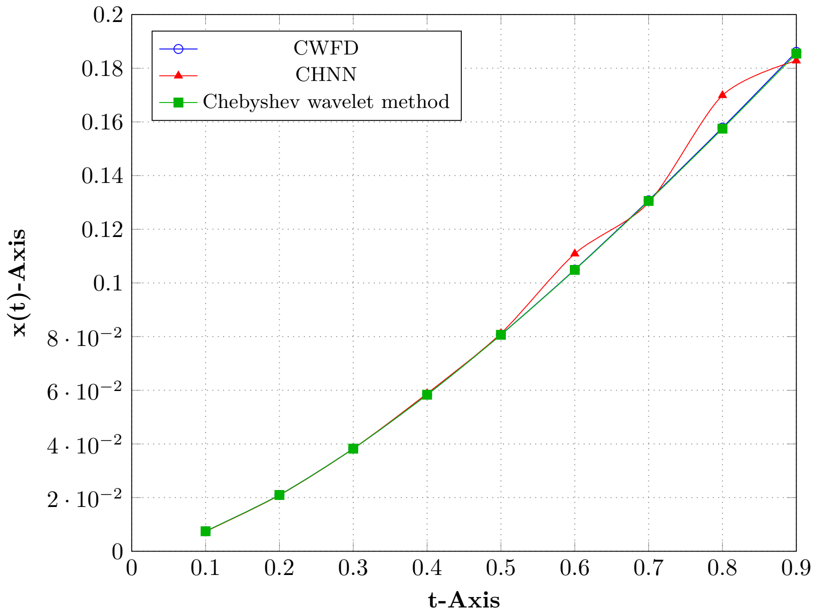

Figure 1. Example 7.1.

tion,

$$D_t^{\delta\delta}x(t) + \frac{2}{t^{\delta}}D_t^{\delta}x(t) + x(t) = 0$$

with the initial conditions $x(0) = 1$, $x'(0) = 0$.

When $\delta = 1$, the exact solution is $\frac{sint}{t}$ (Refer [52]).

The analytical and second kind Chebyshev wavelets solutions for $\delta = 1$ are predicted to be highly consistent, with a smooth curve indicating minimal absolute errors between them. The second kind Chebyshev wavelets method accurately reflects the behaviour of the analytical solution while examining numerical solutions for $\delta = 0.6, 0.7$ and 0.8.

The graphical solution of Example 7.2 shows the $x(t)$ behavior with varying values of the parameter δ employing the Chebyshev wavelet method against the analytical solution for $\delta = 1$. The Chebyshev wavelet solution when $\delta = 1$ agrees very closely with the analytical solution, reaffirming the high accuracy of the method. As δ goes from 1 to 0.6, the function $x(t)$ also decreases for all t, showing a gradual departure from the analytical solution. The $\delta = 0.8$ curve is still closer to $\delta = 1$, but the $\delta = 0.7$ and $\delta = 0.6$ solutions go further down. This indicates that smaller values of δ make $x(t)$ decrease more quickly. The smoothness of all the curves ensures the numerical stability of the Chebyshev Wavelet Method as a sound method for solving fractional differential equations. The behavior indicates that reducing δ reduces the system's response, which may model physical processes like diffusion or damping effects. In general, the Chebyshev wavelet method properly approximates the solutions corresponding to various values of δ and accurately models their effect on the system's behavior.

Example 7.3. By taking the non−homogeneous fractional order Lane−Emden

Table 2. Analytical and Chebyshev wavelets results for $k = 2, M = 8$ (Example 7.2).

Inputs values	Analytical solution for $\delta = 1$	Chebyshev wavelet solution for $\delta = 1$	Absolute error for $\delta = 1$	Chebyshev wavelet solution for $\delta = 0.6$	Chebyshev wavelet solution for $\boldsymbol{\delta} = 0.7$	Chebyshev wavelet solution for $\delta = 0.8$
0.0	1.0000	1.0000	0	1.0000	1.0000	1.0000
0.1	0.9983	0.9983	2.6823e-22	0.9825	0.9900	0.9944
0.2	0.9933	0.9933	1.6522e-21	0.9603	0.9739	0.9832
0.3	0.9851	0.9851	2.2416e-21	0.9363	0.9544	0.9681
0.4	0.9735	0.9735	1.4166e-19	0.9114	0.9326	0.9498
0.5	0.9589	0.9589	3.3094e-19	0.8861	0.9090	0.9289
0.6	0.9411	0.9411	3.3191e-17	0.8605	0.8842	0.9058
0.7	0.9203	0.9203	5.6146e-17	0.8350	0.8583	0.8809
0.8	0.8967	0.8967	7.2941e-17	0.8096	0.8318	0.8543
0.9	0.8704	0.8704	8.5358e-17	0.7846	0.8047	0.8265

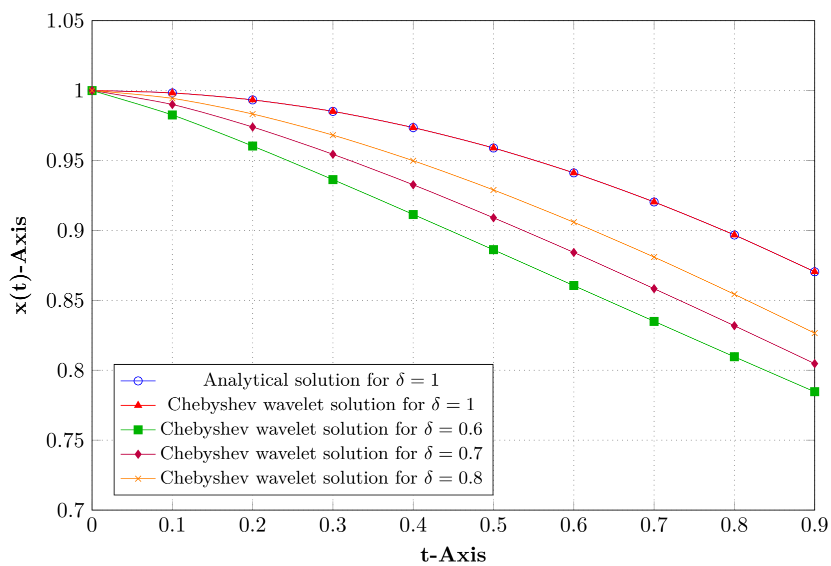

Figure 2. Example 7.2.

type differential equation,

$$D_t^{\delta\delta}x(t) + \frac{2}{t^\delta}D_t^\delta x(t) + x(t) = 6 + 12t^\delta + t^{2\delta} + t^{3\delta}$$

with the initial conditions $x(0) = 0, x'(0) = 0$.

When $\delta = 1$, the analytical solution is $x(t) = t^3 + t^2$ (Refer [56]).

The graphical plots present the performance of the Chebyshev wavelet approach to approximate solutions as δ varies. In the case where $\delta = 1$, the solution using the Chebyshev wavelet has a perfect overlap with the analytical solution, portraying the precision of the approach. When δ varies between 1 and 0.6, solutions have an escalating trend, the highest function value being produced at $\delta = 0.6$ to show a high growth rate. The solutions for $\delta = 0.7$ and $\delta = 0.9$ also differ from the analytical solution but at a slower rate than $\delta = 0.6$. Interestingly, the function values for higher δ values, especially $\delta = 0.9$, are closer to the analytical solution. These findings indicate the effect of δ on the growth of $x(t)$, where smaller δ results in a faster increase. In general, the finding verifies the accuracy of the Chebyshev wavelet approach in approximating solutions and accurately detecting changes in solution behavior for various fractional orders.

Table 3. Analytical and Chebyshev wavelets solution for k = 2, M = 8 and different Values of δ (Example 7.3).

Inputs values	Analytical solution for $\delta = 1$	Chebyshev wavelet solution for $\delta = 1$	Absolute error for $\delta = 1$	Chebyshev wavelet solution for $\delta = 0.6$	Chebyshev wavelet solution for $\delta = 0.7$	Chebyshev wavelet solution for $\delta = 0.9$
0	0	0	0	0	0	0
0.1	0.0110	0.0110	1.2771e-36	0.1442	0.0760	0.0210
0.2	0.0480	0.0480	1.6584e-35	0.3747	0.2261	0.0808
0.3	0.1170	0.1170	6.2343e-35	0.6650	0.4368	0.1828
0.4	0.2240	0.2240	2.7119e-34	1.0055	0.7038	0.3313
0.5	0.3750	0.3750	4.7696e-34	1.3903	1.0244	0.5305
0.6	0.5760	0.5760	2.6276e-33	1.8149	1.3966	0.7847
0.7	0.8330	0.8330	3.9618e-33	2.2758	1.8189	1.0977
0.8	1.1520	1.1520	4.7855e-33	2.7704	2.2898	1.4732
0.9	1.5390	1.5390	5.3714e-33	3.2962	2.8080	1.9149

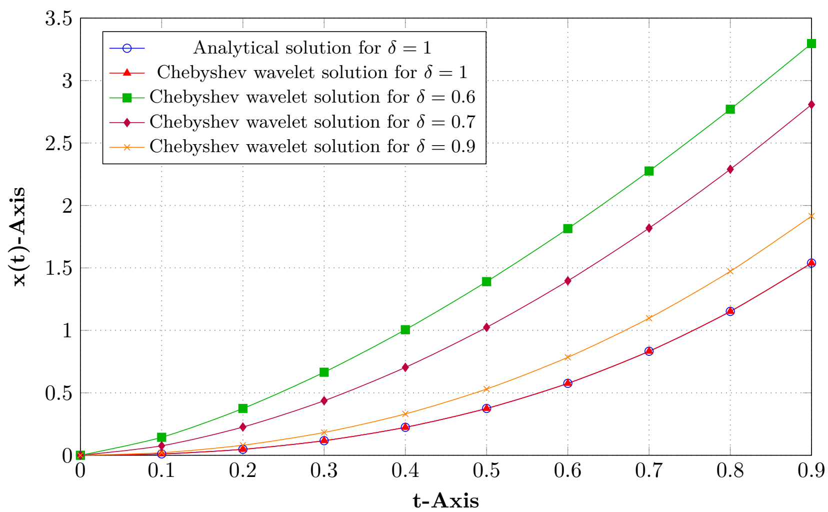

Figure 3. Example 7.3.

Example 7.4. Consider the equation,

$$D_t^{\delta\delta} x(t) + \frac{1}{t^{\delta}} D_t^{\delta} x(t) + \frac{1}{t^{\delta-2}} x(t) = f(t)$$

with the initial conditions $x(0) = 0$, $x'(0) = 0$.

Where

$$f(t) = t^{2-\delta} \left(6t \left(\frac{t^2}{6} + \frac{\Gamma(4-\delta) + \Gamma(4-\beta)}{\Gamma(4-\delta)\Gamma(4-\beta)} \right) - 2 \left(\frac{t^2}{2} + \frac{\Gamma(3-\delta) + \Gamma(3-\beta)}{\Gamma(3-\delta)\Gamma(3-\beta)} \right) \right).$$

The exact solution is

$$x(t) = t^3 - t^2, \quad \text{(Refer [5])}.$$

Table 4. Analytical and Chebyshev wavelets solution for k = 2, M = 8 and different values of δ (Example 7.4).

Inputs values	Analytical solution for $\delta = 1$	Chebyshev wavelet solution for $\delta = 1$	Absolute error for $\delta = 1$	Chebyshev wavelet solution for $\delta = 0.6$	Chebyshev wavelet solution for $\delta = 0.7$	Chebyshev wavelet solution for $\delta = 0.8$
0	0	0	0	0	0	0
0.1	-0.0090	-0.0090	7.4974e-36	-0.0090	-0.0091	-0.0091
0.2	-0.0320	0.0320	2.9776e-35	-0.0320	-0.0321	-0.0321
0.3	-0.0630	-0.0630	2.4855e-35	-0.0630	-0.0631	-0.0631
0.4	-0.0960	-0.0960	4.2542e-35	-0.0960	-0.0960	-0.0961
0.5	-0.1250	-0.1250	6.4845e-35	-0.1250	-0.1251	-0.1251
0.6	-0.1440	-0.1440	2.0832e-34	-0.1440	-0.1441	-0.1442
0.7	-0.1470	-0.1470	3.8843e-34	-0.1470	-0.1471	-0.1472
0.8	-0.1280	-0.1280	5.4597e-34	-0.1280	-0.1281	-0.1282
0.9	-0.0810	-0.0810	6.9804e-34	-0.0810	-0.0811	-0.0812

The graphical solutions illustrate the precision and reliability of the Chebyshev wavelet approach to approximating the analytical solution across various values of δ. Particularly, for $\delta = 1$, the Chebyshev wavelet solution exactly matches the analytical solution, establishing the validity of the approach. For various values of δ (0.6, 0.7, and 0.8), the solutions do not vary much, reflecting that the function $x(t)$ is not significantly different in this interval. The trend of the function is negative all along, reaching its lowest point around $t = 0.7$ before increasing towards $t = 0.9$. These findings indicate that the fractional order δ has a comparatively minimal effect on the behavior of the function in the specified interval. Generally, the Chebyshev wavelet approach successfully describes the behavior of the solution and gives accurate approximations, showing its usability in solving fractional differential equations.

Example 7.5. Let us consider

$$D_t^{\delta\delta} x(t) + \frac{2}{t^\delta} D_t^\delta x(t) + 1 = 0,$$

with the initial conditions $x(0) = 1$, $x'(0) = 0$.

The exact solution is

$$x(t) = 1 - \frac{\Gamma(\delta+1)t^{2\delta}}{\Gamma(2\delta+1)\left[\Gamma(\delta+1)+2\right]}, \quad \text{(Refer [52])}.$$

The graphical outputs reveal the efficiency of the Chebyshev wavelet approach to approximate the analytical solution for varying values of δ. The curve for both analytical and Chebyshev wavelet solutions exhibits a smooth downward trend as t increases. The Chebyshev wavelet solution exactly overlays the analytical solution when $\delta = 1$, proving high accuracy. In the same way, for $\delta = 0.8$ and $\delta = 0.6$, the Chebyshev wavelet solutions are very close to the corresponding analytical solutions,

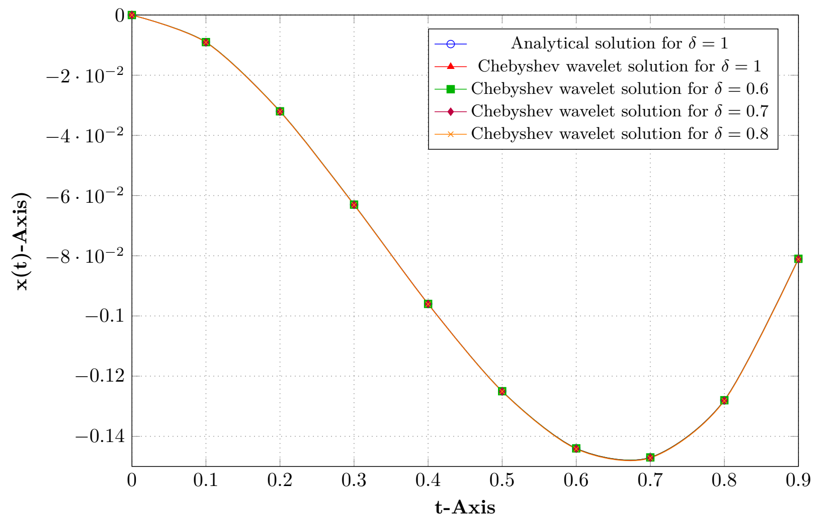

Figure 4. Example 7.4.

illustrating the reliability of the method with varying orders of fractionality. Furthermore, when δ decreases, the function $x(t)$ shows a steeper decline, which means lower orders of fractionality cause a quicker drop in the function values. Generally, the Chebyshev wavelet method is an accurate approach for approximating various orders of fractionality, validating its potential usage in solving fractional differential equations.

Table 5. Analytical and Chebyshev wavelets solution for k = 2, M = 8 and different Values of δ (Example 7.5).

Input	Analytical solution for $\delta=1$	Chebyshev wavelet solution for $\delta=1$	Absolute error for $\delta=1$	Analytical solution for $\delta=0.6$	Chebyshev wavelet solution for $\delta=0.6$	Absolute error for $\delta=0.6$	Analytical solution for $\delta=0.8$	Chebyshev wavelet solution for $\delta=0.8$	Absolute error for $\delta=0.8$
0	1	1	0	1	1	0	1	1	0
0.1	0.9983	0.9983	3.33e-36	0.9823	0.9823	2.69e-18	0.9944	0.9944	3.63e-19
0.2	0.9933	0.9933	1.33e-35	0.9594	0.9594	6.18e-18	0.9831	0.9831	1.10e-18
0.3	0.9850	0.9850	3.00e-35	0.9339	0.9339	1.01e-17	0.9676	0.9676	2.11e-18
0.4	0.9733	0.9733	5.33e-35	0.9067	0.9067	1.42e-17	0.9487	0.9487	3.34e-18
0.5	0.9583	0.9583	1.51e-34	0.8780	0.8780	1.86e-17	0.9267	0.9267	4.77e-18
0.6	0.9400	0.9400	7.54e-35	0.8482	0.8482	2.31e-17	0.9019	0.9019	6.39e-18
0.7	0.9183	0.9183	2.66e-34	0.8173	0.8173	2.78e-17	0.8744	0.8744	8.18e-18
0.8	0.8933	0.8933	4.34e-34	0.7856	0.7856	3.26e-17	0.8445	0.8445	1.01e-17
0.9	0.8650	0.8650	5.83e-34	0.7530	0.7530	3.749e-17	0.8122	0.8122	1.22e-17

8. Conclusion

This work suggests a second kind Chebyshev wavelet method for solving the fractional Lane–Emden equation, improving both accuracy and computational efficiency. By comparing the numerical solution with analytical solution for integer

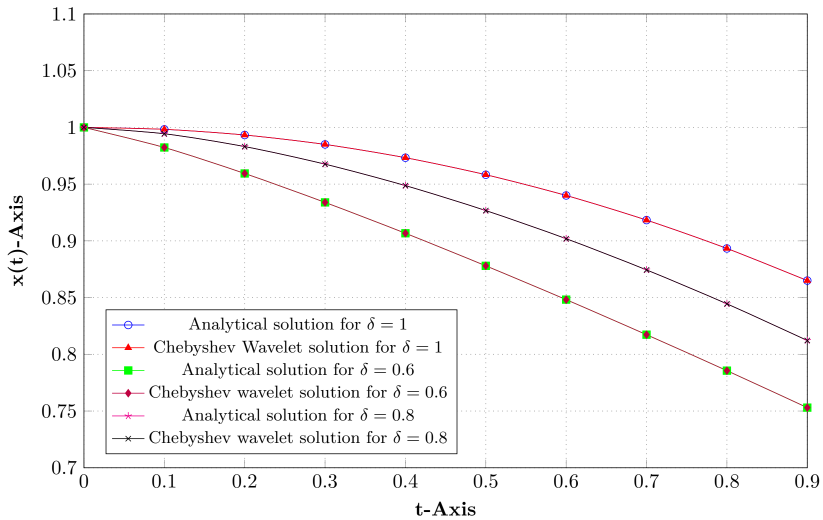

Figure 5. Example 7.5.

orders, we prove that our method had outstanding accuracy and convergence. Further, the method is computationally efficient, making it a viable choice for future solutions to similar differential equations.

The obtained solution pertaining to fractional Lane–Emden differential equation with second kind Chebyshev wavelets provides a viable strategy to address problems related to other fractional differential equations in various domains. This method possesses a broad spectrum of potential applications, and is likely to have a substantial impact on the advancement of numerical analysis in the forthcoming years.

For further research, we can use this approach to investigate multi-dimensional and coupled fractional Lane–Emden equations, which find significant use in astrophysical modeling and stellar dyanmics. Another possibility could be applying this method to nonlinear fractional differential equations from plasma physics and cosmology. Combining machine learning-based optimization methods with wavelet approaches might also improve numerical approximations and solution efficiency. These instructions will also broaden the usage of fractional differential equations in engineering and astrophysical problems.

Acknowledgment

The authors are grateful to the reviewers for a careful reading of the manuscript and making valuable suggestions leading to a better presentation of the paper. The first and second authors would like to thank the management of Sri Sivasubramaniya Nadar College of Engineering for providing research facilities and encouragements to carry out this research work.

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Received December 2024; Accepted May 2025; Available online June 2025.