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EXISTENCE RESULTS OF HILFER FRACTIONAL STOCHASTIC PANTOGRAPH DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES VIA CONDENSING OPERATOR THEORY

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Abstract In this study, we establish the existence of mild solutions for fractional stochastic pantograph differential equations incorporating the Hilfer fractional derivative and non-instantaneous impulses. The analysis is conducted using tools from fractional calculus, semigroup theory, and stochastic analysis under appropriate conditions. Additionally, we employ condensing operator theory, the Hausdorff measure of noncompactness, and Sadovskii's fixed point theorem to derive our existence results. A detailed example, supported by graphical analysis, is presented to illustrate the practical applicability of the theoretical findings.

Keywords Fractional stochastic differential equations, Hilfer fractional derivative, pantograph equation, non-instantaneous impulses, Hausdorff measure of noncompactness, condensing operator theory.

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1. Introduction

Recently, fractional calculus has gained much attention due to its ability to model complex systems with memory and genetic features. These systems are common in many fields, such as biology, physics, engineering, and finance, as several studies have shown [9, 15, 28]. However, it's important to remember that not all fractional derivatives translate into useful physical meanings. The Riemann-Liouville (R-L) and Caputo derivatives are two of the most often used because of their solid mathematical basis. The Hilfer fractional derivative (HFD), which was introduced as a bridge between the R-L and Caputo derivatives, is one very useful derivative in fractional calculus [15]. With its adaptable framework for simulating nonlocal dynamic processes and intermediate-order behaviors, the HFD has proven to be reliable and widely applicable in both theoretical and practical applications, one

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can see [13, 27, 29, 37].

The real world consists of multiple phenomena which contain inherent uncertainty factors that influence human and animal movement as well as natural and engineered systems. Stochastic calculus has progressively become a fundamental analysis tool for random effect predictions because it serves various fields from biology to engineering and economics. Epidemiologists use stochastic models to understand health disease behaviors and choose the most effective population control methods. The methods of stochastic differential operators have become widely used for modeling the spread of infectious diseases by research experts. Shah et al. [33] showed how stochastic processes modify the results of treatment effectiveness. The authors of [17] studied an SEIR-type model of COVID-19 through the implementation of piecewise and stochastic differential operators for evaluating different management strategies. Tahir et al. [36] explored worm transmission within wireless sensor networks through advanced stochastic analytical methods.

Pantograph equations initially studied by Balachandran [4], represent a special class of delay differential equations that are fundamental to numerous scientific applications, including biological modeling, control systems, and electrodynamics. In particular, stochastic effect is added to the traditional pantograph equations to create fractional stochastic pantograph differential equations (FSPDEs). This combination makes it possible to simulate systems that have random fluctuations and memory effects, similar to those that occur in real-world scenarios where uncertainty is essential. FSPDEs are increasingly important in fields like finance, population dynamics, and engineering, where they offer a more comprehensive description of systems influenced by both delays and randomness. The study of fractional stochastic pantograph differential equations FSPDEs, particularly those with delays, plays a crucial role in understanding system dynamics and stability. For instance, Ahmed and Wang [3] investigated the exact null controllability of Hilfer-type stochastic systems with fractional Brownian motion and Poisson jumps. Balachandran et al. [4] explored the existence of solutions for nonlinear fractional pantograph equations. Caraballo and Diop [6] addressed neutral stochastic delay partial functional integrodifferential equations driven by fractional Brownian motion. Makhlouf and Mchiri [23] studied Caputo–Hadamard fractional stochastic differential equations, while their subsequent work [24] focused on the Ulam–Hyers stability of pantograph-type stochastic equations. Mshary et al. [26] examined existence and controllability results for nonlinear evolution equations with HFD, noise, and impulsive effects. Sousa et al. [35] contributed to the theory of mild solutions for impulsive Hilfer-type equations. Vivek et al. [38] established existence results for hybrid stochastic equations involving the ψ-Hilfer derivative. Lastly, Wongcharoen et al. [39] investigated nonlocal boundary value problems for Hilfer-type pantograph equations and inclusions.

For the purpose of modeling phenomena in the social and physical sciences, impulsive fractional differential equations are an effective mathematical tool. There have been significant developments in the theory of impulsive systems, especially impulsive with fixed moments, as described in works like [2, 10, 19, 30]. In physical systems that change over time, sudden changes, called impulses, are common. Impulses are generally divided into two types: instantaneous impulsive, which happen over a very short period relative to the system's overall time span, and non-instantaneous impulses (N-II), which start at a certain time and continue to be active for a limited amount of time. Instantaneous impulsive may explain certain dynamics well, but it does not capture the slow processes that exist in some systems. N-II offers a more realistic modeling framework, for example, in pharmacotherapy, where the delivery and absorption of injectable medications (like insulin) are continuous processes. The concept of N-II was introduced by Hernandez et al. [14] and has since been applied across fields such as medical science, mechanical engineering, and biological systems. Recent work has focused on fractional differential equations with N-II. Gautam and Dabas [12] studied mild solutions for a class of neutral fractional functional differential equations involving N-II. Khalil et al. [20] analyzed the qualitative behavior of impulsive stochastic Hilfer fractional differential equations. Ma et al. [22] established existence and stability results for neutral ψ-Hilfer fractional stochastic systems influenced by fractional noise and N-II. Saravanakumar and Balasubramaniam [31] investigated Hilfer-type stochastic equations driven by fractional Brownian motion under N-II effects.

As of now, no results explicitly investigate the existence of mild solutions for FSPDEs using HFD and N-II. Recent works have explored related topics using different approaches: piecewise equations with non-singular derivatives via fixed point methods [32], fractional delay impulsive systems with Mittag-Leffler laws [1], and fractional models of viral dynamics under piecewise derivatives [34]. Unlike these studies, our work considers a stochastic pantograph system with HFD and N-II, and establishes existence results using condensing operator theory, offering a novel contribution to the field. The problem is described as follows:

$$\begin{cases} {}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}\left(\varrho(\iota)-\displaystyle\sum_{j=1}^{m}\mathfrak{I}^{\lambda_j}_{0^+,\iota}\sigma_j\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right)\right) \\ =\mathcal{A}\varrho(\iota)+\varpi_1\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right) \\ \quad +\varpi_2\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota},\ \iota\in(s_i,\iota_{i+1}]\subset J:=(0,b],\ i=0,1,2,\dots,m, \\ \varrho(\iota)=\Phi_i\left(\iota,\varrho(\iota)\right), \quad \iota\in(\iota_i,s_i], \quad i=1,2,\dots,m, \\ \mathfrak{I}^{1-\gamma}_{0^+,\iota}\left(\varrho(0)-\displaystyle\sum_{j=1}^{m}\mathfrak{I}^{\lambda_j}_{0^+,\iota}\sigma_j\left(0,\varrho(0),\varrho\left(0\right)\right)\right)=\varrho_0, \quad \gamma=p+q-pq, \end{cases} \tag{1.1}$$

where $\mathfrak{I}^{1-\gamma}_{0^+,\iota}$ and $\mathfrak{I}^{\lambda_j}_{0^+,\iota}$ denote the fractional R–L integrals of orders $1-\gamma$ and λ_j, respectively, while ${}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}$ denotes the HFD characterized by the order p and the type q. Here, $0<p<1$, $\frac{1}{2}<q\leq 1$, $\frac{1}{2}<\lambda_j\leq 1$. Let $\mathcal{A}$ be the generator of strongly continuous semigroup $\{\mathcal{S}(\iota):\iota\geq 0\}$ on a Hilbert space $\mathcal{E}$, $\{\mathcal{W}(\iota)\}_{\iota\geq 0}$ denotes the Q-Wiener process defined in the complete probability space $(\Omega,\mathcal{F}_\iota,P)$ with a filteration $(\mathcal{F}_\iota)_{\iota\geq 0}$. $\varpi_1,\sigma_j:J\times\mathcal{E}\times\mathcal{E}\to\mathcal{E}$, $\varpi_2:J\times\mathcal{E}\times\mathcal{E}\to\mathcal{L}^0_2$ are appropriate functions and $0<\kappa<1$. The space $\mathcal{L}^0_2$ will be defined later. $\Phi_i:(\iota_i,s_i]\times\mathcal{E}\to\mathcal{E}$ are measurable for $i=1,2,\dots,m$ and $\varrho\left(\iota_i^+\right)=\lim_{\tau\to 0^+}\varrho\left(\iota_i+\tau\right),\varrho\left(\iota_i^-\right)=\lim_{\tau\to 0^-}\varrho\left(\iota_i-\tau\right)$. ι_i,s_i satisfy $0=s_0=\iota_0<\iota_1\leq s_1<\iota_2<\cdots<\iota_m\leq s_m<\iota_{m+1}=b<\infty$.

2. Preliminaries

Let $\mathbb{L}^2\left(\Omega,\mathcal{F}_\iota,\mathcal{E}\right)=\mathbb{L}^2\left(\Omega,\mathcal{E}\right)$ denote the Hilbert space of real-valued random variables that are square-integrable with respect to the probability measure on $(\Omega,\mathcal{F}_\iota)$.

Let $C(J, \mathbb{L}^2(\Omega, \mathcal{E}))$ be the space of continuous time stochastic processes that are square-integrable with the norm $\|\varrho\|^2 = \sup\left\{\mathbb{E}\|\varrho(\iota)\|^2 : \iota \in J\right\}$, where $\mathbb{E}$ is the mathematical expectation. On the other hand, define the Banach space

$$C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{E})) = \left\{\varrho : J \to \mathbb{L}^2(\Omega, \mathcal{E}) : \iota^{1-\gamma}\varrho(\iota) \in C(J, \mathbb{L}^2(\Omega, \mathcal{E}))\right\}, \quad 0 < \gamma \leq 1.$$

Let

$$\begin{aligned}\mathcal{X} &= PC_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{E})) \\ &= \left\{\begin{array}{l}\varrho : J \to \mathbb{L}^2(\Omega, \mathcal{E}); \quad \varrho \in C_{1-\gamma}\left([\iota_i, \iota_{i+1}], \mathbb{L}^2(\Omega, \mathcal{E})\right), i = 0, \ldots, m, \\ \text{and there exist } \varrho\left(\iota_i^+\right), \varrho\left(\iota_i^-\right), \text{with } \varrho(\iota_i) = \varrho\left(\iota_i^-\right), i = 1, 2, \ldots, m,\end{array}\right\}\end{aligned}$$

using the norm

$$\|\varrho\|_{\mathcal{X}}^2 = \sup_{\iota \in J}\mathbb{E}\left\|\iota^{1-\gamma}\varrho(\iota)\right\|^2.$$

Consider $\mathcal{W} : J \times \Omega \to K$ as a standard Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}_\iota, P)$, with Q being a linear bounded covariance operator such that $\operatorname{Tr} Q < \infty$. This process is associated with the normal filtration $(\mathcal{F}_\iota)_{\iota \in J}$. Suppose there exists a complete orthonormal basis $\{e_n\}_{n \geq 1}$ in K and a sequence of nonnegative real numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ satisfying

$$Qe_n = \lambda_n e_n, \quad \lambda_n \geq 0,\ n = 1, 2, \ldots,$$

as well as a set of independent real-valued Brownian motions $\{\beta_n\}_{n \geq 1}$ such that

$$\langle \mathcal{W}(\iota), e\rangle = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\langle e_n, e\rangle\beta_n(\iota), \quad e \in K,\ \iota \in J.$$

Define the Hilbert space

$$\mathcal{L}_2^0 = \{\mathcal{T} \mid \mathcal{T} \text{ is a Hilbert-Schmidt operator from } Q^{\frac{1}{2}}(K) \text{ to } \mathcal{E}\},$$

with the inner product defined as

$$\langle \psi, \phi\rangle_{\mathcal{L}_2^0} = \operatorname{tr}[\psi Q \phi^*], \quad \psi, \phi \in \mathcal{L}_2^0.$$

Definition 2.1. (Ref. [9], Page No. 231) For $p > 0$, the fractional R-L integral of order p for a function ϱ can be written as

$$\mathfrak{I}^p_{a^+,\iota}\varrho(\iota) = \frac{1}{\Gamma(p)}\int_a^\iota(\iota - s)^{p-1}\varrho(s)ds. \tag{2.1}$$

Definition 2.2. (Ref. [9], Page No. 229) For $n - 1 < p \leq n$, the fractional R-L derivative of order p for a function ϱ is defined by

$$\mathfrak{D}^p_{a^+,\iota}\varrho(\iota) = D^n\mathfrak{I}^{n-p}_{a^+,\iota}\varrho(\iota) = \frac{1}{\Gamma(n-p)}\left(\frac{\mathrm{d}}{\mathrm{d}\iota}\right)^n\int_a^\iota(\iota - s)^{n-p-1}\varrho(s)\mathrm{d}s.$$

Definition 2.3. (Ref. [15], Definition 3.3, Page No. 113) For $0 < p \leq 1$, the HFD of order p and type $0 \leq q \leq 1$ of ϱ is represented as

$${}^H\mathfrak{D}^{p,q}_{a^+,\iota}\varrho(\iota) = \mathfrak{I}^{q(1-p)}_{a^+,\iota}D\mathfrak{I}^{(1-q)(1-p)}_{a^+,\iota}\varrho(\iota) = \mathfrak{I}^{q(1-p)}_{a^+,\iota}\mathfrak{D}^{\gamma}_{a^+,\iota}\varrho(\iota), \quad \gamma = p + q(1-p), \tag{2.2}$$

where $D = \frac{d}{d\iota}$.

Lemma 2.1. (Ref. [11], Lemma 17, Page No. 1619) *Let* $0 < p < 1$ *and* $0 < \gamma \leq 1$. *If* $\varpi \in C_{1-\gamma}[a, b]$ *and* $\mathfrak{I}^{1-p}_{a^+,\iota}\varpi \in C^1_\gamma[a, b]$, *then*

$$\mathfrak{I}^p_{a^+,\iota}\mathfrak{D}^p_{a^+,\iota}\varpi(\iota) = \varpi(\iota) - \frac{\mathfrak{I}^{1-p}_{a^+,\iota}\varpi(a)}{\Gamma(p)}(\iota - a)^{p-1}, \text{ for all } \iota \in (a, b].$$

Lemma 2.2. (Ref. [15], Page No. 13) *Let* $p > 0$ *and* $q > 0$. *Following this,* $\forall \iota \in J$, *there is*

$$\left[\mathfrak{I}^p_{a^+,\iota}(\iota)^{q-1}\right](\iota) = \frac{\Gamma(q)}{\Gamma(q+p)}\iota^{q+p-1}, \quad \left[\mathfrak{D}^p_{a^+,\iota}(\iota)^{p-1}\right](\iota) = 0, \quad 0 < p < 1.$$

Lemma 2.3. (Ref. [18], Page No. 180) *Let* $m \in \mathbb{N}$ *and* $\iota_1, \iota_2, \ldots, \iota_m$ *be nonnegative real numbers, then*

$$\left(\sum_{i=1}^m \iota_i\right)^l \leq m^{l-1}\sum_{i=1}^m \iota_i^l, \quad \text{for } l > 1.$$

Lemma 2.4. (Ref. [21], Proposition 2.12, Page No. 18) *For any predictable process* $\omega(\iota)$ *taking values in* $\mathcal{L}^0_2$ *and defined on the interval* $[\iota_1, \iota_2]$, *which satisfies*

$$\mathbb{E}\left(\int_{\iota_1}^{\iota_2} \|\omega(s)\|^2_{\mathcal{L}^0_2}\, ds\right) < \infty, \quad 0 \leq \iota_1 < \iota_2 \leq b.$$

The following inequality holds:

$$\mathbb{E}\left\|\int_{\iota_1}^{\iota_2} \omega(s)\, d\mathcal{W}(s)\right\|^2 \leq \mathbb{E}\left(\int_{\iota_1}^{\iota_2} \|\omega(s)\|^2_{\mathcal{L}^0_2}\, ds\right).$$

Lemma 2.5. (Ref. [13], Lemma 2.12, Page No. 347) *A stochastic process* $\varrho \in \mathcal{X}$ *is called a mild solution of problem (1.1) if* ϱ *satisfies the following stochastic integral equation*

$$\varrho(\iota) = \begin{cases} \mathcal{S}_{p,q}(\iota)\varrho_0 + \displaystyle\sum_{j=1}^m \frac{1}{\Gamma(\lambda_j)}\int_0^\iota (\iota - s)^{\lambda_j - 1}\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds \\ + \displaystyle\int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\varpi_1(s, \varrho(s), \varrho(\kappa s))ds \\ + \displaystyle\int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\varpi_2(s, \varrho(s), \varrho(\kappa s))d\mathcal{W}(s), \quad \iota \in (0, \iota_1], \\ \Phi_i\left(\iota, \varrho(\iota)\right), \quad \iota \in (\iota_i, s_i], \\ \mathcal{S}_{p,q}(\iota - s_i)\Phi_i\left(s_i, \varrho(s_i)\right) + \displaystyle\sum_{j=1}^m \frac{1}{\Gamma(\lambda_j)}\int_0^{s_i} (s_i - s)^{\lambda_j - 1}\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds \\ + \displaystyle\int_0^{s_i} (s_i - s)^{q-1}P_q(s_i - s)\varpi_1(s, \varrho(s), \varrho(\kappa s))ds \\ + \displaystyle\int_0^{s_i} (s_i - s)^{q-1}P_q(s_i - s)\varpi_2(s, \varrho(s), \varrho(\kappa s))d\mathcal{W}(s) \\ + \displaystyle\int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\varpi_1(s, \varrho(s), \varrho(\kappa s))ds \\ + \displaystyle\int_0^\iota (\iota - s)^{q-1}P_q(\iota - s)\varpi_2(s, \varrho(s), \varrho(\kappa s))d\mathcal{W}(s), \quad \iota \in (s_i, \iota_{i+1}], \end{cases} \tag{2.3}$$

where

$$\mathcal{S}_{p,q}(\iota) = \Im_{0^+,\iota}^{p(1-q)} T_q(\iota), \quad T_q(\iota) = \iota^{q-1} P_q(\iota), \quad P_q(\iota) = \int_0^\infty q\theta M_q(\theta) \mathcal{S}\left(\iota^q \theta\right) d\theta.$$

Lemma 2.6. (Ref. [13], Page No. 349) *Assume that $\mathcal{S}(\iota)$ is continuous in the uniform operator topology for $\iota > 0$ and $\{\mathcal{S}(\iota)\}_{\iota \geq 0}$ is uniformly bounded (i.e., there exists $M > 1$ such that $\sup_{\iota \in [0,\infty)} \|\mathcal{S}(\iota)\| < M$), we have the following properties.*

(i) $P_q(\iota)$, $T_q(\iota)$, and $\mathcal{S}_{p,q}(\iota)$ are linear and bounded operators such that for all $\iota \geq 0, \varrho \in \mathcal{E}$

$$\|P_q(\iota)\varrho\| \leq \frac{M\|\varrho\|}{\Gamma(q)}, \quad \|T_q(\iota)\varrho\| \leq \frac{M\iota^{q-1}\|\varrho\|}{\Gamma(q)} \quad \text{and}$$
$$\|\mathcal{S}_{p,q}(\iota)\varrho\| \leq \frac{M\iota^{\gamma-1}\|\varrho\|}{\Gamma(\gamma)}.$$

(ii) Operators $P_q(\iota)$, $T_q(\iota)$, and $S_{p,q}(\iota)$ are strongly continuous.

Definition 2.4. (Ref. [5], Page No. 2004) The Hausdorff measure of noncompactness for a bounded subset Λ of a Banach space $\mathcal{E}$ is expressed as

$$\vartheta(\Lambda) = \inf\left\{\epsilon > 0 : \Lambda \text{ has a finite } \epsilon - \text{in } \mathcal{E}\right\}.$$

Lemma 2.7. (Ref. [40], Lemma 2.4, Page No. 156) *The Hausdorff measure of noncompactness ϑ defined on the bounded subsets Λ_1 and Λ_2 of a Banach space $\mathcal{E}$ satisfies the following properties:*

(i) Λ_1 is relatively compact set iff $\vartheta(\Lambda_1) = 0$;

(ii) $\vartheta(\Lambda_1 + \Lambda_2) \leq \vartheta(\Lambda_1) + \vartheta(\Lambda_2)$, where $\Lambda_1 + \Lambda_2 = \{a_1 + a_2; a_1 \in \Lambda_1, a_2 \in \Lambda_2\}$;

(iii) For a bounded set $\Lambda \subset \mathcal{E}$, there is a denumerable set $\Lambda_0 \subset \Lambda$ such that $\vartheta(\Lambda_0) \leq \vartheta(\Lambda)$;

(iv) For a bounded and equicontinuous function $\Psi \subset C(J, \mathcal{E})$, the Hausdorff measure of noncompactness $\vartheta(\Psi(\iota))$ is continuous on J and

$$\vartheta(\Psi) = \max_{\iota \in J} \vartheta(\Psi(\iota)).$$

Lemma 2.8. (Ref. [25], Proposition 1.6, Page No. 990) *Let $\Lambda = \{\varrho_n\} \subset C(J, \mathcal{E})$ be a bounded denumerable subset of $\mathcal{E}$. Then, $\vartheta(\Lambda(\iota))$ is Lebesgue integrable on $\mathcal{E}$ and*

$$\vartheta\left(\int \varrho_n(\iota)\, d\iota : n \in \mathbb{N}\right) \leq \int \vartheta(\varrho_n(\iota))\, d\iota.$$

Proposition 2.1. (Ref. [8], Definition 1.9, Page No. 69) *A continuous and bounded map $\Psi : \Lambda \subset \mathcal{E} \longrightarrow \mathcal{E}$ is said to be ϑ-Lipschitz if there exists $r \geq 0$ such that $\vartheta(\Psi(\Lambda_0)) \leq r\vartheta(\Lambda_0)$ for all bounded subsets $\Lambda_0 \subseteq \Lambda$.*

Proposition 2.2. (Ref. [16], Proposition 4, Page No. 4) *If $\Psi : \Lambda \longrightarrow \mathcal{E}$ is Lipschitz with constant r, then Ψ is ϑ-lipschitz with the same constant r.*

Proposition 2.3. (Ref. [8], Definition 1.9, Page No. 69) *A continuous and bounded map $\Psi : \Lambda \subset \mathcal{E} \longrightarrow \mathcal{E}$ is said to be ϑ-condensing if $\vartheta(\Psi(\Lambda_0)) < \vartheta(\Lambda_0)$ for all bounded subsets $\Lambda_0 \subseteq \Lambda$.*

Theorem 2.1. (Ref. [7], Page No. 643) *Let $\Lambda \subset \mathcal{E}$ be closed, bounded, and convex. If the continuous map $\Psi : \mathcal{E} \longrightarrow \mathcal{E}$ is a ϑ-condensing, then Ψ has at least one fixed point.*

3. Main results

This section establishes the criteria for the existence of mild solutions to problem (1.1). To begin our main analysis, we first present the following essential hypotheses. (H_1): $\varpi_1 : J \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $\varpi_2 : J \times \mathcal{E} \times \mathcal{E} \to \mathcal{L}_2^0$ are Carathéodory functions satisfy:

(i) $\exists$ functions $\chi_{f_k}(\iota) \in L^1(J, \mathbb{R}^+)$ and non-decreasing continuous functions Θ_{ϖ_k} : $[0, \infty) \to (0, \infty)$ $(k = 1, 2)$ such that for any $\varrho, \xi \in \mathcal{E}$ and each $\iota \in J$,

$$\|\varpi_k(\iota, \varrho, \xi)\|^2 \leq \chi_{\varpi_k}(\iota)\Theta_{\varpi_k}\left(\iota^{2(1-\gamma)}\left(\|\varrho\|^2 + \|\xi\|^2\right)\right).$$

(ii) For arbitrary $\varrho_1, \varrho_2, \xi_1, \xi_2 \in \mathcal{E}$, there exist positive constants $\mathcal{L}_{\varpi_k}$ ($k = 1, 2$) such that

$$\|\varpi_k(\iota, \varrho_1, \varrho_2) - \varpi_k(\iota, \xi_1, \xi_2)\|^2 \leq \mathcal{L}_{\varpi_k}\iota^{2(1-\gamma)}\left(\|\varrho_1 - \xi_1\|^2 + \|\varrho_2 - \xi_2\|^2\right).$$

(iii) For any bounded subsets $\Lambda_1, \Lambda_2 \subset \mathcal{E}$, there exist functions $\varphi_{\varpi_k}(\iota) \in L^1(J, \mathbb{R}^+)$ and positive constants $\varpi_k^* = \sup_{\iota \in J} \varphi_{\varpi_k}(\iota)$ (for $k = 1, 2$) such that

$$\vartheta(\varpi_k(\iota, \Lambda_1, \Lambda_2)) \leq \varphi_{\varpi_k}(\iota)\left[\vartheta(\Lambda_1) + \vartheta(\Lambda_2)\right].$$

(H_2): For arbitrary $\varrho_1, \varrho_2, \xi_1, \xi_2 \in \mathcal{E}$, there exist positive constants $\mathcal{L}_{\sigma_j}, n_{\sigma_j}$, and m_{σ_j} (for $j = 1, 2, \ldots, m$) such that

$$\|\sigma_j(\iota, \varrho_1, \varrho_2) - \sigma_j(\iota, \xi_1, \xi_2)\|^2 \leq \mathcal{L}_{\sigma_j}\iota^{2(1-\gamma)}\left(\|\varrho_1 - \xi_1\|^2 + \|\varrho_2 - \xi_2\|^2\right),$$
$$\|\sigma_j(\iota, \varrho_1, \varrho_2)\|^2 \leq n_{\sigma_j}\iota^{2(1-\gamma)}\left(\|\varrho_1\|^2 + \|\varrho_2\|^2\right) + m_{\sigma_j}.$$

(H_3): For arbitrary $\varrho, \xi \in \mathcal{E}$, there exist positive constants $\mathcal{L}_{\Phi_i}$ such that for $\iota \in (\iota_i, s_i]$ ($i = 1, 2, \ldots, m$)

$$\|\Phi_i(\iota, \varrho) - \Phi_i(\iota, \xi)\|^2 \leq \mathcal{L}_{\Phi_i}\|\varrho - \xi\|^2.$$

To enhance clarity, we adopt the following notations:

$$\Delta_{1_i} = \sum_{j=1}^m \frac{\mathcal{L}_{\sigma_j}{\iota_{i+1}}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j - 1}, i = 0, 1, \ldots, m,$$
$$\Delta_1 = \max\{\Delta_{1_i}, i = 0, 1, 2, \ldots, m\},$$
$$\mathcal{L}_\Phi = \max\{\mathcal{L}_{\Phi_i}, i = 1, 2, \ldots, m\},\ \Theta_1 = \max\left\{{\iota_{i+1}}^{2-2\gamma+2q}, i = 0, 1, 2, \ldots, m\right\},$$
$$\Theta_2 = \max\left\{{\iota_{i+1}}^{1-2\gamma+2q}, i = 0, 1, 2, \ldots, m\right\},$$
$$\Theta_3 = \max\left\{{\iota_{i+1}}^{2(\gamma-1)}, i = 0, 1, 2, \ldots, m\right\},$$
$$\Theta_4 = \max\{{\iota_{i+1}}^q, i = 0, 1, 2, \ldots, m\}.$$

Theorem 3.1. *Assume that (H_1)-(H_3) are satisfied, then problem (1.1) has at least one mild solution in $\mathcal{X}$, provided*

$$\frac{2M\Theta_4}{q}(\varpi_1^* + \varpi_2^*) + \bar{L} < 1, \text{ where}$$
$$\bar{L} = \max\{2m\Delta_{10}, \mathcal{L}_\Phi, \frac{4M^2\mathcal{L}_\Phi\Theta_3}{\Gamma^2(\gamma)} + \frac{8\mathcal{L}_{\varpi_1}\Theta_1}{\Gamma^2(q)2q - 1} + \frac{8\mathcal{L}_{\varpi_2}\Theta_2}{\Gamma^2(q)2q - 1} + 8m\Delta_1\}.$$

Proof. Let $\mathcal{B}_\tau = \{\varrho \in \mathcal{X} : \|\varrho\|_{\mathcal{X}}^2 \leq \tau, \tau > 0\}$. Clearly, the set $\mathcal{B}_\tau$ is a bounded, closed, and convex subset of $\mathcal{X}$. Moreover, on the bounded set $\mathcal{B}_\tau$ we introduce the operators $\Psi_1, \Psi_2 : \mathcal{B}_\tau \to \mathcal{B}_\tau$ defined as follows

$$\Psi_1\varrho(\iota) = \begin{cases} \mathcal{S}_{p,q}(\iota)\varrho_0 + \displaystyle\sum_{j=1}^{m} \frac{1}{\Gamma(\lambda_j)} \int_0^\iota (\iota - s)^{\lambda_j - 1}\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds, \quad \iota \in (0, \iota_1], \\ \Phi_i\left(\iota, \varrho(\iota)\right), \quad \iota \in (\iota_i, s_i], \\ \mathcal{S}_{p,q}(\iota - s_i)\Phi_i\left(s_i, \varrho(s_i)\right) \\ \quad + \displaystyle\sum_{j=1}^{m} \frac{1}{\Gamma(\lambda_j)} \int_0^{s_i} (s_i - s)^{\lambda_j - 1}\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds \\ \quad + \displaystyle\int_0^{s_i} (s_i - s)^{q-1} P_q(s_i - s)\varpi_1(s, \varrho(s), \varrho(\kappa s)) ds \\ \quad + \displaystyle\int_0^{s_i} (s_i - s)^{q-1} P_q(s_i - s)\varpi_2(s, \varrho(s), \varrho(\kappa s)) d\mathcal{W}(s), \quad \iota \in (s_i, \iota_{i+1}], \end{cases} \tag{3.1}$$

and

$$\Psi_2\varrho(\iota) = \begin{cases} \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_1(s, \varrho(s), \varrho(\kappa s)) ds \\ \quad + \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_2(s, \varrho(s), \varrho(\kappa s)) d\mathcal{W}(s), \quad \iota \in (0, \iota_1], \\ 0, \quad \iota \in (\iota_i, s_i], \\ \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_1(s, \varrho(s), \varrho(\kappa s)) ds \\ \quad + \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_2(s, \varrho(s), \varrho(\kappa s)) d\mathcal{W}(s), \quad \iota \in (s_i, \iota_{i+1}]. \end{cases} \tag{3.2}$$

Additionally, we define $\Psi = \Psi_1 + \Psi_2$, allowing the fractional stochastic integral equation (2.3) to be expressed in operator form

$$\Psi\varrho(\iota) = \Psi_1\varrho(\iota) + \Psi_2\varrho(\iota), \quad \iota \in J.$$

We will now proceed to verify, step by step, that the operator Ψ has a fixed point on $\mathcal{B}_\tau$.

Step 1. Ψ maps $\mathcal{B}_\tau$ into itself.

Let $\varrho \in \mathcal{B}_\tau$, $\iota \in (0, \iota_1]$, and using Lemma 2.3, we have

$$\begin{aligned} &\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2 \\ \leq &4\iota^{2(1-\gamma)}\mathbb{E}\left\|\mathcal{S}_{p,q}(\iota)\varrho_0\right\|^2 \\ &+ 4m\iota^{2(1-\gamma)} \sum_{j=1}^{m} \frac{1}{\Gamma^2(\lambda_j)} \mathbb{E}\left\|\int_0^\iota (\iota - s)^{\lambda_j - 1}\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds\right\|^2 \\ &+ 4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_1\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) ds\right\|^2 \\ &+ 4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s)\varpi_2(s, \varrho(s), \varrho(\kappa s)) d\mathcal{W}(s)\right\|^2. \end{aligned}$$

By applying the Cauchy–Schwarz (C–S) inequality and Lemma 2.4, a straightforward computation yields

$$
\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2\\
\leq&\frac{4M^2}{\Gamma^2(\gamma)}\mathbb{E}\left\|\varrho_0\right\|^2+4m\iota^{2(1-\gamma)}\iota\sum_{j=1}^{m}\frac{1}{\Gamma^2(\lambda_j)}\int_0^\iota(\iota-s)^{2(\lambda_j-1)}\mathbb{E}\left\|\sigma_j\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2ds\\
&+\frac{4M^2\iota^{2(1-\gamma)}\iota}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2ds\\
&+\frac{4M^2\iota^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\mathbb{E}\left\|\varpi_2\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2ds.
\end{aligned}
$$

From assumptions (H_1) and (H_2), we can deduce that

$$
\begin{aligned}
\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2\leq&\frac{4M^2}{\Gamma^2(\gamma)}\mathbb{E}\left\|\varrho_0\right\|^2+4m\sum_{j=1}^{m}\frac{\iota^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\left(m_{\sigma_j}+2n_{\sigma_j}\|\varrho\|_{\mathcal{X}}^2\right)\\
&+\frac{4M^2\iota^{2(1-\gamma)}\iota}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&+\frac{4M^2\iota^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^\iota(\iota-s)^{2(q-1)}\chi_{\varpi_2}(s)\Theta_{\varpi_2}\left(2\tau\right)ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2\leq&\frac{4M^2}{\Gamma^2(\gamma)}\mathbb{E}\left\|\varrho_0\right\|^2+4m\sum_{j=1}^{m}\frac{{\iota_1}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\left(m_{\sigma_j}+2n_{\sigma_j}\|\varrho\|_{\mathcal{X}}^2\right)\\
&+\frac{4M^2{\iota_1}^{2(1-\gamma)}\iota}{\Gamma^2(q)}\int_0^{\iota_1}(\iota_1-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&+\frac{4M^2{\iota_1}^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^{\iota_1}(\iota_1-s)^{2(q-1)}\chi_{\varpi_2}(s)\Theta_{\varpi_2}\left(2\tau\right)ds\\
:=&\mathcal{N}_1.
\end{aligned}
$$

For $\iota\in(\iota_i,s_i]$, $i=1,2,\ldots,m$,

$$
\begin{aligned}
\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2&\leq 2\iota^{2(1-\gamma)}\mathbb{E}\|\Phi_i\left(\iota,\varrho(\iota_i)\right)-\Phi_i\left(\iota,0\right)\|^2+2\iota^{2(1-\gamma)}\mathbb{E}\|\Phi_i\left(\iota,0\right)\|^2\\
&\leq 2\mathcal{L}_{\Phi_i}\iota^{2(1-\gamma)}\mathbb{E}\|\varrho(\iota)\|^2+2\iota^{2(1-\gamma)}\mathbb{E}\|\Phi_i\left(\iota,0\right)\|^2\\
&\leq 2\left(\mathcal{L}_{\Phi}\|\varrho\|_{\mathcal{X}}^2+\mathcal{M}_1\right),
\end{aligned}
$$

where $\mathcal{M}_1=\sup_{\iota\in J}\left\|\Phi_i\left(\iota,0\right)\right\|_{\mathcal{X}}^2$.

It implies that

$$\|\Psi\varrho\|_{\mathcal{X}}^2\leq 2\left(\mathcal{L}_{\Phi}\|\varrho\|_{\mathcal{X}}^2+\mathcal{M}_1\right):=\mathcal{N}_2.$$

For $\iota\in(s_i,\iota_{i+1}]$, $i=1,2,\ldots,m$,

$$
\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2\\
\leq&6\iota^{2(1-\gamma)}\mathbb{E}\left\|\mathcal{S}_{p,q}(\iota-s_i)\Phi_i\left(s_i,\varrho(s_i)\right)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+6m\iota^{2(1-\gamma)}\sum_{j=1}^{m}\frac{1}{\Gamma^2(\lambda_j)}\mathbb{E}\left\|\int_0^{s_i}(s_i-s)^{\lambda_j-1}\sigma_j\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)ds\right\|^2\\
&+6\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_i}(s_i-s)^{q-1}P_q(s_i-s)\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)ds\right\|^2\\
&+6\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_i}(s_i-s)^{q-1}P_q(s_i-s)\varpi_2(s,\varrho(s),\varrho(\kappa s))d\mathcal{W}(s)\right\|^2\\
&+6\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)ds\right\|^2\\
&+6\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\varpi_2(s,\varrho(s),\varrho(\kappa s))d\mathcal{W}(s)\right\|^2\\
\leq&\sum_{k=1}^{6}\mathcal{I}_k.
\end{aligned}
$$

By Lemma 2.6, we get

$$
\begin{aligned}
\mathcal{I}_1\leq&\frac{12M^2s_i^{2(1-\gamma)}}{\Gamma^2(\gamma)}\left(\mathcal{L}_\Phi\|\varrho\|^2_{\mathcal{X}}+\mathcal{M}_1\right)\\
\leq&\frac{12M^2s_i^{2(1-\gamma)}}{\Gamma^2(\gamma)}\left(\mathcal{L}_\Phi\tau+\mathcal{M}_1\right).
\end{aligned}
$$

Applying C-S inequality and (H_2), we arrive at

$$
\begin{aligned}
\mathcal{I}_2\leq&6m\iota^{2(1-\gamma)}s_i\sum_{j=1}^{m}\frac{1}{\Gamma^2(\lambda_j)}\int_0^{s_i}(s_i-s)^{2(\lambda_j-1)}\mathbb{E}\left\|\sigma_j\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2ds\\
\leq&6m\sum_{j=1}^{m}\frac{{\iota_{i+1}}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\left(m_{\sigma_j}+2n_{\sigma_j}\|\varrho\|^2_{\mathcal{X}}\right)\\
\leq&6m\sum_{j=1}^{m}\frac{{\iota_{i+1}}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\left(m_{\sigma_j}+2n_{\sigma_j}\tau\right).
\end{aligned}
$$

Using C-S inequality and $(H_1)\,(i)$, we get

$$
\begin{aligned}
\mathcal{I}_3\leq&\frac{6M^2\iota^{2(1-\gamma)}s_i}{\Gamma^2(q)}\int_0^{s_i}(s_i-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
\leq&\frac{6M^2{\iota_{i+1}}^{2(1-\gamma)}\iota_{i+1}}{\Gamma^2(q)}\int_0^{\iota_{i+1}}(\iota_{i+1}-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_5\leq&\frac{6M^2\iota^{2(1-\gamma)}\iota}{\Gamma^2(q)}\int_0^{\iota}(\iota-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
\leq&\frac{6M^2{\iota_{i+1}}^{2(1-\gamma)}\iota_{i+1}}{\Gamma^2(q)}\int_0^{\iota_{i+1}}(\iota_{i+1}-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds.
\end{aligned}
$$

Using Lemma 2.4 and $(H_1)\,(i)$, we get

$$
\mathcal{I}_4\leq\frac{6M^2\iota^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^{s_i}(s_i-s)^{2(q-1)}\chi_{\varpi_2}(s)\Theta_{\varpi_2}\left(2\tau\right)ds
$$

$$\leq \frac{6M^2 {\iota_{i+1}}^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1}-s)^{2(q-1)} \chi_{\varpi_2}(s) \Theta_{\varpi_2}(2\tau)\, ds,$$

and

$$\begin{aligned}\mathcal{I}_6 \leq & \frac{6M^2 \iota^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^{\iota} (\iota-s)^{2(q-1)} \chi_{\varpi_2}(s) \Theta_{\varpi_2}(2\tau)\, ds \\ \leq & \frac{6M^2 {\iota_{i+1}}^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1}-s)^{2(q-1)} \chi_{\varpi_2}(s) \Theta_{\varpi_2}(2\tau)\, ds.\end{aligned}$$

Combining these estimates, $\mathcal{I}_1 - \mathcal{I}_6$ yields

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}\Psi\varrho(\iota)\right\|^2 \\ \leq & \frac{12M^2 s_i^{2(1-\gamma)}}{\Gamma^2(\gamma)}\left(\mathcal{L}_\Phi \tau + \mathcal{M}_1\right) + 6m\sum_{j=1}^{m} \frac{{\iota_{i+1}}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\left(m_{\sigma_j} + 2n_{\sigma_j}\tau\right) \\ & + \frac{12M^2 {\iota_{i+1}}^{2(1-\gamma)}\iota_{i+1}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1}-s)^{2(q-1)} \chi_{\varpi_1}(s) \Theta_{\varpi_1}(2\tau)\, ds \\ & + \frac{12M^2 {\iota_{i+1}}^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1}-s)^{2(q-1)} \chi_{\varpi_2}(s) \Theta_{\varpi_2}(2\tau)\, ds \\ := & \mathcal{N}_3.\end{aligned}$$

Let $\mathcal{N} = \max\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$. As a result, for every $\varrho \in \mathcal{B}_\tau$, it follows that $\|\Psi\varrho\|_{\mathcal{X}}^2 \leq \mathcal{N}$.

Step 2. Ψ_1 is ϑ-Lipschitz.

Take $\varrho, \xi \in \mathcal{X}$, $\iota \in (0, \iota_1]$, and using Lemma 2.3, we have

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \\ \leq & \iota^{2(1-\gamma)} m \sum_{j=1}^{m} \frac{1}{\Gamma^2(\lambda_j)} \mathbb{E}\left\|\int_0^{\iota} (\iota-s)^{\lambda_j-1}\left[\sigma_j\left(s,\varrho(s),\varrho(\kappa s)\right) - \sigma_j\left(s,\xi(s),\xi(\kappa s)\right)\right] ds\right\|^2.\end{aligned}$$

By applying the C-S inequality and (H_2), we arrive at

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \\ \leq & \iota^{2(1-\gamma)} m\iota \sum_{j=1}^{m} \frac{\mathcal{L}_{\sigma_j}}{\Gamma^2(\lambda_j)} \int_0^{\iota} (\iota-s)^{2(\lambda_j-1)}\Big[s^{2(1-\gamma)}\mathbb{E}\left\|\varrho(s) - \xi(s)\right\|^2 \\ & + s^{2(1-\gamma)}\mathbb{E}\left\|\varrho(\kappa s) - \xi(\kappa s)\right\|^2\Big] ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \leq 2m\sum_{j=1}^{m} \frac{\mathcal{L}_{\sigma_j}{\iota_1}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\|\varrho - \xi\|_{\mathcal{X}}^2.$$

Consequently,

$$\|\Psi_1\varrho - \Psi_1\xi\|_{\mathcal{X}}^2 \leq 2m\Delta_{10}\|\varrho - \xi\|_{\mathcal{X}}^2.$$

For $\iota \in (\iota_i, s_i]$, $i = 1, 2, \ldots, m$,

$$\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \leq \mathcal{L}_{\Phi_i}\|\varrho - \xi\|_{\mathcal{X}}^2.$$

Consequently,

$$\|\Psi_1\varrho - \Psi_1\xi\|_{\mathcal{X}}^2 \leq \mathcal{L}_{\Phi}\|\varrho - \xi\|_{\mathcal{X}}^2.$$

For $\iota \in (s_i, \iota_{i+1}]$, $i = 1, 2, \ldots, m$,

By using Lemma 2.3, we get

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \\
\leq &4\iota^{2(1-\gamma)}\mathbb{E}\left\|\mathcal{S}_{p,q}(\iota - s_i)\left\{\Phi_i\left(s_i, \varrho(s_i)\right) - \Phi_i\left(s_i, \xi(s_i)\right)\right\}\right\|^2 \\
&+ 4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_i}(s_i - s)^{q-1}P_q(s_i - s)\left[\varpi_1\left(s, \varrho(s), \varrho\left(\kappa s\right)\right)\right.\right. \\
&\left.\left.-\varpi_1\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right]ds\right\|^2 \\
&+ 4\iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_i}(s_i - s)^{q-1}P_q(s_i - s)\left[\varpi_2\left(s, \varrho(s), \varrho\left(\kappa s\right)\right)\right.\right. \\
&\left.\left.-\varpi_2\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right]d\mathcal{W}(s)\right\|^2 \\
&+ 4m\iota^{2(1-\gamma)}\sum_{j=1}^{m}\frac{1}{\Gamma^2(\lambda_j)}\mathbb{E}\left\|\int_0^{s_i}(s_i - s)^{\lambda_j-1}\left[\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right)\right.\right. \\
&\left.\left.-\sigma_j\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right]ds\right\|^2.
\end{aligned}$$

By applying the C-S inequality and Lemma 2.4, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \\
\leq &4\iota^{2(1-\gamma)}\mathbb{E}\left\|\mathcal{S}_{p,q}(\iota - s_i)\left\{\Phi_i\left(s_i, \varrho(s_i)\right) - \Phi_i\left(s_i, \xi(s_i)\right)\right\}\right\|^2 \\
&+ \frac{4M^2\iota^{2(1-\gamma)}s_i}{\Gamma^2(q)}\int_0^{s_i}(s_i - s)^{2(q-1)}\mathbb{E}\left\|\varpi_1\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) - \varpi_1\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right\|^2 ds \\
&+ \frac{4M^2\iota^{2(1-\gamma)}}{\Gamma^2(q)}\int_0^{s_i}(s_i - s)^{2(q-1)}\mathbb{E}\left\|\varpi_2\left(s, \varrho(s), \varrho\left(\kappa s\right)\right) - \varpi_2\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right\|^2 ds \\
&+ 4m\iota^{2(1-\gamma)}\iota\sum_{j=1}^{m}\frac{1}{\Gamma^2(\lambda_j)}\int_0^{\iota}(\iota - s)^{2(\lambda_j-1)}\mathbb{E}\left\|\sigma_j\left(s, \varrho(s), \varrho\left(\kappa s\right)\right)\right. \\
&\left.-\sigma_j\left(s, \xi(s), \xi\left(\kappa s\right)\right)\right\|^2 ds.
\end{aligned}$$

It follows from (H_1), (H_2), and (H_3) that

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_1\varrho(\iota) - \Psi_1\xi(\iota)\right)\right\|^2 \\
\leq &\frac{4M^2\mathcal{L}_{\Phi_i}{\iota_{i+1}}^{2(\gamma-1)}}{\Gamma^2(\gamma)}\|\varrho - \xi\|_{\mathcal{X}}^2 + \frac{8\mathcal{L}_{\varpi_1}{\iota_{i+1}}^{2-2\gamma+2q}}{\Gamma^2(q)2q-1}\|\varrho - \xi\|_{\mathcal{X}}^2 \\
&+ \frac{8\mathcal{L}_{\varpi_2}{\iota_{i+1}}^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}\|\varrho - \xi\|_{\mathcal{X}}^2 + 8m\sum_{j=1}^{m}\frac{\mathcal{L}_{\sigma_j}{\iota_{i+1}}^{2-2\gamma+2\lambda_j}}{\Gamma^2(\lambda_j)2\lambda_j-1}\|\varrho - \xi\|_{\mathcal{X}}^2.
\end{aligned}$$

Then,

$$\|\Psi_1\varrho - \Psi_2\xi\|_{\mathcal{X}}^2 \leq \left(\frac{4M^2\mathcal{L}_{\Phi}\Theta_3}{\Gamma^2(\gamma)} + \frac{8\mathcal{L}_{\varpi_1}\Theta_1}{\Gamma^2(q)2q-1} + \frac{8\mathcal{L}_{\varpi_2}\Theta_2}{\Gamma^2(q)2q-1} + 8m\Delta_1\right)\|\varrho - \xi\|_{\mathcal{X}}^2.$$

Consequently,

$$\|\Psi_1\varrho - \Psi_2\xi\|_{\mathcal{X}}^2 \leq \bar{L}\|\varrho - \xi\|_{\mathcal{X}}^2.$$

Thus, Ψ_1 satisfies the Lipschitz condition with the constant $\bar{L}$. By Proposition 2.2, Ψ_1 is ϑ-Lipschitz with the same constant $\bar{L}$.

Step 3. Ψ_2 is continuous.

For $\iota \in (0, \iota_1]$ and $\iota \in (s_i, \iota_{i+1}]$, $i = 1, 2, \ldots, m$.

Consider a sequence $\varrho_n \to \varrho$ in $\mathcal{X}$. Then, under the assumption (H_1), it follows that

$$\lim_{n\to\infty} \varpi_j\left(s, \varrho_n(s), \varrho_n(\kappa s)\right) = \varpi_j\left(s, \varrho(s), \varrho(\kappa s)\right), \quad \text{for} \quad j = 1, 2,$$

and

$$\mathbb{E}\left\|\varpi_j\left(s, \varrho_n(s), \varrho_n(\kappa s)\right) - \varpi_j\left(s, \varrho(s), \varrho(\kappa s)\right)\right\|^2 \leq 4\chi_{\varpi_j}(s)\Theta_{\varpi_j}(2\tau), \quad \text{for} \quad j = 1, 2.$$

By the C-S inequality and Lemma 2.4, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\left(\Psi_2\varrho_n(\iota) - \Psi_2\varrho(\iota)\right)\right\|^2 \\
\leq& \frac{4M^2\iota^{2(1-\gamma)}\iota}{\Gamma^2(q)} \int_0^\iota (\iota - s)^{2(q-1)} \mathbb{E}\left\|\varpi_1\left(s, \varrho_n(s), \varrho_n(\kappa s)\right) - \varpi_1\left(s, \varrho(s), \varrho(\kappa s)\right)\right\|^2 ds \\
&+ \frac{4M^2\iota^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^\iota (\iota - s)^{2(q-1)} \mathbb{E}\left\|\varpi_2\left(s, \varrho_n(s), \varrho_n(\kappa s)\right) - \varpi_2\left(s, \varrho(s), \varrho(\kappa s)\right)\right\|^2 ds.
\end{aligned}$$

Then,

$$\begin{aligned}
&\|\Psi_2\varrho_n - \Psi_2\varrho\|^2_{\mathcal{X}} \\
\leq& \frac{4M^2\iota_{i+1}^{2(1-\gamma)}\iota_{i+1}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1} - s)^{2(q-1)} \mathbb{E}\left\|\varpi_1\left(s, \varrho_n(s), \varrho_n(\kappa s)\right)\right. \\
&\left.-\varpi_1\left(s, \varrho(s), \varrho(\kappa s)\right)\right\|^2 ds \\
&+ \frac{4M^2\iota_{i+1}^{2(1-\gamma)}}{\Gamma^2(q)} \int_0^{\iota_{i+1}} (\iota_{i+1} - s)^{2(q-1)} \mathbb{E}\left\|\varpi_2\left(s, \varrho_n(s), \varrho_n(\kappa s)\right)\right. \\
&\left.-\varpi_2\left(s, \varrho(s), \varrho(\kappa s)\right)\right\|^2 ds.
\end{aligned}$$

By the Lebesgue dominated convergence theorem

$$\|\Psi_2\varrho_n - \Psi_2\varrho\|^2_{\mathcal{X}} \to 0 \quad \text{as} \quad n \to \infty.$$

For $\iota \in (\iota_i, s_i]$, $i = 1, 2, \ldots, m$.

$$\|\Psi_2\varrho_n - \Psi_2\varrho\|^2_{\mathcal{X}} = 0.$$

Consequently,

$$\|\Psi_2\varrho_n - \Psi_2\varrho\|^2_{\mathcal{X}} \to 0 \quad \text{as} \quad n \to \infty,$$

which confirms that Ψ_2 is continuous.

Step 4. Ψ_2 maps bounded sets into equicontinuous sets of $\mathcal{B}_\tau$.

Let $\tau_1, \tau_2 \in (s_i, \iota_{i+1}]$, $i = 0, 1, \ldots, m$, $\tau_1 < \tau_2$, and $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\left\|\tau_2^{1-\gamma}\Psi_2\varrho(\tau_2) - \tau_1^{1-\gamma}\Psi_2\varrho(\tau_1)\right\|^2 \\
\leq& 2\mathbb{E}\left\|\tau_2^{2(1-\gamma)} \int_0^{\tau_2} (\tau_2 - s)^{q-1} P_q(\tau_2 - s)\varpi_1\left(s, \varrho(s), \varrho(\kappa s)\right) ds\right.
\end{aligned}$$

$$
\begin{aligned}
&-\tau_1^{2(1-\gamma)}\int_0^{\tau_1}(\tau_1-s)^{q-1}P_q(\tau_1-s)\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)ds\Big\|^2\\
&+2\mathbb{E}\Big\|\tau_2^{2(1-\gamma)}\int_0^{\tau_2}(\tau_2-s)^{q-1}P_q(\tau_2-s)\varpi_2\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)d\mathcal{W}(s)\\
&-\tau_1^{2(1-\gamma)}\int_0^{\tau_1}(\tau_1-s)^{q-1}P_q(\tau_1-s)\varpi_2\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)d\mathcal{W}(s)\Big\|^2\\
&\leq \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}
$$

By Lemma 2.3, we get

$$
\begin{aligned}
&\mathcal{I}_1\\
\leq&8\mathbb{E}\left\|\int_0^{\tau_1}\left[\tau_2^{2(1-\gamma)}(\tau_2-s)^{q-1}-\tau_1^{2(1-\gamma)}(\tau_1-s)^{q-1}\right]P_q(\tau_2-s)\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2\\
&+8\mathbb{E}\left\|\int_0^{\tau_1}\tau_1^{2(1-\gamma)}(\tau_1-s)^{q-1}\left[P_q(\tau_2-s)-P_q(\tau_1-s)\right]\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2\\
&+4\mathbb{E}\left\|\int_{\tau_1}^{\tau_2}\tau_2^{2(1-\gamma)}(\tau_2-s)^{q-1}P_q(\tau_2-s)\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right)\right)\right\|^2.
\end{aligned}
$$

Using C-S inequality and (H_1), we get

$$
\begin{aligned}
&\mathcal{I}_1\\
&\leq\frac{8M^2\tau_1}{\Gamma^2(q)}\int_0^{\tau_1}\left[\tau_2^{2(1-\gamma)}(\tau_2-s)^{q-1}-\tau_1^{2(1-\gamma)}(\tau_1-s)^{q-1}\right]^2\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&\quad+16\tau_1^{2(1-\gamma)}\tau_1\left(\sup_{s\in[0,\tau_1-\varepsilon]}\left\|P_q\left(\tau_2-s\right)-P_q\left(\tau_1-s\right)\right\|\right)^2\\
&\quad\times\int_0^{\tau_1-\epsilon}(\tau_1-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&\quad+\frac{64M^2\tau_1^{2(1-\gamma)}\epsilon}{\Gamma^2(q)}\int_{\tau_1-\epsilon}^{\tau_1}(\tau_1-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&\quad+\frac{4M^2\tau_2^{2(1-\gamma)}(\tau_2-\tau_1)}{\Gamma^2(q)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2(q-1)}\chi_{\varpi_1}(s)\Theta_{\varpi_1}\left(2\tau\right)ds\\
&\longrightarrow 0,\quad\text{as}\quad\tau_1\longrightarrow\tau_2,\epsilon\longrightarrow 0.
\end{aligned}
$$

Similar, by applying Lemma 2.4, we can get

$$
\begin{aligned}
\mathcal{I}_2\leq&\frac{8M^2}{\Gamma^2(q)}\int_0^{\tau_1}\left[\tau_2^{2(1-\gamma)}(\tau_2-s)^{q-1}-\tau_1^{2(1-\gamma)}(\tau_1-s)^{q-1}\right]^2\chi_{\varpi_2}(s)\,\Theta_{\varpi_2}(2\tau)\,ds\\
&+16\tau_1^{2(1-\gamma)}\left(\sup_{s\in[0,\tau_1-\varepsilon]}\left\|P_q(\tau_2-s)-P_q(\tau_1-s)\right\|\right)^2\\
&\times\int_0^{\tau_1-\varepsilon}(\tau_1-s)^{2(q-1)}\chi_{\varpi_2}(s)\,\Theta_{\varpi_2}(2\tau)\,ds\\
&+\frac{64M^2\tau_1^{2(1-\gamma)}}{\Gamma^2(q)}\int_{\tau_1-\varepsilon}^{\tau_1}(\tau_1-s)^{2(q-1)}\chi_{\varpi_2}(s)\,\Theta_{\varpi_2}(2\tau)\,ds
\end{aligned}
$$

$$+\frac{4M^2\tau_2^{2(1-\gamma)}}{\Gamma^2(q)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{2(q-1)}\chi_{\varpi_2}(s)\,\Theta_{\varpi_2}(2\tau)\,ds$$
$$\longrightarrow 0, \quad \text{as } \tau_1\longrightarrow\tau_2,\ \varepsilon\longrightarrow 0.$$

Therefore, we find that

$$\mathbb{E}\left\|\tau_2^{1-\gamma}\Psi_2\varrho(\tau_2)-\tau_1^{1-\gamma}\Psi_2\varrho(\tau_1)\right\|^2$$

approaches zero as $\tau_1\to\tau_2$, independently of $\varrho\in\mathcal{B}_\tau$. This implies that $\{\Psi_2\varrho,\varrho\in\mathcal{B}_\tau\}$ is equicontinuous.

Step 5. Ψ is ϑ-condensing.

From Step 2, we established that Ψ_1 is ϑ-Lipschitz with the constant $\bar{L}$. Consequently, for any bounded set $\Lambda\subset\mathcal{B}_\tau$, we have

$$\vartheta(\Psi_1(\Lambda))\le\bar{L}\vartheta(\Lambda).$$

Now, by the Lemma 2.7, since Ψ_2 is bounded, continuous, and equicontinuous map, for any bounded set $\Lambda\subset\mathcal{B}_\tau$, there exists a countable set $\Lambda_0=\{\varrho_n\}\subset\Lambda$ such that

$$\begin{aligned}\vartheta(\Psi_2(\Lambda))&=\vartheta(\Psi_2(\Lambda_0))\\&=\max_{\iota\in(s_i,\iota_{i+1}]}\vartheta(\Psi_2(\Lambda_0)(\iota)),\quad i=0,1,\dots,m,\\\vartheta(\Psi_2(\Lambda_0)(\iota))&\le\vartheta\left(\int_0^\iota(\iota-s)^{q-1}P_q(\iota-s)\varpi_1(s,\varrho_n(s),\varrho_n(\kappa s))ds\right.\\&\quad\left.+\int_0^\iota(\iota-s)^{q-1}P_q(\iota-s)\varpi_2(s,\varrho_n(s),\varrho_n(\kappa s))d\mathcal{W}(s)\right)\\&\le\frac{M}{\Gamma(q)}\int_0^\iota(\iota-s)^{q-1}\vartheta\left(\varpi_1(s,\varrho_n(s),\varrho_n(\kappa s))\right)ds\\&\quad+\frac{M}{\Gamma(q)}\int_0^\iota(\iota-s)^{q-1}\vartheta\left(\varpi_2(s,\varrho_n(s),\varrho_n(\kappa s))\right)ds.\end{aligned}$$

By (H_1) (iii), we have

$$\begin{aligned}\vartheta(\Psi_2(\Lambda_0)(\iota))\le&\frac{2M}{\Gamma(q)}\vartheta(\Lambda)\int_0^\iota(\iota-s)^{q-1}\varphi_{\varpi_1}(s)ds\\&+\frac{2M}{\Gamma(q)}\vartheta(\Lambda)\int_0^\iota(\iota-s)^{q-1}\varphi_{\varpi_2}(s)ds\\\le&\frac{2M\iota_{i+1}^q}{q}\left(\varpi_1^*+\varpi_2^*\right)\vartheta(\Lambda).\end{aligned}$$

Consequently,

$$\begin{aligned}\vartheta(\Psi(\Lambda))&\le\vartheta(\Psi_1(\Lambda))+\vartheta(\Psi_2(\Lambda))\\&\le\left(\frac{2M\iota_{i+1}^q}{q}\left(\varpi_1^*+\varpi_2^*\right)+\bar{L}\right)\vartheta(\Lambda)\\&<\vartheta(\Lambda).\end{aligned}$$

Thus, Ψ is a ϑ-condensing map. Hence, the hypotheses of Theorem 2.1 are satisfied, which implies that problem (1.1) has at least one mild solution. □

4. An example

Consider the following FSPDEs with HFD and N-II

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{0.5,0.83}\left(\varrho(\iota,z)-\mathfrak{I}_{0^+,\iota}^{0.85}\sigma_1\left(\iota,\varrho(\iota,z),\varrho\left(\kappa\iota,z\right)\right)\right) \\ =\dfrac{\partial^2}{\partial\xi^2}\varrho(\iota,z)+\varpi_1\left(\iota,\varrho(\iota,z),\varrho\left(\kappa\iota,z\right)\right) \\ \quad+\varpi_2\left(\iota,\varrho(\iota,z),\varrho\left(\kappa\iota,z\right)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota}, \quad \iota\in[0,\dfrac{1}{2}]\cup[\dfrac{3}{4},1],\ z\in[0,\pi], \\ \varrho(\iota,z)=\Phi_1\left(\iota,\varrho(\iota,z)\right), \quad \iota\in[\dfrac{1}{2},\dfrac{3}{4}], \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma}\left(\varrho(0,z)-\mathfrak{I}_{0^+,\iota}^{0.85}\sigma_1\left(0,\varrho(0,z),\varrho\left(0,z\right)\right)\right)=\varrho_0(z). \end{cases} \tag{4.1}$$

Define the operator $\mathcal{A}$ by $\mathcal{A}\varsigma=\frac{\partial^2\varsigma}{\partial\xi^2}$, where

$$D(\mathcal{A})=\left(\varsigma\in\mathcal{E}:\varsigma \text{ and } \frac{\partial\varsigma}{\partial\xi} \text{ are absolutely continuous, } \frac{\partial^2\varsigma}{\partial\xi^2}\in\mathcal{E},\ \varsigma(0)=\varsigma(\pi)=0\right).$$

It is easy to check that $\mathcal{A}$ generates a strongly continuous semigroup $\{\mathcal{S}(\iota)\}_{\iota\geq 0}$ which is compact, analytic, and self-adjoint.

We can rewrite the above problem into the abstract form

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+,\iota}^{0.5,0.83}\left(\varrho(\iota)-\mathfrak{I}_{0^+,\iota}^{0.85}\sigma_1\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right)\right) \\ =\dfrac{\partial^2}{\partial\xi^2}x(\iota)+\varpi_1\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right) \\ \quad+\varpi_2\left(\iota,\varrho(\iota),\varrho\left(\kappa\iota\right)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota}, \quad \iota\in[0,\dfrac{1}{2}]\cup[\dfrac{3}{4},1], \\ \varrho(\iota)=\Phi_1\left(\iota,\varrho(\iota)\right), \quad \iota\in[\dfrac{1}{2},\dfrac{3}{4}], \\ \mathfrak{I}_{0^+,\iota}^{1-\gamma}\left(\varrho(0)-\mathfrak{I}_{0^+,\iota}^{0.85}\sigma_1\left(0,\varrho(0),\varrho\left(0\right)\right)\right)=\varrho_0, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} &\varrho(\iota)(z)=\varrho(\iota,z),\\ &\varpi_1(\iota,\varrho(\iota),\varrho(\kappa\iota))(z)=\varpi_1(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z)),\\ &\varpi_2(\iota,\varrho(\iota),\varrho(\kappa\iota))(z)=\varpi_2(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z)),\\ &\sigma_1(\iota,\varrho(\iota),\varrho(\kappa\iota))(z)=\sigma_1(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z)),\\ &\varrho_0(z)=\sin(z), \end{aligned} \tag{4.3}$$

where $\mathcal{E}=L^2\left([0,\pi],\mathbb{R}\right)$, with the norm $\|\cdot\|$.

Let

$$\begin{aligned} \varpi_1(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z))&=\frac{e^{-2\iota}}{15\sqrt{2}}\sin(\varrho(\kappa\iota,z)),\\ \varpi_2(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z))&=\frac{|\varrho(\kappa\iota,z)|}{8\sqrt{8}},\\ \sigma_1(\iota,\varrho(\iota,z),\varrho(\kappa\iota,z))&=\frac{e^{-2\iota}}{11\left(\iota^2+2\right)}+\frac{\cos(\varrho(\kappa\iota,z))}{10\sqrt{3}}, \end{aligned}$$

$$\Phi_1(\iota, \varrho(\iota, z)) = \frac{|\varrho(\iota, z)|}{10\,(1 + |\varrho(\iota, z)|)}.$$

Then, for any bounded set $\Lambda \subset \mathcal{E}$, we estime

$$\vartheta\left(\varpi_k(\iota, \Lambda(\iota, z), \Lambda(\kappa\iota, z))\right) \leq \varphi_{\varpi_k}(\iota)\left[\vartheta(\Lambda(\iota, z)) + \Lambda(\kappa\iota, z))\right], \quad k = 1, 2.$$

Here, $p = 0.5$, $q = 0.83$, $\kappa = 0.8$, $\lambda_1 = 0.85$, $s_0 = \iota_0 = 0$, $\iota_1 = \frac{1}{2}$, $s_1 = \frac{3}{4}$, $\iota_2 = 1$. The assumptions (H_1), (H_2), and (H_3) are satisfied with $\mathcal{L}_{\varpi_1} = \frac{1}{450}$, $\mathcal{L}_{\varpi_2} = \frac{1}{512}$, $\mathcal{L}_\sigma = \frac{1}{300}$, $\mathcal{L}_\Phi = \frac{1}{100}$, $\varpi_1^* = \frac{1}{15\sqrt{2}}$, and $\varpi_2^* = \frac{1}{8\sqrt{8}}$. Additionally, we find $\bar{L} = 0.1105$ and $\frac{2M\Theta_4}{q}\left(\varpi_1^* + \varpi_2^*\right) + \bar{L} = 0.3306 < 1$. Thus, by Theorem 3.1, problem (4.1) has at least one mild solution.

Next, we present the numerical results for $\frac{2M\Theta_4}{q}\left(\varpi_1^* + \varpi_2^*\right) + \bar{L}$. These results are detailed in Table 1 and visualized in Figure 1. The Table 1 demonstrates how $\frac{2M\Theta_4}{q}\left(\varpi_1^* + \varpi_2^*\right) + \bar{L}$ decrease significantly when the orders increases and is less than 1.

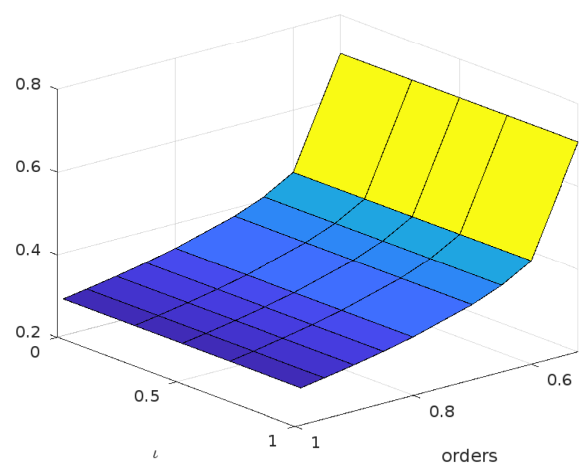

Figure 1. Lipschitz constant with different values of orders.

Table 1. Lipschitz constant for various order values.

Time	q = 0.52	q = 0.60	q = 0.65	q = 0.70	q = 0.83	q = 0.85	q = 0.90	q = 0.95
0.00	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.50	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.60	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.70	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.75	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.80	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.85	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.90	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
0.95	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981
1.00	0.7070	0.4495	0.4093	0.3811	0.3306	0.3246	0.3106	0.2981

5. Conclusion

In this paper, we investigated the existence of mild solutions for a class of FSPDEs involving the HFD and N-II. By employing tools such as the Hausdorff measure of noncompactness, Sadovskii's fixed point theorem, condensing operator theory, semigroup theory, and stochastic analysis, we established sufficient conditions for the existence of solutions under appropriate assumptions.

To support the theoretical findings, an illustrative example was provided, highlighting how variations in the fractional order influence the Lipschitz constant and the behavior of the solutions. Graphical simulations further confirmed the theoretical results and demonstrated the role of fractional parameters in the solution structure.

The inclusion of non-instantaneous impulses, pantograph terms, and stochastic components enhances the realism of modeling complex dynamical systems. Future research may extend this framework to study uniqueness, controllability, or numerical approximation methods for such systems.

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