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LIOUVILLE THEOREM FOR INDEFINITE FRACTIONAL PARABOLIC EQUATION INVOLVING PSEUDO-RELATIVISTIC SCHRÖDINGER OPERATORS

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Abstract In this paper, we study the following indefinite fractional parabolic equation involving pseudo-relativistic Schrödinger operators

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta + m^2)^s u(x,t) = a(x_1) f(u(x,t)), \quad \text{in } \mathbb{R}^N \times \mathbb{R},$$

where $0 < s < 1$ and the mass $m > 0$. We first prove the monotonicity of positive bounded solutions by using the method of moving planes. Moreover, the nonexistence of positive bounded solutions is established.

Keywords Indefinite nonlinearity, pseudo-relativistic Schrödinger operators, Liouville theorem, methods of moving planes.

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1. Introduction

In this paper, we study the monotonicity and nonexistence of positive solutions to the following problem

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta + m^2)^s u(x,t) = a(x_1) f(u(x,t)), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $0 < s < 1$ and $m > 0$. The nonlocal pseudo-relativistic Schrödinger operator $(-\Delta + m^2)^s$ is defined as

$$\begin{aligned} &(-\Delta + m^2)^s u(x,t) \\ &= C_{N,s} m^{\frac{N}{2}+s} P.V. \int_{\mathbb{R}^N} \frac{u(x,t) - u(y,t)}{|x-y|^{\frac{N}{2}+s}} K_{\frac{N}{2}+s}(m|x-y|) dy + m^{2s} u(x,t), \end{aligned} \tag{1.2}$$

where $P.V.$ denotes the Cauchy principal value and

$$C_{N,s} = 2^{1-\frac{N}{2}+s} \pi^{-\frac{N}{2}} \frac{s(1-s)}{\Gamma(2-s)},$$

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see [15]. To ensure that the integral (1.2) is well-defined, we assume that $u \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s$ with

$$\mathcal{L}_s = \left\{ u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \mid \int_{\mathbb{R}^N} \frac{e^{-|x|}|u(x,t)|}{1+|x|^{\frac{N+1}{2}+s}} dx < \infty \right\}.$$

When $m \to 0^+$, $(-\Delta + m^2)^s$ becomes the linear fractional Laplacian $(-\Delta)^s$.

The kernel K_ν denotes the modified Bessel function with order ν. It satisfies the following equation

$$r^2 K_\nu^{''} + rK' - (r^2 + \nu^2)K_\nu = 0.$$

For $\nu > 0$, we have

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu},$$

as $r \to 0$ and $K_\nu = K_{-\nu}$. For $\nu > 0$, we have

$$K_\nu(r) \sim \frac{\sqrt{\pi}}{\sqrt{2}} r^{-\frac{1}{2}} e^{-r},$$

as $r \to \infty$ (see [14,15]). That is, there exists a small constant $r_0 > 0$ and constants $C_0 > c_0 > 0$ such that

$$\frac{c_0}{r^\nu} \le K_\nu(r) \le \frac{C_0}{r^\nu} \quad \text{for } r \ge r_0.$$

Also, there exists a large constant $R_\infty > 0$ and constants $C_\infty > c_\infty > 0$ such that

$$\frac{c_\infty}{r^{\frac{1}{2}} e^r} \le K_\nu(r) \le \frac{C_\infty}{r^{\frac{1}{2}} e^r} \quad \text{for } r \ge R_\infty.$$

The operator $(-\Delta + m^2)^s$ have many applications in anomalous diffusion, finance, optimization and others (see [1,3,12,20]). In particular, $(-\Delta + m^2)^s - m^2$ can be seen as an infinitesimal generator of a Levy process called the $2s$-stable relativistic process. For $s = \frac{1}{2}$, it is related to the Hamiltonian $\mathcal{H} = \sqrt{p^2c^2 + m^2c^4}$ of a free relativistic particle of momentum p and mass m.

It is worth noting that some classical results valid for local operators fail to extend to the nonlocal operator $(-\Delta + m^2)^s$ $(0 < s < 1,\ m \ge 0)$ due to its nonlocal nature. To overcome this difficulty, Caffarelli and Silvestre [2] introduced an extension method, which reduces the nonlocal problems into a local one in higher dimensions. Another useful approach to deal with such a nonlocal problem is to derive and investigate its equivalent integral equation, see [2,6,7]. Later, a direct method of moving planes was introduced in [5], which can be applied directly to the nonlocal problems involving $(-\Delta)^s$ to obtain symmetry and monotonicity of positive solutions.

In recent years, there have been a large number of results about the monotonicity and nonexistence of positive solutions to the nonlocal problems involving $(-\Delta+m^2)^s$ $(m \ge 0)$ in bounded and unbounded domains, see [4,8,10,11,13,16,17,21,22]. We focus on the typical example for indefinite fractional equation involving $(-\Delta+m^2)^s$ $(m \ge 0)$, especially the nonexistence of positive solutions. It is well known that the Liouville theorem plays crucial role in deriving a prior estimate and uniqueness of solutions. Recently, Guo and Peng [17] studied the following indefinite fractional equation

$$(-\Delta+m^2)^s u(x)=a(x_1)f(u,\nabla u), \quad x\in\mathbb{R}^N, \tag{1.3}$$

where $0<s<1$ and $m>0$. Under certain assumptions on $a(x_1)$ and $f(u,\nabla u)$, the authors proved the monotonicity and nonexistence of positive bounded solutions for equation (1.3) via the method of moving planes. In [21], Wang investigated the following nonlocal pseudo-relativistic equation

$$(-\Delta+m^2)^s u(x)=f(x,u), \quad x\in\mathbb{R}^N,$$

and obtained the radial symmetry and monotonicity of positive solutions. In the case of $m=0$ and $f(u,\nabla u)=f(u)$, equation (1.3) is reduced to the following problem

$$(-\Delta)^s u(x)=a(x_1)f(u), \quad x\in\mathbb{R}^N.$$

When $\frac{1}{2}<s<1$ and $a(x_1)f(u)=x_1u^p$ $(1<p<\infty)$, Chen and Zhu [10] obtained the nonexistence of positive solutions of (1). Later, Chen, Li and Zhu [8] extended the results of [10] to the general case $0<s<1$ and indefinite nonlinearity $a(x_1)f(u)$.

In the case of indefinite parabolic problem, Poláik and Quittner [18] investigated the monotonicity and nonexistence of positive bounded solutions to the following equation

$$\frac{\partial u}{\partial t}(x,t)-\Delta u(x,t)=a(x_1)u^p(x,t), \quad \text{in } \mathbb{R}^N\times\mathbb{R}.$$

Recently, Chen, Wu and Wang in [9] extended the results of [18] to the following fractional parabolic problem with indefinite nonlinearities:

$$\frac{\partial u}{\partial t}(x,t)+(-\Delta)^s u(x,t)=x_1u^p(x,t), \quad \text{in } \mathbb{R}^N\times\mathbb{R}. \tag{1.4}$$

In contrast, not much is known for the fractional parabolic pseudo-relativistic Schrödinger equations since the modified Bessel function $K_{\frac{N}{2}+s}$ leads the inhomogeneity of operator $(-\Delta+m^2)^s$. This makes that the invariance properties under Kelvin-type or scaling transforms are not valid for the operator $(-\Delta+m^2)^s$. In this context, our focus is on the monotonicity and nonexistence of positive solutions to (1.1).

To show that the positive solutions of (1.1) is monotonically increasing in x_1-direction, we employ the method of moving planes. In order to state our main results, we introduce some well known notations.

For $\lambda\in\mathbb{R}$, let

$$T_\lambda=\{x\in\mathbb{R}^N \mid x_1=\lambda\}$$

be the moving plane. Let

$$\Sigma_\lambda=\{x\in\mathbb{R}^N \mid x_1<\lambda\}$$

be the region to the left of T_λ, and let

$$x_\lambda=(2\lambda-x_1,x_2...,x_N)$$

be the the reflection of the point x about the plane T_λ. To compare the value of $u_\lambda(x,t)$ and $u(x,t)$, we denote

$$w_\lambda(x,t)=u_\lambda(x,t)-u(x,t),$$

where $u_\lambda(x,t) = u(x_\lambda, t)$.

Now we state our main results.

Theorem 1.1. *Let* $u(x,t) \in (C^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s) \times C^1(\mathbb{R})$ *be a positive bounded solution of equation* (1.1)*. Assume that*
$(H1)$ $a(x_1) \in C^s(\mathbb{R})$ *and* $a(x_1)$ *is nondecreasing in* x_1*;*
$(H2)$ $a(x_1) > 0$ *somewhere for* $x_1 > 0$ *and* $a(x_1) < 0$ *for* $x_1 < 0$*;*
$(H3)$ $f(u)$ *is locally Lipschitz continuous and nondecreasing with respect to* u*,* $f(0) = 0$ *and* $f > 0$ *in* $(0, +\infty)$*.*

Then $u(x,t)$ *is monotonically increasing in* x_1*.*

Remark 1.1. In case of $m = 0$, $a(x_1) = x_1$ and $f(u) = u^p$ $(1 < p < \infty)$, the equation (1.1) is reduced to (1.4).

Motivated by the recent work of [17] and [9], we establish the monotonicity result of equation (1.1). The main tool we use is the direct method of moving planes, which was introduced in [5]. Recently, Chen, Wu and Wang [9] introduced some new ideas such that the direct method of moving planes can be applied to the indefinite fractional parabolic problem. We will use it to prove Theorem 1.1 with some modifications.

Based on above result, we prove the nonexistence of positive bounded solution to equation (1.1).

Theorem 1.2. *Assume that* $u(x,t) \in (C^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s) \times C^1(\mathbb{R})$ *is a positive solution of equation* (1.1) *and satisfies conditions* $(H1) - (H3)$*. Suppose further that there exists a constant* $C > 0$ *such that* $\lim\limits_{x \to +\infty} f(u) \geq Cu^\kappa$ *with* $\kappa \geq 1$*, and* $\lim\limits_{x_1 \to +\infty} a(x_1) = +\infty$*. Then equation* (1.1) *has no positive bounded solution.*

In case of $a(x_1) = x_1$ and $f(u) = u^p$, we also obtain the monotonicity and nonexistence results for the following equation

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta + m^2)^s u(x,t) = x_1 u^p(x,t), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.5}$$

where $0 < s < 1$, $1 < p < \infty$ and $m > 0$.

Theorem 1.3. *Let* $0 < s < 1$*,* $1 < p < \infty$ *and* $m > 0$*. Assume that* $u(x,t) \in (C^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s) \times C^1(\mathbb{R})$ *is a positive bounded solution of equation* (1.5)*, then* $u(x,t)$ *is monotonically increasing with respect to* x_1*.*

Theorem 1.4. *Let* $0 < s < 1$*,* $1 < p < \infty$ *and* $m > 0$*. Suppose that* $u(x,t) \in (C^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s) \times C^1(\mathbb{R})$ *is a positive solution of equation* (1.5)*, then equation* (1.1) *possesses no positive bounded solution.*

Notice that we use x_1 and u^p instead of $a(x_1)$ and $f(u)$ for equation (1.5) in $\mathbb{R}^N \times \mathbb{R}$, the methods in Section 2 and Section 3 are still holds here with small modifications, so we present only the results.

This paper is organized as follows. In Section 2, we prove the monotonicity of positive solutions of equation (1.1) by using the method of moving planes. In Section 3, we show the nonexistence of positive bounded solutions.

In this paper, positive constants are denoted by c, C (with subscripts in some cases) and may vary within lines or formulas.

2. Monotonicity of positive solutions

In this section, we prove Theorem 1.1 via the method of moving planes.

Proof of Theorem 1.1. We carry out the proof in two steps.

Step 1. We show that for $\lambda \leq 0$, we have

$$w_\lambda(x,t) \geq 0, \quad \text{in } \Sigma_\lambda \times \mathbb{R}. \tag{2.1}$$

By the assumption $(H1)$, we have

$$\begin{aligned} &\frac{\partial w_\lambda}{\partial t}(x,t) + (-\Delta + m^2)^s w_\lambda(x,t) \\ &= a((x_1)_\lambda) f(u_\lambda) - a(x_1) f(u) \\ &= a((x_1)_\lambda) f(u_\lambda) - a(x_1) f(u_\lambda) + a(x_1)(f(u_\lambda) - f(u)) \\ &\geq a(x_1) c_\lambda(x,t) w_\lambda(x,t), \quad \text{in } \Sigma_\lambda \times \mathbb{R}, \end{aligned} \tag{2.2}$$

where

$$c_\lambda(x,t) = \frac{f(u_\lambda(x,t)) - f(u(x,t))}{u_\lambda(x,t) - u(x,t)}.$$

Since $f(u)$ is locally Lipschitz continuous, there exists a positive constant b such that $\| c_\lambda(x,t) \|_{L^\infty(\Sigma_\lambda)} \leq b$.

Now we introduce an auxiliary function

$$g(x) = |x - (\lambda+1)e_1|^\sigma, \tag{2.3}$$

where $e_1 = (1, 0, ..., 0)$ and $\sigma > 0$ is a small number to be chosen later. We define

$$\bar{w}_\lambda(x,t) = \frac{w_\lambda(x,t)}{g(x)}. \tag{2.4}$$

It follows from (2.4) that $\bar{w}_\lambda(x,t)$ and $w_\lambda(x,t)$ have the same sign and

$$\lim_{|x| \to +\infty} \bar{w}_\lambda(x,t) = 0. \tag{2.5}$$

Then we know from (2.5) that for each fixed $t \in \mathbb{R}$, there exists a point $x(t)$ such that

$$\bar{w}_\lambda(x(t),t) = \inf_{\Sigma_\lambda} \bar{w}_\lambda(x,t).$$

Using similar arguments as (2.11) in [17], we know that if $w_\lambda(x,t) < 0$, then $c_\lambda(x,t) \geq 0$ and

$$(-\Delta + m^2)^s w_\lambda(x,t) \leq C w_\lambda(x,t) \frac{1}{|x_1 - \lambda|^{2s}} + m^{2s} w_\lambda(x,t). \tag{2.6}$$

By the assumption $(H2)$, we deduce that for $\forall \lambda \leq 0$,

$$a(x_1) \leq 0, \quad \forall\ (x,t) \in \Sigma_\lambda \times \mathbb{R}.$$

Combining (2.2), (2.4), (2.6) with above estimate, one can conclude that for each fixed $t \in \mathbb{R}$, if $\bar{w}_\lambda(x(t),t) < 0$, then

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(t),t) \geq -\frac{C\bar{w}_\lambda(x(t),t)}{|x_1(t) - \lambda|^{2s}} + (a(x_1(t)) c_\lambda(x(t),t) - m^{2s}) \bar{w}_\lambda(x(t),t)$$

$$\geq -\frac{C}{|x_1(t)-\lambda|^{2s}}\bar{w}_\lambda(x(t),t). \tag{2.7}$$

To prove (2.1), it suffices to show that

$$\bar{w}_\lambda(x(t),t) \geq 0, \quad \forall\, t \in \mathbb{R}. \tag{2.8}$$

Suppose (2.8) is not true, then there exists a $t_0 \in \mathbb{R}$ such that

$$-\gamma_0 = \bar{w}_\lambda(x(t_0),t_0) < 0.$$

From Lemma 2.1 in [9], we know that

$$\frac{C}{|x_1(t_0)-\lambda|^{2s}} > c_0 > 0. \tag{2.9}$$

For any $\bar{t} < t_0$, we define

$$z(t) = -Me^{-c_0(t-\bar{t})} \quad \text{and} \quad -M = \inf_{\Sigma_\lambda\times\mathbb{R}} \bar{w}_\lambda(x,t),$$

where c_0 is a positive constant defined as in Lemma 2.1 of [9].

Next we will prove $z(t)$ is a subsolution, i.e.,

$$\bar{w}_\lambda(x,t) \geq z(t) \ \text{ in } \overline{\Sigma_\lambda}\times[\bar{t},t_0]. \tag{2.10}$$

Set

$$v(x,t) = \bar{w}_\lambda(x,t) - z(t) \ \text{ in } \overline{\Sigma_\lambda}\times[\bar{t},t_0].$$

By the definition of $z(t)$, we have

$$v(x,\bar{t}) = \bar{w}_\lambda(x,\bar{t}) - z(\bar{t}) = \bar{w}_\lambda(x,\bar{t}) - (-M) \geq 0, \quad \forall\, x \in \overline{\Sigma_\lambda}$$

and

$$v(x,t) = \bar{w}_\lambda(x,t) - z(t) = -z(t) \geq 0, \quad \forall\, (x,t) \in T_\lambda\times[\bar{t},t_0].$$

If (2.10) is false, then there exists a point $(x(\hat{t}),\hat{t}) \in \Sigma_\lambda\times(\bar{t},t_0]$ such that

$$v(x(\hat{t}),\hat{t}) = \inf_{\Sigma_\lambda\times[\bar{t},t_0]} v(x,t) < 0. \tag{2.11}$$

It follows

$$\frac{\partial v}{\partial t}(x(\hat{t}),\hat{t}) \leq 0. \tag{2.12}$$

On one hand, we have

$$\bar{w}_\lambda(x(\hat{t}),\hat{t}) = \inf_{\Sigma_\lambda} \bar{w}_\lambda(x,\hat{t}) < z(\hat{t}) < 0.$$

Then by (2.7), we have

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(\hat{t}),\hat{t}) \geq -\frac{C}{|x_1(\hat{t})-\lambda|^{2s}}\bar{w}_\lambda(x(\hat{t}),\hat{t}). \tag{2.13}$$

On the other hand, using (2.11) and the monotonicity of $z(t)$, we know

$$v(x(\hat{t}),\hat{t}) \leq v(x(t_0),t_0).$$

Therefore, one has

$$\bar{w}_\lambda(x(\hat{t}),\hat{t}) \leq \bar{w}_\lambda(x(t_0),t_0) = -\gamma_0.$$

Using similar arguments as (2.9), we conclude that

$$\frac{C}{|x_1(\hat{t})-\lambda|^{2s}} > c_0 > 0. \tag{2.14}$$

By (2.13) and (2.14), we have

$$\frac{\partial \bar{w}_\lambda}{\partial t}(x(\hat{t}),\hat{t}) \geq -c_0\bar{w}_\lambda(x(\hat{t}),\hat{t}).$$

Then using (2.12), one can deduce that

$$-c_0 z(\hat{t}) = \frac{\partial z}{\partial t}(\hat{t}) \geq \frac{\partial \bar{w}_\lambda}{\partial t}(x(\hat{t}),\hat{t}) \geq -c_0\bar{w}_\lambda(x(\hat{t}),\hat{t}),$$

which implies

$$v(x(\hat{t}),\hat{t}) = \bar{w}_\lambda(x(\hat{t}),\hat{t}) - z(\hat{t}) \geq 0.$$

This contradicts $v(x(\hat{t}),\hat{t}) < 0$. Therefore, (2.10) holds.

For any t, we have $z(t) \to 0$ as $t \to -\infty$. Then by (2.10), we have

$$\bar{w}_\lambda(x,t) \geq 0, \quad \forall\ (x,t) \in \overline{\Sigma_\lambda} \times (-\infty, t_0].$$

Step 2. Step 1 provides the starting point to move the plane. Then we move the plane T_λ continuously up as long as (2.1) holds to its limiting position. Define

$$\lambda_0 = \sup\{\lambda \mid w_\mu(x,t) \geq 0 \text{ in } \Sigma_\mu \times \mathbb{R} \quad \text{for all} \quad \mu \leq \lambda\}.$$

We will show that

$$\lambda_0 = +\infty.$$

To this end, we argue by contradiction. Suppose that $0 < \lambda_0 < +\infty$. Then there exists a sequence $\lambda_k \searrow \lambda_0$ such that

$$\inf_{\Sigma_{\lambda_k}\times\mathbb{R}} w_{\lambda_k}(x,t) < 0.$$

Set

$$\bar{w}_{\lambda_k}(x,t) = \frac{w_{\lambda_k}(x,t)}{g(x)},$$

where $g(x)$ is defined as (2.3). Then we have

$$-\gamma_k = \inf_{\Sigma_{\lambda_k}\times\mathbb{R}} \bar{w}_{\lambda_k}(x,t) < 0.$$

To avoid the attainability of the minimum value of $\bar{w}_{\lambda_k}(x,t)$ at infinity with respect to t, we choose a sequence t_k, and corresponding $x(t_k)$ and ϵ_k such that

$$\bar{w}_{\lambda_k}(x(t_k),t_k) = \inf_{\Sigma_{\lambda_k}} \bar{w}_{\lambda_k}(\cdot,t_k) = -\gamma_k + \epsilon_k\gamma_k.$$

Let $\eta(t)$ be a smooth function in $\mathbb{R}$ satisfying $|\eta'(t)| \leq 1$ and

$$\eta(t) = \begin{cases} 1, & \text{if} \quad |t| < \dfrac{1}{2}, \\ 0, & \text{if} \quad |t| \geq 2. \end{cases}$$

Now we introduce a sequence of auxiliary functions

$$\tilde{w}_{\lambda_k}(x,t) = \bar{w}_{\lambda_k}(x,t) - \epsilon_k \gamma_k \eta_k(t),$$

where $\eta_k(t) = \eta(t - t_k)$. It is easy to see that for $|t - t_k| \geq 2$,

$$\tilde{w}_{\lambda_k}(x,t) = \bar{w}_{\lambda_k}(x,t) \geq -\gamma_k.$$

Notice that

$$\tilde{w}_{\lambda_k}(x(t_k),t_k) = \bar{w}_{\lambda_k}(x(t_k),t_k) - \epsilon_k \gamma_k = -\gamma_k.$$

Therefore, there exists $(x(\tilde{t}_k),\tilde{t}_k) \in \Sigma_{\lambda_k} \times (t_k - 2, t_k + 2)$ such that

$$\tilde{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) = \inf_{\Sigma_{\lambda_k} \times \mathbb{R}} \tilde{w}_{\lambda_k}(x,t) < 0.$$

Then we have

$$\frac{\partial \tilde{w}_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k) = 0,$$

and hence

$$\left| \frac{\partial \bar{w}_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k) \right| = \left| \epsilon_k \gamma_k \frac{\partial \eta_k}{\partial t}(t) \right| \leq \epsilon_k \gamma_k. \tag{2.15}$$

Since $\tilde{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \leq \tilde{w}_{\lambda_k}(x(t_k),t_k)$, we have

$$-\gamma_k \leq \bar{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \leq \bar{w}_{\lambda_k}(x(t_k),t_k) = -\gamma_k + \epsilon_k \gamma_k. \tag{2.16}$$

By the definition of $\tilde{w}_{\lambda_k}(x,t)$, we obtain

$$\bar{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) = \inf_{\Sigma_{\lambda_k}} \bar{w}_{\lambda_k}(x,\tilde{t}_k) < 0. \tag{2.17}$$

Using similar arguments as Step 1, we know from (2.17) that for $\lambda_k > 0$, $x_1(\tilde{t}_k) > 0$. Hence we assume $0 < x_1(\tilde{t}_k) < \lambda_0 + 1$. Then by a similar calculation as (2.11) in [17], we obtain

$$(-\Delta + m^2)^s w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \leq C w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \frac{1}{|x_1(\tilde{t}_k) - \lambda_k|^{2s}} + m^{2s} w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k). \tag{2.18}$$

By the assumption $(H1)$ and $(H2)$, there exists a positive constant C_1 such that

$$a(x_1(\tilde{t}_k)) c_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \leq C_1,$$

where

$$c_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) = \frac{f(u_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)) - f(u(x(\tilde{t}_k),\tilde{t}_k))}{u_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) - u(x(\tilde{t}_k),\tilde{t}_k)}.$$

By (2.2), (2.18) and $w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) < 0$, we have

$$\begin{aligned}
&\frac{\partial w_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k) + C w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \frac{1}{|x_1(\tilde{t}_k) - \lambda_k|^{2s}} \\
\geq &(a(x_1(\tilde{t}_k)) c_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) - m^{2s}) w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k) \\
\geq &C_1 w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k).
\end{aligned} \tag{2.19}$$

Then dividing $g(x(\tilde{t}_k),\tilde{t}_k)$ on both side of inequality (2.19), we obtain

$$\begin{aligned}&\frac{\partial \bar{w}_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k)+C\bar{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)\frac{1}{|x_1(\tilde{t}_k)-\lambda_k|^{2s}}\\ \geq& C_1\bar{w}_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k).\end{aligned}\tag{2.20}$$

Combining (2.15), (2.16) with (2.20), dividing $-\gamma_k$ on both side of (2.20), one can deduce that

$$\frac{C}{|x_1(\tilde{t}_k)-\lambda_k|^{2s}}\leq\frac{C_1}{2},\tag{2.21}$$

since ϵ_k is small. This implies that if k is sufficiently large, we have

$$|x_1(\tilde{t}_k)-\lambda_k|\geq C_2$$

and

$$|x_1(\tilde{t}_k)-\lambda_0|\geq\frac{C_2}{2}>0.\tag{2.22}$$

It is easy to see that $|x_1(\tilde{t}_k)-\lambda_k|\leq\lambda_k\leq\lambda_0+1$ for sufficiently large k. Therefore, there exists a positive constant C_3 such that

$$u(x(\tilde{t}_k),\tilde{t}_k)\geq C_3.\tag{2.23}$$

Using (2.2), (2.22) and (2.23), we have, for sufficiently large k,

$$\begin{aligned}&\frac{\partial w_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k)+Cw_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)\frac{1}{|x_1(\tilde{t}_k)-\lambda_k|^{2s}}\\ \geq& [a(x_1^{\lambda_k}(\tilde{t}_k))-a(x_1(\tilde{t}_k))]f(u_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k))\\ &+[a(x_1(\tilde{t}_k))c_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)-m^{2s}]w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)\\ \geq& C_4\\ >&0,\end{aligned}\tag{2.24}$$

where we have used the fact that $u_{\lambda_k}(x,t)$ is uniformly Hölder continuous and

$$u_{\lambda_k}(x,t)\rightrightarrows u_{\lambda_0}(x,t)\text{ and }w_{\lambda_k}(x,t)\rightrightarrows w_{\lambda_0}(x,t)\geq 0.$$

Since $w_{\lambda_k}(x(\tilde{t}_k),\tilde{t}_k)<0$, we know from (2.24) that

$$\frac{\partial w_{\lambda_k}}{\partial t}(x(\tilde{t}_k),\tilde{t}_k)\geq C_4>0.\tag{2.25}$$

Define

$$\hat{w}_{\lambda_k}(x,t)=w_{\lambda_k}(x+x(\tilde{t}_k),t+\tilde{t}_k).$$

It follows from (2.25) that

$$\frac{\partial \hat{w}_{\lambda_k}}{\partial t}(0,0)\geq C_4>0.$$

Since u_{λ_k} is bounded, by regularity theory for parabolic equations [19], there exists a subsequence of $(x(\tilde{t}_k),\tilde{t}_k)$ (still denoted by $(x(\tilde{t}_k),\tilde{t}_k)$) such that

$$\hat{w}_{\lambda_k}(x,t)\to\hat{w}_{\lambda_0}(x,t)\ \text{ and }\ \frac{\partial \hat{w}_{\lambda_k}}{\partial t}(x,t)\to\frac{\partial \hat{w}_{\lambda_0}}{\partial t}(x,t)\ \text{ as }k\to\infty.$$

Since $0 < x_1(\tilde{t}_k) \leq \lambda_k$ and $\lambda_k \to \lambda_0$ as $k \to \infty$, there exists a subsequence of $x_1(\tilde{t}_k)$ (still denoted by $x_1(\tilde{t}_k)$) and $0 \leq x_1^0 \leq \lambda_0$ such that

$$x_1(\tilde{t}_k) \to x_1^0 \ \text{ as } k \to \infty.$$

Now we consider the function $\hat{w}_{\lambda_0}(x,t)$. It is easy to see that

$$\hat{w}_{\lambda_0}(x,t) \geq 0 \ \text{ in } \Sigma_{\lambda_0 - x_1^0} \times \mathbb{R}.$$

Since $w_{\lambda_k}(x(\tilde{t}_k), \tilde{t}_k) < 0$, we conclude that

$$\hat{w}_{\lambda_0}(0,0) = 0 = \inf_{\Sigma_{\lambda_0 - x_1^0} \times \mathbb{R}} \hat{w}_{\lambda_0}(x,t).$$

It implies that

$$\frac{\partial \hat{w}_{\lambda_0}}{\partial t}(0,0) = 0,$$

which contradicts (2.25). Therefore, we have $\lambda_0 = +\infty$. Hence, $u(x,t)$ is monotonically increasing with respect to x_1 in $\mathbb{R}^N \times \mathbb{R}$. □

3. Nonexistence of positive solution

In this section, we will show the nonexistence of positive bounded solution for equation (1.1). To prove Theorem 1.2, we need the following result.

Lemma 3.1. *Let $\phi(x) \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s$ be the first eigenfunction associated with $(-\Delta + m^2)^s$ in $B_1(0)$ satisfying*

$$\begin{cases} (-\Delta + m^2)^s \phi(x) = \lambda_1 \phi(x), & \text{in } B_1(0), \\ \phi(x) = 0, & \text{in } B_1^c(0). \end{cases}$$

Let $\rho(x) \in C_0^\infty(B_1(0))$ be such that $\int_{B_1(0)} \rho(x)dx = 1$. Then we have

$$\int_{\mathbb{R}^N} (-\Delta + m^2)^s \phi(z) \rho(x-z) dz = \int_{\mathbb{R}^N} \phi(z) (-\Delta + m^2)^s \rho(x-z) dz$$

and

$$(-\Delta + m^2)^s \phi_1(x) \leq \lambda_1 \phi_1(x) \quad \text{in } \mathbb{R}^N,$$

where $\phi_1(x) = (\phi * \rho)(x)$ and $*$ denotes the convolution.

The proof of Lemma 3.1 is similar to Lemma A.1 in [9] with small modifications, so here we omit the details.

Remark 3.1. From the proof process of Lemma 3.1, we note that if $u \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) \cap \mathcal{L}_s$ and $v \in C_0^\infty(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} (-\Delta + m^2)^s u(x) v(x) dx = \int_{\mathbb{R}^N} u(x) (-\Delta + m^2)^s v(x) dx.$$

Now we are ready to give a complete proof of Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. Suppose there exists a positive bounded solution $u(x,t)$ of equation (1.1). Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} (-\Delta + m^2)^s \phi(x) = \lambda_1 \phi(x), & \text{in } B_1(Re_1), \\ \phi(x) = 0, & \text{in } B_1^c(Re_1), \end{cases}$$

where $B_1(Re_1)$ is the unit ball centered at $Re_1 = (R, 0, ..., 0)$ and $R > 0$ will be chosen sufficiently large later.

Let $\phi_1(x) = (\phi * \rho)(x) \in C_0^\infty(\mathbb{R}^N)$, where $\rho(x) \in C_0^\infty(B_1(Re_1))$ satisfies $\int_{\mathbb{R}^N} \rho(x)dx = 1$. By Lemma 3.1, we have

$$(-\Delta + m^2)^s \phi_1(x) \le \lambda_1 \phi_1(x) \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Now we assume

$$\int_{\mathbb{R}^N} \phi_1(x) dx = 1.$$

Obviously, suppϕ_1 is contained in $B_2(Re_1)$. Set

$$\psi_R(t) = \int_{\mathbb{R}^N} u(x,t)\phi_1(x)dx = \int_{B_2(Re_1)} u(x,t)\phi_1(x)dx.$$

Using Jensen's inequality, Remark 3.1, $(H1)$ and (3.1), we have, for sufficiently large R, that

$$\begin{aligned} \frac{\partial}{\partial t}\psi_R(t) &= -\int_{\mathbb{R}^N} (-\Delta + m^2)^s u(x,t)\phi_1(x)dx + \int_{\mathbb{R}^N} a(x_1) f(u)\phi_1(x)dx \\ &= -\int_{\mathbb{R}^N} u(x,t)(-\Delta + m^2)^s \phi_1(x)dx + \int_{\mathbb{R}^N} a(x_1) f(u)\phi_1(x)dx \\ &\ge -\lambda_1 \int_{\mathbb{R}^N} u(x,t)\phi_1(x)dx + a(R-2)\int_{\mathbb{R}^N} f(u)\phi_1(x)dx \\ &\ge -\lambda_1 \psi_R(t) + a(R-2)C\int_{\mathbb{R}^N} u^\kappa(x,t)\phi_1(x)dx \\ &\ge -\lambda_1 \psi_R(t) + a(R-2)C\left(\int_{\mathbb{R}^N} u(x,t)\phi_1(x)dx\right)^\kappa \\ &= \left(a(R-2)C\psi_R^{\kappa-1}(t) - \lambda_1\right)\psi_R(t). \end{aligned} \tag{3.2}$$

It follows from Theorem 1.1 that for any fixed $t \in \mathbb{R}$, $\psi_R(t)$ is monotonically increasing with respect to R. Therefore, we obtain

$$\psi_R(0) \ge 2c_0 := \psi_0(0). \tag{3.3}$$

If $t \ge 0$ such that $\psi_R(t) \ge c_0$, we can choose R sufficiently large such that

$$a(R-2)C\psi_R^{\kappa-1}(t) - \lambda_1 \ge 1.$$

Then by (3.2), we have

$$\frac{\partial}{\partial t}\psi_R(t) \ge \psi_R(t).$$

Thus we deduce from (3.3) that

$$\psi_R(t) \ge 2c_0 \mathrm{e}^t. \tag{3.4}$$

Now we use a contradiction argument to verify

$$\psi_R(t) \geq c_0. \tag{3.5}$$

Suppose (3.5) is false, then there exists a point $t_0 > 0$ such that

$$\psi_R(t_0) = c_0 \text{ and } \psi_R(t) > c_0, \quad \forall\, t \in [0, t_0).$$

Using (3.4), we know that

$$\psi_R(t) \geq 2c_0 \mathrm{e}^t \geq 2c_0, \quad \forall\, t \in [0, t_0),$$

which contradicts $\psi_R(t_0) = c_0$. Therefore, (3.5) holds.

From (3.4), one can conclude that

$$\psi_R(t) \to +\infty \text{ as } t \to +\infty.$$

This contradicts the boundedness of $u(x, t)$. This completes the proof of Theorem 1.2. □

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