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ON A SYSTEM OF DIFFERENCE EQUATIONS DEFINED BY THE CONTINUOUS AND STRICTLY MONOTONE FUNCTIONS

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Abstract In this paper, we solve the following difference equations system

$$\begin{cases} \omega_{n+1} = g^{-1}\left(g\left(\omega_n\right)\dfrac{\zeta_1 h\left(\vartheta_n\right)+\eta_1 h\left(\vartheta_{n-1}\right)}{\mu_1 h\left(\vartheta_n\right)+\xi_1 h\left(\vartheta_{n-1}\right)}\right), \\ \vartheta_{n+1} = h^{-1}\left(h\left(\vartheta_n\right)\dfrac{\zeta_2 g\left(\omega_n\right)+\eta_2 g\left(\omega_{n-1}\right)}{\mu_2 g\left(\omega_n\right)+\xi_2 g\left(\omega_{n-1}\right)}\right), \end{cases} \quad n \in \mathbb{N}_0,$$

where the coefficients $\mu_k^2+\xi_k^2 \neq 0$, ζ_k, η_k, μ_k, ξ_k, for $k \in \{1,2\}$ are real numbers, the initial values ω_{-j}, ϑ_{-j}, for $j \in \{0,1\}$ are real numbers, g and h are continuous and strictly monotone functions, $g\left(\mathbb{R}\right)=\mathbb{R}$, $h\left(\mathbb{R}\right)=\mathbb{R}$, $g\left(0\right)=0$, $h\left(0\right)=0$, in explicit form depending on whether or not the parameters are equal to 0.

Keywords Difference equations systems, solution, monotone functions.

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1. Introduction

The notation of $a=\overline{b,c}$ stands for $\{a \in \mathbb{Z} : b \leq a \leq c\}$ if $b, c \in \mathbb{Z},\ b \leq c$. In addition, the other notations of $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, mean set of natural, non-negative integer, integer and real numbers, respectively.

Difference equations and systems of difference equations appear in many branches of mathematics and science, where they model real and abstract phenomena. There is a growing interest in some topics in this area, their solvability, stability, invariants and applications [2, 5, 7–9, 16, 19–26]. Many solvable difference equations and systems of difference equations can be transformed into well-known solvable equations with suitable variable changes. For example, the author of [4] has solved special cases of the following difference equation

$$z_{n+1} = \alpha z_n + \frac{\beta z_n^2}{\gamma z_n + \delta z_{n-1}},\ n \in \mathbb{N}_0, \tag{1.1}$$

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where the initial values z_{-1}, z_0 are arbitrary positive real numbers and the coefficients α, β, γ, δ are positive constants. In [18], the authors solved the following difference equation

$$z_{n+1} = h^{-1}\left(h\left(z_n\right)\frac{ah\left(z_n\right)+bh\left(z_{n-1}\right)}{ch\left(z_n\right)+dh\left(z_{n-1}\right)}\right),\ n\in\mathbb{N}_0, \tag{1.2}$$

where the initial conditions z_{-k}, for $k\in\{0,1\}$, and the parameters a,b,c,d are real numbers such that $c^2+d^2\neq 0$ and h is a strictly monotone and continuous function such that $h\left(\mathbb{R}\right)=\mathbb{R}$, $h\left(0\right)=0$, by using transformation. Note that equation (1.2) is a more general form of equation (1.1).

The natural question is if both equation (1.1) and equation (1.2) transform into a more general system of difference equations. We give a favourable answer. In this paper, we extend equations in (1.1) and (1.2) to the following two-dimensional system of difference equations

$$\begin{cases}\omega_{n+1} = g^{-1}\left(g\left(\omega_n\right)\dfrac{\zeta_1 h\left(\vartheta_n\right)+\eta_1 h\left(\vartheta_{n-1}\right)}{\mu_1 h\left(\vartheta_n\right)+\xi_1 h\left(\vartheta_{n-1}\right)}\right),\\ \vartheta_{n+1} = h^{-1}\left(h\left(\vartheta_n\right)\dfrac{\zeta_2 g\left(\omega_n\right)+\eta_2 g\left(\omega_{n-1}\right)}{\mu_2 g\left(\omega_n\right)+\xi_2 g\left(\omega_{n-1}\right)}\right),\end{cases}\quad n\in\mathbb{N}_0, \tag{1.3}$$

where the initial conditions ω_{-t}, ϑ_{-t}, for $t\in\{0,1\}$ are real numbers, the parameters ζ_k, η_k, μ_k, ξ_k, for $k\in\{1,2\}$ are real numbers, g and h are continuous and strictly monotone functions, $g\left(\mathbb{R}\right)=\mathbb{R}$, $h\left(\mathbb{R}\right)=\mathbb{R}$, $g\left(0\right)=0$, $h\left(0\right)=0$. We achieved the solutions of system (1.3) according to whether the parameters are equal to zero or non-zero. Appropriate variable change was used when obtaining solutions in this paper. Moreover, solutions were found depending on the generalized Fibanocci sequence in some cases.

Many studies related to number sequences can be found in the literature [1, 6, 11–13]. In addition, it is possible to find general difference equations or systems of difference equations similar to the system (1.3) in the literature [10, 14, 17].

The following second order linear difference equation

$$y_{n+2} = \gamma y_{n+1}+\delta y_n,\ n\in\mathbb{N}_0, \tag{1.4}$$

was solved by De Moivre in [15]. The solution of (1.4) is given by

$$y_n = \frac{\left(y_1-\lambda_2 y_0\right)\lambda_1^n-\left(y_1-\lambda_1 y_0\right)\lambda_2^n}{\lambda_1-\lambda_2},\ n\in\mathbb{N}_0, \tag{1.5}$$

if $\gamma\neq 0$ and $\gamma^2+4\delta\neq 0$, and

$$y_n = \left(\left(y_1-\lambda_1 y_0\right)n+\lambda_1 y_0\right)\lambda_1^{n-1},\ n\in\mathbb{N}_0, \tag{1.6}$$

while if $\gamma\neq 0$ and $\gamma^2+4\delta=0$, where $\lambda_{1,2}=\frac{\gamma\pm\sqrt{\gamma^2+4\delta}}{2}$ are the roots of the polynomial $P\left(\lambda\right)=\lambda^2-\gamma\lambda-\delta=0$.

We will use the following very well-known results; see, for example, [3].

Lemma 1.1. *Consider the linear difference equation*

$$w_{rn+j} = a_n w_{r(n-1)+j}+b_n,\ n\in\mathbb{N}_0,$$

where a_n and b_n are real number sequences and $j \in \{0, 1, \ldots, r-1\}$. Then, the general solution of variable coefficients linear difference equation is given by the following formula

$$w_{rn+j} = \left(\prod_{j=0}^{n} a_j\right) w_{j-r} + \sum_{k=0}^{n} \left(\prod_{j=k+1}^{n} a_j\right) b_k,$$

where the next standard conventions $\prod_{i=j}^{l} \gamma_i = 1$ and $\sum_{i=j}^{l} \eta_i = 0$, for all $l < j$, are utilized here. Moreover if a_n and b_n are constants, that is, $a_n = a$ and $b_n = b$, then the general solution to constant coefficient linear difference equation is given by the following formula

$$w_{rn+j} = \begin{cases} a^{n+1} w_{j-r} + \dfrac{a^{n+1}-1}{a-1} b, & a \neq 1, \\ w_{j-r} + (n+1)\, b, & a = 1, \end{cases} \quad n \in \mathbb{N}_0.$$

Definition 1.1. Consider the following homogeneous second-order linear difference equation with constant coefficients, that is, the following one:

$$s_{n+2} = \alpha s_{n+1} + \beta s_n, \; n \in \mathbb{N}_0, \tag{1.7}$$

where the initial values $s_0 = 0$, $s_1 = 1$ and the parameters α, β are real numbers. $\lambda^2 - \alpha\lambda - \beta = 0$ is the characteristic equation of (1.7), where $\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$ are the roots of the characteristic equation. It is clear that Binet formula for (1.7) is

$$s_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \; n \in \mathbb{N}_0. \tag{1.8}$$

The sequence $(s_n)_{n \geq 0}$ is called the generalized Fibonacci sequence in the literature.

2. Solutions of system (1.3) in explicit-form

Here, we demonstrate that system (1.3) is a specific case of a solvable system of difference equation.

Theorem 2.1. *Suppose that ζ_i, η_i, μ_i, $\xi_i \in \mathbb{R}$, for $i \in \{1, 2\}$, such that $\mu_i^2 + \xi_i^2 \neq 0$, g and h are continuous and strictly monotone functions, $g(\mathbb{R}) = \mathbb{R}$, $h(\mathbb{R}) = \mathbb{R}$, $g(0) = 0$, $h(0) = 0$. So, the general system (1.3) is solvable in explicit-form.*

Proof. If at least one of the initial values $\omega_{-j} = 0$ or $\vartheta_{-j} = 0$, for $j \in \{0, 1\}$, then the solution of system (1.3) is not defined. Moreover, assume that $\omega_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. Then from system (1.3) we have $\omega_{n_0+1} = 0$. These facts along with (1.3) imply that ϑ_{n_0+2} is not defined. Similarly, suppose that $\vartheta_{n_1} = 0$ for some $n_1 \in \mathbb{N}_0$. Then from system (1.3) we have $\vartheta_{n_1+1} = 0$. These facts along with (1.3) imply that ω_{n_1+2} is not defined. Hence, for every well-defined solution of system (1.3), we have

$$\omega_n \vartheta_n \neq 0, \; n \geq -1. \tag{2.1}$$

Firstly, since $g(\mathbb{R}) = \mathbb{R}$, $h(\mathbb{R}) = \mathbb{R}$, $g(0) = 0$, $h(0) = 0$ and $g, h : \mathbb{R} \to \mathbb{R}$ are continuous and strictly monotone functions, g, h are one to one functions. The only root of the functions g and h is 0. These functions are homomorphism on $\mathbb{R}$. Taking this property of the functions into consideration, the solutions of system (1.3) according to the states of the parameters will be examined as follows:

Case 1. $\zeta_k\xi_k = \eta_k\mu_k$ for $k \in \{1,2\}$: Under this assumptions, there are also additional subcases to take into account.

Subcase 1.1. $\zeta_k = 0 = \eta_k$, $\mu_k\xi_k \neq 0$, for $k \in \{1,2\}$: In this case, system (1.3) is

$$\omega_{n+1} = g^{-1}(0),\ \vartheta_{n+1} = h^{-1}(0),\ n \in \mathbb{N}_0.$$

By using the properties of functions g and h in the last equations, the solution of system (1.3) under these conditions is as follows

$$\omega_n = 0,\ \vartheta_n = 0,\ n \in \mathbb{N}. \tag{2.2}$$

Subcase 1.2. $\zeta_k = 0$, $\eta_k\xi_k \neq 0$, for $k \in \{1,2\}$: Under these conditions, we have $\mu_k = 0$, for $k \in \{1,2\}$ and also

$$\omega_{n+1} = g^{-1}\left(\frac{\eta_1}{\xi_1}g(\omega_n)\right),\ \vartheta_{n+1} = h^{-1}\left(\frac{\eta_2}{\xi_2}h(\vartheta_n)\right),\ n \in \mathbb{N}_0, \tag{2.3}$$

from which it follows that

$$g(\omega_{n+1}) = \frac{\eta_1}{\xi_1}g(\omega_n),\ h(\vartheta_{n+1}) = \frac{\eta_2}{\xi_2}h(\vartheta_n),\ n \in \mathbb{N}_0. \tag{2.4}$$

Since the equations in (2.4) are solvable, we define new variables as following forms

$$\varsigma_n = g(\omega_n),\ \varrho_n = h(\vartheta_n)\ n \in \mathbb{N}_0. \tag{2.5}$$

By substituting the new variables to equations in (2.4), we obtain the first-order linear difference equations as follows:

$$\varsigma_{n+1} = \frac{\eta_1}{\xi_1}\varsigma_n,\ \varrho_{n+1} = \frac{\eta_2}{\xi_2}\varrho_n,\ n \in \mathbb{N}_0. \tag{2.6}$$

By using Lemma 1.1 for $r = 1$, we can write the general solutions of (2.6) as in the following form

$$\varsigma_n = \left(\frac{\eta_1}{\xi_1}\right)^n \varsigma_0,\ \varrho_n = \left(\frac{\eta_2}{\xi_2}\right)^n \varrho_0,\ n \in \mathbb{N}_0. \tag{2.7}$$

Relations (2.5) and (2.7) yield

$$\omega_n = g^{-1}\left(\left(\frac{\eta_1}{\xi_1}\right)^n g(\omega_0)\right),\ \vartheta_n = h^{-1}\left(\left(\frac{\eta_2}{\xi_2}\right)^n h(\vartheta_0)\right),\ n \in \mathbb{N}_0. \tag{2.8}$$

Subcase 1.3. $\eta_k = 0, \zeta_k\mu_k \neq 0$, for $k \in \{1,2\}$: Under these conditions, we get $\xi_k = 0$, for $k \in \{1,2\}$ and also

$$\omega_{n+1} = g^{-1}\left(\frac{\zeta_1}{\mu_1}g(\omega_n)\right),\ \vartheta_{n+1} = h^{-1}\left(\frac{\zeta_2}{\mu_2}h(\vartheta_n)\right),\ n \in \mathbb{N}_0, \tag{2.9}$$

from which it follows

$$g(\omega_{n+1}) = \frac{\zeta_1}{\mu_1}g(\omega_n),\ h(\vartheta_{n+1}) = \frac{\zeta_2}{\mu_2}h(\vartheta_n),\ n \in \mathbb{N}_0. \tag{2.10}$$

By using transforms in (2.5), we get the first-order linear difference equations as follows:

$$\varsigma_{n+1} = \frac{\zeta_1}{\mu_1}\varsigma_n,\ \varrho_{n+1} = \frac{\zeta_2}{\mu_2}\varrho_n,\ n \in \mathbb{N}_0. \tag{2.11}$$

By using Lemma 1.1 for $r = 1$, then the solutions to equations in (2.11) are

$$\varsigma_n = \left(\frac{\zeta_1}{\mu_1}\right)^n \varsigma_0,\ \varrho_n = \left(\frac{\zeta_2}{\mu_2}\right)^n \varrho_0,\ n \in \mathbb{N}_0. \tag{2.12}$$

From (2.5) and the solutions in (2.12), we see that the general solution to system (2.9) in these cases $\eta_k = 0, \zeta_k\mu_k \neq 0$, for $k \in \{1, 2\}$, is

$$\omega_n = g^{-1}\left(\left(\frac{\zeta_1}{\mu_1}\right)^n g(\omega_0)\right),\ \vartheta_n = h^{-1}\left(\left(\frac{\zeta_2}{\mu_2}\right)^n h(\vartheta_0)\right),\ n \in \mathbb{N}_0. \tag{2.13}$$

Subcase 1.4. $\xi_k = 0$, for $k \in \{1, 2\}$: In this case, with these conditions, we immediately obtain $\mu_k \neq 0$, for $k \in \{1, 2\}$ and hereby $\eta_k = 0$, for $k \in \{1, 2\}$. Then there are two cases to be considered. These cases are either $\zeta_k = 0$ or $\zeta_k \neq 0$, for $k \in \{1, 2\}$. Further, they were investigated in sub-case 1 and sub-case 3, respectively.

Subcase 1.5. $\mu_k = 0$, for $k \in \{1, 2\}$: In this case, from these conditions, we have $\xi_k \neq 0$, for $k \in \{1, 2\}$ and also $\zeta_k = 0$, for $k \in \{1, 2\}$. Similarly as in the previous case, there are two-cases to be considered. These cases are either $\eta_k = 0$ or $\eta_k \neq 0$, for $k \in \{1, 2\}$. They were investigated in sub-case 1 and sub-case 2, respectively.

Subcase 1.6. $\zeta_k\eta_k\mu_k\xi_k \neq 0$, for $k \in \{1, 2\}$: In this case, with these conditions, we obtain $\zeta_k = \frac{\eta_k\mu_k}{\xi_k}$. Then, system (1.3) reduce to system (2.3), whose solution is given by formulas (2.8).

Case 2. $\zeta_k\xi_k \neq \eta_k\mu_k$ for $k \in \{1, 2\}$: In this case, from (2.1) and the monotonicity of g and h, we obtain

$$g(\omega_n) \neq 0,\ h(\vartheta_n) \neq 0,\ n \geq -1. \tag{2.14}$$

Then, the system (1.3) can be written in the following form

$$\frac{g(\omega_{n+1})}{g(\omega_n)} = \frac{\zeta_1 \frac{h(\vartheta_n)}{h(\vartheta_{n-1})} + \eta_1}{\mu_1 \frac{h(\vartheta_n)}{h(\vartheta_{n-1})} + \xi_1},\ \frac{h(\vartheta_{n+1})}{h(\vartheta_n)} = \frac{\zeta_2 \frac{g(\omega_n)}{g(\omega_{n-1})} + \eta_2}{\mu_2 \frac{g(\omega_n)}{g(\omega_{n-1})} + \xi_2},\ n \in \mathbb{N}_0. \tag{2.15}$$

Let

$$\varsigma_n = \frac{g(\omega_n)}{g(\omega_{n-1})},\ \varrho_n = \frac{h(\vartheta_n)}{h(\vartheta_{n-1})},\ n \in \mathbb{N}_0. \tag{2.16}$$

By using (2.16) in (2.15), we get the first-order riccati difference equations as follows:

$$\varsigma_{n+1} = \frac{\zeta_1\varrho_n + \eta_1}{\mu_1\varrho_n + \xi_1},\ \varrho_{n+1} = \frac{\zeta_2\varsigma_n + \eta_2}{\mu_2\varsigma_n + \xi_2},\ n \in \mathbb{N}_0. \tag{2.17}$$

Then, there are two-cases to be considered.

Subcase 2.1. $\mu_k = 0$, for $k \in \{1, 2\}$: In this case, system (2.17) is presented by

$$\varsigma_{n+1} = \frac{\zeta_1}{\xi_1}\varrho_n + \frac{\eta_1}{\xi_1},\ \varrho_{n+1} = \frac{\zeta_2}{\xi_2}\varsigma_n + \frac{\eta_2}{\xi_2},\ n \in \mathbb{N}_0. \tag{2.18}$$

By considering system (2.18) with replacing n by $n-1$ in (2.18) and by substituting the first equation in (2.18) into the second ones and the second equation in (2.18) into the first ones respectively, we obtain the second-order linear difference equations as follows:

$$\varsigma_{n+1}=\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\varsigma_{n-1}+\frac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2},\ \varrho_{n+1}=\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\varrho_{n-1}+\frac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2},\ n\in\mathbb{N}. \quad (2.19)$$

Now the subcases $\frac{\zeta_1\zeta_2}{\xi_1\xi_2}=1$ and $\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\neq 1$ will be considered separately.

Subsubcase 2.1.1. $\zeta_1\zeta_2=\xi_1\xi_2$: In this case, by using Lemma 1.1 for $r=2$, the solutions of equations in (2.19) can be written in the following form

$$\varsigma_{2n+i}=\varsigma_i+\frac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2}n,\ \varrho_{2n+i}=\varrho_i+\frac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2}n,\ n\in\mathbb{N}_0, \quad (2.20)$$

for $i\in\{0,1\}$. Also, from (2.16), we have

$$\begin{cases} g\left(\omega_{2n+i}\right)=\left(\dfrac{g\left(\omega_i\right)}{g\left(\omega_{i-1}\right)}+\dfrac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2}n\right) \\ \qquad\times\left(\dfrac{g\left(\omega_{1-i}\right)}{g\left(\omega_{-i}\right)}+\dfrac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2}\left(n+i-1\right)\right)g\left(\omega_{2(n-1)+i}\right), \\ h\left(\vartheta_{2n+i}\right)=\left(\dfrac{h\left(\vartheta_i\right)}{h\left(\vartheta_{i-1}\right)}+\dfrac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2}n\right) \\ \qquad\times\left(\dfrac{h\left(\vartheta_{1-i}\right)}{h\left(\vartheta_{-i}\right)}+\dfrac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2}\left(n+i-1\right)\right)h\left(\vartheta_{2(n-1)+i}\right), \end{cases} \quad (2.21)$$

for $n\in\mathbb{N}$, $i\in\{0,1\}$. Thus, the general solutions of equations in (2.19) are

$$\begin{cases} \omega_{2n+i}=g^{-1}\left(g\left(\omega_i\right)\displaystyle\prod_{j=1}^{n}\left(\dfrac{g\left(\omega_i\right)}{g\left(\omega_{i-1}\right)}+\dfrac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2}j\right)\right. \\ \qquad\left.\times\left(\dfrac{g\left(\omega_{1-i}\right)}{g\left(\omega_{-i}\right)}+\dfrac{\zeta_1\eta_2+\eta_1\xi_2}{\xi_1\xi_2}\left(j+i-1\right)\right)\right), \\ \vartheta_{2n+i}=h^{-1}\left(h\left(\vartheta_i\right)\displaystyle\prod_{j=1}^{n}\left(\dfrac{h\left(\vartheta_i\right)}{h\left(\vartheta_{i-1}\right)}+\dfrac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2}j\right)\right. \\ \qquad\left.\times\left(\dfrac{h\left(\vartheta_{1-i}\right)}{h\left(\vartheta_{-i}\right)}+\dfrac{\zeta_2\eta_1+\eta_2\xi_1}{\xi_1\xi_2}\left(j+i-1\right)\right)\right), \end{cases} \quad (2.22)$$

for $n\in\mathbb{N}_0$, $i\in\{0,1\}$.

Subsubcase 2.1.2. $\zeta_1\zeta_2\neq\xi_1\xi_2$: In this case, by using Lemma 1.1 for $r=2$, we can write the solutions of equations in (2.19) as follows

$$\begin{cases} \varsigma_{2n+i}=\left(\dfrac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n\varsigma_i+\left(\zeta_1\eta_2+\eta_1\xi_2\right)\dfrac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n}{\xi_1\xi_2-\zeta_1\zeta_2}, \\ \varrho_{2n+i}=\left(\dfrac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n\varrho_i+\left(\zeta_2\eta_1+\eta_2\xi_1\right)\dfrac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n}{\xi_1\xi_2-\zeta_1\zeta_2}, \end{cases} \quad n\in\mathbb{N}_0, \quad (2.23)$$

for $i \in \{0,1\}$. By taking into account (2.16), we have

$$\begin{cases} g\left(\omega_{2n+i}\right) = \left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n \frac{g\left(\omega_i\right)}{g\left(\omega_{i-1}\right)} + \left(\zeta_1\eta_2+\eta_1\xi_2\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n}{\xi_1\xi_2-\zeta_1\zeta_2}\right) \\ \qquad \times\left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{n+i-1} \frac{g\left(\omega_{1-i}\right)}{g\left(\omega_{-i}\right)} + \left(\zeta_1\eta_2+\eta_1\xi_2\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{n+i-1}}{\xi_1\xi_2-\zeta_1\zeta_2}\right) \\ \qquad \times g\left(\omega_{2(n-1)+i}\right), \\ h\left(\vartheta_{2n+i}\right) = \left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n \frac{h\left(\vartheta_i\right)}{h\left(\vartheta_{i-1}\right)} + \left(\zeta_2\eta_1+\eta_2\xi_1\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^n}{\xi_1\xi_2-\zeta_1\zeta_2}\right) \\ \qquad \times\left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{n+i-1} \frac{h\left(\vartheta_{1-i}\right)}{h\left(\vartheta_{-i}\right)} + \left(\zeta_2\eta_1+\eta_2\xi_1\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{n+i-1}}{\xi_1\xi_2-\zeta_1\zeta_2}\right) \\ \qquad \times h\left(\vartheta_{2(n-1)+i}\right), \end{cases} \tag{2.24}$$

for $n \in \mathbb{N}_0$, $i \in \{0,1\}$, and consequently

$$\begin{cases} \omega_{2n+i} = g^{-1}\Bigg[g\left(\omega_i\right)\prod_{j=1}^{n}\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^j \frac{g\left(\omega_i\right)}{g\left(\omega_{i-1}\right)} + \left(\zeta_1\eta_2+\eta_1\xi_2\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^j}{\xi_1\xi_2-\zeta_1\zeta_2} \\ \qquad \times\left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{j+i-1} \frac{g\left(\omega_{1-i}\right)}{g\left(\omega_{-i}\right)} + \left(\zeta_1\eta_2+\eta_1\xi_2\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{j+i-1}}{\xi_1\xi_2-\zeta_1\zeta_2}\right)\Bigg], \\ \vartheta_{2n+i} = h^{-1}\Bigg[h\left(\vartheta_i\right)\prod_{j=1}^{n}\left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^j \frac{h\left(\vartheta_i\right)}{h\left(\vartheta_{i-1}\right)} + \left(\zeta_2\eta_1+\eta_2\xi_1\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^j}{\xi_1\xi_2-\zeta_1\zeta_2}\right) \\ \qquad \times\left(\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{n+i-1} \frac{h\left(\vartheta_{1-i}\right)}{h\left(\vartheta_{-i}\right)} + \left(\zeta_2\eta_1+\eta_2\xi_1\right)\frac{1-\left(\frac{\zeta_1\zeta_2}{\xi_1\xi_2}\right)^{j+i-1}}{\xi_1\xi_2-\zeta_1\zeta_2}\right)\Bigg], \end{cases} \tag{2.25}$$

for $n \in \mathbb{N}_0$, $i \in \{0,1\}$.

Subcase 2.2. $\mu_k \neq 0$ for $k \in \{1,2\}$: In this case, by employing the first equation in (2.17) into the second one and the second equation in (2.17) into the first one respectively, we obtain Riccati-type difference equations as follows:

$$\begin{aligned} \varsigma_{n+1} &= \frac{\left(\zeta_1\zeta_2+\eta_1\mu_2\right)\varsigma_{n-1}+\zeta_1\eta_2+\eta_1\xi_2}{\left(\zeta_2\mu_1+\mu_2\xi_1\right)\varsigma_{n-1}+\eta_2\mu_1+\xi_1\xi_2}, \\ \varrho_{n+1} &= \frac{\left(\zeta_1\zeta_2+\eta_2\mu_1\right)\varrho_{n-1}+\zeta_2\eta_1+\eta_2\xi_1}{\left(\zeta_1\mu_2+\mu_1\xi_2\right)\varrho_{n-1}+\eta_1\mu_2+\xi_1\xi_2}, \end{aligned} \tag{2.26}$$

for $n \in \mathbb{N}$. If we apply the decomposition of indexes $n \to 2n+i-1$ for $n \in \mathbb{N}$ and $i \in \{0,1\}$ to (2.26), then they become

$$\varsigma_{2n+i} = \frac{(\zeta_1\zeta_2 + \eta_1\mu_2)\,\varsigma_{2(n-1)+i} + \zeta_1\eta_2 + \eta_1\xi_2}{(\zeta_2\mu_1 + \mu_2\xi_1)\,\varsigma_{2(n-1)+i} + \eta_2\mu_1 + \xi_1\xi_2},$$
$$\varrho_{2n+i} = \frac{(\zeta_1\zeta_2 + \eta_2\mu_1)\,\varrho_{2(n-1)+i} + \zeta_2\eta_1 + \eta_2\xi_1}{(\zeta_1\mu_2 + \mu_1\xi_2)\,\varrho_{2(n-1)+i} + \eta_1\mu_2 + \xi_1\xi_2}. \tag{2.27}$$

Let $\varsigma_{2n+i} = \varsigma_n^{(i)} = \widehat{\varsigma}_n$, $\varrho_{2n+i} = \varrho_n^{(i)} = \widehat{\varrho}_n$ and $n \to n+1$ for $n \in \mathbb{N}_0$, $i \in \{0,1\}$. Then equations in (2.27) can be written in the following form

$$\widehat{\varsigma}_{n+1} = \frac{(\zeta_1\zeta_2 + \eta_1\mu_2)\,\widehat{\varsigma}_n + \zeta_1\eta_2 + \eta_1\xi_2}{(\zeta_2\mu_1 + \mu_2\xi_1)\,\widehat{\varsigma}_n + \eta_2\mu_1 + \xi_1\xi_2},$$
$$\widehat{\varrho}_{n+1} = \frac{(\zeta_1\zeta_2 + \eta_2\mu_1)\,\widehat{\varrho}_n + \zeta_2\eta_1 + \eta_2\xi_1}{(\zeta_1\mu_2 + \mu_1\xi_2)\,\widehat{\varrho}_n + \eta_1\mu_2 + \xi_1\xi_2}, \tag{2.28}$$

for $n \in \mathbb{N}_0$, which are well-known first-order Riccati difference equations with constant coefficients. By employing the following change of variables for $n \in \mathbb{N}_0$, then we have

$$(\zeta_2\mu_1 + \mu_2\xi_1)\,\widehat{\varsigma}_n + \eta_2\mu_1 + \xi_1\xi_2 = \frac{\alpha_{n+1}}{\alpha_n},$$
$$(\zeta_1\mu_2 + \mu_1\xi_2)\,\widehat{\varrho}_n + \eta_1\mu_2 + \xi_1\xi_2 = \frac{\beta_{n+1}}{\beta_n}, \tag{2.29}$$

equations in (2.28) become the following second-order constant coefficients linear difference equations

$$\begin{aligned}\alpha_{n+2} =& (\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2)\,\alpha_{n+1}\\ &- (\zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2)\,\alpha_n,\\ \beta_{n+2} =& (\zeta_1\zeta_2 + \eta_2\mu_1 + \eta_1\mu_2 + \xi_1\xi_2)\,\beta_{n+1}\\ &- (\zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2)\,\beta_n,\end{aligned} \tag{2.30}$$

for $n \in \mathbb{N}_0$. Since the equations given in (2.30) have the same recurrence relation, the characteristic equation of the equations in (2.30) is given by

$$\begin{aligned}&\lambda^2 - (\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2)\,\lambda\\ &+ (\zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2) = 0.\end{aligned} \tag{2.31}$$

From (2.31), we see that there are two different cases for the solutions to the equations in (2.30), depending of whether or not is $\Delta = (\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2)^2 - 4\,(\zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2) = 0$.

Subsubcase 2.2.1. $\Delta \neq 0$: In this case, since the roots of the characteristic equation in (2.31) are different, we easily get the roots of characteristic equation as follows

$$\lambda_{1,2} = \frac{(\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2) \pm \sqrt{\Delta}}{2}. \tag{2.32}$$

Further, the general solutions to the equations in (2.30) in terms of the initial conditions α_0, α_1, β_0, β_1 are given by

$$\alpha_n = \frac{(\alpha_1 - \alpha_0\lambda_2)\,\lambda_1^n + (\alpha_0\lambda_1 - \alpha_1)\,\lambda_2^n}{\lambda_1 - \lambda_2},$$

$$\beta_n = \frac{(\beta_1 - \beta_0\lambda_2)\,\lambda_1^n + (\beta_0\lambda_1 - \beta_1)\,\lambda_2^n}{\lambda_1 - \lambda_2}, \tag{2.33}$$

for $n \in \mathbb{N}_0$, from which along with (2.29), it follows that

$$\begin{aligned}(\zeta_2\mu_1 + \mu_2\xi_1)\,\widehat{\varsigma}_n + \eta_2\mu_1 + \xi_1\xi_2 &= \frac{(\alpha_1 - \alpha_0\lambda_2)\,\lambda_1^{n+1} + (\alpha_0\lambda_1 - \alpha_1)\,\lambda_2^{n+1}}{(\alpha_1 - \alpha_0\lambda_2)\,\lambda_1^{n} + (\alpha_0\lambda_1 - \alpha_1)\,\lambda_2^{n}} \\ &= \frac{\left(\frac{\alpha_1}{\alpha_0} - \lambda_2\right)\lambda_1^{n+1} - \left(\frac{\alpha_1}{\alpha_0} - \lambda_1\right)\lambda_2^{n+1}}{\left(\frac{\alpha_1}{\alpha_0} - \lambda_2\right)\lambda_1^{n} - \left(\frac{\alpha_1}{\alpha_0} - \lambda_1\right)\lambda_2^{n}},\end{aligned} \tag{2.34}$$

$$\begin{aligned}(\zeta_1\mu_2 + \mu_1\xi_2)\,\widehat{\varrho}_n + \eta_1\mu_2 + \xi_1\xi_2 &= \frac{(\beta_1 - \beta_0\lambda_2)\,\lambda_1^{n+1} + (\beta_0\lambda_1 - \beta_1)\,\lambda_2^{n+1}}{(\beta_1 - \beta_0\lambda_2)\,\lambda_1^{n} + (\beta_0\lambda_1 - \beta_1)\,\lambda_2^{n}} \\ &= \frac{\left(\frac{\beta_1}{\beta_0} - \lambda_2\right)\lambda_1^{n+1} - \left(\frac{\beta_1}{\beta_0} - \lambda_1\right)\lambda_2^{n+1}}{\left(\frac{\beta_1}{\beta_0} - \lambda_2\right)\lambda_1^{n} - \left(\frac{\beta_1}{\beta_0} - \lambda_1\right)\lambda_2^{n}}.\end{aligned} \tag{2.35}$$

After the necessary arrangements in (2.34) and (2.35), we get

$$\begin{cases} \widehat{\varsigma}_n = \dfrac{1}{\Phi_1}\dfrac{(\Phi_1\widehat{\varsigma}_0 + \Phi_2)\left(\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}\right) - \lambda_1\lambda_2\left(\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}\right)}{(\Phi_1\widehat{\varsigma}_0 + \Phi_2)\left(\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}\right) - \lambda_1\lambda_2\left(\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}\right)} - \dfrac{\Phi_2}{\Phi_1}, \\ \widehat{\varrho}_n = \dfrac{1}{\Psi_1}\dfrac{(\Psi_1\widehat{\varrho}_0 + \Psi_2)\left(\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}\right) - \lambda_1\lambda_2\left(\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}\right)}{(\Psi_1\widehat{\varrho}_0 + \Psi_2)\left(\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}\right) - \lambda_1\lambda_2\left(\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{\lambda_1-\lambda_2}\right)} - \dfrac{\Psi_2}{\Psi_1}, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.36}$$

where $\zeta_2\mu_1 + \mu_2\xi_1 = \Phi_1$, $\eta_2\mu_1 + \xi_1\xi_2 = \Phi_2$, $\zeta_1\mu_2 + \mu_1\xi_2 = \Psi_1$, $\eta_1\mu_2 + \xi_1\xi_2 = \Psi_2$, from which along with using Definition 1.1, it follows that

$$\begin{cases} \varsigma_{2n+i} = \dfrac{1}{\Phi_1}\dfrac{(\Phi_1\varsigma_i + \Phi_2)\,s_{n+1} - \Upsilon s_n}{(\Phi_1\varsigma_i + \Phi_2)\,s_n - \Upsilon s_{n-1}} - \dfrac{\Phi_2}{\Phi_1}, \\ \varrho_{2n+i} = \dfrac{1}{\Psi_1}\dfrac{(\Psi_1\varrho_i + \Psi_2)\,s_{n+1} - \Upsilon s_n}{(\Psi_1\varrho_i + \Psi_2)\,s_n - \Upsilon s_{n-1}} - \dfrac{\Psi_2}{\Psi_1}, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.37}$$

for $i \in \{0,1\}$, where $\Upsilon = \zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2$. Moreover, by using (2.16), we can write

$$\begin{cases} g\left(\omega_{2n+i}\right) = \left(\dfrac{1}{\Phi_1}\dfrac{\left(\Phi_1\frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right)s_{n+1} - \Upsilon s_n}{\left(\Phi_1\frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right)s_n - \Upsilon s_{n-1}} - \dfrac{\Phi_2}{\Phi_1}\right) \\ \qquad\qquad \times\left(\dfrac{1}{\Phi_1}\dfrac{\left(\Phi_1\frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)s_{n+i} - \Upsilon s_{n+i-1}}{\left(\Phi_1\frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)s_{n+i-1} - \Upsilon s_{n+i-2}} - \dfrac{\Phi_2}{\Phi_1}\right) g\left(\omega_{2(n-1)+i}\right), \\ h\left(\vartheta_{2n+i}\right) = \left(\dfrac{1}{\Psi_1}\dfrac{\left(\Psi_1\frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right)s_{n+1} - \Upsilon s_n}{\left(\Psi_1\frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right)s_n - \Upsilon s_{n-1}} - \dfrac{\Psi_2}{\Psi_1}\right) \\ \qquad\qquad \times\left(\dfrac{1}{\Psi_1}\dfrac{\left(\Psi_1\frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)s_{n+i} - \Upsilon s_{n+i-1}}{\left(\Psi_1\frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)s_{n+i-1} - \Upsilon s_{n+i-2}} - \dfrac{\Psi_2}{\Psi_1}\right) h\left(\vartheta_{2(n-1)+i}\right), \end{cases} \tag{2.38}$$

for $n \in \mathbb{N}_0$, $i \in \{0,1\}$, and consequently

$$\begin{cases} \omega_{2n+i} \\ = g^{-1}\Bigg[g(\omega_i) \displaystyle\prod_{j=1}^{n} \left(\frac{1}{\Phi_1} \frac{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right) s_{j+1} - \Upsilon s_j}{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right) s_j - \Upsilon s_{j-1}} - \frac{\Phi_2}{\Phi_1} \right) \\ \quad \times \left(\frac{1}{\Phi_1} \frac{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right) s_{j+i} - \Upsilon s_{j+i-1}}{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right) s_{j+i-1} - \Upsilon s_{j+i-2}} - \frac{\Phi_2}{\Phi_1} \right) \Bigg], \\ \vartheta_{2n+i} \\ = h^{-1}\Bigg[h(\vartheta_i) \displaystyle\prod_{j=1}^{n} \left(\frac{1}{\Psi_1} \frac{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right) s_{j+1} - \Upsilon s_j}{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right) s_j - \Upsilon s_{j-1}} - \frac{\Psi_2}{\Psi_1} \right) \\ \quad \times \left(\frac{1}{\Psi_1} \frac{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right) s_{j+i} - \Upsilon s_{j+i-1}}{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right) s_{j+i-1} - \Upsilon s_{j+i-2}} - \frac{\Psi_2}{\Psi_1} \right) \Bigg], \end{cases} \tag{2.39}$$

for $n \in \mathbb{N}_0$ and $i \in \{0,1\}$.

Subsubcase 2.2.2. $\Delta = 0$: In this case, similarly as in the previous case, since the roots of the characteristic equation in (2.31) are same, we immediately get the roots of characteristic equation as follows

$$\lambda_{3,4} = \frac{\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2}{2}. \tag{2.40}$$

Then, the general solutions to the equations in (2.30) in terms of the initial conditions α_0, α_1, β_0, β_1 are given by

$$\begin{aligned} \alpha_n &= \alpha_0\lambda_3^n + (\alpha_1 - \alpha_0\lambda_3)\, n\lambda_3^{n-1}, \\ \beta_n &= \beta_0\lambda_3^n + (\beta_1 - \beta_0\lambda_3)\, n\lambda_3^{n-1},\ n \in \mathbb{N}_0. \end{aligned} \tag{2.41}$$

By using (2.29), we get

$$\begin{cases} \Phi_1\widehat{\varsigma}_n + \Phi_2 = \dfrac{\alpha_0\lambda_3^{n+1} + (\alpha_1 - \alpha_0\lambda_3)(n+1)\lambda_3^n}{\alpha_0\lambda_3^n + (\alpha_1 - \alpha_0\lambda_3)\, n\lambda_3^{n-1}}, \\ \Psi_1\widehat{\varrho}_n + \Psi_2 = \dfrac{\beta_0\lambda_3^{n+1} + (\beta_1 - \beta_0\lambda_3)(n+1)\lambda_3^n}{\beta_0\lambda_3^n + (\beta_1 - \beta_0\lambda_3)\, n\lambda_3^{n-1}}, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.42}$$

where $\zeta_2\mu_1 + \mu_2\xi_1 = \Phi_1$, $\eta_2\mu_1 + \xi_1\xi_2 = \Phi_2$, $\zeta_1\mu_2 + \mu_1\xi_2 = \Psi_1$, $\eta_1\mu_2 + \xi_1\xi_2 = \Psi_2$, from which along with $\varsigma_{2n+i} = \varsigma_n^{(i)} = \widehat{\varsigma}_n$, $\varrho_{2n+i} = \varrho_n^{(i)} = \widehat{\varrho}_n$, it follows that

$$\begin{cases} \varsigma_{2n+i} = \dfrac{1}{\Phi_1} \dfrac{(\Phi_1\varsigma_i + \Phi_2)(n+1)\lambda_3^n - n\lambda_3^{n+1}}{(\Phi_1\varsigma_i + \Phi_2)\, n\lambda_3^{n-1} - (n-1)\lambda_3^n} - \dfrac{\Phi_2}{\Phi_1}, \\ \varrho_{2n+i} = \dfrac{1}{\Psi_1} \dfrac{(\Psi_1\varrho_i + \Psi_2)(n+1)\lambda_3^n - n\lambda_3^{n+1}}{(\Psi_1\varrho_i + \Psi_2)\, n\lambda_3^{n-1} - (n-1)\lambda_3^n} - \dfrac{\Psi_2}{\Psi_1}, \end{cases} \quad n \in \mathbb{N}_0, \tag{2.43}$$

for $i \in \{0,1\}$. Also, from (2.16), we can write

$$\begin{cases} g\left(\omega_{2n+i}\right) \\ = \left(\dfrac{1}{\Phi_1} \dfrac{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right)(n+1)\lambda_3^n - n\lambda_3^{n+1}}{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right) n\lambda_3^{n-1} - (n-1)\lambda_3^n} - \dfrac{\Phi_2}{\Phi_1} \right) \\ \quad \times \left(\dfrac{1}{\Phi_1} \dfrac{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)(n+i)\lambda_3^{n+i-1} - (n+i-1)\lambda_3^{n+i}}{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)(n+i-1)\lambda_3^{n+i-2} - (n+i-2)\lambda_3^{n+i-1}} - \dfrac{\Phi_2}{\Phi_1} \right) \\ \quad \times g\left(\omega_{2(n-1)+i}\right), \\ h\left(\vartheta_{2n+i}\right) \\ = \left(\dfrac{1}{\Psi_1} \dfrac{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right)(n+1)\lambda_3^n - n\lambda_3^{n+1}}{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right) n\lambda_3^{n-1} - (n-1)\lambda_3^n} - \dfrac{\Psi_2}{\Psi_1} \right) \\ \quad \times \left(\dfrac{1}{\Psi_1} \dfrac{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)(n+i)\lambda_3^{n+i-1} - (n+i-1)\lambda_3^{n+i}}{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)(n+i-1)\lambda_3^{n+i-2} - (n+i-2)\lambda_3^{n+i-1}} - \dfrac{\Psi_2}{\Psi_1} \right) \\ \quad \times h\left(\vartheta_{2(n-1)+i}\right), \end{cases} \tag{2.44}$$

for $n \in \mathbb{N}_0$, $i \in \{0,1\}$, and consequently

$$\begin{cases} \omega_{2n+i} \\ = g^{-1}\Bigg[g\left(\omega_i\right) \displaystyle\prod_{j=1}^{n} \left(\dfrac{1}{\Phi_1} \dfrac{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right)(j+1)\lambda_3^j - j\lambda_3^{j+1}}{\left(\Phi_1 \frac{g(\omega_i)}{g(\omega_{i-1})} + \Phi_2\right) j\lambda_3^{j-1} - (j-1)\lambda_3^j} - \dfrac{\Phi_2}{\Phi_1} \right) \\ \quad \times \left(\dfrac{1}{\Phi_1} \dfrac{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)(j+i)\lambda_3^{j+i-1} - (j+i-1)\lambda_3^{j+i}}{\left(\Phi_1 \frac{g(\omega_{1-i})}{g(\omega_{-i})} + \Phi_2\right)(j+i-1)\lambda_3^{j+i-2} - (j+i-2)\lambda_3^{j+i-1}} - \dfrac{\Phi_2}{\Phi_1} \right) \Bigg], \\ \vartheta_{2n+i} \\ = h^{-1}\Bigg[h\left(\vartheta_i\right) \displaystyle\prod_{j=1}^{n} \left(\dfrac{1}{\Psi_1} \dfrac{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right)(j+1)\lambda_3^j - j\lambda_3^{j+1}}{\left(\Psi_1 \frac{h(\vartheta_i)}{h(\vartheta_{i-1})} + \Psi_2\right) j\lambda_3^{j-1} - (j-1)\lambda_3^j} - \dfrac{\Psi_2}{\Psi_1} \right) \\ \quad \times \left(\dfrac{1}{\Psi_1} \dfrac{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)(j+i)\lambda_3^{j+i-1} - (j+i-1)\lambda_3^{j+i}}{\left(\Psi_1 \frac{h(\vartheta_{1-i})}{h(\vartheta_{-i})} + \Psi_2\right)(j+i-1)\lambda_3^{j+i-2} - (j+i-2)\lambda_3^{j+i-1}} - \dfrac{\Psi_2}{\Psi_1} \right) \Bigg], \end{cases} \tag{2.45}$$

for $n \in \mathbb{N}_0$, $i \in \{0,1\}$. □

Corollary 2.1. *Consider system (1.3) with the parameters ζ_k, η_k, μ_k, ξ_k, for $k \in \{1,2\}$ and the initial values ω_{-j}, ϑ_{-j}, for $j \in \{0,1\}$, which are real numbers. Then the following statements are true.*

a) If $\zeta_k\xi_k = \eta_k\mu_k$, $\zeta_k = \eta_k = 0$ and $\mu_k\xi_k \neq 0$ for $k \in \{1,2\}$, then the general solution to system (1.3) is given by (2.2).

b) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\zeta_k = 0$ *and* $\eta_k\xi_k \neq 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.8)*.*

c) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\eta_k = 0$ *and* $\zeta_k\mu_k \neq 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.13)*.*

d) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\xi_k = 0$ *and* $\zeta_k = 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.2)*.*

e) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\xi_k = 0$ *and* $\zeta_k \neq 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.13)*.*

f) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\mu_k = 0$ *and* $\eta_k = 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.2)*.*

g) If $\zeta_k\xi_k = \eta_k\mu_k$*,* $\mu_k = 0$ *and* $\eta_k \neq 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.8)*.*

h) If $\zeta_k\xi_k = \eta_k\mu_k$ *and* $\zeta_k\eta_k\mu_k\xi_k \neq 0$ *for* $k \in \{1, 2\}$*, then the general solution to system* (1.3) *is given by* (2.8)*.*

i) If $\zeta_k\xi_k \neq \eta_k\mu_k$*,* $\mu_k = 0$*, for* $k \in \{1, 2\}$ *and* $\zeta_1\zeta_2 = \xi_1\xi_2$*, then the general solution to system* (1.3) *is given by* (2.22)*.*

j) If $\zeta_k\xi_k \neq \eta_k\mu_k$*,* $\mu_k = 0$*, for* $k \in \{1, 2\}$ *and* $\zeta_1\zeta_2 \neq \xi_1\xi_2$*, then the general solution to system* (1.3) *is given by* (2.25)*.*

k) If $\zeta_k\xi_k \neq \eta_k\mu_k$*,* $\mu_k \neq 0$*, for* $k \in \{1, 2\}$ *and* $\Delta \neq 0$*, then the general solution to system* (1.3) *is given by* (2.39)*.*

l) If $\zeta_k\xi_k \neq \eta_k\mu_k$*,* $\mu_k \neq 0$*, for* $k \in \{1, 2\}$ *and* $\Delta = 0$*, then the general solution to system* (1.3) *is given by* (2.45)*.*

Where

$$\Delta = (\zeta_1\zeta_2 + \eta_1\mu_2 + \eta_2\mu_1 + \xi_1\xi_2)^2 - 4(\zeta_1\zeta_2\xi_1\xi_2 + \eta_1\eta_2\mu_1\mu_2 - \zeta_1\eta_2\mu_2\xi_1 - \zeta_2\eta_1\mu_1\xi_2).$$

3. Conclusion

In this paper, we investigated the solutions of the following two dimensional system of difference equations

$$\begin{cases} \omega_{n+1} = g^{-1}\left(g(\omega_n)\dfrac{\zeta_1 h(\vartheta_n) + \eta_1 h(\vartheta_{n-1})}{\mu_1 h(\vartheta_n) + \xi_1 h(\vartheta_{n-1})}\right), \\ \vartheta_{n+1} = h^{-1}\left(h(\vartheta_n)\dfrac{\zeta_2 g(\omega_n) + \eta_2 g(\omega_{n-1})}{\mu_2 g(\omega_n) + \xi_2 g(\omega_{n-1})}\right), \end{cases} \quad n \in \mathbb{N}_0,$$

where the parameters ζ_k, η_k, μ_k, ξ_k, for $k \in \{1, 2\}$ are real numbers, the initial values ω_{-l}, ϑ_{-l}, for $l \in \{0, 1\}$ are real numbers, g and h are continuous and strictly monotone functions, $g(\mathbb{R}) = \mathbb{R}$, $h(\mathbb{R}) = \mathbb{R}$, $g(0) = 0$, $h(0) = 0$.

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