Journal of Applied Analysis and Computation
Volume 15, Number 6, December 2025, 3704–3727
Website:http://www.jaac-online.com
DOI:10.11948/20250062

STOCHASTIC BIFURCATION ANALYSIS IN A SIV EPIDEMIC MODEL WITH POPULATION MIGRATION

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Abstract In this paper, we propose a stochastic SIV epidemic model with population migration and analyze its stochastic stability and bifurcation. By utilizing polar coordinate transformation, stochastic averaging method and singular boundary theory, we prove the stochastic local and global stability of the system. Moreover, we derive sufficient conditions for the system to undergo the stochastic pitchfork bifurcation and Hopf bifurcation. Finally, numerical simulations are performed to verify the theoretical results.

Keywords Stochastic bifurcation, stochastic stability, epidemic model, invariant measure.

MSC(2010) 60H10, 65C20.

1. Introduction

Rubella is a highly contagious and serious human disease caused by the rubella virus. Pregnant women and children are at high risk of contracting rubella [21,32]. If a pregnant woman contracts the virus during the first trimester of pregnancy, it can potentially result in premature birth, miscarriage or even fetal abnormalities. Typical symptoms of rubella infection in children are fever, rash, loss of appetite and can even lead to serious complications. Therefore, it is crucial for children and pregnant women to receive the MMR (measles, mumps and rubella) vaccine to prevent this viral disease.

Mathematicians have been using mathematical means to study infectious diseases for over a century. These studies [13, 17] have provided important theoretical foundations for understanding the mechanisms of disease transmission and for developing effective prevention and control strategies. In this process, they have continuously attempted to construct various mathematical models to explore the dynamics of disease spreading, and have analyzed the asymptotic behavior of these

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epidemic models. In 1932, for example, Kermack and Mckendrick [11] created the classical compartment model, which has served as a crucial theoretical basis for subsequent infectious disease research and is still widely used today. Additionally, Alexander and Moghadas [1] introduced an SIV model with a nonlinear incidence rate that includes the model with susceptible individuals, infected individuals and perfect vaccination as follows

$$\begin{cases} \mathrm{d}S(t) = [(1-p)\Pi - \beta\left(1+\nu I(t)\right)I(t)S(t) - \mu S(t)]\,\mathrm{d}t, \\ \mathrm{d}I(t) = [\beta\left(1+\nu I(t)\right)I(t)S(t) - (\mu+\alpha)\,I(t)]\,\mathrm{d}t, \\ \mathrm{d}V(t) = [p\Pi + \alpha I - \mu V]\,\mathrm{d}t. \end{cases} \tag{1.1}$$

The authors demonstrated that model (1.1) experiences a supercritical Hopf bifurcation and a subcritical Hopf bifurcation, respectively. It is worth noting that such an infectious disease spreads unidirectionally within a population, and thus this model is suitable for describing diseases with permanent immunity. Considering the characteristics of various infectious diseases and the effectiveness of vaccines, the disease described by the model is consistent with rubella (see [32]). To better reflect the actual spread of rubella in a population, we introduce the birth rate of the susceptible individuals, denoted as b. Moreover, since infected individuals who become pregnant usually choose to terminate the pregnancy, we propose the following model that neglects the birth rate of infected individuals

$$\begin{cases} \mathrm{d}S(t) = [(1-p)\Pi + bS(t) - \beta\left(1+\nu I(t)\right)I(t)S(t) - \mu S(t)]\,\mathrm{d}t, \\ \mathrm{d}I(t) = [\beta\left(1+\nu I(t)\right)I(t)S(t) - (\mu+\alpha)\,I(t)]\,\mathrm{d}t, \\ \mathrm{d}V(t) = [p\Pi + \alpha I - \mu V]\,\mathrm{d}t, \end{cases} \tag{1.2}$$

where $S(t)$, $I(t)$ and $V(t)$ represent susceptible, infective and vaccinated individuals, respectively. The significance of parameters in model (1.2) is shown in Table 1. Based on the biological significance, all the parameters of system (1.2) are positive constants. Further, the flowchart diagram of model (1.2) is shown in Figure 1.

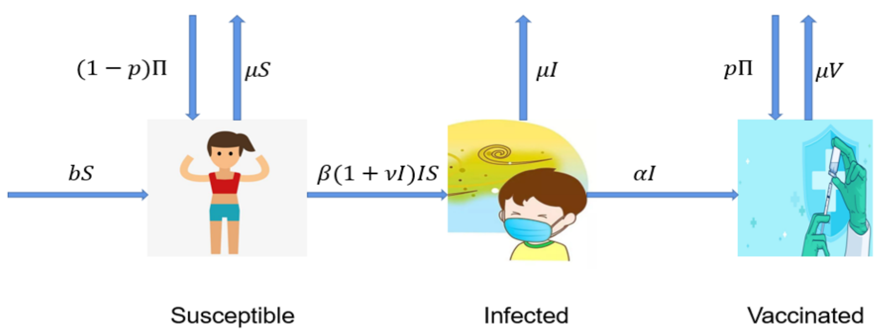


Figure 1. Flowchart diagram of model (1.2).

For convenience, define the basic reproduction number $R_0 = \dfrac{\beta(1-p)\Pi}{(\mu-b)(\mu+\alpha)}$ and

$$\Delta = \frac{\left(-\mu-\alpha+(1-p)\,\Pi\nu\right)^2\beta + 4\nu\left(\alpha+\mu\right)^2(b-\mu) + 4\beta(\mu+\alpha)\nu(1-p)\Pi}{\beta}$$

Table 1. The significance of parameters in model (1.2).

Parameter	Significance
b	the birth rate of susceptible individuals
α	the recovery rate
μ	the natural death rate
p	the proportion of immigrants who are vaccinated
Π	the number of immigrants per unit of time
β	the rate of disease infection

in system (1.2).

Lemma 1.1. *In model* (1.2)*, if* $b < \mu$*, then*

(i) *There is a disease-free equilibrium point* $E_0\left(\frac{(1-p)\Pi}{\mu-b}, 0, \frac{p\Pi}{\mu}\right)$. *When* $R_0 < 1$, E_0 *is locally stable, and when* $R_0 > 1$, E_0 *is unstable.*

(ii) *When* $\Delta > 0$ *and* $\nu > \max\{\frac{\sqrt{\Delta}-\mu-\alpha}{\Pi(1-p)}, \frac{\alpha+\mu-\sqrt{\Delta}}{\Pi(1-p)}\}$*, the endemic equilibrium point* $E_1^*(S_1^*, I_1^*, V_1^*)$ *exists, where* $S_1^* = \frac{\sqrt{\Delta}-\mu-\alpha-(1-p)\Pi\nu}{2\nu(b-\mu)}$, $I_1^* = \frac{\sqrt{\Delta}-\mu-\alpha+(1-p)\Pi\nu}{2\nu(\alpha+\mu)}$, $V_1^* = \frac{p\Pi+\alpha I_1^*}{\mu}$.

In particular, we only discuss the stochastic bifurcation at the endemic equilibrium point $E_1^*(S_1^*, I_1^*, V_1^*)$ in this paper.

First, we perform an equivalence transformation on system (1.2) with respect to the equilibrium point E_1^* to obtain the following system

$$\begin{cases} \mathrm{d}S(t) = [b(S(t)-S_1^*)+\beta(1+\nu I_1^*)I_1^*S_1^*-\beta(1+\nu I(t))I(t)S(t) \\ \qquad\qquad -\mu(S(t)-S_1^*)]\,\mathrm{d}t, \\ \mathrm{d}I(t) = [\beta(1+\nu I(t))I(t)S(t)-\beta(1+\nu I_1^*)I_1^*S_1^*-\mu(I(t)-I_1^*) \\ \qquad\qquad -\alpha(I(t)-I_1^*)]\,\mathrm{d}t, \\ \mathrm{d}V(t) = [\alpha(I(t)-I_1^*)-\mu(V(t)-V_1^*)]\,\mathrm{d}t. \end{cases} \tag{1.3}$$

However, the spread of diseases is often affected by various uncertain factors such as temperature, weather and natural disasters. These factors make the study of the disease transmission complicated and the results difficult to predict. To better simulate this process, many researchers have considered stochastic effects in their modeling and proposed stochastic models [4–6, 18, 19, 22, 30, 37, 39]. Therefore, for system (1.3), we assume that the birth rate of susceptible population and recovery rate of the infective population are influenced by white noise, i.e., $b \longrightarrow b+\sigma_1\dot{B}_1(t)$, $-\alpha \longrightarrow -\alpha+\sigma_2\dot{B}_2(t)$, where $B_i(t)$ $(i=1,2)$ are mutually independent Brownian motions, and σ_i $(i=1,2)$ represent the white noise intensity. As a result, we yield the following stochastic model

$$\begin{cases} \mathrm{d}S(t) = [(1-p)\Pi+bS(t)-\beta(1+\nu I(t))I(t)S(t)-\mu S(t)]\,\mathrm{d}t \\ \qquad\qquad +\sigma_1(S(t)-S_1^*)\,\mathrm{d}B_1(t), \\ \mathrm{d}I(t) = [\beta(1+\nu I(t))I(t)S(t)-(\mu+\alpha)I(t)]\,\mathrm{d}t+\sigma_2(I(t)-I_1^*)\,\mathrm{d}B_2(t), \\ \mathrm{d}V(t) = [p\Pi+\alpha I(t)-\mu V(t)]\,\mathrm{d}t-\sigma_2(I(t)-I_1^*)\,\mathrm{d}B_2(t). \end{cases} \tag{1.4}$$

It is easy to find that the first two equations and the third equation of model (1.4) are independent. Therefore, we only need to discuss the first two equations of model (1.4).

$$
\begin{cases}
\mathrm{d}S(t) = \left[(1-p)\Pi + bS(t) - \beta\left(1+\nu I(t)\right)I(t)S(t) - \mu S(t)\right]\mathrm{d}t \\
\qquad\qquad + \sigma_1\left(S(t) - S_1^*\right)\mathrm{d}B_1(t), \\
\mathrm{d}I(t) = \left[\beta\left(1+\nu I(t)\right)I(t)S(t) - \left(\mu+\alpha\right)I(t)\right]\mathrm{d}t \\
\qquad\qquad + \sigma_2\left(I(t) - I_1^*\right)\mathrm{d}B_2(t).
\end{cases}
\tag{1.5}
$$

For the deterministic SIV models, many scholars have studied their dynamical behavior and bifurcation phenomenon, such as [16, 38, 41]. While the theory of bifurcation in deterministic systems is well-established and has been extensively explored over a long period of time [7, 8, 14, 31], the theory of stochastic bifurcation is still in its infancy. And based on the existing literature, there are currently no scholars focusing on the bifurcation phenomena of the stochastic model (1.4). But Arnold [2] made significant contributions to the development of the theory of random dynamical systems (RDS). This theory has been applied in many fields such as biology, mathematics, physics and economics to study stochastic bifurcations and stochastic stability [3, 9, 10, 15, 20, 23, 26, 27, 29, 34, 36, 43]. Although the development of the RDS theory has contributed to the progress of stochastic bifurcation theory, it has not yet formed a complete framework until now. On the one hand, stochastic bifurcation [42] includes only two types: dynamical bifurcation (D-bifurcation) and phenomenological bifurcation (P-bifurcation). The system undergoes a D-bifurcation if the sign of the maximum Lyapunov exponent suddenly changes, and from this perspective, the D-bifurcation is a dynamical concept. The system undergoes a P-bifurcation, which refers to a change in the shape of the stationary probability density function and hence the P-bifurcation is a static concept. On the other hand, scholars have not established a standardized approach to judge the stochastic bifurcation of most models. There are primarily two methods for studying stochastic bifurcation. One approach is to approximate a two-dimensional Markov diffusion process by stochastic averaging method and to determine whether the approximated process undergoes bifurcation. An alternative approach is to determine it directly by definition. However, not all equations have analytical solutions corresponding to the Fokker-Planck-Kolmogorov (FPK) equation. Nia and Akrami [25] considered a vocal fold model perturbed by white noise. They employed polar coordinate transformation, Taylor expansion, and stochastic averaging method to obtain the conclusion of the existence of P-bifurcation in the stochastic model. Zhang and Yuan [40] investigated the bifurcation in a stochastic logistic model with distributed delays in the weak kernel. By solving the FPK equation for the original equation, they derived the sufficient conditions for the P-bifurcation in the system.

The rest of the paper is arranged as follows. In Section 2, we provide some mathematical definitions. In Section 3 we obtain a two-dimensional Markov process that weakly converges from the original process. Section 4 is devoted to studying the stochastic local and global stability of the original system (1.4) by dimension reduction, stochastic averaging method, the maximum Lyapunov exponent and the singular boundary theory. In Section 5, we derive sufficient conditions for the stochastic pitchfork bifurcation and Hopf bifurcation to occur in system (1.4). In Section 6, we use numerical examples to verify the theoretical results in this paper. The last section ends with a conclusion.

2. Mathematical preliminaries

In this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. We let $\mathbb{E}$ represent the probability expectation with respect to $\mathbb{P}$. Next, we give the definitions of stochastic bifurcation, see [9].

Definition 2.1. (D-bifurcation) Dynamical bifurcation is concerned with a family RDS which is differential and has invariant measure μ_α. If there exist a constant α_D satisfying in any neighborhood of α_D, there exist another constant α and the corresponding invariant measure $\nu_\alpha \neq \mu_\alpha$ satisfying $\nu_\alpha \to \mu_\alpha$ as $\alpha \to \alpha_D$. Then, the constant α_D is a point of D-bifurcation.

Definition 2.2. (P-bifurcation) Phenomenological bifurcation is concerned with the change in the shape of stationary probability density of a family RDS as the change of the parameter. If there exists a constant α_P satisfying in any neighborhood of α_P, there exist other two constant α_1, α_2 and their corresponding invariant measures p_{α_1}, p_{α_2} satisfying p_{α_1} and p_{α_2} are not equivalent. Then the constant α_P is a point of P-bifurcation.

Definition 2.3. (Stochastic pitchfork bifurcation) In the viewpoint of P-bifurcation: The stationary solution of the FPK equation corresponding to the stochastic differential equation changes from one peak to two peaks.

In the viewpoint of D-bifurcation: If there exists a constant α_0 satisfying the following conditions.

(i) When $\alpha < \alpha_0$, the stochastic differential equation has only one invariant measure μ_0, moreover μ_0 is stable.

(ii) When $\alpha = \alpha_0$, the invariant measure μ_0 loses its stability and becomes unstable.

(iii) When $\alpha > \alpha_0$, the stochastic differential equation has three invariant measures μ_0, μ_1 and μ_2, both μ_1 and μ_2 are stable.

If the stochastic bifurcation of a stochastic differential equation has the above characteristic, then the stochastic differential equation occurs stochastic pitchfork bifurcation at $\alpha = \alpha_0$.

Definition 2.4. (Stochastic Hopf bifurcation) In the viewpoint of P-bifurcation: The stationary solution of the FPK equation which is corresponded with the stochastic differential equation changes from peak to crater.

In the viewpoint of D-bifurcation: If one of the invariant measures of the stochastic differential equation loses its stability and becomes unstable. Meanwhile there at least appears one new invariant measure.

If the stochastic bifurcation of a stochastic differential equation has the above characteristic, then the stochastic differential equation occurs stochastic Hopf bifurcation.

3. Model analysis

Without loss of generality, we discuss the properties of equilibrium point (S_1^*, I_1^*) by translation transformations for system (1.5). Let $X(t) = S(t) - S_1^*$, $Y(t) =$

$I(t) - I_1^*$. Then we have that

$$\begin{cases} \mathrm{d}X(t) = [(-\beta(\nu I_1^* + 1) I_1^* + b - \mu) X(t) - \beta(2\nu I_1^* + 1) S_1^* Y(t) \\ \qquad - \beta(2\nu I_1^* + 1) X(t)Y(t) - \beta\nu S_1^* Y^2(t) - \beta\nu X(t)Y^2(t)]\mathrm{d}t \\ \qquad + \sigma_1 X(t)\mathrm{d}B_1(t), \\ \mathrm{d}Y(t) = [\beta(\nu I_1^* + 1) I_1^* X(t) + (\beta(2\nu I_1^* + 1) S_1^* - \mu - \alpha) Y(t) \\ \qquad + \beta(2\nu I_1^* + 1) X(t)Y(t) + \beta\nu S_1^* Y^2(t) + \beta\nu X(t)Y^2(t)]\mathrm{d}t \\ \qquad + \sigma_2 Y(t)\mathrm{d}B_2(t). \end{cases} \tag{3.1}$$

Next, we perform a stretching transformation of equation (3.1) by

$$a_{ijk} = \varepsilon\bar{a}_{ijk},\ b_{imn} = \sqrt{\varepsilon}\bar{b}_{imn},\ i \in \{1,2\},\ j,k \in \{0,1,2\},\ m,n \in \{0,1\},$$

where a_{ijk} and b_{imn} represent the coefficients of the drift term X^jY^k and the diffusion term X^mY^n of equation i, respectively. ε is a sufficient small positive number. For convenience, we drop the bars from the scaled variables and have that

$$\begin{cases} \mathrm{d}X(t) = \varepsilon[(-\beta(\nu I_1^* + 1) I_1^* + b - \mu) X(t) - \beta(2\nu I_1^* + 1) S_1^* Y(t) \\ \qquad - \beta(2\nu I_1^* + 1) X(t)Y(t) - \beta\nu S_1^* Y^2(t) \\ \qquad - \beta\nu X(t)Y^2(t)]\mathrm{d}t + \sqrt{\varepsilon}\sigma_1 X(t)\mathrm{d}B_1(t), \\ \mathrm{d}Y(t) = \varepsilon[\beta(\nu I_1^* + 1) I_1^* X(t) + (\beta(2\nu I_1^* + 1) S_1^* - \mu - \alpha) Y(t) + \beta(2\nu I_1^* + 1) \\ \qquad \times X(t)Y(t) + \beta\nu S_1^* Y^2(t) + \beta\nu X(t)Y^2(t)]\mathrm{d}t + \sqrt{\varepsilon}\sigma_2 Y(t)\mathrm{d}B_2(t). \end{cases} \tag{3.2}$$

We then employ the polar transformation $X = r\cos\theta$, $Y = r\sin\theta$ to the stochastic system (3.2), which means applying Itô's formula to $r = \sqrt{X^2 + Y^2}$, $\theta = \arctan(Y/X)$, respectively. We can obtain

$$\begin{cases} \mathrm{d}r = \varepsilon[\beta\nu(\cos^4\theta - \sin\theta\cos^3\theta - \cos^2\theta + \sin\theta\cos\theta)r^3 + \beta((1 + \nu(2I_1^* - S_1^*))\cos\theta \\ \qquad + \nu S_1^*\sin\theta - (1 + \nu(2I_1^* - S_1^*))\cos^3\theta - (1 + \nu(2I_1^* + S_1^*))\sin\theta\cos^2\theta)r^2 \\ \qquad + ((-\frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2})\cos^4\theta + (\beta(-\nu(I_1^* + 2S_1^*)I_1^* - I_1^* - S_1^*) + \frac{\sigma_2^2}{2} + \alpha + \frac{\sigma_1^2}{2} + b)\cos^2\theta \\ \qquad + \beta(\nu(I_1^* - 2S_1^*)I_1^* + I_1^* - S_1^*)\sin\theta\cos\theta + \beta(2\nu I_1^* S_1^* + S_1^*) - \alpha - \mu)r]\mathrm{d}t \\ \qquad + \sqrt{\varepsilon}\sigma_1 r\cos^2\theta\mathrm{d}B_1(t) + \sqrt{\varepsilon}\sigma_2 r\sin^2\theta\mathrm{d}B_2(t), \\ \mathrm{d}\theta = \varepsilon[-\beta\nu(\cos^4\theta + \sin\theta\cos^3\theta - \cos^2\theta - \sin\theta\cos\theta)r^2 - \beta((1 + \nu(2I_1^* + S_1^*))\cos^3\theta \\ \qquad - (1 + \nu(2I_1^* - S_1^*))\sin\theta\cos^2\theta) - (1 + \nu(2I_1^* + S_1^*))\cos\theta - \nu S_1^*\sin\theta)r \\ \qquad - ((-\sigma_1^2 - \sigma_2^2)\sin\theta\cos^3\theta - \beta(\nu I_1^*(I_1^* - 2S_1^*) + I_1^* - S_1^*)\cos^2\theta \\ \qquad + (\beta(\nu(-I_1^*(I_1^* + 2S_1^*) - I_1^* - S_1^*) + \sigma_2^2 + b + \alpha)\sin\theta\cos\theta - \beta(2\nu I_1^* + 1)S_1^*)]\mathrm{d}t \\ \qquad - \sqrt{\varepsilon}\sigma_1\sin\theta\cos\theta\mathrm{d}B_1(t) + \sqrt{\varepsilon}\sigma_2\sin\theta\cos\theta\mathrm{d}B_2(t). \end{cases} \tag{3.3}$$

It is very difficult to compute the analytical solution of the FPK equation corresponding to the two-dimensional system (3.3). Therefore, in this case, we choose the stochastic averaging method to achieve the dimension reduction of system (3.3). According to Khasminskii limiting theorem [12], when the intensity of the white noise or ε is sufficiently small, the process $(r(t), \theta(t))$ weakly converge to a two-dimensional Markov diffusion process. We can yield the Itô stochastic differential equation by utilizing stochastic averaging method as follows

$$\begin{cases} \mathrm{d}r = m_1(r)\,\mathrm{d}t + \sigma_1(r)\,\mathrm{d}B_r(t), \\ \mathrm{d}\theta = m_2(r)\,\mathrm{d}t + \sigma_2(r)\,\mathrm{d}B_\theta(t), \end{cases} \tag{3.4}$$

where $B_r(t)$, $B_\theta(t)$ are mutually independent Brownian motions. Moreover, $\begin{bmatrix} m_1(r) \\ m_2(r) \end{bmatrix}$ is the drift coefficient vector, $\begin{bmatrix} \sigma_1(r) & 0 \\ 0 & \sigma_2(r) \end{bmatrix}$ is the diffusion coefficient matrix,

$$\begin{aligned} m_1(r) &= \varepsilon\left(\frac{1}{2}\left(\beta\left(-\nu I_1^{*2} + (2\nu S_1^* - 1)\, I_1^* + S_1^*\right) + b - 2\mu - \alpha\right)\right. \\ &\quad \left. + \frac{11}{16}\left(\sigma_1^2 + \sigma_2^2\right)\right) r - \frac{1}{8}\varepsilon\beta\nu r^3, \\ m_2(r) &= \frac{1}{2}\varepsilon\beta\left(\nu I_1^{*2} + (2\nu S_1^* + 1)\, I_1^* + S^*\right) + \frac{1}{8}\varepsilon\beta\nu r^2, \\ \sigma_1^2(r) &= \frac{3}{8}\varepsilon\left(\sigma_1^2 + \sigma_2^2\right) r^2, \\ \sigma_2^2(r) &= \frac{1}{8}\varepsilon\left(\sigma_1^2 + \sigma_2^2\right). \end{aligned}$$

As a matter of convenience, we define the following variables:

$$\begin{aligned} \tau_1 &:= \varepsilon\left(\beta\left(-\nu I_1^{*2} + (2\nu S_1^* - 1)\, I_1^* + S_1^*\right) + b - 2\mu - \alpha\right), \\ \tau_2 &:= 11\varepsilon\left(\sigma_1^2 + \sigma_2^2\right), \\ \tau_3 &:= -\varepsilon\beta\nu, \\ \tau_4 &:= 3\varepsilon\left(\sigma_1^2 + \sigma_2^2\right), \\ \tau_5 &:= \varepsilon\beta\left(\nu I_1^{*2} + (2\nu S_1^* + 1)\, I_1^* + S_1^*\right), \\ \tau_6 &:= \varepsilon\left(\sigma_1^2 + \sigma_2^2\right). \end{aligned}$$

Thus, we can rewrite system (3.4) as

$$\begin{cases} \mathrm{d}r = \left[\left(\frac{1}{2}\tau_1 + \frac{1}{16}\tau_2\right) r + \frac{1}{8}\tau_3 r^3\right] \mathrm{d}t + \left(\frac{1}{8}\tau_4 r^2\right)^{\frac{1}{2}} \mathrm{d}B_r(t), \\ \mathrm{d}\theta = \left[\frac{1}{2}\tau_5 - \frac{1}{8}\tau_3 r^2\right] \mathrm{d}t + \left(\frac{1}{8}\tau_6\right)^{\frac{1}{2}} \mathrm{d}B_\theta(t). \end{cases} \tag{3.5}$$

It is easy to notice that the first equation in system (3.5) does not contain θ. Therefore, we just need to focus on the following equation

$$\mathrm{d}r = \left[\left(\frac{1}{2}\tau_1 + \frac{1}{16}\tau_2\right) r + \frac{1}{8}\tau_3 r^3\right] \mathrm{d}t + \left(\frac{1}{8}\tau_4 r^2\right)^{\frac{1}{2}} \mathrm{d}B_r(t). \tag{3.6}$$

Because this paper mainly focuses on the stability and bifurcation of the stochastic system (3.2), it is necessary to ensure that $\tau_4 \neq 0$. Therefore, we can see that neither

τ_2 nor τ_6 is zero. System (3.5) obtained by weak convergence of the stochastic averaging method is not only formally simplified, but more importantly, we obtain a system with two equations that are independent of each other, which provides a great convenience for the later study of stochastic bifurcation. Furthermore, in [42], it is mentioned that the pth moment of $(r(t), \theta(t))$ in system (3.3) converge to the corresponding the pth moment in system (3.5). In the circumstances where $\xi(t) := \mathrm{d}B(t)/\mathrm{d}t$ is ergodic, this convergence also holds as t approaches infinity. For sufficiently small ε, the stability and invariant measure of system (3.3) can be obtained by analyzing system (3.5). Thus, we will only investigate the stability and existence of invariant measure of system (3.6).

4. Stochastic stability

From the discussion in the previous section, we know that the stability of the equilibrium point $E_1^*(S_1^*, I_1^*, V_1^*)$ of system (1.4) is equivalent to the equilibrium point $r = 0$ of system (3.6) by dimension reduction and stochastic averaging method. Therefore, we will only explore the stochastic local and global stability of the equilibrium point $r = 0$ of system (3.6).

4.1. Stochastic local stability

In this subsection, we first determine the stochastic local stability of the equilibrium point $r = 0$ of the linear equation corresponding to system (3.6). Then, we compute the maximum Lyapunov exponent of the solution process $r(t)$ and derive the theorem based on its robustness as follows.

Theorem 4.1. *The following conclusions hold.*

(i) When $\tau_1/2 + \tau_2/16 - \tau_4/16 < 0$, *i.e.*

$$\sigma_1^2 + \sigma_2^2 < -\left(\beta\left(-\nu I_1^{*2} + (2\nu S_1^* - 1)\, I_1^* + S_1^*\right) + b - 2\mu - \alpha\right),$$

the stochastic system (3.6) *is stable at the equilibrium point* $r = 0$. *Then the stochastic system* (1.4) *is stable at the endemic equilibrium point* $E_1^*(S_1^*, I_1^*, V_1^*)$.

(ii) When $\tau_1/2 + \tau_2/16 - \tau_4/16 > 0$, *i.e.*

$$\sigma_1^2 + \sigma_2^2 > -\left(\beta\left(-\nu I_1^{*2} + (2\nu S_1^* - 1)\, I_1^* + S_1^*\right) + b - 2\mu - \alpha\right),$$

the stochastic system (3.6) *is unstable at the equilibrium point* $r = 0$. *Then the stochastic system* (1.4) *is unstable at the endemic equilibrium point* $E_1^*(S_1^*, I_1^*, V_1^*)$.

Proof. As for (3.6), the corresponding homogeneous linear equation at $r=0$ is

$$\mathrm{d}r = \left(\frac{1}{2}\tau_1 + \frac{1}{16}\tau_2\right) r\mathrm{d}t + \left(\frac{1}{8}\tau_4 r^2\right)^{\frac{1}{2}} \mathrm{d}B_r(t). \tag{4.1}$$

Let $V(t) = \ln r(t)$, utilizing Itô's formula to V yields that

$$\mathrm{d}V(t) = \left(\frac{1}{2}\tau_1 + \frac{1}{16}\tau_2 - \frac{1}{16}\tau_4\right) \mathrm{d}t + \left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} \mathrm{d}B_r(t). \tag{4.2}$$

Thus, integrating from 0 to t on the both sides of (4.2), that is

$$V(t) - V(0) = \left(\frac{1}{2}\tau_1 + \frac{1}{16}\tau_2 - \frac{1}{16}\tau_4\right)t + M(t), \tag{4.3}$$

where $M(t) = \int_0^t (\tau_4/8)^{\frac{1}{2}}\,\mathrm{d}B_r(s)$. Hence, according to the strong law of large numbers, we have

$$\limsup_{t\to\infty} \frac{M(t)}{t} = 0. \tag{4.4}$$

Compute maximum Lyapunov exponent

$$\gamma = \limsup_{t\to\infty} \frac{1}{t}\ln r(t) = \frac{1}{2}\tau_1 + \frac{1}{16}\tau_2 - \frac{1}{16}\tau_4. \tag{4.5}$$

We notice that $\gamma < 0$ when $\tau_1/2 + \tau_2/16 - \tau_4/16 < 0$. Based on Oseledec's multiplicative ergodic theorem [28], this linear equation (4.1) is asymptotically stable with probability one. By Khasminskii's conclusion [35] on the stability and asymptotic stability of higher order perturbations, we can conclude that the equilibrium point $r = 0$ of system (3.6) is stable. Similarly, we can prove the unstable case in the same way. The proof is complete. □

4.2. Stochastic global stability

Next, applying the singular boundary theory, we obtain the sufficient condition for the stochastic global stability of the equilibrium point $E_1^*(S_1^*, I_1^*, V_1^*)$ of system (1.4).

Theorem 4.2. *When* $8\tau_1 + \tau_2 - \tau_4 < 0$, *i.e.*

$$\sigma_1^2 + \sigma_2^2 < -\left(\beta\left(-\nu {I_1^*}^2 + (2\nu S_1^* - 1)I_1^* + S_1^*\right) + b - 2\mu - \alpha\right),$$

the stochastic system (3.6) *is globally stable at the equilibrium point* $r = 0$. *Then the stochastic system* (1.4) *is globally stable at the endemic equilibrium point* $E_1^*(S_1^*, I_1^*, V_1^*)$.

Proof. According to the classification of singular boundaries, when $r = 0$, the diffusion coefficient $\sigma_1(r) = 0$. Therefore, $r = 0$ is the first kind of singular boundary of system (3.6). α_l, β_l and c_l represent the diffusion exponent, drift exponent and character value of the left boundary $r = 0$, respectively. By a simple calculation, we can get

$$\alpha_l = 2,\ \beta_l = 1,\ c_l = \lim_{r\to 0^+} \frac{2[(\tau_1/2 + \tau_2/16)r + \tau_3 r^3/8]r^{2-1}}{\tau_4 r^2/8} = \frac{8\tau_1 + \tau_2}{\tau_4}.$$

By referring to Table 2.8-2 of [42], we can derive the following conclusions:

- if $8\tau_1 + \tau_2 - \tau_4 < 0$, i.e. $c_l < 1$, the left boundary $r = 0$ is attractively natural;
- if $8\tau_1 + \tau_2 - \tau_4 = 0$, i.e. $c_l = 1$, the left boundary $r = 0$ is strictly natural;
- if $8\tau_1 + \tau_2 - \tau_4 > 0$, i.e. $c_l > 1$, the left boundary $r = 0$ is repulsively natural.

Moreover, when $r = +\infty$, the drift coefficient $m_1(r) = +\infty$. Therefore, $r = +\infty$ is the second kind of singular boundary at infinity of system (3.6). α_r, β_r and c_r represent the diffusion exponent, drift exponent and character value of the right boundary $r = +\infty$, respectively. We can easily compute

$$\alpha_r = 2,\ \beta_r = 3,\ c_r = \lim_{r\to+\infty} -\frac{2[(\tau_1/2+\tau_2/16)r+\tau_3 r^3/8]r^{2-3}}{\tau_4 r^2/8} = -\frac{2\tau_3}{\tau_4}.$$

Due to $m_1(+\infty) = -\infty$, $\sigma_1^2(+\infty) = +\infty$, $\beta_r > \alpha_r - 1$ and $\beta_r > 1$, by Table 2.8-4 of [42], we conclude that the right boundary $r = +\infty$ belongs to the entrance boundary. Hence, the global stability of the solution of system (3.6) is determined by the left boundary. If the left boundary is an attractive boundary, all solution curves of system (3.6) will approach the left boundary as they enter from the right boundary. This implies that the equilibrium point $r = 0$ of system (3.6) is stochastically globally stable. This completes the proof. □

5. Stochastic bifurcation

In this section we will discuss the stochastic Hopf bifurcation in the viewpoint of the P-bifurcation and analyze the stochastic pitchfork bifurcation in the viewpoint of the D-bifurcation. We will also provide sufficient conditions for system (3.6) to undergo the stochastic bifurcation.

5.1. Stochastic Hopf bifurcation

Firstly, we consider the following FPK equation corresponding to system (3.6).

$$\frac{\partial P(r,t)}{\partial t} = -\frac{\partial}{\partial r}\left[\left(\left(\frac{1}{2}\tau_1+\frac{1}{16}\tau_2\right)r+\frac{1}{8}\tau_3 r^3\right)P(r,t)\right]+\frac{1}{2}\frac{\partial^2}{\partial r^2}\left[\frac{1}{8}\tau_4 r^2 P(r,t)\right]. \tag{5.1}$$

Hence, the stationary probability density $P(r)$ is the solution of the following degenerate system, i.e.,

$$0 = -\frac{\partial}{\partial r}\left[\left(\left(\frac{1}{2}\tau_1+\frac{1}{16}\tau_2\right)r+\frac{1}{8}\tau_3 r^3\right)P(r)\right]+\frac{1}{2}\frac{\partial^2}{\partial r^2}\left[\frac{1}{8}\tau_4 r^2 P(r)\right]. \tag{5.2}$$

The solution of FPK equation (5.2) is

$$P(r) = \begin{cases} \delta(r), & \tau_4 \geq 8\tau_1+\tau_2, \\ \dfrac{r^{\frac{8\tau_1+\tau_2-2\tau_4}{\tau_4}}\cdot\exp\left(\frac{\tau_3}{\tau_4}r^2\right)}{\Gamma\left(\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}\right)\left(-\frac{\tau_4}{\tau_3}\right)^{\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}}}, & \tau_4 < 8\tau_1+\tau_2, \end{cases} \tag{5.3}$$

where $\Gamma(\cdot)$ is Gamma function. Furthermore, define

$$N_\tau := \left[\Gamma\left(\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}\right)\left(-\frac{\tau_4}{\tau_3}\right)^{\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}}\right]^{-1}.$$

By solving $P'(r) = 0$, namely,

$$N_\tau \cdot \left(\frac{8\tau_1 + \tau_2 - 2\tau_4}{\tau_4} + \frac{2\tau_3}{\tau_4} r^2 \right) r^{\frac{8\tau_1+\tau_2-3\tau_4}{\tau_4}} \cdot \exp\left(\frac{\tau_3}{\tau_4} r^2 \right) = 0. \tag{5.4}$$

We can obtain the extreme value of the probability density function $P(r)$. It is easy to know that the quantity of the extreme value is determined by the sign of the parameter $2\tau_4 - 8\tau_1 - \tau_2$. If $2\tau_4 - 8\tau_1 - \tau_2 < 0$, then there are two extreme values at point $r_1^* = 0$ and

$$r_2^* = \sqrt{\frac{2\tau_4 - 8\tau_1 - \tau_2}{2\tau_3}},$$

otherwise only one extreme value at point $r_1^* = 0$. Therefore, we yield the following result.

Lemma 5.1. *The following conclusions hold for the probability density function* $P(r)$*:*

- *(i) If* $4\tau_1 + \tau_2/2 < \tau_4 < 8\tau_1 + \tau_2$, $P(r)$ *is monotonically decreasing in the interval* $(0, +\infty)$ *and tends to infinity as* $r \to 0^+$. *In this situation, the solution trajectories of system* (3.6) *are concentrated in a neighborhood of* $r = 0$.
- *(ii) If* $(8\tau_1 + \tau_2)/3 < \tau_4 \le 4\tau_1 + \tau_2/2$, $P(r)$ *reaches a minimum value at point* $r = 0$ *and a maximum value at point* $r = r_2^*$. *In addition, the derivative of* $P(r)$ *does not exist at point* $r = 0$. *In this case, the solution trajectories of system* (3.6) *are concentrated in a neighborhood of* $r = r_2^*$.
- *(iii) If* $0 < \tau_4 \le (8\tau_1 + \tau_2)/3$, $P(r)$ *has the minimum value at point* $r = 0$ *and maximum value at point* $r = r_2^*$. *Moreover, the derivative of* $P(r)$ *exists at point* $r = 0$.

Then, we give the following theorem.

Theorem 5.1. *System* (3.6) *undergoes the stochastic P-bifurcation as the parameter* τ_4 *passes through* $4\tau_1 + \tau_2/2$ *and* $(8\tau_1 + \tau_2)/3$. *Furthermore, considering the effect of the noise intensity on the P-bifurcation, the critical values for the bifurcation are* $\sigma_1^2 + \sigma_2^2 = -8\tau_1/5\varepsilon$ *and* $\sigma_1^2 + \sigma_2^2 = -4\tau_1/\varepsilon$, *when* $\tau_1 < 0$ *is satisfied.*

Next, we also explore the P-bifurcation of the stochastic system (3.1). Thus, we will discuss its joint probability density function $\tilde{P}(X, Y)$. In previous section, we perform the polar coordinate transformation in order to apply the stochastic averaging method. Consequently, $\tilde{P}(X, Y) = |J| P(r, \theta)$, where the determinant of the Jacobian matrix J of the polar coordinate transformation is given by $|J| = 1/r$. Combining with the marginal probability density function $P(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P(r, \theta)\,\mathrm{d}\theta$ of joint probability density function $\tilde{P}(X, Y)$ (see [33]), we can calculate

$$\tilde{P}(X, Y) = \frac{\left(X^2 + Y^2\right)^{\frac{8\tau_1+\tau_2-3\tau_4}{2\tau_4}} \cdot \exp\left(\frac{\tau_3}{\tau_4} \left(X^2 + Y^2\right) \right)}{\pi \Gamma\left(\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4} \right) \left(-\frac{\tau_4}{\tau_3} \right)^{\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}}}. \tag{5.5}$$

Further, define $M_\tau := \left[\pi \Gamma\left(\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4} \right) \left(-\frac{\tau_4}{\tau_3} \right)^{\frac{8\tau_1+\tau_2-\tau_4}{2\tau_4}} \right]^{-1}$. Similar to the approach discussed in Lemma 5.1, we focus on describing the variation of the probability density function as the parameter τ_4 changes. In addition, we determine the extreme value points of system (3.1) by calculating $\nabla\tilde{P}(X,Y)=\mathbf{0}$, where ∇ denotes the gradient in $\mathbb{R}^2$ and

$$\begin{cases} \dfrac{\partial\tilde{P}(X,Y)}{\partial X} \\ = 2M_\tau X\left(X^2+Y^2\right)^{\frac{8\tau_1+\tau_2-5\tau_4}{2\tau_4}}\left(\dfrac{\tau_3}{\tau_4}(X^2+Y^2)+\dfrac{8\tau_1+\tau_2-3\tau_4}{2\tau_4}\right)\exp\left(\dfrac{\tau_3}{\tau_4}\left(X^2+Y^2\right)\right), \\ \dfrac{\partial\tilde{P}(X,Y)}{\partial Y} \\ = 2M_\tau Y\left(X^2+Y^2\right)^{\frac{8\tau_1+\tau_2-5\tau_4}{2\tau_4}}\left(\dfrac{\tau_3}{\tau_4}(X^2+Y^2)+\dfrac{8\tau_1+\tau_2-3\tau_4}{2\tau_4}\right)\exp\left(\dfrac{\tau_3}{\tau_4}\left(X^2+Y^2\right)\right). \end{cases}$$

As a consequence, we obtain the following lemma.

Lemma 5.2. *The following conclusions hold for the joint probability density function $\tilde{P}(X,Y)$.*

- *(i) If $\tau_4 \geq (8\tau_1+\tau_2)/3$, the joint probability density function $\tilde{P}(X,Y)$ tends to infinity as $(X,Y)\to(0,0)$.*
- *(ii) If $(8\tau_1+\tau_2)/5 < \tau_4 < (8\tau_1+\tau_2)/3$, $\tilde{P}(X,Y)$ reaches a minimum value at point $(0,0)$ and a maximum value on cycle $X^2+Y^2=(3\tau_4-8\tau_1-\tau_2)/2\tau_3$. Moreover, the partial derivatives of $\tilde{P}(X,Y)$ are discontinuous at point $(0,0)$.*
- *(iii) If $0<\tau_4\leq(8\tau_1+\tau_2)/5$, $\tilde{P}(X,Y)$ has a minimum value at point $(0,0)$ and a maximum value on cycle $X^2+Y^2=(3\tau_4-8\tau_1-\tau_2)/2\tau_3$. In addition, $\tilde{P}(X,Y)$ is continuously differentiable at $(0,0)$.*

Remark 5.1. In the viewpoint of the biological significance in Lemma 5.2, for case (i), $\tilde{P}(X,Y)$ tends to infinity at point $(0,0)$. According to Namachchivaya's theory [24] yields that the sample paths $(X(t),Y(t))$ eventually stay near $(0,0)$, which implies that the susceptible individuals $S(t)$ and infected individuals $I(t)$ eventually remain near S_1^* and I_1^* with high probability, respectively, thus leading to the persistence of the disease.

Theorem 5.2. *System (3.1) undergoes stochastic P-bifurcation as the parameter τ_4 passes through $(8\tau_1+\tau_2)/5$ and $(8\tau_1+\tau_2)/3$.*

5.2. Stochastic pitchfork bifurcation

Let $u_t=(-\tau_3/8)^{\frac{1}{2}}r$, and according to the Itô formula yields that

$$\mathrm{d}u_t=\left[\left(\frac{1}{2}\tau_1+\frac{1}{16}\tau_2\right)u_t-u_t^3\right]\mathrm{d}t+\left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}}u_t\mathrm{d}B_r(t), \tag{5.6}$$

which is equivalent to the following Stratonovich stochastic differential equation

$$\mathrm{d}u_t=\left[\left(\frac{1}{2}\tau_1+\frac{1}{16}\tau_2-\frac{1}{16}\tau_4\right)u_t-u_t^3\right]\mathrm{d}t+\left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}}u_t\circ\mathrm{d}B_r(t). \tag{5.7}$$

Set $\alpha=\tau_1/2+\tau_2/16-\tau_4/16$. We can rewrite system (5.7) as

$$\mathrm{d}u_t=\left(\alpha u_t-u_t^3\right)\mathrm{d}t+\left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}}u_t\circ\mathrm{d}B_r(t), \tag{5.8}$$

which is solved by

$$u \to \varphi_\alpha(t,\omega)\, u = \frac{u \cdot \exp\left(\alpha t + \left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} B_r(t)\right)}{\left(1 + 2u^2 \int_0^t \exp\left(2\alpha s + 2\left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s\right)^{\frac{1}{2}}}, \tag{5.9}$$

where u is the initial value of u_t. Before studying the stochastic pitchfork bifurcation, we need some preparatory work as detailed on page 101 of reference [2]. Thus, we will transform system (3.6) into the following local RDS $\varphi_\alpha(t,\omega)$.

Lemma 5.3. *For the local RDS $\varphi_\alpha(t,\omega) : D_t^\alpha(\omega) \to R_t^\alpha(\omega)$ generated by (3.6), the domain $D_t^\alpha(\omega)$ and range $R_t^\alpha(\omega)$ of $\varphi_\alpha(t,\omega)$ can be defined as follows,*

$$D_t^\alpha(\omega) = \begin{cases} \mathbb{R}, & t \geq 0, \\ \left(-d_t^\alpha(\omega), d_t^\alpha(\omega)\right), & t < 0, \end{cases} \tag{5.10}$$

and

$$R_t^\alpha(\omega) = D_{-t}^\alpha(\vartheta_t \omega) = \begin{cases} \left(-r_t^\alpha(\omega), r_t^\alpha(\omega)\right), & t > 0, \\ \mathbb{R}, & t \leq 0, \end{cases} \tag{5.11}$$

where ϑ_t means a flow of Ω and

$$d_t^\alpha(\omega) = \frac{1}{\left(2\left|\int_0^t \exp\left(2\alpha s + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s\right|\right)^{\frac{1}{2}}} > 0, \tag{5.12}$$

and

$$r_t^\alpha(\omega) = d_{-t}^\alpha(\vartheta_t \omega) = \frac{\exp\left(\alpha t + \left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} B_r(t)\right)}{\left(2\int_0^t \exp\left(2\alpha s + 2\left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s\right)^{\frac{1}{2}}} > 0. \tag{5.13}$$

Define $D^\alpha(\omega) := \bigcap_{t\in\mathbb{R}} D_t^\alpha(\omega)$, which is the collection of initial value u ensuring the non-explosion of RDS $\varphi_\alpha(t,\omega)\, u$. Therefore,

$$D^\alpha(\omega) = \begin{cases} \left(-d^\alpha(\omega), d^\alpha(\omega)\right), & \alpha > 0, \\ \{0\}, & \alpha \leq 0, \end{cases} \tag{5.14}$$

where

$$0 < d^\alpha(\omega) = \frac{1}{\left(2\int_{-\infty}^0 \exp\left(2\alpha s + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s\right)^{\frac{1}{2}}} < \infty. \tag{5.15}$$

Theorem 5.3. *Let $\alpha = \tau_1/2 + \tau_2/16 - \tau_4/16$, the family of RDS $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ generated by (3.6) undergoes the stochastic pitchfork bifurcation at $\alpha_D = 0$, More precisely:*

(i) $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ has an invariant measure $\kappa_{1,\omega}^\alpha = \delta_0$ with Lyapunov exponent $\lambda(\kappa_{1,\omega}^\alpha) = \alpha$. Therefore, the invariant measure $\kappa_{1,\omega}^\alpha$ loses stability at $\alpha_D = 0$. If $\alpha < \alpha_D$, δ_0 is the unique invariant measure.

(ii) If $\alpha > \alpha_D$, $(\varphi_\alpha)_{\alpha\in\mathbb{R}}$ has two additional invariant measures $\kappa_{2,\omega}^\alpha = \delta_{d_\alpha(\omega)}$ and $\kappa_{3,\omega}^\alpha = \delta_{-d_\alpha(\omega)}$. Both measures are $\mathcal{F}_{-\infty}^0$ measurable and have negative Lyapunov exponents $\lambda(\kappa_{i,\omega}^\alpha) = -2\alpha, (i = 2, 3)$.

For $\varphi_\alpha-$invariant measure $\kappa^{\alpha}_{i,\omega}(i=1,2,3)$, which are all Markov measures, whose probability density functions are the solution of the corresponding FPK equations.

Proof. The existence of the ergodic invariant measures is discussed in [2]. Further, we calculate the Lyapunov exponent to judge the stability of the invariant measure.

The linearized RDS $\zeta_t = D\varphi_\alpha(t,\omega,u)\,\zeta$ satisfies

$$\mathrm{d}\zeta_t = [\alpha - 3(\varphi_\alpha(t,\omega)u)^2]\mathrm{d}t + \left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} \zeta_t \circ \mathrm{d}B_r(t),$$

whose solution ζ_t is given by

$$D\varphi_\alpha(t,\omega,u)\zeta = \zeta \exp\left(\alpha t + \left(\frac{1}{8}\tau_4\right)^{\frac{1}{2}} B_r\,(t) - 3\int_0^t (\varphi_\alpha(s,\omega)u)^2\,\mathrm{d}s\right).$$

Consequently, when $\kappa^{\alpha}_{i,\omega} = \delta_{u_i(\omega)}(i=1,2,3)$ are $\varphi_\alpha-$invariant measures, their Lyapunov exponents are

$$\begin{aligned}\lambda\left(\kappa^{\alpha}_{i,\omega}\right) &= \lim_{t\to\infty}\frac{1}{t}\ln\|D\varphi_\alpha(t,\omega,u)\zeta\| \\ &= \alpha - 3\lim_{t\to\infty}\frac{1}{t}\int_0^t(\varphi_\alpha(s,\omega)u)^2\mathrm{d}s \\ &= \alpha - 3\mathbb{E}u_i^2,\end{aligned} \tag{5.16}$$

provided that the integrability conditions (IC) $u_i^2 \in L^1(\mathbb{P})$ hold.

(i) For $\alpha \in \mathbb{R}$, the IC for $\kappa^{\alpha}_{1,\omega} = \delta_0$ is trivially satisfied and we can obtain $\lambda(\kappa^{\alpha}_{1,\omega}) = \alpha$. It follows that $\kappa^{\alpha}_{1,\omega}$ is stable for $\alpha < 0$ and unstable for $\alpha > 0$.

(ii) For $\alpha > 0$, the invariant measure $\kappa^{\alpha}_{2,\omega} = \delta_{d_\alpha(\omega)}$ is $\mathcal{F}^0_{-\infty}$ measurable. The probability density function $p_1^\alpha(x)$ of $\kappa^{\alpha}_{2,\omega}$ satisfies the FPK equation as follows:

$$0 = -\frac{\partial}{\partial x}\left[\left(\left(\alpha + \frac{1}{16}\tau_4\right)x - x^3\right)p_1^\alpha(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\frac{1}{8}\tau_4 x^2 p_1^\alpha\,(x)\right], \tag{5.17}$$

which possesses the unique probability density solution

$$p_1^\alpha\,(x) = \begin{cases} 0, & x \le 0, \\ N_\alpha x^{\frac{16\alpha}{\tau_4}-1}\exp\left(\dfrac{-8x^2}{\tau_4}\right), & x > 0, \end{cases} \tag{5.18}$$

with the normalizing parameter N_α satisfying $N_\alpha^{-1} = \Gamma\left(8\alpha/\tau_4\right)\left(8/\tau_4\right)^{8\alpha/\tau_4}$, where $\Gamma\left(\cdot\right)$ is Gamma function. Because

$$\mathbb{E}u^2 = \mathbb{E}\left(d^\alpha\left(\omega\right)\right)^2 = \int_0^\infty u^2 p_1^\alpha(u)\mathrm{d}u < \infty,$$

then IC is satisfied. Moreover,

$$\left(d^\alpha\left(\vartheta_t\omega\right)\right)^2 = \frac{\exp\left(2\alpha t + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}}B_r(t)\right)}{2\int_{-\infty}^t \exp\left(2\alpha s + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}}B_r(s)\right)\mathrm{d}s}.$$

Let $\Phi(t) = \int_{-\infty}^{t} \exp\left(2\alpha s + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s$. Then

$$(d^{\alpha}(\vartheta_t\omega))^2 = \frac{\exp\left(2\alpha t + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}} B_r(t)\right)}{2\int_{-\infty}^{t} \exp\left(2\alpha s + 2(\frac{1}{8}\tau_4)^{\frac{1}{2}} B_r(s)\right) \mathrm{d}s} = \frac{\Phi'(t)}{2\Phi(t)}.$$

Applying the ergodic theorem leads to

$$\mathbb{E}(d^{\alpha}(\omega))^2 = \lim_{t\to\infty} \frac{1}{t}\int_0^t (d^{\alpha}(\vartheta_t\omega))^2 \mathrm{d}s = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\ln\Phi(t) = \alpha. \tag{5.19}$$

Thus, by (5.16), we have $\lambda\left(\kappa_{2,\omega}^{\alpha}\right) = -2\alpha$. Therefore, the invariant measure $\kappa_{2,\omega}^{\alpha}$ is stable for $\alpha > 0$.

(iii) For $\alpha > 0$, the invariant measure $\kappa_{3,\omega}^{\alpha} = \delta_{-d_{\alpha}(\omega)}$ is $\mathcal{F}_{-\infty}^{0}$ measurable, and its probability density $p_2^{\alpha}(x)$ satisfied with $p_2^{\alpha}(x) = p_1^{\alpha}(-x)$. In addition, we have $\mathbb{E}(-d^{\alpha}(\omega))^2 = \mathbb{E}(d^{\alpha}(\omega))^2 = \alpha$ and hence $\lambda\left(\kappa_{3,\omega}^{\alpha}\right) = -2\alpha$. Similarly, we conclude that the invariant measure $\kappa_{3,\omega}^{\alpha}$ is stable for $\alpha > 0$.

Therefore, system (1.5) undergoes a D-bifurcation. Similar to the analysis of (5.1), we can prove that two families of densities $p_1^{\alpha}(x)$ and $p_2^{\alpha}(x)$ undergo the P-bifurcation at $\tau_4/16$. Hence, the stochastic system (1.5) undergoes the stochastic pitchfork bifurcation. As a result, the proof is complete. □

Next, Figure 2 describes the change in the number and stability of the invariant measures for a stochastic pitchfork bifurcation.

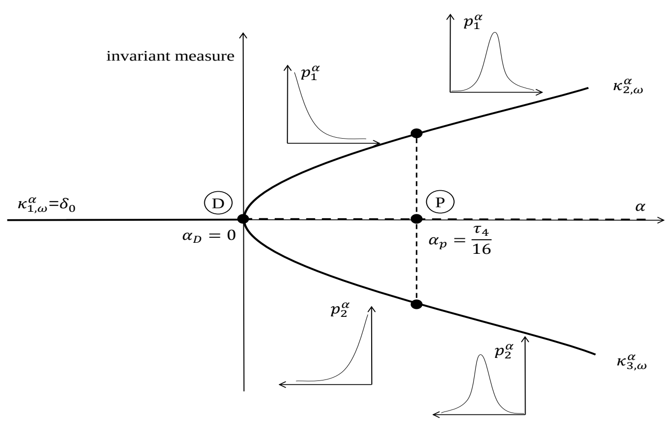


Figure 2. Bifurcation diagram of the stochastic pitchfork bifurcation.

Remark 5.2. If we choose the intensity of white noise as the bifurcation parameter value, then when

$$\sigma_1^2 + \sigma_2^2 = -\left(\beta\left(-\nu {I_1^*}^2 + (2\nu S_1^* - 1) I_1^* + S_1^*\right) + b - 2\mu - \alpha\right),$$

the stochastic system (3.6) undergoes a stochastic pitchfork bifurcation.

6. Numerical simulations

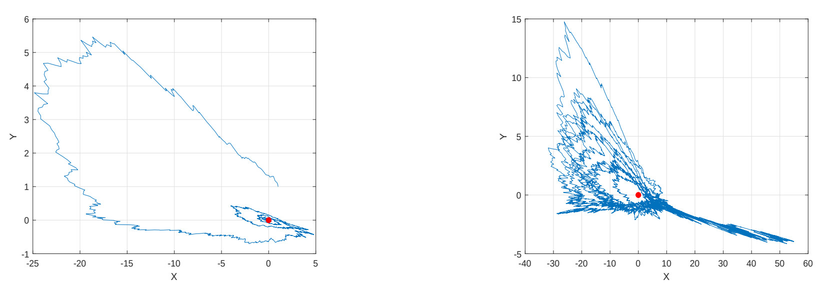
In this section, we verify the theorems of the stochastic model (3.1) by several numerical examples. Firstly, we employ the Euler-Maruyama method to provide numerical solutions for model (3.1), resulting in the following discretized equations

$$\begin{cases} X_{i+1} = X_i + [(-\beta(\nu I_1^* + 1) I_1^* + b - \mu) X_i - (\beta(2\nu I_1^* + 1) S_1^*) Y_i \\ \qquad - \beta(2\nu I_1^* + 1) X_i Y_i - \beta\nu S_1^* {Y_i}^2 - \beta\nu X_i {Y_i}^2]\Delta t + \sigma_1 X_i \sqrt{\Delta t}\varsigma_i, \\ Y_{i+1} = Y_i + [\beta(\nu I_1^* + 1) I_1^* X_i + (\beta(2\nu I_1^* + 1) S_1^* - \mu - \alpha) Y_i + \beta(2\nu I_1^* + 1) X_i Y_i \\ \qquad + \beta\nu S_1^* {Y_i}^2 + \beta\nu X_i {Y_i}^2]\Delta t + \sigma_2 Y_i \sqrt{\Delta t}\varrho_i, \end{cases} \tag{6.1}$$

where ς_i, $\varrho_i (i = 1, 2, \dots)$ are independent Gaussian random variables which obey the norm distribution $N(0, 1)$.

Example 6.1. Let $\Pi = 50$, $\beta = 0.03$, $\mu = 0.6$, $\alpha = 0.7$, $p = 0.9$, $\nu = 0.2$, $\varepsilon = 0.001$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$ and the initial values $X(0) = 1$ and $Y(0) = 1$ in system (3.1). We simulate the stochastic stability of point $(0, 0)$ for system (3.1).

Let $b = 0.4884$ such that $\tau_1/2 + \tau_2/16 - \tau_4/16 < 0$ satisfies the condition (i) of Theorem 4.1. Then, the trivial solution of system (3.1) is stable with probability one. From Figure 3(a), we can see that sample path from the point $(1, 1)$ converges to the origin point $(0, 0)$. Furthermore, let $b = 0.5$ such that $\tau_1/2+\tau_2/16-\tau_4/16 > 0$ satisfies the condition (ii) of Theorem 4.1. Hence, the origin point $(0, 0)$ is unstable with probability one. Figure 3(b) depicts that sample path from the point $(1, 1)$ stays away from the origin point $(0, 0)$.

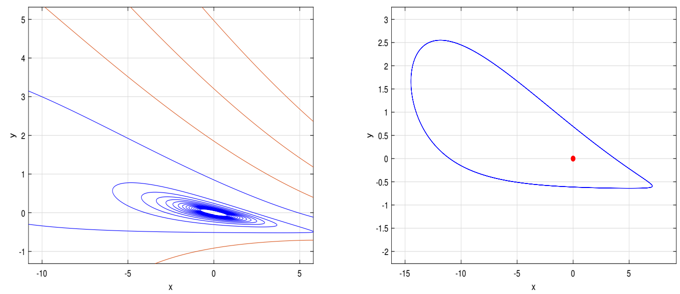


(a) $b = 0.4884$ (b) $b = 0.5$

Figure 3. Phase portraits for the stable and unstable solutions of system (3.1), respectively.

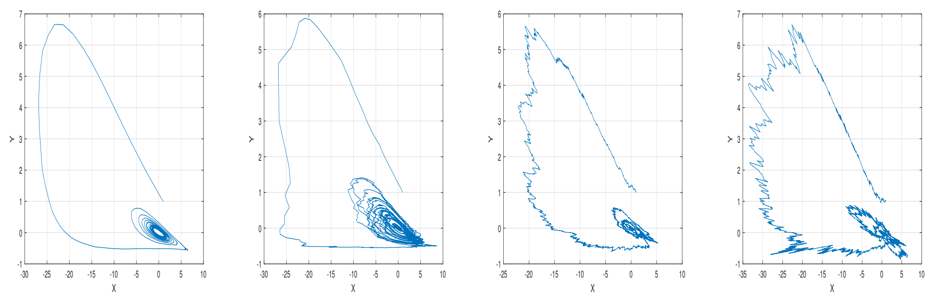
Example 6.2. For system (3.1) with parameters $\Pi = 50$, $\beta = 0.03$, $\mu = 0.6$, $\alpha = 0.7$, $p = 0.9$, $\nu = 0.2$, $\varepsilon = 0.001$, $\sigma_1 = 0$, $\sigma_2 = 0$ and the initial values $X(0) = 1$ and $Y(0) = 1$, we consider the effect of noise on the phase portraits.

When $\sigma_1 = \sigma_2 = 0$, the stochastic system (3.1) degenerates the deterministic system. The phase portrait of the deterministic system is shown in Figure 4(a), which possesses a limit cycle in this situation. However, the shape of the limit cycle changes slightly for small noise perturbations in Figure 5, but the limit cycle



(a) $b = 0.49$ (b) $b = 0.492$

Figure 4. Phase portraits for system (3.1) when $\sigma_1 = \sigma_2 = 0$.



(a) $\sigma_1 = 0.01, \sigma_2 = 0.01$ (b) $\sigma_1 = 0.1, \sigma_2 = 0.01$ (c) $\sigma_1 = 0.1, \sigma_2 = 0.1$ (d) $\sigma_1 = 0.2, \sigma_2 = 0.2$

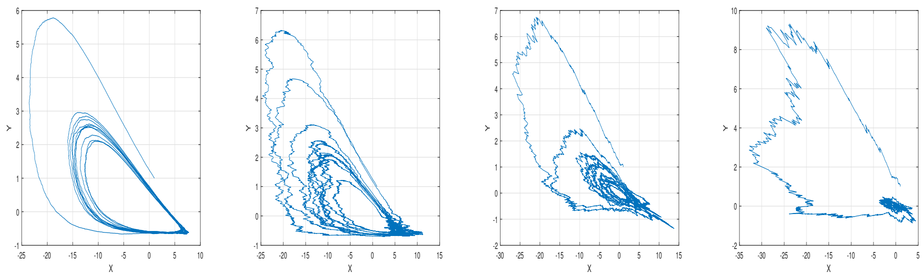
Figure 5. For $b = 0.49$, phase portraits of system (3.1) as the noise intensity increases.

disappears when the noise intensity is continuously increasing, and the solution of system (3.1) tends to be stable under large noise. Furthermore, in Figure 4(b) and Figure 6, when other parameters are unchanged, we only vary $b = 0.492$ to observe the changes of phase portrait and obtain the similar results to Figure 4(a) and Figure 5. Consequently, the limit cycle will break when the noise intensity increases to a certain value, i.e., the dynamic behavior of system (3.1) will change with the addition of noise.

Example 6.3. Consider the qualitative changes of probability density function $P(r)$ when $\Pi = 10$, $\beta = 0.04$, $\mu = 0.055$, $\alpha = 0.4$, $p = 0.75$, $\nu = 0.015$, $b = 0.04$, $\varepsilon = 0.001$ and the initial values $X(0) = 1$ and $Y(0) = 1$.

With the help of Maple, we can calculate $S_1^* \approx 10.5598$, $I_1^* \approx 5.1464$ and $\tau_1 = -2.0414 \times 10^{-4}$. Next, we discuss the shape of the probability density function $P(r)$ by varying noise intensity as follows.

(i) Let $\sigma_1 = 0.2$ and $\sigma_2 = 0.5$. Therefore, we have $-\tau_1/\varepsilon < \sigma_1^2 + \sigma_2^2 \leq -8\tau_1/5\varepsilon$.



(a) $\sigma_1 = 0.01, \sigma_2 = 0.01$ (b) $\sigma_1 = 0.1, \sigma_2 = 0.01$ (c) $\sigma_1 = 0.1, \sigma_2 = 0.1$ (d) $\sigma_1 = 0.2, \sigma_2 = 0.2$

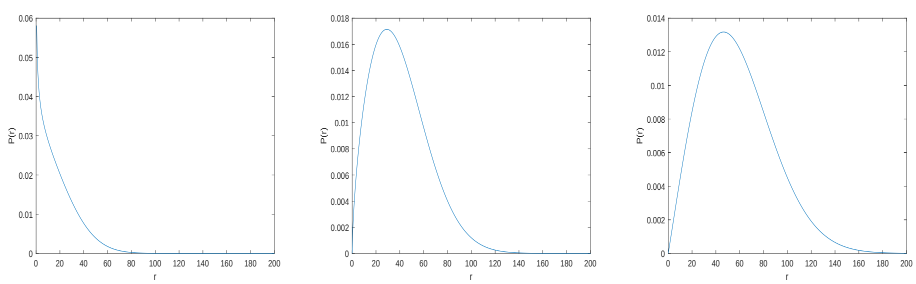
Figure 6. For $b = 0.492$, phase portraits of system (3.1) as the noise intensity increases.

Then, $\lim\limits_{t\to 0^+} P(r) = +\infty$. The probability density function $P(r)$ is shown in Figure 7(a).

(ii) Let $\sigma_1 = 0.5$ and $\sigma_2 = 0.7$, and we can yield $-8\tau_1/5\varepsilon < \sigma_1^2 + \sigma_2^2 \leq -4\tau_1/\varepsilon$. Then $P(r)$ has minimum value point $r_1^* = 0$ and maximum value point $r_2^* \approx 41.5019$. In addition, the derivative of $P(r)$ at $r = 0$ does not exist and the probability density function $P(r)$ is shown in Figure 7(b).

(iii) Let $\sigma_1 = \sigma_2 = 0.65$ such that $\sigma_1^2 + \sigma_2^2 > -4\tau_1/\varepsilon$. $P(r)$ possesses minimum value point $r_1^* = 0$ and maximum value point $r_2^* \approx 46.4748$. Figure 7(c) shows the probability density function $P(r)$ and the derivative of $P(r)$ exists at $r = 0$.

Figure 8 shows the stochastic Hopf bifurcation graph, which region I, region II and region III represent the range of noise intensity in case (i), case (ii) and case (iii), respectively.



(a) $\sigma_1 = 0.2,\ \sigma_2 = 0.5$ (b) $\sigma_1 = 0.5,\ \sigma_2 = 0.7$ (c) $\sigma_1 = 0.65,\ \sigma_2 = 0.65$

Figure 7. The probability density function $P(r)$ for system (3.6) under different noise intensity.

Similar to Theorem 5.1, we want to utilize noise intensity as bifurcation parameters to simulate the change in the shape of the probability density function $\tilde{P}(X, Y)$ in Theorem 5.2. Unfortunately, choosing the noise intensity as parameters may lead to some inequalities that do not hold in Theorem 5.2. Therefore, we choose b as a stochastic Hopf bifurcation parameter and give the following example.

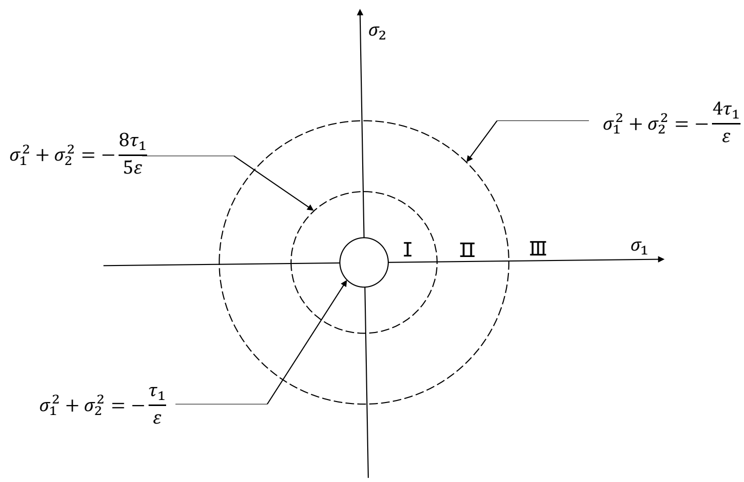


Figure 8. The range of noise intensity of the stochastic Hopf bifurcation for system (3.6).

Example 6.4. Consider the qualitative changes of the probability density function $P(X, Y)$ when $\Pi = 50$, $\beta = 0.03$, $\mu = 0.6$, $\alpha = 0.7$, $p = 0.9$, $\nu = 0.2$, $\varepsilon = 0.001$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$ and the initial values $X(0) = 1$ and $Y(0) = 1$.

The Hopf bifurcation values $\tau_4 = (8\tau_1 + \tau_2)/5$ and $\tau_4 = (8\tau_1 + \tau_2)/3$ have been obtained in Section 5. Therefore, we can compute all parameters of the Hopf bifurcation in the stochastic system (3.1) as shown in Table 2.

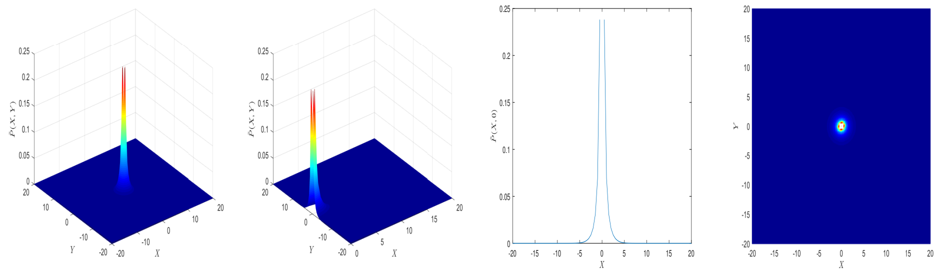
Table 2. The main parameters of the Hopf bifurcation in the stochastic system (3.1).

the rate of birth	S_1^*	I_1^*	τ_1	τ_2	τ_4	$(8\tau_1 + \tau_2)/3$	$(8\tau_1 + \tau_2)/5$
$b = 0.489$	39.7223	0.4544	-1.7571×10^{-5}	2.2×10^{-4}	6×10^{-5}	2.6478×10^{-5}	1.5887×10^{-5}
$b = 0.490$	39.1463	0.5323	-2.5952×10^{-6}	2.2×10^{-4}	6×10^{-5}	6.6413×10^{-5}	3.9848×10^{-5}
$b = 0.492$	38.1891	0.6735	2.3399×10^{-5}	2.2×10^{-4}	6×10^{-5}	1.3573×10^{-4}	8.1439×10^{-5}

(i) If $b = 0.489$, then $\tau_4 > (8\tau_1 + \tau_2)/3$. In system (3.1), Figure 9 shows the limit of the probability density function $\tilde{P}(X, Y)$ does not exist at $(0, 0)$ and the sample paths of $(X(t), Y(t))$ eventually approach point $(0, 0)$ in Figure 12(a).

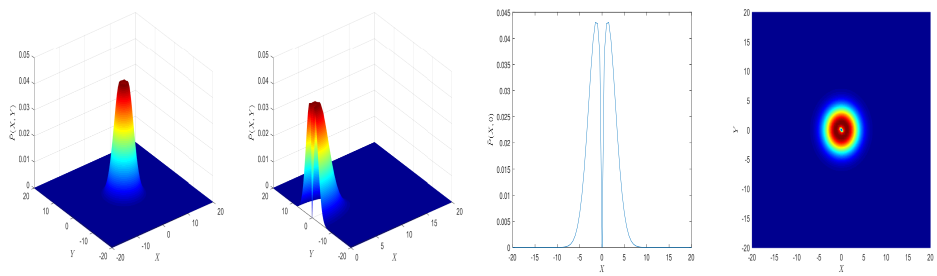
(ii) If $b = 0.490$, then $(8\tau_1 + \tau_2)/5 < \tau_4 < (8\tau_1 + \tau_2)/3$. For system (3.1), we find the partial derivatives of joint probability density function $\tilde{P}(X, Y)$ do not exist at point $(0, 0)$ in Figure 10 and the sample paths of $(X(t), Y(t))$ is shown in Figure 12(b).

(iii) If $b = 0.492$, therefore, $0 < \tau_4 < (8\tau_1 + \tau_2)/5$. In system (3.1), it can be seen that the partial derivatives of joint probability density function $\tilde{P}(X, Y)$ exist at point $(0, 0)$ in Figure 11 and Figure 12(c) illustrates the sample paths of $(X(t), Y(t))$.



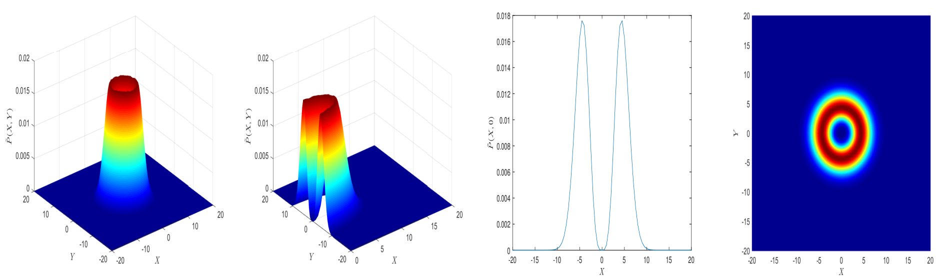
(a) (b) (c) (d)

Figure 9. In system (3.1), for $b = 0.489$, (a) joint probability density function $\tilde{P}(X, Y)$, (b) the cross section of $\tilde{P}(X, Y)$, (c) joint probability density function $\tilde{P}(X, 0)$, (d) the projection of $\tilde{P}(X, Y)$.



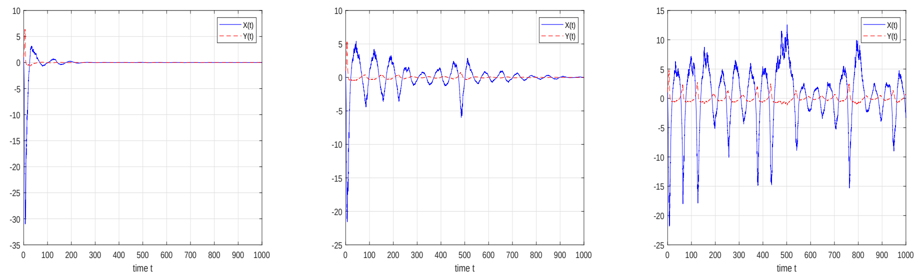
(a) (b) (c) (d)

Figure 10. In system (3.1), for $b = 0.490$, (a) joint probability density function $\tilde{P}(X, Y)$, (b) the cross section of $\tilde{P}(X, Y)$, (c)joint probability density function $\tilde{P}(X, 0)$, (d) the projection of $\tilde{P}(X, Y)$.



(a) (b) (c) (d)

Figure 11. In system (3.1), for $b = 0.492$, (a) joint probability density function $\tilde{P}(X, Y)$, (b) the cross section of $\tilde{P}(X, Y)$, (c) joint probability density function $\tilde{P}(X, 0)$, (d) the projection of $\tilde{P}(X, Y)$.



(a) $b = 0.489$ (b) $b = 0.490$ (c) $b = 0.492$

Figure 12. The sample paths of $X(t)$ and $Y(t)$ of system (3.1).

7. Conclusion

In this paper, we focus on the stability and bifurcation of the stochastic SIV epidemic model. Firstly, we investigate the stochastic stability of the equilibrium point E_1^* by stochastic averaging method, maximum Lyapunov exponent and singular boundary theory. It can be seen from Theorem 4.1 and Theorem 4.2 that when the noise intensity is large enough, the endemic equilibrium point is unstable, while when the noise intensity is small, the endemic equilibrium point is stochastically stable. Next, we find the shape of the stationary probability density function changes from peak to crater, which implies that system (3.1) undergoes the stochastic P-bifurcation (see Theorem 5.2). That is to say, it is also the stochastic Hopf bifurcation in viewpoint of P-bifurcation. Then, according to the existence and stability of the invariant measure, we prove system (3.6) undergoes the stochastic pitchfork bifurcation in Theorem 5.3. By numerical simulations, we know that the noises can change the dynamical behavior of system (1.4). Therefore, according to the noise intensity, we can adjust the prevention and control strategies, such as increasing the coverage rate of vaccination to enhance the immunity of the population and strengthening the isolation and treatment of rubella patients to prevent the spread of the virus to others, etc. In addition, it is necessary to promptly monitor the bifurcation parameters of model (1.4) to avoid the occurrence of the stochastic bifurcations, which keep the number of patients fluctuating within the range of available medical resources.

Acknowledgments

The authors would like to thank the editor and referees for their valuable suggestions, which have greatly helped us to improve our manuscript.

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Received February 2025; Accepted June 2025; Available online June 2025.