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THEORIES AND APPLICATIONS OF ADOMIAN DECOMPOSITION $\mathcal{J}$-TRANSFORM METHOD WITH THEORETICAL ANALYSIS AND SIMULATION

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Abstract This work focuses on using the Adomian decomposition $\mathcal{J}$-transform method ($\mathcal{ADJM}$) to solve both linear and nonlinear differential equations, with the aim of obtaining exact solutions for various types of differential equations, such as the Bratu equations and second-order linear partial telegraph equation. We present comprehensive proofs for new theorems associated with the $\mathcal{J}$-transform method. This approach combines the $\mathcal{J}$-transform method ($\mathcal{JTM}$) and the Adomian decomposition method ($\mathcal{ADM}$). We carry out a theoretical analysis of $\mathcal{ADJM}$ applied to certain nonlinear differential equations and give proofs to the existence and uniqueness theorems along with error estimates. The solutions obtained are compared with exact solutions from other established methods in the literature. The study emphasizes the notable advantages of $\mathcal{ADJM}$, highlighting its effectiveness in solving both ODEs and PDEs. In order to demonstrate the unique advantages of the employed method, we give exact solutions in the form of convergent power series with easily obtainable coefficients. Some of the symbolic and numerical calculations were executed using Mathematica software 13.

Keywords Bratu equation, telegraph equation, Adomian decomposition method, $\mathcal{J}$-transform method, Banach fixed point theorem.

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1. Introduction

Integral transforms play an important role in the solution of differential and integral equations with initial and boundary value conditions. One of the most popular methods for solving these equations is the integral transform method [11, 15–17, 20, 26, 27]. The most popular transform in the literature is the Laplace transform [8, 10]. The variable s in the Laplace transform is regarded as a dummy variable, transforming the function $f(t)$ in the t domain into the function $\mathbb{F}(s)$. The Laplace-Carson transform, also known as the p-multiplied form of the ordinary Laplace transform, was first proposed by Watugala in 1993 [3]. The Sumudu transform is closely related to it and has been used to solve controlled engineering problems

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[2, 28, 29]. Similar to the Laplace-Carson and Sumudu integral transforms, the $\mathbb{N}$-transform, commonly referred to as the natural transform, was first presented in 2008. By modifying the variables, the problem of unstable fluid flow through a flat wall has been solved effectively using the $\mathbb{N}$-transform, which offers both Laplace and Sumudu integral transformations (see [22, 24]).

Linear and nonlinear differential equations may accurately represent a wide range of linear and nonlinear phenomena across a variety of scientific fields, including population dynamics, fluid mechanics, solid-state physics, plasma physics, and chemical kinetics [12,13,33–35]. As a result, the need to obtain exact or approximate solutions for these equations remains a critical problem in applied mathematics and physics, necessitating the development of new methods [5, 14, 18]. Several powerful mathematical approaches have been proposed to address this, including the Adomian decomposition method [4,31], the variational iteration method [9,32], the reduced differential transform method [21,30], the natural Adomian decomposition method [7,23], the Yang-Abdel-Cattani derivative method [6], and the Laplace transform residual power series method [19].

Most physical phenomena are nonlinear and can be represented using partial differential equations (PDEs) or ordinary differential equations (ODEs) of integer order. Analyzing these equations, both analytically and numerically, is challenging. Consequently, a variety of integral transform techniques have recently been developed to solve nonlinear PDEs and ODEs. Integral transforms are particularly effective and valuable for finding both exact and analytical approximate solutions to PDEs and ODEs. They are crucial because they avoid perturbations or long-lasting polynomials. Motivated and inspired by recent studies in this area, we introduce a new approach for solving both linear and nonlinear differential equations, which we call the Adomian decomposition $\mathcal{J}$-transform method, outlined in [1, 25].

The rest of this research is organized as follows: In Section 2, we give some background on the theory of integral transform, including definitions and important properties of the $\mathcal{J}-$transform method. Section 3 is devoted to the theories of $\mathcal{J}-$transform with detailed proofs. In Section 4, we give proofs for the theoretical analysis of the $\mathcal{ADJM}$, including the uniqueness and existence along with the error estimates. We give exact solutions to nonlinear Bratu equations in Section 5. Section 6 is devoted to giving exact solutions to the linear telegraph equation. Finally, in Section 7, we give the conclusion of our work.

The original contribution of this research is mainly comprised of detailed proofs of theorems related to the $\mathcal{J}-$transform in Section 3. Moreover, we give detailed proofs to the theoretical analysis of $\mathcal{ADJM}$ in Section 4. Additionally, exact solutions to four applications in Sections 5 and 6 are presented.

2. Adomian polynomials and $\mathcal{J}$-transform: An overview

In this section, we provide some background material on the $\mathcal{J}-$transform method; refer to [4, 31]. The function $\Omega(\epsilon),\ \epsilon \in \mathbb{R}$ is assumed to exist. The following is the definition of the general integral transform:

$$\mathbb{F}\left[\Omega(\epsilon)\right](\theta) = \int_{-\infty}^{\infty} \mathbb{K}(\theta,\epsilon)\Omega(\epsilon)\, d\epsilon, \tag{2.1}$$

where θ is a real (or complex) number that is independent of ϵ and $\mathbb{K}(\theta,\epsilon)$ is the transform's kernel. It should be noted that Eq. (2.1) corresponds to the Laplace, Hankel, and Mellin transforms, respectively, when $\mathbb{K}(\theta,\epsilon)$ is $e^{-\theta\epsilon}$, $\epsilon J_n(\theta\epsilon)$, or $\epsilon^{\theta-1}$. Now, for $\Omega(\epsilon)$, $\epsilon\in(-\infty,\infty)$, consider the integral transforms defined by:

$$\mathbb{F}\left[\Omega(\epsilon)\right](\mu)=\int_{-\infty}^{\infty}\mathbb{K}(\epsilon)\Omega(\mu\epsilon)\,d\epsilon, \tag{2.2}$$

and

$$\mathbb{F}\left[\Omega(\epsilon)\right](\theta,\mu)=\int_{-\infty}^{\infty}\mathbb{K}(\theta,\epsilon)\Omega(\mu\epsilon)\,d\epsilon. \tag{2.3}$$

When $\mathbb{K}(\epsilon)=e^{-\epsilon}$, Eq. (2.2) converges to the Sumudu integral transform, where the parameter θ is substituted with μ, it is important to note. Additionally, the generalized Laplace and Sumudu transformations are defined by the following for any value of n:

$$\mathcal{L}\left[\Omega(\epsilon)\right]=\Psi(\theta)=\theta^{n}\int_{0}^{\infty}e^{-\theta^{n+1}\epsilon}\Omega(\theta^{n}\epsilon)\,d\epsilon, \tag{2.4}$$

and

$$\mathcal{S}\left[\Omega(\epsilon)\right]=G(\mu)=\mu^{n}\int_{0}^{\infty}e^{-\mu^{n}\epsilon}\Omega(\mu^{n+1}\epsilon)\,d\epsilon. \tag{2.5}$$

Note that when n, then Eq. (2.4) and Eq. (2.5) become the Laplace and Sumudu transform, respectively.

Basic definitions: $\mathcal{J}-$transform method

The $\mathcal{J}-$transform method was introduced by Shehu Maitama and W. Zhao [25]. By using the $\mathcal{J}-$transform method, we do not require any unnecessary linearization, discretization or taking some restrictive assumption as in the case of using (HPM). The new computational algorithm of $\mathcal{J}-$transform method drastically reduces the size of the computational work and round-off error is avoided.

Definition 2.1. Let $\Omega(\epsilon)$ be a piece-wise continuous function over $\mathbb{R}$ with $M,p>0$ along with the characteristic function $\chi_{(0,\infty)}(\epsilon)$, and $\mathcal{A}=\{\Omega(\epsilon):|\Omega(\epsilon)|<Me^{p\,\epsilon}$ $\chi_{(0,\infty)}(\epsilon)\}$.

So, $|\Omega(\epsilon)|\leq Me^{p\,\epsilon}$ for $\epsilon\longrightarrow\infty$, and given $\Omega(\epsilon)\in\mathcal{A}$, where$\theta,\mu>0$, then we have:

$$\begin{aligned}\left|\int_{0}^{\infty}e^{-\theta\epsilon}\Omega(\epsilon\mu)d\epsilon\right|&\leq M\int_{0}^{\infty}e^{-\theta\epsilon}e^{p|\epsilon\mu|}d\epsilon\\&=M\int_{0}^{\infty}e^{(p\mu-\theta)\epsilon}d\epsilon.\end{aligned}$$

The above is convergent if $p\mu-\theta<0$. Hence, $\Omega(\epsilon)$ is of exponential order.

The $\mathcal{J}-$transformation is then provided as follows:

$$\mathcal{J}\left(\Omega(\epsilon)\right)=\Psi(\theta,\mu)=\mu\int_{0}^{\infty}e^{\frac{-\theta\epsilon}{\mu}}\Omega(\epsilon)d\epsilon,\ \theta,\mu>0, \tag{2.6}$$

where θ,μ are the $\mathcal{J}-$transform variables.

Therefore, Eq. (2.6) can be expressed as:

$$\mathcal{J}\left(\Omega(\epsilon)\right)=\Psi(\theta,\mu)=\mu^{2}\int_{0}^{\infty}e^{-\theta\epsilon}\Omega(\mu\epsilon)\,d\epsilon,\ \theta,\mu\in(0,\infty). \tag{2.7}$$

Before we show how to use the $\mathcal{J}$-transform, it is important to comprehend its inverse property. We start by going over the following significant theorems.

Theorem 2.1. [1] *Assume that σ is a simple closed curve and that the function $\Omega(\epsilon)$ is analytic on an area that contains σ and its interior. Assume that σ is oriented counterclockwise. Next, we have the following for each ϵ_0 inside σ:*

$$\Omega(\epsilon_0) = \frac{1}{2\pi i}\int_\sigma \frac{\Omega(\epsilon)}{\epsilon-\epsilon_0}d\epsilon.$$

Theorem 2.2. [1] *If σ is a simple closed, positively orientated contour and Ω is analytic on $\mathbb{C}$, with the exception of a few points inside it, $\epsilon_1, \epsilon_2, ..., \epsilon_n$, then*

$$\oint_\sigma \Omega(\epsilon)\, d\epsilon = 2\pi i \sum_{k=1}^{n} Res\left(\Omega, \epsilon_k\right).$$

Definition 2.2. (Inverse $\mathcal{J}$-transform) [30]. Assume that $\mathcal{J}^{-1}$ is the inverse $\mathcal{J}-$transform of $\Psi(\theta,\mu)$, where $\Psi(\theta,\mu)$ is the $\mathcal{J}-$transform of the function $\Omega(\epsilon)$. Subsequently, that

$$\mathcal{J}^{-1}[\Psi(\theta,\mu)] = \Omega(\epsilon), \text{ for } \epsilon \geqslant 0.$$

Equivalently, based on Theorem 2.1 and Theorem 2.2, the complex inverse $\mathcal{J}-$transform is defined as follows:

$$\begin{aligned}\Omega(\epsilon) &= \lim_{\beta\to\infty}\frac{1}{2\pi i}\int_{\alpha-i\beta}^{\alpha+i\beta}\frac{1}{\mu^2}e^{\left(\frac{\theta s}{\mu}\right)}\Psi(\theta,\mu)d\theta \\ &= \sum \text{ residues of } \frac{1}{\mu^2}e^{\left(\frac{\theta s}{\mu}\right)}\Psi(\theta,\mu) \text{ at the poles of } \Psi(\theta,\mu).\end{aligned}$$

$\mathcal{J}$-transform's properties

Here we introduce some properties of the $\mathcal{J}$-transform ($\mathcal{JT}$), which we will use throughout this work [25].

1. If we have two functions, $\Omega(\epsilon)$ and $\omega(\epsilon)$, and two constants, α and β, then the $\mathcal{J}$-transform has the following linear property:

$$\mathcal{J}[\alpha\Omega(\epsilon)+\beta\omega(\epsilon)] = \alpha\mathcal{J}[\Omega(\epsilon)] + \beta\mathcal{J}[\omega(\epsilon)].$$

2. If we multiply a function $\Omega(\epsilon)$ by an exponential term $e^{\alpha\epsilon}$, where α is a constant, the $\mathcal{J}$-transform of this product is given by:

$$\mathcal{J}[e^{\alpha\epsilon}\Omega(\epsilon)] = \frac{\theta-\alpha\mu}{\theta}\Psi\left(\theta, \frac{\theta\mu}{\theta-\alpha\mu}\right).$$

3. Let $\Psi(\theta,\mu) = \frac{\mu^{n+2}}{\theta^{n+1}}$, $\mu,\ \theta > 0$, n=0, 1, 2, 3,..., then the inverse $\mathcal{J}-$transform transform ($\mathcal{IJT}$) is given by:

$$\mathcal{J}^{-1}\left[\frac{\mu^{n+2}}{\theta^{n+1}}\right] = \frac{\epsilon^n}{n!} = \frac{\epsilon^n}{\Gamma(n+1)},\ n = 0,1,2,3,....$$

4. For the n-th derivative of the function $\Omega(\epsilon)$, denoted as $\Omega^{(n)}(\epsilon)$, the $\mathcal{J}$-transform is:

$$\Psi_n(\theta,\mu) = \mathcal{J}[\Omega^{(n)}(\epsilon)] = \frac{\theta^n}{\mu^n}\Psi(\theta,\mu) - \sum_{k=0}^{n-1}\frac{\theta^{n-(k+1)}}{\mu^{n-(k+2)}}\Omega^{(k)}(0).$$

5. If n is a non-negative integer and $\Omega(\epsilon)$, $\epsilon\Omega^{(n)}(\epsilon)$, and $\epsilon^2\Omega^{(n)}(\epsilon)$ are valid functions in set $\mathcal{A}$, then: The $\mathcal{J}$-transform of $\epsilon\Omega^{(n)}(\epsilon)$ is:

$$\mathcal{J}[\epsilon\Omega^{(n)}(\epsilon)] = \frac{\mu^2}{\theta}\frac{d}{d\mu}[\Psi_n(\theta,\mu)] - \frac{\mu}{\theta}[\Psi_n(\theta,\mu)].$$

The $\mathcal{J}$-transform of $\epsilon^2\Omega^{(n)}(\epsilon)$ is:

$$\mathcal{J}[\epsilon^2\Omega^{(n)}(\epsilon)] = \frac{\mu^4}{\theta^2}\frac{d^2}{d\mu^2}[\Psi_n(\theta,\mu)].$$

Computational Adomian polynomials

An invaluable tool for effectively decomposing a complicated nonlinear component into smaller, more manageable, and maybe integrable components is the Adomian polynomials. The following is a representation of the unknown function Θ:

$$\Theta = \sum_{m=0}^{\infty} \Theta_m, \tag{2.8}$$

where the establishment of a recursive relation is necessary to determine Θ_m, $m \geq 0$. It is possible to define $G(\Theta)$ as an infinite series when working with nonlinear terms or Adomian polynomials B_m, using the formula below:

$$G(\Theta) = \sum_{m=0}^{\infty} B_m(\Theta_0,\, \Theta_1, ..., \Theta_m). \tag{2.9}$$

Additionally, the nonlinear term B_m of $G(\Theta)$ can be obtained using the formula in [23]:

$$B_m = \frac{1}{m!}\frac{d^m}{d\eta^m}\left[G\left(\sum_{i=0}^{m}\eta^i\Theta_i\right)\right]_{\eta=0},\ m = 0, 1, 2, ... \tag{2.10}$$

The following is an expression for the general formula for Eq. (2.10): Let $G(\Theta)$ be the nonlinear function, for instance. The following outcomes can be achieved by applying Eq. (2.9) and the definition of an Adomian polynomial:

$$\begin{aligned} B_0 &= G(\Theta_0), \\ B_1 &= \Theta_1 G'(\Theta_0), \\ B_2 &= \Theta_2 G'(\Theta_0) + \frac{1}{2!}\Theta_1^2 G''(\Theta_0). \end{aligned} \tag{2.11}$$

Finally, the other terms can be constructed using a similar procedure. The polynomials previously presented in Eq. (2.9) provide two significant observations. While B_2 depends solely on Θ_0, Θ_1, and Θ_2, etc., B_0 and Θ_0 are the only variables on which B_1, B_0, and Θ_1 rely.

By changing Eq. (2.9) to Eq. (2.10), one can observe:

$$\begin{aligned} G(\Theta) &= B_0 + B_1 + B_2 + ... \\ &= G(\Theta_0) + (\Theta_1 + \Theta_2 + \Theta_3 + ...)G'(\Theta_0) \\ &\quad + \frac{1}{2!}(\Theta_1^2 + 2\Theta_1\Theta_2 + 2\Theta_1\Theta_3 + \Theta_2^2 + ...)G''(\Theta_0) \end{aligned}$$

$$+\frac{1}{3!}(\Theta_1^3+3\Theta_1^2\Theta_2+3\Theta_1^2\Theta_3+6\Theta_1\Theta_2\Theta_3+...)G'''(\Theta_0)+...$$
$$=G(\Theta_0)+(\Theta-\Theta_0)G'(\Theta_0)+\frac{1}{2!}(\Theta-\Theta_0)^2G''(\Theta_0)+....$$

3. Theories of $\mathcal{J}-$transform with detailed proofs

In this section, we give detailed proofs to the $\daleth-$transform for some useful theorems. We shall use these theorems later on to solve linear and nonlinear ODEs and PDEs.

Theorem 3.1. *Let* $\Omega(\epsilon)=\frac{e^{\beta\epsilon}\sinh(\alpha\epsilon)}{\alpha}\in\mathcal{A}$, *where* $\alpha\in\mathbb{R}^+$ *and* $\beta\in\mathbb{R}$. *Then its* $\daleth-$*transform is given by:*

$$\daleth\left[\frac{e^{\beta\epsilon}\sinh(\alpha\epsilon)}{\alpha}\right]=\frac{\mu^3}{(\theta-\beta\mu)^2-\alpha^2\mu^2}.$$

Proof. By using the $\daleth-$transform definition, we get:

$$\begin{aligned}\daleth\left[\frac{e^{\beta\epsilon}\sinh(\alpha\epsilon)}{\alpha}\right]&=\frac{\mu^2}{\alpha}\int_0^\infty e^{-\theta\epsilon}e^{\beta\epsilon\mu}\sinh(\alpha\epsilon\mu)\,d\epsilon\\&=\frac{\mu^2}{2\alpha}\int_0^\infty e^{-\theta\epsilon}e^{\beta\epsilon\mu}(e^{\alpha\mu\epsilon}-e^{-\alpha\mu\epsilon})\,d\epsilon\\&=\frac{\mu^2}{2\alpha}\int_0^\infty\left(e^{-\epsilon(\theta-\beta\mu-\alpha\mu)}-e^{-\epsilon(\theta-\beta\mu-\alpha\mu)}\right)d\epsilon\\&=\frac{\mu^2}{2\alpha}\left[\left(\frac{-e^{-\epsilon(\theta-\beta\mu-\alpha\mu)}}{\theta-\beta\mu-\alpha\mu}+\frac{e^{-\epsilon(\theta-\beta\mu+\alpha\mu)}}{\theta-\beta\mu+\alpha\mu}\right)\Bigg|_0^\infty\right]\\&=\frac{\mu^2}{2\alpha}\left(\frac{1}{\theta-\beta\mu-\alpha\mu}-\frac{1}{\theta-\beta\mu+\alpha\mu}\right)\\&=\frac{\mu^3}{(\theta-\beta\mu)^2-\alpha^2\mu^2}.\end{aligned}$$

□

Theorem 3.2. *Let* $\Omega(\epsilon)=\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}\in\mathcal{A}$, *where* $\alpha\in\mathbb{R}\setminus\{0\}$. *Then its* $\daleth-$*transform is given by*

$$\daleth[\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}]=\frac{\mu^4\theta}{(\theta^2+\alpha^2\mu^2)^2}.$$

Proof. Applying property 5 of $\daleth-$transform, we have

$$\begin{aligned}\daleth\left[\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}\right]&=\frac{\mu^2}{\theta}\frac{d}{d\mu}\left(\frac{\mu^3}{2(\theta^2+\alpha^2\mu^2)}\right)-\frac{\mu}{\theta}\frac{\mu^3}{2(\theta^2+\alpha^2\mu^2)}\\&=\frac{\mu^2}{2\theta}\left(\frac{3\mu^2(\theta^2+\alpha^2\mu^2)-2\alpha^2\mu^4}{(\theta^2+\alpha^2\mu^2)^2}\right)-\frac{\mu^4}{2\theta(\theta^2+\alpha^2\mu^2)}\\&=\left(\frac{3\mu^4(\theta^2+\alpha^2\mu^2)-2\alpha^2\mu^6}{2\theta(\theta^2+\alpha^2\mu^2)^2}\right)-\frac{\mu^4}{2\theta(\theta^2+\alpha^2\mu^2)}\times\frac{(\theta^2+\alpha^2\mu^2)}{(\theta^2+\alpha^2\mu^2)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{3\mu^4\theta^2 + 3\alpha^2\mu^6 - 2\alpha^2\mu^6 - \mu^4\theta^2 - \alpha^2\mu^6}{2\theta(\theta^2 + \alpha^2\mu^2)^2} \\
&= \frac{2\mu^4\theta^2}{2\theta(\theta^2 + \alpha^2\mu^2)^2} \\
&= \frac{\mu^4\theta}{(\theta^2 + \alpha^2\mu^2)^2}.
\end{aligned}
$$

Hence,

$$
\daleth\left[\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}\right] = \frac{\mu^4\theta}{(\theta^2 + \alpha^2\mu^2)^2}.
$$

□

Theorem 3.3. *Let* $\Omega(\epsilon) = \frac{\sin(\alpha\epsilon) + \alpha\epsilon\cos(\alpha\epsilon)}{2\alpha} \in \mathcal{A}$, *where* $\alpha \in \mathbb{R} \setminus \{0\}$. *Then its* $\daleth-$*transform is given by*

$$
\daleth\left[\frac{\sin(\alpha\epsilon) + \alpha\epsilon\cos(\alpha\epsilon)}{2\alpha}\right] = \frac{\mu^3\theta^2}{(\theta^2 + \mu^2\alpha^2)^2}.
$$

Proof. Using linearity property and the $\daleth-$transform of $\frac{\sin(\alpha\epsilon)}{\alpha}$, and $\epsilon\cos\alpha\epsilon$, we obtain:

$$
\begin{aligned}
\daleth\left[\frac{\sin(\alpha\epsilon) + \alpha\epsilon\cos(\alpha\epsilon)}{2\alpha}\right] &= \frac{1}{2}\daleth\left[\frac{\sin\alpha\epsilon}{\alpha}\right] + \frac{1}{2}\daleth[\,\epsilon\cos\alpha\epsilon] \\
&= \frac{1}{2}\left(\frac{\mu^3}{\theta^2 + \alpha^2\mu^2} + \frac{\mu^3(\theta^2 - \alpha^2\mu^2)}{(\theta^2 + \alpha^2\mu^2)^2}\right) \\
&= \frac{\mu^3}{2}\left(\frac{(\theta^2 + \alpha^2\mu^2)}{(\theta^2 + \alpha^2\mu^2)^2} + \frac{(\theta^2 - \alpha^2\mu^2)}{(\theta^2 + \alpha^2\mu^2)^2}\right) \\
&= \frac{\mu^3\theta^2}{(\theta^2 + \mu^2\alpha^2)^2}.
\end{aligned}
$$

Hence,

$$
\daleth\left[\frac{\sin(\alpha\epsilon) + \alpha\epsilon\cos(\alpha\epsilon)}{2\alpha}\right] = \frac{\mu^3\theta^2}{(\theta^2 + \mu^2\alpha^2)^2}.
$$

□

Theorem 3.4. *Let* $\Omega(\epsilon) = \cos(\alpha\epsilon) - \frac{\alpha\epsilon\sin(\alpha\epsilon)}{2} \in \mathcal{A}$, *where* $\alpha \in \mathbb{R} \setminus \{0\}$. *Its* $\daleth-$*transform is then provided by:*

$$
\daleth\left[\cos(\alpha\epsilon) - \frac{\alpha\epsilon\sin(\alpha\epsilon)}{2}\right] = \frac{\mu^2\theta^3}{(\theta^2 + \mu^2\alpha^2)^2}.
$$

Proof. Using linearity property and the $\daleth-$transform of $\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}, \cos(\alpha\epsilon)$, we obtain:

$$\beth\left[\cos(\alpha\epsilon)-\frac{\alpha\epsilon\sin(\alpha\epsilon)}{2}\right]=\beth\left[\cos(\alpha\epsilon)\right]-\alpha^2\mathbb{J}\left[\frac{\epsilon\sin(\alpha\epsilon)}{2\alpha}\right]$$
$$=\left(\frac{\mu^2\theta}{\theta^2+\alpha^2\mu^2}-\alpha^2\left(\frac{\mu^4\theta}{(\theta^2+\alpha^2\mu^2)^2}\right)\right)$$
$$=\left(\frac{\mu^2\theta(\theta^2+\alpha^2\mu^2)}{(\theta^2+\alpha^2\mu^2)^2}-\alpha^2\left(\frac{\mu^4\theta}{(\theta^2+\alpha^2\mu^2)^2}\right)\right)$$
$$=\frac{\mu^2\theta^3+\mu^4\alpha^2\theta-\mu^4\alpha^2\theta}{(\theta^2+\alpha^2\mu^2)^2}$$
$$=\frac{\mu^2\theta^3}{(\theta^2+\alpha^2\mu^2)^2}.$$

Hence,

$$\beth\left[\cos(\alpha\epsilon)-\frac{\alpha\epsilon\sin(\alpha\epsilon)}{2}\right]=\frac{\mu^2\theta^3}{(\theta^2+\mu^2\alpha^2)^2}.$$

□

4. Convergence analysis using $\mathcal{ADJM}$

In this section, we demonstrate the convergence and uniqueness theorems, and we will provide an estimate of the error determined using the $\mathcal{ADJM}$. Consider the nonlinear ODEs:

$$^cD_\tau^\sigma\varphi(\tau)+N(\varphi(\tau))+L(\varphi(\tau))=\psi(\tau). \tag{4.1}$$

Accompanied with its I.C:

$$\varphi(0)=\varphi_0. \tag{4.2}$$

Note that the non-linear part is $N(\varphi(\tau))$, the linear term is $L(\varphi(\tau))$, and $\psi(\eta,\tau)$ is the source term. Next, we use the $\mathcal{J}$-transformation and property 4 on Eq. (4.1):

$$\Phi(r,u)=\frac{\varphi_0}{r}-\left(\frac{u}{r}\right)\beth\left[L(\varphi(\tau))+N(\varphi(\tau))-\psi(\tau)\right]. \tag{4.3}$$

Next, we use the inverse $\beth$-transform to interpret Eq. (4.3) and obtain:

$$\varphi(\tau)=\Psi(\tau)+\beth^{-1}\left[\left(\frac{u}{r}\right)\beth\left[L(\varphi(\tau))+N(\varphi(\tau))\right]\right]. \tag{4.4}$$

$\Psi(\tau)$ denotes the nonhomogeneous part as well as the I.C. Make the assumption that there is an infinite series solution to the unknown function, $\varphi(\tau)$, as follows:

$$\varphi(\tau)=\sum_{k=0}^{\infty}\varphi_k(\tau). \tag{4.5}$$

The Adomian polynomials are A_j in the nonlinear term $N(\varphi(\tau))=\sum_{j=0}^{\infty}A_j$. We rewrite Eq. (4.4) as follows using Eq. (4.5):

$$\sum_{j=0}^{\infty}\varphi_j(\tau)=\Psi(\tau)+\beth^{-1}\left[\left(\frac{u}{r}\right)\beth\left[\sum_{j=0}^{\infty}A_j+\sum_{j=0}^{\infty}\varphi_j\right]\right]. \tag{4.6}$$

The result of comparing the two sides of Eq. (4.6) is $\varphi_0(\tau) = \Psi(\tau)$. The following general relation can therefore be produced:

$$\varphi_{j+1}(\tau) = \daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[A_j + \varphi_j\right]\right],\ j \geq 0. \tag{4.7}$$

The following is the last assertion for the expected exact solution:

$$\varphi(\tau) = \sum_{j=0}^{\infty} \varphi_j(\tau). \tag{4.8}$$

Theorem 4.1. (Uniqueness theorem). *If $0 < \mu < 1$, then there exists a unique solution to Eq. (4.1), with $\mu = (C_1 + C_2)\,\tau$, $\forall \tau \in [0, \beta]$.*

Proof. Consider the Banach space of every continuous function on $\Delta = [0, \beta]$ is $\mathbb{K} = (C[\Delta], \|.\|)$ and consider the norm $\|.\|$, then we define $\zeta : \mathbb{K} \to \mathbb{K}$ by

$$\varphi_{k+1}(\tau) = \Psi(\tau) + \daleth^{-1}\left[\left(\frac{u}{r}\right)^{\xi}\daleth\left[N\left(\varphi_k(\tau)\right) + L\left(\varphi_k(\tau)\right)\right]\right].$$

Suppose that $L[\varphi(\tau)] = \varphi(\tau)$ and $N[\varphi(\tau)] = N(\varphi(\tau))$. Further, let $|N(\varphi) - N(\tilde{\varphi})| < C_1|\varphi - \tilde{\varphi}|$ and $|L(\varphi) - L(\tilde{\varphi})| < C_2|\varphi - \tilde{\varphi}|$, where C_1, C_2 are the constants of Lipschitz with $0 \leq C_1, C_2 < 1$ and $\varphi, \tilde{\varphi}$ are distinct solutions of Eq. (4.1). Then,

$$\begin{aligned}
\|\zeta(\varphi) - \zeta(\tilde{\varphi})\| &= \max_{\tau\in\Delta}\left|\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[L(\varphi) + N(\varphi)\right]\right] - \daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[L(\tilde{\varphi}) + N(\tilde{\varphi})\right]\right]\right| \\
&= \max_{\tau\in\Delta}\left|\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[L(\varphi) - L(\tilde{\varphi})\right]\right] + \daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[M(\varphi) - M(\tilde{\varphi})\right]\right]\right| \\
&\leq \max_{\tau\in\Delta}\left[C_1\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[|\varphi - \tilde{\varphi}|\right]\right] + C_2\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[|\varphi - \tilde{\varphi}|\right]\right]\right] \\
&\leq \max_{\tau\in\Delta}\left(C_1 + C_2\right)\left[\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[|\varphi - \tilde{\varphi}|\right]\right]\right] \\
&\leq \left(C_1 + C_2\right)\left[\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[\|\varphi(\tau) - \tilde{\varphi}(\tau)\|\right]\right]\right] \\
&= \|\varphi - \tilde{\varphi}\|\left(C_1 + C_2\right)\tau.
\end{aligned}$$

Consequently, since $0 < \mu < 1$, then there exists a unique solution for Eq. (4.1). The Banach fixed-point theorem for contraction suggests that ζ is a contraction mapping. This leads to the proof of Theorem 4.1. □

Theorem 4.2. (Convergence theorem). *Eq. (4.8) of Eq. (4.1) has a convergent series solution for every $|\varphi_1| < \infty$ and $0 < \mu < 1$.*

Proof. Consider $q_i = \sum_{k=0}^{i}\varphi_k(\tau)$. We shall show that $\{q_i\}$ is a Cauchy sequence in the Banach space Δ. Consider the Adomian polynomial in its most recent version (see [1]). Let $N(q_i) = \tilde{A}_i + \sum_{k=0}^{m-1}\tilde{A}_k$, $i \geq n$ and choose two partial sums q_n and q_i. Then,

$$\begin{aligned}
&\|q_i - q_n\| \\
&= \max_{\tau\in\Delta}|q_i - q_n| \\
&= \max_{\tau\in\Delta}\left|\sum_{k=n+1}^{i}\tilde{\varphi}_k(\tau)\right|,\ i = 1, 2, ...
\end{aligned}$$

$$
\begin{aligned}
&\le \max_{\tau\in\Delta}\left|\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[C\left(\sum_{k=n+1}^{i}\varphi_{k-1}(\tau)\right)\right]\right]+\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[\sum_{k=n+1}^{i}A_{i-1}(\tau)\right]\right]\right|\\
&= \max_{\tau\in\Delta}\left|\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[C\left(\sum_{k=n}^{i-1}\varphi_{k}(\tau)\right)\right]\right]+\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[\sum_{k=n}^{i-1}A_{i}(\tau,\tau)\right]\right]\right|\\
&\le \max_{\tau\in\Delta}\left|\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[C(q_{i-1})-C(q_{n-1})\right]\right]+\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[N(q_{i-1})-N(q_{n-1})\right]\right]\right|\\
&\le C_1\max_{\tau\in\Delta}\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[\|q_{i-1}-q_{n-1}\|\right]\right]+C_2\max_{\tau\in\Delta}\daleth^{-1}\left[\left(\frac{u}{r}\right)\daleth\left[\|q_{i-1}-q_{n-1}\|\right]\right]\\
&= (C_1+C_2)\,\tau\,\|q_{i-1}-q_{n-1}\|\,.
\end{aligned}
$$

Now, $\|q_i - q_n\| \le \mu \|q_{i-1} - q_{n-1}\|$. Choose $i = n+1$, then

$$\|q_{n+1}-q_n\| \le \mu\|q_n-q_{n-1}\| \le \mu^2\|q_{n-1}-q_{n-2}\| \le \dots \le \mu^n\|q_1-q_0\|\,.$$

Additionally, the triangle inequality can be utilized to determine:

$$
\begin{aligned}
\|q_i-q_n\| &\le \|q_{n+1}-q_n\|+\|q_{n+2}-q_{n+1}\|+\dots+\|q_i-q_{i-1}\|\\
&\le \left[\mu^n+\mu^{n+1}+\dots+\mu^{i-1}\right]\|q_1-q_0\|\\
&\le \mu^n\left[\frac{1-\mu^{i-n}}{1-\mu}\right]\|\varphi_1\|\,.
\end{aligned}
$$

But, $0<\mu<1$, then $1-\mu^{i-n}<1$. So,

$$\|q_i-q_n\| \le \frac{\mu^n}{1-\mu}\max_{\tau\in\Delta}|\varphi_1|\,. \tag{4.9}$$

Note $|\varphi_1|<\infty$, since $\varphi(\tau)$ is bounded. Thus, $\|q_i-q_n\|\to 0$ as $n\to\infty$. Consequently, in $\mathbb{K}$, the sequence $\{q_i\}$ is a Cauchy sequence. Hence, $\varphi(\tau)=\sum_{k=0}^{\infty}\varphi_k(\tau)$ converges as a result. We've established Theorem 4.2. □

Theorem 4.3. (Error estimate). *The series solution in equations (4.8) to (4.1) should have the following maximum absolute error:*

$$\max_{\tau\in\Delta}\left|\varphi(\tau)-\sum_{i=0}^{n}\varphi_i(\tau)\right| \le \frac{\mu^n}{1-\mu}\max_{\tau\in\Delta}|\varphi_1|\,.$$

Proof. Using Eq. (4.9) above, we can arrive at: $\|q_i-q_n\|\le\frac{\mu^n}{1-\mu}\max_{\tau\in\Delta}|\varphi_1|$. So as $i\to\infty$, we have $q_i\to\varphi(\tau)$. So, $\|\varphi(\tau)-q_n\|\le\frac{\mu^n}{1-\mu}\max_{\tau\in\Delta}|\varphi_1(\tau)|$. Based on this, the maximum absolute truncation error for Δ is:

$$\max_{\tau\in\Delta}\left|\varphi(\tau)-\sum_{i=0}^{n}\varphi_i(\tau)\right| \le \max_{\tau\in\Delta}\frac{\mu^n}{1-\mu}|\varphi_1(\tau)| = \frac{\mu^n}{1-\mu}\|\varphi_1(\tau)\|\,.$$

We've established Theorem 4.3. □

5. Applications of $\mathcal{ADJM}$

This section contains applications of the $\mathcal{ADJM}$ for a class of linear and nonlinear ordinary differential equations. Let's look at a general nonlinear ordinary differential equation:

$$L\Omega + R(\Omega) + F(\Omega) = g(\epsilon), \tag{5.1}$$

with the initial condition:

$$\Omega(0) = h(\epsilon), \tag{5.2}$$

where: L is the highest-order derivative operator, R is the rest of the differential operator, $g(\epsilon)$ is a term that doesn't depend on Ω, $F(\Omega)$ is the nonlinear term. If L is a first-order differential operator, applying the $\daleth$-transform to this equation gives us:

$$\frac{\theta^2\Psi(\theta,\mu)}{\mu} - \mu\Omega(0) + \daleth[R(\Omega)] + \daleth[F(\Omega)] = \daleth[g(\epsilon)]. \tag{5.3}$$

Substitute $\Omega(0)$ into the above equation, to get:

$$\Psi(\theta,\mu) = \frac{\mu^2 h(\epsilon)}{\theta^2} + \frac{\mu}{\theta^2}\daleth[g(\epsilon)] - \frac{\mu}{\theta^2}\daleth[R(\Omega) + F(\Omega)]. \tag{5.4}$$

Taking the inverse $\daleth$-transform, we obtain:

$$\Omega(\epsilon) = G(\epsilon) - \daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth[R(\Omega) + F(\Omega)]\right], \tag{5.5}$$

where $G(\epsilon)$ is the source term. We assume that the solution $\Omega(\epsilon)$ can be written as an infinite series:

$$\Omega(\epsilon) = \sum_{n=0}^{\infty} \Omega_n(\epsilon). \tag{5.6}$$

Substituting this series into the equation, we have:

$$\sum_{n=0}^{\infty} \Omega_n(\epsilon) = G(\epsilon) - \daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth\left[R\sum_{n=0}^{\infty}\Omega_n(\epsilon) + \sum_{n=0}^{\infty} A_n(\epsilon)\right]\right], \tag{5.7}$$

where the nonlinear term is represented by the Adomian polynomials $A_n(\epsilon)$. To find $\Omega_n(\epsilon)$, we use these recursive relations:

$$\begin{aligned}
\Omega_0(\epsilon) &= G(\epsilon),\\
\Omega_1(\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth[R\Omega_0(\epsilon) + A_0(\epsilon)]\right],\\
\Omega_2(\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth[R\Omega_1(\epsilon) + A_1(\epsilon)]\right],\\
\Omega_3(\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth[R\Omega_2(\epsilon) + A_2(\epsilon)]\right].
\end{aligned}$$

In general, the recursive relation is:

$$\Omega_{n+1}(\epsilon) = -\daleth^{-1}\left[\frac{\mu}{\theta^2}\daleth[R\Omega_n(\epsilon) + A_n(\epsilon)]\right], \quad n \geq 0. \tag{5.8}$$

So, the solution (either exact or approximate) is:

$$\Omega(\epsilon) = \sum_{n=0}^{\infty} \Omega_n(\epsilon). \tag{5.9}$$

Example 5.1. Consider the nonlinear differential equation of the first order [5]:

$$\frac{d\Omega}{d\epsilon} - e^{\Omega} = 0, \tag{5.10}$$

with the initial condition:

$$\Omega(0) = 1. \tag{5.11}$$

Solution. Applying the ℶ-transform to Eq. (5.10):

$$\frac{\theta}{\mu}\Psi(\theta,\mu) - \mu\Omega(0) - ℶ[e^{\Omega}] = 0. \tag{5.12}$$

Using $\Omega(0) = 1$, then Eq. (5.12) becomes:

$$\Psi(\theta,\mu) = \frac{\mu^2}{\theta} + \frac{\mu}{\theta}ℶ\left[e^{\Omega}\right]. \tag{5.13}$$

Applying the inverse ℶ-transform to Eq. (5.11), one conclude:

$$\Omega(\epsilon) = 1 + ℶ^{-1}\left[\frac{\mu}{\theta}ℶ\left[e^{\Omega}\right]\right]. \tag{5.14}$$

We assume $\Omega(\epsilon)$ can be written as an infinite series:

$$\Omega(\epsilon) = \sum_{n=0}^{\infty}\Omega_n(\epsilon). \tag{5.15}$$

Substituting the above series into Eq. (5.14), we get:

$$\sum_{n=0}^{\infty}\Omega_n(\epsilon) = 1 + ℶ^{-1}\left[\frac{\mu}{\theta}ℶ\left[\sum_{n=0}^{\infty}A_n(\epsilon)\right]\right]. \tag{5.16}$$

Here, $A_n(\epsilon)$ represents the nonlinear term e^{Ω}.

Comparing both sides of Eq. (5.16), we get:

$$\begin{aligned}
&\Omega_0(\epsilon) = 1,\\
&\Omega_1(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[A_0(\epsilon)]]\right],\\
&\Omega_2(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[A_1(\epsilon)]]\right],\\
&\Omega_3(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[A_2(\epsilon)]]\right].
\end{aligned} \tag{5.17}$$

The general relation is

$$\Omega_{n+1}(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[A_n(\epsilon)]]\right],\ \forall n \geq 0. \tag{5.18}$$

Now, we compute the components:

$$\begin{aligned}
&\Omega_1(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[e]]\right] = e\epsilon,\\
&\Omega_2(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[e^2\epsilon]]\right] = \frac{e^2\epsilon^2}{2},\\
&\Omega_3(\epsilon) = ℶ^{-1}\left[\frac{\mu}{\theta}[ℶ[e^3\epsilon^2]]\right] = \frac{e^3\epsilon^3}{3}.
\end{aligned} \tag{5.19}$$

So, the approximate solution is

$$\Omega(\epsilon) = 1 + e\epsilon + \frac{(e\epsilon)^2}{2} + \frac{(e\epsilon)^3}{3} + \cdots, \quad -1 \leq e\epsilon < 1. \tag{5.20}$$

Since we cannot solve for Ω explicitly in terms of ϵ, the implicit solution is

$$\Omega(\epsilon) = 1 - \ln(1 - e\epsilon), \quad -1 \leq e\epsilon < 1. \tag{5.21}$$

This result is the same as the exact solution obtained using ADM, see [5].

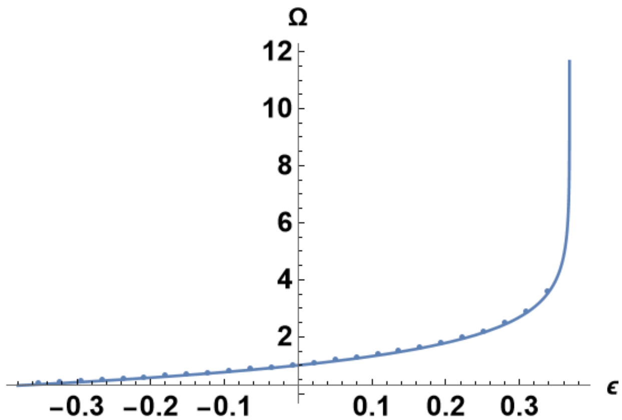

Figure 1. Exact solution of $\Omega(\epsilon)$ for Example 5.1.

Example 5.2. Consider the nonlinear second order Bratu differential equation [5]:

$$\frac{d^2\Omega}{d\epsilon^2} - 2e^{\Omega} = 0, \tag{5.22}$$

with initial conditions:

$$\Omega(0) = 0, \quad \Omega'(0) = 0. \tag{5.23}$$

Solution. Apply the $\gimel$-transform to Eq. (5.22), to obtain:

$$\frac{\theta^2\Psi(\theta,\mu)}{\mu^2} - \theta\Omega(0) - \mu\Omega'(0) - \gimel\left[2e^{\Omega}\right] = 0. \tag{5.24}$$

Substituting Eq. (5.23) into the above equation, to arrive:

$$\Psi(\theta,\mu) = \frac{\mu^2}{\theta^2}\gimel\left[2e^{\Omega}\right]. \tag{5.25}$$

Next, we take the inverse $\gimel$-transform of this result:

$$\Omega(\epsilon) = \gimel^{-1}\left[\frac{\mu^2}{\theta^2}\gimel\left[2e^{\Omega}\right]\right]. \tag{5.26}$$

Assume that the solution $\Omega(\epsilon)$ can be expressed as an infinite series:

$$\Omega(\epsilon) = \sum_{n=0}^{\infty}\Omega_n(\epsilon). \tag{5.27}$$

Using the above Eq. (5.27) along with Eq. (5.26), we get:

$$\sum_{n=0}^{\infty} \Omega_n(\epsilon) = \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.28}$$

The Adomian polynomials for the nonlinear term $2\Omega\frac{d\Omega}{d\epsilon}$ are denoted by A_n. The recursive relation can be obtained by comparing the two sides of Eq. (5.28):

$$\Omega_{n+1}(\epsilon) = \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_n\right]\right], \quad \forall n \geq 0. \tag{5.29}$$

Using this relation in Eq. (5.29), we can find the components of $\Omega(\epsilon)$:

$$\Omega_0(\epsilon) = 0, \tag{5.30}$$

$$\Omega_1(\epsilon) = \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_0\right]\right], \tag{5.31}$$

$$\Omega_2(\epsilon) = \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_1\right]\right]. \tag{5.32}$$

The general recursive relation is:

$$\Omega_{n+1}(\epsilon) = \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_n\right]\right], \quad \forall n \geq 0. \tag{5.33}$$

Using Eq. (5.33), we can find more components of $\Omega(\epsilon)$:

$$\begin{aligned}
\Omega_1(\epsilon) &= \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_0\right]\right] \\
&= \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[2e^{\Omega_0}\right]\right] \\
&= 2\beth^{-1}\left[\frac{\mu^4}{\theta^3}\right] \\
&= 2\frac{\epsilon^2}{2!}, \\
\Omega_2(\epsilon) &= \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[A_1\right]\right] \\
&= \beth^{-1}\left[\frac{\mu^2}{\theta^2}\beth\left[2\Omega_1 e^{\Omega_0}\right]\right] \\
&= 4\beth^{-1}\left[\frac{\mu^6}{\theta^5}\right] \\
&= 4\frac{\epsilon^4}{4!}.
\end{aligned}$$

Therefore, the approximate series solution is:

$$\begin{aligned}
\Omega(\epsilon) &= \sum_{n=0}^{\infty} \Omega_n(\epsilon) \\
&= 0 + 2\frac{\epsilon^2}{2!} + 4\frac{\epsilon^4}{4!} + \cdots
\end{aligned}$$

$$= -2\left(-\frac{\epsilon^2}{2} - \frac{\epsilon^4}{12} - \cdots\right).$$

Thus, the exact solution is:

$$\Omega(\epsilon) = -2\ln(\cos(\epsilon)). \tag{5.34}$$

This is the exact solution that matches the result obtained by the ADM in [5].

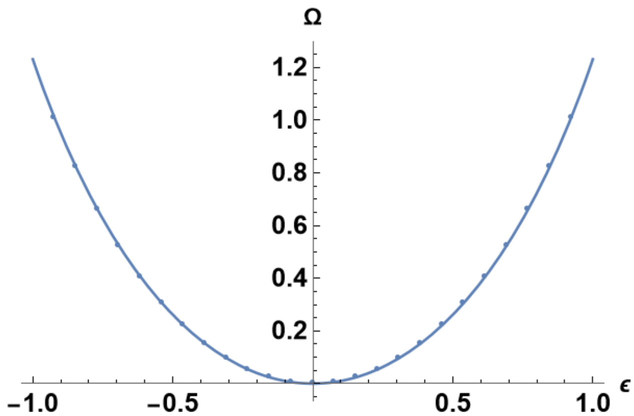

Figure 2. Exact solution of $\Omega(\epsilon)$ for Example 5.2.

Example 5.3. Consider the nonlinear second order differential equation [5]:

$$\frac{d^2\Omega}{d\epsilon^2} - 2\Omega\frac{d\Omega}{d\epsilon} = 0, \tag{5.35}$$

with initial conditions:

$$\Omega(0) = 0, \quad \Omega'(0) = 1. \tag{5.36}$$

Solution. Apply ℷ-transform to Eq. (5.35), to obtain:

$$\frac{\theta^2\Psi(\theta,\mu)}{\mu^2} - \theta\Omega(\omega,0) - \mu\Omega'(\omega,0) - ℷ\left[2\Omega\frac{d\Omega}{d\epsilon}\right] = 0. \tag{5.37}$$

Substituting Eq. (5.36) into the above equation, to arrive:

$$\Psi(\theta,\mu) = \frac{\mu^3}{\theta^2} + \frac{\mu^2}{\theta^2}ℷ\left[2\Omega\frac{d\Omega}{d\epsilon}\right]. \tag{5.38}$$

Next, we take the inverse ℷ-transform of this result:

$$\Omega(\epsilon) = \epsilon + ℷ^{-1}\left[\frac{\mu^2}{\theta^2}ℷ\left[2\Omega\frac{d\Omega}{d\epsilon}\right]\right]. \tag{5.39}$$

Assume that the solution $\Omega(\epsilon)$ can be expressed as an infinite series:

$$\Omega(\epsilon) = \sum_{n=0}^{\infty}\Omega_n(\epsilon). \tag{5.40}$$

Using the above Eq. (5.40) along with Eq. (5.39), we get:

$$\sum_{n=0}^{\infty} \Omega_n(\epsilon) = \epsilon + \daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{5.41}$$

The Adomian polynomials for the nonlinear term $2\Omega\frac{d\Omega}{d\epsilon}$ are denoted by A_n. The recursive relation can be obtained by comparing the two sides of Eq. (5.41):

$$\Omega_{n+1}(\epsilon) = -\daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_n\right]\right], \quad \forall n \geq 0. \tag{5.42}$$

Using Eq. (5.42), we can find the components of $\Omega(\epsilon)$:

$$\begin{aligned}
\Omega_0(\epsilon) &= \frac{\epsilon}{1!}, \\
\Omega_1(\epsilon) &= -\daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_0\right]\right], \\
\Omega_2(\epsilon) &= -\daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_1\right]\right], \\
\Omega_3(\epsilon) &= -\daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_2\right]\right].
\end{aligned}$$

The general recursive relation is:

$$\Omega_{n+1}(\epsilon) = -\daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_n\right]\right], \quad \forall n \geq 0. \tag{5.43}$$

Using Eq. (5.43), we can find more components of $\Omega(\epsilon)$:

$$\begin{aligned}
\Omega_1(\epsilon) &= \daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[A_0\right]\right] \\
&= \daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[2\Omega_0\Omega_0'\right]\right] \\
&= 2\frac{\epsilon^3}{3!} + \cdots.
\end{aligned}$$

Canceling the noise terms between $\Omega_0(\epsilon)$ and $\Omega_1(\epsilon)$, we find that the remaining term satisfies Eq. (5.35), leading to the exact solution:

$$\Omega(\epsilon) = \tan(\epsilon). \tag{5.44}$$

This is the exact solution that matches the result obtained by the ADM in [5].

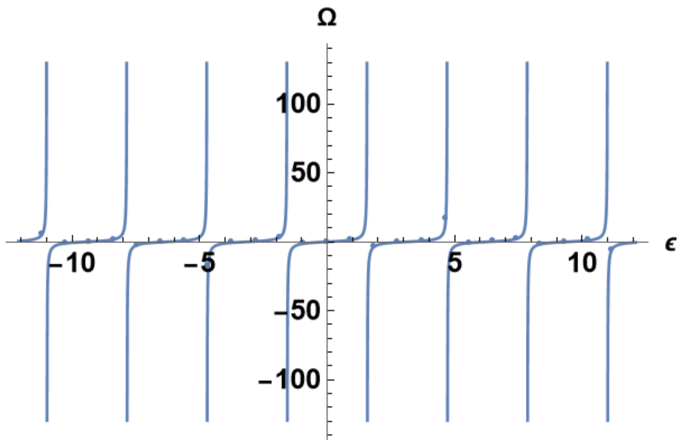

Figure 3. Exact solution of $\Omega(\epsilon)$ for Example 5.2.

6. Using the $\mathcal{J}$−transform for solving linear PDEs

In this section, we explain the $\daleth$−transform methodology for solving linear PDEs.

Methodology: Consider the following PDE:

$$\Omega'(\omega,\epsilon)+R\Omega(\omega,\epsilon)+L\Omega(\omega,\epsilon)=g(\omega,\epsilon), \tag{6.1}$$

with the initial condition

$$\Omega(\omega,0)=f(\omega), \tag{6.2}$$

where R is a lower-order derivative that is always invertible, L is a linear differential operator, and g is the source term.

Applying the $\daleth$−transform to both sides of Eq. (6.1), we get:

$$\frac{\theta}{\mu}\Psi(\omega,\theta,\mu)-\mu\Psi(\omega,0)+\daleth\left[L\Omega\right]=\daleth\left[g\right]. \tag{6.3}$$

Substituting Eq. (6.2) into Eq. (6.3), to get:

$$\Psi(\omega,\theta,\mu)=\frac{\mu^2}{\theta}f(\omega)+\frac{\mu}{\theta}\daleth\left[g\right]-\frac{\mu}{\theta}\daleth\left[L\Omega\right]. \tag{6.4}$$

Taking the inverse $\daleth$-transform of Eq. (6.4), we obtain:

$$\Omega(\omega,\epsilon)=G(\omega,\epsilon)-\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\Omega\right]\right], \tag{6.5}$$

where $G(\omega,\epsilon)$ comes from the source term. Assuming a series solution for the unknown function $\Omega(\omega,\epsilon)$:

$$\Omega(\omega,\epsilon)=\sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon). \tag{6.6}$$

Combining Eq. (6.6) and Eq. (6.5), one conclude:

$$\sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon)=G(\omega,\epsilon)-\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\sum_{n=0}^{\infty}\Omega_n\right]\right]. \tag{6.7}$$

From Eq. (6.7), we derive the general recursive relation:

$$\begin{aligned}\Omega_0(\omega,\epsilon) &= G(\omega,\epsilon),\\ \Omega_1(\omega,\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\Omega_0\right]\right],\\ \Omega_2(\omega,\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\Omega_1\right]\right],\\ \Omega_3(\omega,\epsilon) &= -\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\Omega_2\right]\right].\end{aligned}$$

Thus, the general recursive formula is:

$$\Omega_{n+1}(\omega,\epsilon) = -\daleth^{-1}\left[\frac{\mu}{\theta}\daleth\left[L\Omega_n\right]\right], \quad \forall n \geq 0. \tag{6.8}$$

So, the approximate solution to the given nonhomogeneous PDE is:

$$\Omega(\omega,\epsilon) = \sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon). \tag{6.9}$$

Example 6.1. Consider the second order linear partial telegraph equation [5]:

$$\Omega_{\omega\omega} = \Omega_{\epsilon\epsilon} + \Omega_{\epsilon} - \Omega, \tag{6.10}$$

with initial and boundary conditions:

$$\Omega(0,\epsilon) = e^{-2\epsilon},\ \Omega_{\omega}(0,\epsilon) = e^{-2\epsilon},\ \Omega(\omega,0) = e^{\omega},\ \Omega_{\epsilon}(\omega,0) = -2e^{\omega}. \tag{6.11}$$

Solution. Apply $\daleth$-transform to Eq. (6.10), we conclude:

$$\frac{\theta^2}{\mu^2}\Psi(\omega,\epsilon) - \theta\Omega(0,\epsilon) - \mu\Omega_{\omega}(0,\epsilon) = \daleth[\Omega_{\epsilon\epsilon} + \Omega_{\epsilon} - \Omega]. \tag{6.12}$$

Substituting Eq. (6.11) into Eq. (6.12), we get:

$$\Psi(\omega,\epsilon) = \frac{\mu^2}{\theta}e^{-2\epsilon} + \frac{\mu^3}{\theta^2}e^{-2\epsilon} + \frac{\mu^2}{\theta^2}\daleth[\Omega_{\epsilon\epsilon} + \Omega_{\epsilon} - \Omega]. \tag{6.13}$$

Taking the inverse $\daleth$-transform of Eq. (6.13), we obtain:

$$\Omega(\omega,\epsilon) = e^{-2\epsilon} + \omega e^{-2\epsilon} + \daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth[\Omega_{\epsilon\epsilon} + \Omega_{\epsilon} - \Omega]\right]. \tag{6.14}$$

Assume a series solution for $\Omega(\omega,\epsilon)$:

$$\Omega(\omega,\epsilon) = \sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon). \tag{6.15}$$

Combining Eq. (6.15) and Eq. (6.14), we conclude:

$$\sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon) = e^{-2\epsilon} + \omega e^{-2\epsilon} + \daleth^{-1}\left[\frac{\mu^2}{\theta^2}\daleth\left[\sum_{n=0}^{\infty}\left((\Omega_n)_{\epsilon\epsilon} + (\Omega_n)_{\epsilon} - \Omega_n\right)\right]\right]. \tag{6.16}$$

We determine the recursive relation by comparing the two sides:

$$\Omega_{n+1}(\omega,\epsilon) = \mathfrak{I}^{-1}\left[\frac{\mu^2}{\theta^2}\mathfrak{I}\left[(\Omega_n)_{\epsilon\epsilon} + (\Omega_n)_\epsilon - (\Omega_n)\right]\right], \quad \forall n \geq 0. \tag{6.17}$$

Compute the components:

$$\begin{aligned}
\Omega_0(\omega,\epsilon) &= e^{-2\epsilon} + \omega e^{-2\epsilon},\\
\Omega_1(\omega,\epsilon) &= \mathfrak{I}^{-1}\left[\frac{\mu^2}{\theta^2}\mathfrak{I}[(\Omega_0)_{\epsilon\epsilon} + (\Omega_0)_\epsilon - (\Omega_0)]\right]\\
&= e^{-2\epsilon}\left(\frac{\omega^2}{2!} + \frac{\omega^3}{3!}\right),\\
\Omega_2(\omega,\epsilon) &= \mathfrak{I}^{-1}\left[\frac{\mu^2}{\theta^2}\mathfrak{I}[(\Omega_1)_{\epsilon\epsilon} + (\Omega_1)_\epsilon - (\Omega_1)]\right]\\
&= e^{-2\epsilon}\left(\frac{\omega^4}{4!} + \frac{\omega^5}{5!}\right).
\end{aligned}$$

Therefore, the approximate series solution is:

$$\begin{aligned}
\Omega(\omega,\epsilon) &= \sum_{n=0}^{\infty}\Omega_n(\omega,\epsilon)\\
&= e^{-2\epsilon}\left(1 + \omega + \frac{\omega^2}{2!} + \frac{\omega^3}{3!} + \frac{\omega^4}{4!} + \frac{\omega^5}{5!} + \cdots\right).
\end{aligned}$$

Hence, the exact solution is:

$$\Omega(\omega,\epsilon) = e^{\omega-2\epsilon}. \tag{6.18}$$

This matches the result obtained by the ADM in [5].

The exact solution in Eq. (6.18) is in closed form. The numerical results for different values of ω, ϵ are shown in Figures 4 and 5 below. It is evident that the curves are affected differently by the values of ω, ϵ, and all of these curves have no cusps, which indicates the effectiveness of the $\mathcal{ADJM}$.

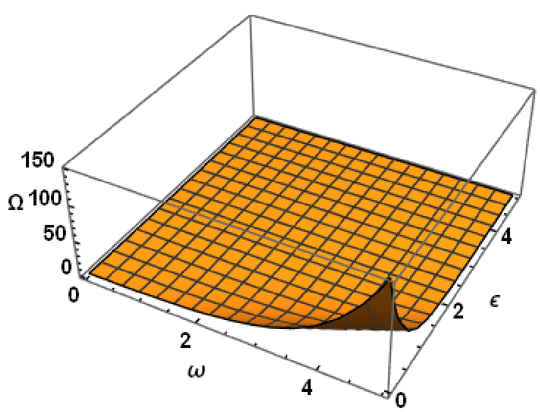

Figure 4. Exact solution of $\Omega(\omega,\epsilon)$ for Example 6.1.

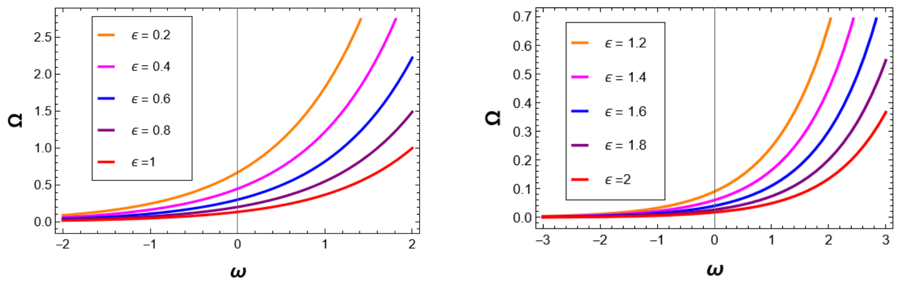

Figure 5. Numerical solutions of $\Omega(\omega)$ for Example 6.1 for multiple values of ϵ.

7. Conclusion

In this work, we employed the Adomian decomposition $\mathcal{J}$-transform method for solving linear and nonlinear differential equations. The $\mathcal{ADJM}$ consistently provides exact solutions in many cases with elegant computational terms. It can be concluded that the $\mathcal{ADJM}$ demonstrates a high level of improvement over existing methods due to its flexibility, accuracy, and simplicity. Using $\mathcal{ADJM}$, we successfully found exact solutions for several linear and nonlinear differential equations, such as the Bratu equations and second order linear partial telegraph equation. The results obtained through $\mathcal{ADJM}$ were compared with those from existing methods. The applicability of $\mathcal{ADJM}$ has proven its significance in the fields of applied science and engineering. Therefore, it is reasonable to think of the $\mathcal{J}$-transform as a modification of both the Sumudu and natural transforms. It is relatively easy to extend the $\mathcal{J}$-transform to investigate a wide range of physical research and engineering applications. We intend to investigate the extended properties and applications of the suggested integral transform in the near future. Therefore, the Adomian decomposition $\mathcal{J}$-transform method is a viable alternative to existing methods.

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Declarations

Funding. There is currently no funding available for the study project.

Compliance with ethical standards.

Conflicts of interest. The authors declare that they have no conflict of interest concerning the publication of this manuscript.

Availability of data and materials. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Code availability. (Not applicable).

Ethics approval. (Not applicable).

Consent to participate. Participants is aware that they can contact the Jordan University of Science and Technology Ethics Officer if they have any concerns or complaints regarding the way in which the research is or has been conducted.

Consent for publication. (Not applicable).

Availability of data and materials. Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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