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DYNAMICS OF A DIFFUSIVE SINGLE-SPECIES MODEL WITH NONLOCALITY AND DISTRIBUTED MEMORY*

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Abstract Cognitive abilities and memorized information are significant for "smart" animals to make movement decisions. This work establishes a diffusive single-species model with nonlocality and distributed memory. We consider weak and strong temporal kernels in the memory-based diffusion term and find that the model exhibits complex dynamical behaviour produced by nonlocality and distributed memory. For either temporal kernel, mode-n Turing (or Hopf) bifurcation can emerge and double Turing bifurcation can be generated by mode-n and mode-m ($n \neq m$) Turing bifurcations. Additionally, Hopf bifurcation is possible to occur for small random diffusion or large repulsive memory-based diffusion in considerations of weak or strong kernels, respectively. Note that the critical Turing bifurcation curve can be mode-n ($n \geq 2$), which differs from that in the model with nonlocal spatial average, while Turing, Turing-Hopf and double Turing bifurcations do not appear in the single-species model involving only distributed memory delay. An application to our theoretical findings is presented and Turing-Hopf bifurcation and stability switches are found in numerical exploration by considering weak and strong kernels, respectively.

Keywords Nonlocality, distributed memory, Turing bifurcation, Hopf bifurcation.

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1. Introduction

The movement of substances and highly developed species often follow Brownian and non-Brownian movement, respectively. Theoretical studies and numerical observations reveal the interaction between movement process and memorized information of "smart" species. For one thing, animals benefiting from their perceptual or cognitive abilities can acquire spatiotemporal information in movement process; for another, travelling experience (or memorized information) is also significant for animals to make movement decisions [3, 9]. Episodic-like memory of animals can be described by the revised reaction-diffusion equation with a delayed diffusion term [14]. In addition, the single-species model involving random diffusion,

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memory-based diffusion and maturation delay is considered in [13] and it has been demonstrated that memory and maturation delays result in the appearance of Hopf bifurcation, with spatially nonhomogeneous periodic solutions being born.

In [15], knowledge transfer between animals is taken into account and the single-species model with spatiotemporal delayed diffusion has been proposed by Shi *et al*, where the memory-based diffusion term describes how spatiotemporal factors affect memory of animals. In [16], Song *et al.* have discussed the single-species model with spatiotemporal delays in diffusion and reaction, which is possible to undergo Turing-Hopf bifurcation generated by two delays. In [17], Wang and Wang have studied the diffusive two-species model with spatiotemporal memory delay and have demonstrated global existence and uniqueness of the solution and the existence of Turing and Hopf bifurcations. In [18], Wu and Song have investigated the diffusive single-species model with spatiotemporal memory delay and discrete maturation delay, in which codimension 3 and 4 bifurcations are found.

In [6], Lin and Song have established the diffusive single population model incorporating distributed memory delay to analyse the temporal impacts of memory on animal movements. A reaction-diffusion population model with Dirichlet boundary condition and distributed memory has been considered and delay-induced stability switch and spatiotemporal patterns are obtained in [12]. A heterogeneous memory-based diffusive model involving distributed memory delay has been investigated in [5], where spatial heterogeneity and distributed memory can stabilize or destabilize spatially nonhomogeneous steady state by considering the weak or strong temporal kernels, respectively. Besides, distributed memory has also been incorporated into diffusive resource-consumer models [10, 11].

In the present paper, we study the diffusive single-species model with nonlocality and distributed memory that is given by

$$\begin{cases} u_t(x,t) = d_1 u_{xx}(x,t) + d_2(u(x,t)v_x(x,t))_x + g(u(x,t),h(x,t)), 0<x<\ell\pi, t>0, \\ u_x(0,t) = u_x(\ell\pi,t) = 0, \quad t>0, \end{cases} \tag{1.1}$$

with

$$d_1>0,\ d_2\in\mathbb{R},\ h(x,t)=\int_0^{\ell\pi} K(x,y)u(y,t)dy,$$
$$v(x,t)=\int_{-\infty}^{t}\frac{(t-s)^{k-1}}{\tau^k}e^{-\frac{t-s}{\tau}}u(x,s)ds,\ k=1,2.$$

Here, u and g refer to the population density and biological process of the species, respectively; d_1 and d_2 are the diffusion coefficients based on random and memory-based diffusion, respectively. When it comes to $v(x,t)$, weak and strong temporal kernels (i.e., $\frac{1}{\tau}e^{-\frac{t}{\tau}}$ and $\frac{t}{\tau^2}e^{-\frac{t}{\tau}}$) are taken into account. The former kernel shows that the influence of memory of animals is in decline as time passes, and the latter kernel indicates that the influence of memory of animals increases until the maximum is reached and then it decreases [6, 15]. Moreover, $h(x,t)$ illustrates the nonlocal intraspecific competition of the species with $K(x,y)$ being the Green function of the operator $-d_1K_{xx}+I$ subject to the Neumann boundary condition.

Suppose $u_\star>0$ is the stable steady state of the equation $\frac{du(t)}{dt}=g(u(t),u(t))$, which implies it is also a spatially homogeneous steady state of system (1.1). Namely, $u_\star>0$ satisfies $g(u_\star,u_\star)=0$ and $A+B<0$, where $A=\frac{\partial g(u_\star,u_\star)}{\partial u}$ and $B=\frac{\partial g(u_\star,u_\star)}{\partial h}$. When considering the weak kernel, we have the following conclusion

(I) if $A \leq 0$, then memory delay has no influence on the stability of the spatially homogeneous steady state;

(II) if either $0 < A \leq 1$, or $A > 1$ and $B \leq -\frac{(A+1)^2}{4}$, then memory delay has no influence on the stability of the spatially homogeneous steady state, and Turing and double Turing bifurcations can emerge for attractive memory-based diffusion;

(III) if $A > 1$ and $B > -\frac{(A+1)^2}{4}$, then memory delay and diffusion coefficients result in the appearance of Turing, double Turing and Hopf bifurcations and Hopf bifurcation is possible to occur for small random diffusion.

When considering the strong kernel, we see that

(IV) if $A \leq 0$, then the spatially homogeneous steady state remains stable for small memory-based diffusion and Hopf bifurcation may occur for repulsive memory-based diffusion;

(V) if either $0 < A \leq 1$, or $A > 1$ and $B \leq -\frac{(A+1)^2}{4}$, then the spatially homogeneous steady state remains stable for small memory-based diffusion and Turing and Hopf bifurcations can emerge for attractive and repulsive memory-based diffusion, respectively;

(VI) if $A > 1$ and $B > -\frac{(A+1)^2}{4}$, then Turing, double Turing and Hopf bifurcations are possible to emerge.

In the next section, we show that the dynamics of system (1.1) are complex due to the effects of the nonlocality and distributed memory and derive the conditions for the appearance of bifurcations. Application for numerical exploration and discussion for the present paper are presented in Sections 3 and 4, respectively.

2. Stability and bifurcation analysis

Linearizing model (1.1) about the steady state u_* yields

$$\begin{cases} u_t = d_1 u_{xx} + d_2 u_* v_{xx} + Au + Bh, & 0 < x < \ell\pi,\ t > 0, \\ u_x(0,t) = u_x(\ell\pi, t) = 0, & t > 0, \end{cases} \tag{2.1}$$

where $v_{xx} = \int_{-\infty}^{t} \frac{(t-s)^{k-1}}{\tau^k} e^{-\frac{t-s}{\tau}} u_{xx}(x,s)ds$, $k = 1, 2$. Letting $u(x,t) = e^{\mu t}\cos\frac{nx}{\ell}$, we obtain the following characteristic equation

$$\mu + \frac{d_1 n^2}{\ell^2} + \frac{d_2 u_* n^2}{\ell^2}\int_{-\infty}^{t} \frac{(t-s)^{k-1}}{\tau^k} e^{-\left(\frac{1}{\tau}+\mu\right)(t-s)} ds - A - \frac{B\ell^2}{\ell^2 + d_1 n^2} = 0, \tag{2.2}$$

where $n \geq 0$ are integers and $k = 1, 2$. Similar to [6], we see that $\lim_{s\to-\infty} e^{(\frac{1}{\tau}+\mu)s}$ exists provided that $\mathrm{Re}\mu > -\frac{1}{\tau}$ or $\mu = -\frac{1}{\tau}$, while $\lim_{s\to-\infty} s e^{(\frac{1}{\tau}+\mu)s}$ exists if and only if $\mathrm{Re}\mu > -\frac{1}{\tau}$. Notice that if $\mu = -\frac{1}{\tau}$, then $\int_{-\infty}^{t} e^{-\left(\frac{1}{\tau}+\mu\right)(t-s)} ds = +\infty$. Thus, we have $\mathrm{Re}\mu > -\frac{1}{\tau}$ and $\int_{-\infty}^{t} \frac{(t-s)^{k-1}}{\tau^k} e^{-\left(\frac{1}{\tau}+\mu\right)(t-s)} ds = \frac{1}{(\tau\mu+1)^k}$, $k = 1, 2$. It follows that Eq. (2.2) is given by

$$\mu + \frac{d_1 n^2}{\ell^2} + \frac{d_2 u_* n^2}{(\tau\mu+1)^k \ell^2} - A - \frac{B\ell^2}{\ell^2 + d_1 n^2} = 0, \tag{2.3}$$

where $n \geq 0$ and $k = 1, 2$. By Eq. (2.3), we have $\mu = A + B < 0$ for $n = 0$. For any $n \geq 1$, mode-n Turing (or Hopf) bifurcation occurs if 0 is a simple root (or $\pm i\sigma$ is a pair of imaginary roots) of Eq. (2.3) with the associated transversality condition holding.

2.1. The weak kernel

For $k = 1$, Eq. (2.3) is equivalent to

$$\tau\mu^2 + \left(1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B \ell^2}{\ell^2 + d_1 n^2}\right)\mu + \frac{(d_1 + d_2 u_*) n^2}{\ell^2} - A - \frac{B\ell^2}{\ell^2 + d_1 n^2} = 0. \quad (2.4)$$

We suppose $\mu = 0$ is a simple root for some $n \geq 1$ and obtain $d_2 = d_2^T(d_1, n)$ and $1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2} \neq 0$, where

$$d_2^T(d_1, n) = -\frac{d_1^2 n^4 + d_1 n^2 \ell^2 (1 - A) - \ell^4 (A + B)}{u_* n^2 (\ell^2 + d_1 n^2)}, \ n \geq 1. \quad (2.5)$$

Additionally, we suppose $\mu = i\sigma_n$ ($\sigma_n > 0$) and separate real and imaginary parts of Eq. (2.4) to obtain

$$\begin{cases} \sigma_n \left(1 + \dfrac{\tau d_1 n^2}{\ell^2} - \tau A - \dfrac{\tau B\ell^2}{\ell^2 + d_1 n^2}\right) = 0, \\ -\tau\sigma_n^2 + \dfrac{(d_1 + d_2 u_*) n^2}{\ell^2} - A - \dfrac{B\ell^2}{\ell^2 + d_1 n^2} = 0, \end{cases}$$

from which it follows that

$$\begin{cases} 1 + \dfrac{\tau d_1 n^2}{\ell^2} - \tau A - \dfrac{\tau B\ell^2}{\ell^2 + d_1 n^2} = 0, \\ \sigma_n^2 = \dfrac{(d_1 + d_2 u_*) n^2}{\tau \ell^2} - \dfrac{A}{\tau} - \dfrac{B\ell^2}{\tau(\ell^2 + d_1 n^2)} > 0. \end{cases} \quad (2.6)$$

We next establish the following lemmas for $d_2^T(d_1, n)$ and $1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2}$, which correspond to Turing and Hopf bifurcation curves, respectively.

Lemma 2.1. *Suppose $A + B < 0$ and $d_2^T(d_1, n)$ is defined in* (2.5).

(I) For fixed $n \geq 1$, we have

- *(i)* $\lim_{d_1 \to +\infty} d_2^T(d_1, n) = -\infty$, $\lim_{d_1 \to 0^+} d_2(d_1, n) = \frac{\ell^2(A+B)}{u_* n^2} < 0$ *and* $\lim_{d_1 \to 0^+} -\frac{d_1}{u_*} = 0$*;*
- *(ii) if $B < -1$, then $d_2^T(d_1, n)$ is strictly increasing for $0 < d_1 < \frac{\ell^2(\sqrt{-B}-1)}{n^2}$ and is strictly decreasing for $d_1 > \frac{\ell^2(\sqrt{-B}-1)}{n^2}$; if $B \geq -1$, then $d_2^T(d_1, n)$ is strictly decreasing with respect to d_1;*
- *(iii) if $A \leq 0$, then $d_2^T(d_1, n) < -\frac{d_1}{u_*}$ for $d_1 > 0$; if either $0 < A \leq 1$, or $A > 1$ and $B \leq -\frac{(A+1)^2}{4}$, then $d_2^T(d_1, n) < -\frac{d_1}{u_*}$ for $0 < d_1 < -\frac{\ell^2(A+B)}{An^2}$, and $-\frac{d_1}{u_*} < d_2^T(d_1, n) \leq 0$ for $d_1 > -\frac{\ell^2(A+B)}{An^2}$; if $A > 1$ and $B > -\frac{(A+1)^2}{4}$, then $d_2^T(d_1, n) < -\frac{d_1}{u_*}$ for $0 < d_1 < -\frac{\ell^2(A+B)}{An^2}$, $-\frac{d_1}{u_*} < d_2^T(d_1, n) < 0$ for*

$-\frac{\ell^2(A+B)}{An^2} < d_1 < \widehat{d}_{1,n}$ *or* $d_1 > \widetilde{d}_{1,n}$, *and* $d_2^T(d_1,n) > 0$ *for* $\widehat{d}_{1,n} < d_1 < \widetilde{d}_{1,n}$, *where*

$$\widehat{d}_{1,n} = \frac{\ell^2(A-1-\sqrt{(A+1)^2+4B})}{2n^2}, \tag{2.7}$$

and

$$\widetilde{d}_{1,n} = \frac{\ell^2(A-1+\sqrt{(A+1)^2+4B})}{2n^2}. \tag{2.8}$$

(II) For fixed d_1, *we have*

(i) $\lim\limits_{n\to+\infty} d_2^T(d_1,n) = -\frac{d_1}{u_*}$;

(ii) if $A \le 0$, *then* $d_2^T(d_1,n) < d_2^T(d_1,n+1)$; *if* $A > 0$, *then* $d_2^T(d_1,n) < d_2^T(d_1,n+1)$ *or* $d_2^T(d_1,n) > d_2^T(d_1,n+1)$ *provided that* $0 < d_1 < d_{1,n}^*$ *or* $d_1 > d_{1,n}^*$, *respectively, where* $d_{1,n}^* > d_{1,n+1}^*$, $d_{1,n}^* > -\frac{\ell^2(A+B)}{An^2}$ *and*

$$\begin{aligned} d_{1,n}^* = &-\frac{\ell^2(A+B)}{2A}\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right) \\ &+\frac{\ell^2}{2A}\sqrt{(A+B)^2\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}\right)^2-\frac{4A(A+B)}{n^2(n+1)^2}}. \end{aligned} \tag{2.9}$$

Proof. For fixed n, $d_2^T(d_1,n)$ can be regarded as the function of d_1. Then, the statement of (i) of (I) can be derived from direct calculation. Moreover, the partial derivative of $d_2^T(d_1,n)$ with respect to d_1 is given by

$$\frac{\partial d_2^T(d_1,n)}{\partial d_1} = -\frac{d_1^2n^4+2d_1n^2\ell^2+\ell^4(1+B)}{u_*(\ell^2+d_1n^2)^2}. \tag{2.10}$$

It is clear that $\frac{\partial d_2^T(d_1,n)}{\partial d_1} < 0$ when $B \ge -1$. Assuming that $B < -1$ and letting $\frac{\partial d_2^T(d_1,n)}{\partial d_1} = 0$, we see that $\frac{\partial d_2^T(d_1,n)}{\partial d_1} > 0$ for $0 < d_1 < \frac{\ell^2(\sqrt{-B}-1)}{n^2}$ and $\frac{\partial d_2^T(d_1,n)}{\partial d_1} < 0$ for $d_1 > \frac{\ell^2(\sqrt{-B}-1)}{n^2}$, which leads to the statement of (ii) of (I).

Since $\ell^2 + d_1n^2 > 0$ and

$$d_2^T(d_1,n) - \left(-\frac{d_1}{u_*}\right) = \frac{Ad_1n^2\ell^2+\ell^4(A+B)}{u_*n^2(\ell^2+d_1n^2)},$$

we see that the signs of $d_2^T(d_1,n)-\left(-\frac{d_1}{u_*}\right)$ and $d_2^T(d_1,n)$ are determined by $Ad_1n^2\ell^2+\ell^4(A+B)$ and $-d_1^2n^4-d_1n^2\ell^2(1-A)+\ell^4(A+B)$, respectively. Therefore, it follows that

$$d_2^T(d_1,n) - \left(-\frac{d_1}{u_*}\right)\begin{cases} <0, \text{ if either } A\le 0, \text{ or } A>0 \text{ and } 0<d_1<-\dfrac{\ell^2(A+B)}{An^2}, \\ =0, \text{ if } A>0 \text{ and } d_1 = -\dfrac{\ell^2(A+B)}{An^2}, \\ >0, \text{ if } A>0 \text{ and } d_1 > -\dfrac{\ell^2(A+B)}{An^2}. \end{cases}$$

Moreover, when either $A \le 1$, or $A > 1$ and $B < -\frac{(A+1)^2}{4}$, we have $d_2^T(d_1,n) < 0$; when $A > 1$ and $B = -\frac{(A+1)^2}{4}$, we obtain $d_2^T(d_1,n) \le 0$; when $A > 1$ and $B >$

$-\frac{(A+1)^2}{4}$, we know that

$$d_2^T(d_1,n)\begin{cases} <0, & \text{if } 0<d_1<\widehat{d}_{1,n} \text{ or } d_1>\widetilde{d}_{1,n},\\ =0, & \text{if } d_1=\widehat{d}_{1,n} \text{ or } d_1=\widetilde{d}_{1,n},\\ >0, & \text{if } \widehat{d}_{1,n}<d_1<\widetilde{d}_{1,n}.\end{cases}$$

This proves (iii) of (I).

Notice that $A+B<0$ and

$$\begin{aligned} &d_2^T(d_1,n)-d_2^T(d_1,n+1)\\ &=\frac{\ell^2(2n+1)\left(Ad_1^2n^2(n+1)^2+d_1\ell^2\left(n^2+(n+1)^2\right)(A+B)+\ell^4(A+B)\right)}{u_*n^2(n+1)^2(\ell^2+d_1n^2)\left(\ell^2+d_1(n+1)^2\right)}.\end{aligned}$$

When $A\le 0$, we have $d_2^T(d_1,n)<d_2^T(d_1,n+1)$. When $A>0$, we see that $d_2^T(d_1,n)<d_2^T(d_1,n+1)$ if and only if $0<d_1<d_{1,n}^*$, and $d_2^T(d_1,n)>d_2^T(d_1,n+1)$ if and only if $d_1>d_{1,n}^*$, where $d_{1,n}^*$ is defined in (2.9). Then, straightforward calculation gives $d_{1,n}^*>d_{1,n+1}^*$ and $d_{1,n}^*>-\frac{\ell^2(A+B)}{An^2}$. This completes the proof of (II). □

Lemma 2.2. *Suppose* $A+B<0$.

(I) For any $n\ge 1$, $1+\frac{\tau d_1n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}>0$ *if one of the following hypotheses holds:*

- *(H1)* $A\le 1$;
- *(H2)* $A>1$ *and* $B<-\frac{(A+1)^2}{4}$;
- *(H3)* $A>1$, $B=-\frac{(A+1)^2}{4}$ *and* $d_1\ne\frac{\ell^2(A-1)}{2n^2}$;
- *(H4)* $A>1$, $B>-\frac{(A+1)^2}{4}$ *and* $0<d_1<\widehat{d}_{1,n}$;
- *(H5)* $A>1$, $B>-\frac{(A+1)^2}{4}$ *and* $d_1>\widetilde{d}_{1,n}$.

(II) For any $n\ge 1$, $1+\frac{\tau d_1n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}=1$ *if one of the following hypotheses holds:*

- *(H6)* $A>1$, $B=-\frac{(A+1)^2}{4}$ *and* $d_1=\frac{\ell^2(A-1)}{2n^2}$;
- *(H7)* $A>1$, $B>-\frac{(A+1)^2}{4}$ *and* $d_1=\widehat{d}_{1,n}$;
- *(H8)* $A>1$, $B>-\frac{(A+1)^2}{4}$ *and* $d_1=\widetilde{d}_{1,n}$.

(III) For any $n\ge 1$, *if the following hypothesis holds:*

- *(H9)* $A>1$, $B>-\frac{(A+1)^2}{4}$ *and* $\widehat{d}_{1,n}<d_1<\widetilde{d}_{1,n}$,

then

$$1+\frac{\tau d_1n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}>0\Longleftrightarrow\tau<\tau(d_1,n),$$

where

$$\tau(d_1,n)=-\frac{\ell^2(\ell^2+d_1n^2)}{d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)}. \tag{2.11}$$

Moreover, $\tau(d_1,n)$ *is strictly decreasing for* $\widehat{d}_{1,n}<d_1<\frac{\ell^2(\sqrt{-B}-1)}{n^2}$ *and is strictly increasing for* $\frac{\ell^2(\sqrt{-B}-1)}{n^2}<d_1<\widetilde{d}_{1,n}$.

Proof. Since $1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2} = 1+\frac{\tau\left(d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)\right)}{\ell^2(\ell^2+d_1n^2)}$ and $\ell^2(\ell^2+d_1n^2)>0$, it is clear that the sign of $1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}$ depends on $d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)$. Similarly to the proof of (iii) of (I) of Lemma 2.1, we can conclude the statement on the sign of $1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}$. It remains to prove the monotonicity of $\tau(d_1,n)$ under the assumption that $A>1$ and $B>-\frac{(A+1)^2}{4}$. For fixed n, $\tau(d_1,n)$ can be regarded as the function of d_1 and the derivative of $\tau(d_1,n)$ is given by

$$\frac{\partial\tau(d_1,n)}{\partial d_1}=\frac{\ell^2n^2\left(d_1^2n^4+2d_1n^2\ell^2+\ell^4(1+B)\right)}{\left(d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)\right)^2}.$$

Using $A+B<0$ and $A>1$ yields $B<-1$, which gives

$$\frac{\partial\tau(d_1,n)}{\partial d_1}\begin{cases}<0, \text{ if } \widehat{d}_{1,n}<d_1<\dfrac{\ell^2(\sqrt{-B}-1)}{n^2},\\ =0, \text{ if } d_1=\dfrac{\ell^2(\sqrt{-B}-1)}{n^2},\\ >0, \text{ if } \dfrac{\ell^2(\sqrt{-B}-1)}{n^2}<d_1<\widetilde{d}_{1,n}.\end{cases}$$

This completes the proof. □

Now, we are ready to establish the result on the stability of u_* by using Lemmas 2.1 and 2.2.

Theorem 2.1. *Suppose that* $v(x,t)=\int_{-\infty}^{t}\frac{1}{\tau}e^{-\frac{t-s}{\tau}}u(x,s)ds$ *and* $A+B<0$. *Let* $d_2^T(d_1,n)$, $\tau(d_1,n)$ *and* $d_{1,n}^*$ *be denoted by* (2.5), (2.11) *and* (2.9), *respectively.*

(I) If $A\le 0$, *then* u_* *is asymptotically stable for* $d_2\ge-\frac{d_1}{u_*}$ *and is unstable for* $d_2<-\frac{d_1}{u_*}$, *and* $\tau>0$ *has no influence on the stability of* u_*.

(II) If $0<A\le 1$ *or* $A>1$ *and* $B\le-\frac{(A+1)^2}{4}$, *then*

- *(i) if* $d_1>d_{1,1}^*$, *then* u_* *is asymptotically stable for* $d_2>d_2^T(d_1,1)$ *and is unstable for* $d_2<d_2^T(d_1,1)$, *and mode-1 Turing bifurcation occurs at* $d_2=d_2^T(d_1,1)$;
- *(ii) if* $d_{1,N}^*<d_1<d_{1,N-1}^*$ *for some positive integer* $N\ge 2$, *then* u_* *is asymptotically stable for* $d_2>d_2^T(d_1,N)$ *and is unstable for* $d_2<d_2^T(d_1,N)$, *and mode-N Turing bifurcation occurs at* $d_2=d_2^T(d_1,N)$;
- *(iii) if* $d_1=d_{1,N}^*$ *for some positive integer* $N\ge 1$, *then* u_* *is asymptotically stable for* $d_2>d_2^T(d_1,N)$ *and is unstable for* $d_2<d_2^T(d_1,N)$, *and double Turing bifurcation generated by mode-N and mode-N+1 Turing bifurcations occurs at* $d_2=d_2^T(d_1,N)$;
- *(iv)* $\tau>0$ *has no influence on the stability of* u_*.

(III) If $A>1$, $B>-\frac{(A+1)^2}{4}$, *and* $\widehat{d}_{1,n}$ *and* $\widetilde{d}_{1,n}$ *are defined in* (2.7) *and* (2.8), *respectively, then*

- *(i) if* $d_1>d_{1,1}^*$, *then* u_* *is unstable for* $d_2<d_2^T(d_1,1)$ *and there is no mode-1 Hopf bifurcation for* $d_2\le d_2^T(d_1,1)$; *if further* $0<d_1\le\widehat{d}_{1,1}$ *or* $d_1\ge\widetilde{d}_{1,1}$, *then mode-1 Turing bifurcation occurs at* $d_2=d_2^T(d_1,1)$, *while if further* $\widehat{d}_{1,1}<d_1<\widetilde{d}_{1,1}$, *then mode-1 Turing bifurcation occurs at* $d_2=d_2^T(d_1,1)$ *provided that* $\tau\ne\tau(d_1,1)$;

(ii) *if* $d_{1,N}^* < d_1 < d_{1,N-1}^*$ *for some positive integer* $N \geq 2$, *then* u_* *is unstable for* $d_2 < d_2^T(d_1, N)$ *and there is no mode-N Hopf bifurcation for* $d_2 \leq d_2^T(d_1, N)$; *if further* $0 < d_1 \leq \widehat{d}_{1,N}$ *or* $d_1 \geq \widetilde{d}_{1,N}$, *then mode-N Turing bifurcation occurs at* $d_2 = d_2^T(d_1, N)$, *while if further* $\widehat{d}_{1,N} < d_1 < \widetilde{d}_{1,N}$, *mode-N Turing bifurcation occurs at* $d_2 = d_2^T(d_1, N)$ *provided that* $\tau \neq \tau(d_1, N)$;

(iii) *if* $d_1 = d_{1,N}^*$ *for some positive integer* $N \geq 1$, *then* u_* *is unstable for* $d_2 < d_2^T(d_1, N)$ *and there is no mode-N (or mode-N+1) Hopf bifurcation for* $d_2 \leq d_2^T(d_1, N)$; *if further* $0 < d_1 \leq \widehat{d}_{1,N+1}$ *or* $d_1 \geq \widetilde{d}_{1,N}$, *then double Turing bifurcation generated by mode-N and mode-N+1 Turing bifurcations occurs at* $d_2 = d_2^T(d_1, N)$, *while if further* $\widehat{d}_{1,N} < d_1 < \widetilde{d}_{1,N}$ *(or* $\widehat{d}_{1,N+1} < d_1 < \widetilde{d}_{1,N+1}$*), then mode-N (or mode-N+1) Turing bifurcation occurs at* $d_2 = d_2^T(d_1, N)$ *provided that* $\tau \neq \tau(d_1, N)$ *(or* $\tau \neq \tau(d_1, N+1)$*)*;

(iv) *if either* $d_1 > d_{1,1}^*$ *and* $d_2 > d_2^T(d_1, 1)$, *or* $d_{1,N}^* < d_1 \leq d_{1,N-1}^*$ *and* $d_2 > d_2^T(d_1, N)$ *for some positive integer* $N \geq 2$, *then we have*

(a) *if further* $d_1 \geq \widetilde{d}_{1,1}$, *then for any* $\tau > 0$, u_* *is asymptotically stable;*

(b) *if further* $d_1 < \widetilde{d}_{1,1}$ *and there is no* $n \geq 1$ *such that*

$$\sqrt{\frac{\ell^2\left(A-1-\sqrt{(A+1)^2+4B}\right)}{2d_1}} < n < \sqrt{\frac{\ell^2\left(A-1+\sqrt{(A+1)^2+4B}\right)}{2d_1}},$$

then for any $\tau > 0$, u_* *is asymptotically stable;*

(c) *if further* $d_1 < \widetilde{d}_{1,1}$ *and there exists* $n \geq 1$ *such that*

$$\sqrt{\frac{\ell^2\left(A-1-\sqrt{(A+1)^2+4B}\right)}{2d_1}} < n < \sqrt{\frac{\ell^2\left(A-1+\sqrt{(A+1)^2+4B}\right)}{2d_1}},$$

then u_* *is asymptotically stable for* $0 < \tau < \tau_\sharp$ *and is unstable for* $\tau > \tau_\sharp$, *where* $\tau_\sharp$ *is the minimum value of* $\tau(d_1, n)$ *with* n *satisfying* $\sqrt{\frac{\ell^2\left(A-1-\sqrt{(A+1)^2+4B}\right)}{2d_1}} < n < \sqrt{\frac{\ell^2\left(A-1+\sqrt{(A+1)^2+4B}\right)}{2d_1}}$, *and mode-n Hopf bifurcation occurs at* $\tau = \tau(d_1, n)$.

Proof. In (2.6), $\sigma_n^2 > 0$ if and only if $d_2 > d_2^T(d_1, n)$. For any $n \geq 1$, Eq. (2.4) has a simple zero root at $d_2 = d_2^T(d_1, n)$ provided that $1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B \ell^2}{\ell^2 + d_1 n^2} \neq 0$, while Eq. (2.4) has a pair of purely imaginary roots if and only if $d_2 > d_2^T(d_1, n)$ and $1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B \ell^2}{\ell^2 + d_1 n^2} = 0$. Regarding d_2 as a bifurcation parameter and noting that $\mu(d_2) = 0$ for $d_2 = d_2^T(d_1, n)$, it follows that for $1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B \ell^2}{\ell^2 + d_1 n^2} \neq 0$,

$$\left.\frac{d\mu(d_2)}{dd_2}\right|_{d_2 = d_2^T(d_1, n)} = -\frac{u_* n^2(\ell^2 + d_1 n^2)}{(\ell^2 + d_1 n^2)(\ell^2 + \tau d_1 n^2 - \tau A \ell^2) - \tau B \ell^4} \neq 0. \tag{2.12}$$

Regarding τ as a bifurcation parameter and noting that $\mu(\tau) = \pm i\sigma_n$ for $\tau = \tau(d_1, n)$, it follows that

$$\left.\frac{d\mathrm{Re}\mu(\tau)}{d\tau}\right|_{\tau = \tau(d_1, n)} = \frac{1}{2\tau^2} > 0. \tag{2.13}$$

Uing (iii) of (I) of Lemma 2.1, (II) of Lemma 2.1, and (I) of Lemma 2.2 allows us to prove (I).

When $0 < A \leq 1$ or $A > 1$, we utilize (ii) of (II) of Lemma 2.1 to obtain the following statements: if $d_1 > d_{1,1}^*$, then $d_2^T(d_1, 1) > d_2^T(d_1, n)$ for $n \neq 1$; if $d_{1,N}^* < d_1 < d_{1,N-1}^*$ for some positive integer $N \geq 2$, then $d_2^T(d_1, N) > d_2^T(d_1, n)$ for $n \neq N$; if $d_1 = d_{1,N}^*$ for some positive integer $N \geq 1$, then $d_2^T(d_1, N) = d_2^T(d_1, N+1)$ and $d_2^T(d_1, N) > d_2^T(d_1, n)$ for $n \neq N, N+1$. More specifically, it should be mentioned that if $d_{1,N}^* < d_1 < d_{1,N-1}^*$, we have $d_2^T(d_1, N) > d_2^T(d_1, n)$ for $n > N$ and $n < N$, since $d_1 > d_{1,N}^* > d_{1,n}^*$ for $n > N$ and $d_1 < d_{1,n}^*$ for $n \leq N-1$. Similarly, if $d_1 = d_{1,N}^*$, we have $d_{1,N+1}^* < d_1 < d_{1,N-1}^*$ and thus $d_2^T(d_1, N) > d_2^T(d_1, n)$ for $n \neq N, N+1$. Then, we use the transversality condition (2.12) and Lemma 2.2 to obtain the conclusion of (II) and (i)–(iii) of (III).

We next concentrate on $A > 1$ and $B > -\frac{(A+1)^2}{4}$. It is clear that $\widehat{d}_{1,n} > \widehat{d}_{1,n+1}$ and $\widetilde{d}_{1,n} > \widetilde{d}_{1,n+1}$ for each $n \geq 1$. For fixed d_1, we have $\lim_{n \to +\infty} \tau(d_1, n) = 0$ and $\tau(d_1, n) < 0$ for sufficiently large $n \geq 1$. It is easy to verify that $\tau(d_1, n) > 0$ for $\sqrt{\frac{\ell^2\left(A-1-\sqrt{(A+1)^2+4B}\right)}{2d_1}} < n < \sqrt{\frac{\ell^2\left(A-1+\sqrt{(A+1)^2+4B}\right)}{2d_1}}$. Then, we can use the transversality condition (2.13) and Lemma 2.2 to conclude (iv) of (III). $\square$

2.2. The strong kernel

For $k = 2$, Eq. (2.3) is equivalent to

$$\begin{aligned} &\tau^2\mu^3 + \tau\left(2 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2}\right)\mu^2 \\ &+ \left(1 + \frac{2\tau d_1 n^2}{\ell^2} - 2\tau A - \frac{2\tau B\ell^2}{\ell^2 + d_1 n^2}\right)\mu \\ &+ \frac{(d_1 + d_2 u_*)n^2}{\ell^2} - A - \frac{B\ell^2}{\ell^2 + d_1 n^2} \\ = {}& 0, \quad n \geq 0. \end{aligned} \tag{2.14}$$

If $\mu = 0$ for some $n \geq 1$, then we can derive $d_2 = d_2^T(d_1, n)$ and $\frac{1}{2} + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2} \neq 0$, where $d_2^T(d_1, n)$ is given by (2.5). In addition, if $\mu = i\sigma_n$ ($\sigma_n > 0$), then we separate real and imaginary parts of Eq. (2.14) and obtain

$$\begin{cases} \tau^2\sigma_n^2 = \left(1 + \dfrac{2\tau d_1 n^2}{\ell^2} - 2\tau A - \dfrac{2\tau B\ell^2}{\ell^2 + d_1 n^2}\right) > 0, \\ \tau\sigma_n^2\left(2 + \dfrac{\tau d_1 n^2}{\ell^2} - \tau A - \dfrac{\tau B\ell^2}{\ell^2 + d_1 n^2}\right) = \dfrac{(d_1 + d_2 u_*)n^2}{\ell^2} - A - \dfrac{B\ell^2}{\ell^2 + d_1 n^2}, \end{cases} \tag{2.15}$$

from which it follows that $\frac{1}{2} + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2} > 0$ and $d_2 = d_2^H(d_1, \tau, n)$, where

$$d_2^H(d_1, \tau, n) = \frac{2\ell^2}{\tau u_* n^2}\left(1 + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B\ell^2}{\ell^2 + d_1 n^2}\right)^2 > 0. \tag{2.16}$$

Now we establish the following lemma for $d_2^H(d_1, \tau, n)$, which is associated with the Hopf bifurcation curve.

Lemma 2.3. *Suppose that $A+B<0$. Let $d_2^H(d_1,\tau,n)$ be defined in (2.16) and (H1)–(H9) be defined in Lemma 2.2.*

(I) For fixed d_1 and $n \geq 1$, we have

(i) if one of (H1)–(H5) holds, then

$$\lim_{\tau\to 0^+} d_2^H(d_1,\tau,n)=+\infty, \quad \lim_{\tau\to+\infty} d_2^H(d_1,\tau,n)=+\infty,$$

and $d_2^H(d_1,\tau,n)$ is strictly decreasing for $0<\tau<-\tau(d_1,n)$ and is strictly increasing for $\tau>-\tau(d_1,n)$, where $\tau(d_1,n)$ is defined in (2.11);

(ii) if one of (H6)–(H8) holds, then

$$\lim_{\tau\to 0^+} d_2^H(d_1,\tau,n)=+\infty, \quad \lim_{\tau\to+\infty} d_2^H(d_1,\tau,n)=0,$$

and $d_2^H(d_1,\tau,n)$ is strictly decreasing with respect to τ;

(iii) if (H9) holds, then

$$\lim_{\tau\to 0^+} d_2^H(d_1,\tau,n)=+\infty, \quad \lim_{\tau\to+\infty} d_2^H(d_1,\tau,n)=+\infty,$$

and $d_2^H(d_1,\tau,n)$ is strictly decreasing for $0<\tau<\tau(d_1,n)$ and is strictly increasing for $\tau>\tau(d_1,n)$, where $\tau(d_1,n)$ is defined in (2.11).

(II) For fixed d_1 and τ, we define

$$d_2^H(p)=\frac{2\ell^2}{\tau u_* p}\left(1+\frac{\tau d_1 p}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1 p}\right)^2, \ p>0,$$

and we have

(i) there must be at least one $p_\star>0$ and at most three $p_\star>0$ satisfying $-1+\frac{\tau d_1 p}{\ell^2}+\tau A+\frac{\tau B\ell^2}{\ell^2+d_1p}+\frac{2\tau B\ell^2 d_1 p}{(\ell^2+d_1p)^2}=0$ if one of the following conditions holds: $A\leq 1$; $A>1$ and $B\leq -\frac{(A+1)^2}{4}$; $A>1$, $B>-\frac{(A+1)^2}{4}$ and $0<\tau\leq\frac{1}{1+A-2\sqrt{-B}}$;

(ii) there must be at least three $p_\star>0$ and at most five $p_\star>0$ satisfying $\frac{dd_2^H(p_\star)}{dp}=0$ if $A>1$, $B>-\frac{(A+1)^2}{4}$ and $\tau>\frac{1}{1+A-2\sqrt{-B}}$;

(iii) denote

$$d_2^\star=\min\left\{d_2^H\left(d_1,\tau,[\sqrt{p_\star}]\right), d_2^H\left(d_1,\tau,[\sqrt{p_\star}]+1\right)\right\}, \tag{2.17}$$

where $p_\star$ is mentioned above, and we have $\min\limits_{n\geq 1}\{d_2^H(d_1,\tau,n)\}=d_2^\star$ and $\lim\limits_{n\to+\infty} d_2^H(d_1,\tau,n)=+\infty$.

Proof. For fixed d_1 and $n\geq 1$, $d_2^H(d_1,\tau,n)$ is regarded as the function of τ. In addition, the partial derivative of $d_2^H(d_1,\tau,n)$ with respect to τ yields

$$\begin{aligned}&\frac{\partial d_2^H(d_1,\tau,n)}{\partial\tau}\\&=\frac{2\ell^2}{u_* n^2\tau^2}\left(1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}\right)\left(-1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}\right).\end{aligned}$$

Since $1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}=1+\frac{\tau\left(d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)\right)}{\ell^2(\ell^2+d_1n^2)}$ and $-1+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}=-1+\frac{\tau\left(d_1^2n^4+d_1n^2\ell^2(1-A)-\ell^4(A+B)\right)}{\ell^2(\ell^2+d_1n^2)}$, it follows that if one of (H1)–(H5) holds, then $\lim\limits_{\tau\to0^+} d_2^H(d_1,\tau,n)=+\infty$, $\lim\limits_{\tau\to+\infty} d_2^H(d_1,\tau,n)=+\infty$ and

$$\frac{\partial d_2^H(d_1,\tau,n)}{\partial\tau}\begin{cases}<0, & \text{if } 0<\tau<-\tau(d_1,n),\\ =0, & \text{if } \tau=-\tau(d_1,n),\\ >0, & \text{if } \tau>-\tau(d_1,n),\end{cases}$$

if one of (H6)–(H8) is satisfied, then $\lim\limits_{\tau\to0^+} d_2^H(d_1,\tau,n)=+\infty$, $\lim\limits_{\tau\to+\infty} d_2^H(d_1,\tau,n)=0$ and $\frac{\partial d_2^H(d_1,\tau,n)}{\partial\tau}=-\frac{2\ell^2}{u_*n^2\tau^2}<0$; if (H9) holds, then $\lim\limits_{\tau\to0^+} d_2^H(d_1,\tau,n)=+\infty$, $\lim\limits_{\tau\to+\infty} d_2^H(d_1,\tau,n)=+\infty$ and

$$\frac{\partial d_2^H(d_1,\tau,n)}{\partial\tau}\begin{cases}<0, & \text{if } 0<\tau<\tau(d_1,n),\\ =0, & \text{if } \tau=\tau(d_1,n),\\ >0, & \text{if } \tau>\tau(d_1,n).\end{cases}$$

For fixed d_1 and τ, it is easy to obtain the following partial derivative of $d_2^H(p)$ with respect to p

$$\begin{aligned}&\frac{dd_2^H(p)}{dp}\\ &=\frac{2\ell^2}{\tau u_*p^2}\left(1+\frac{\tau d_1p}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1p}\right)\\ &\quad\times\left(-1+\frac{\tau d_1p}{\ell^2}+\tau A+\frac{\tau B\ell^2}{\ell^2+d_1p}+\frac{2\tau B\ell^2d_1p}{(\ell^2+d_1p)^2}\right).\end{aligned}$$

Note that $1+\frac{\tau d_1p}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1p}=\frac{\tau d_1^2p^2+d_1p\ell^2(\tau-\tau A+1)+\ell^4(1-\tau A-\tau B)}{\ell^2(\ell^2+d_1p)}$ and $-1+\frac{\tau d_1p}{\ell^2}+\tau A+\frac{\tau B\ell^2}{\ell^2+d_1p}+\frac{2\tau B\ell^2d_1p}{(\ell^2+d_1p)^2}=\frac{\tau d_1^3p^3+d_1^2p^2\ell^2(2\tau+\tau A-1)+d_1p\ell^4(\tau+2\tau A+3\tau B-2)+\ell^6(\tau A+\tau B-1)}{\ell^2(\ell^2+d_1p)^2}$. Since $\tau A+\tau B-1<0$, it follows that $-1+\frac{\tau d_1p}{\ell^2}+\tau A+\frac{\tau B\ell^2}{\ell^2+d_1p}+\frac{2\tau B\ell^2d_1p}{(\ell^2+d_1p)^2}=0$ has at least one positive real root and has at most three positive real roots. In addition, $1+\frac{\tau d_1p}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1p}\geq0$ for any $p>0$ if one of the following conditions holding: $A\leq1$; $A>1$ and $B\leq-\frac{(A+1)^2}{4}$; $A>1$, $B>-\frac{(A+1)^2}{4}$ and $0<\tau\leq\frac{1}{1+A-2\sqrt{-B}}$; while $1+\frac{\tau d_1p}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1p}=0$ has two different positive real toots if $A>1$, $B>-\frac{(A+1)^2}{4}$ and $\tau>\frac{1}{1+A-2\sqrt{-B}}$. This completes the proof. □

Next, we use Lemmas 2.2 and 2.3 and establish the following theorem for the strong kernel.

Theorem 2.2. *Suppose that* $v(x,t)=\int_{-\infty}^t\frac{t-s}{\tau^2}e^{-\frac{t-s}{\tau}}u(x,s)ds$ *and* $A+B<0$. *Let* $d_2^T(d_1,n)$, $\tau(d_1,n)$, $d_2^H(d_1,\tau,n)$, $d_{1,n}^*$ *and* $d_2^\star$ *be denoted by* (2.5), (2.11), (2.16), (2.9) *and* (2.17), *respectively.*

(I) If $A\leq0$, *then* u_* *is asymptotically stable for* $-\frac{d_1}{u_*}\leq d_2<d_2^\star$ *and is unstable for* $d_2<-\frac{d_1}{u_*}$ *or* $d_2>d_2^\star$, *and mode-n Hopf bifurcation occurs at*

$d_2 = d_2^H(d_1,\tau,n)$; moreover, double Hopf bifurcation generated by mode-n and mode-m Hopf bifurcations occurs at $d_2 = d_2^H(d_1,\tau,n)$ provided that $d_2^H(d_1,\tau,n) = d_2^H(d_1,\tau,m)$ for some $n,m \in \mathbb{N}$ satisfying $n \neq m$.

(II) If $0 < A \le 1$ or $A > 1$ and $B \le -\frac{(A+1)^2}{4}$, then

- *(i) if $d_1 > d_{1,1}^*$, then u_* is asymptotically stable for $d_2^T(d_1,1) < d_2 < d_2^\star$ and is unstable for $d_2 < d_2^T(d_1,1)$ or $d_2 > d_2^\star$, and mode-1 Turing bifurcation occurs at $d_2 = d_2^T(d_1,1)$;*
- *(ii) if $d_{1,N}^* < d_1 < d_{1,N-1}^*$ for some positive integer $N \ge 2$, then u_* is asymptotically stable for $d_2^T(d_1,N) < d_2 < d_2^\star$ and is unstable for $d_2 < d_2^T(d_1,N)$ or $d_2 > d_2^\star$, and mode-N Turing bifurcation occurs at $d_2 = d_2^T(d_1,N)$;*
- *(iii) if $d_1 = d_{1,N}^*$ for some positive integer $N \ge 1$, then u_* is asymptotically stable for $d_2^T(d_1,N) < d_2 < d_2^\star$ and is unstable for $d_2 < d_2^T(d_1,N)$ or $d_2 > d_2^\star$, and double Turing bifurcation generated by mode-N and mode-$N+1$ Turing bifurcations occurs at $d_2 = d_2^T(d_1,N)$;*
- *(iv) when $d_2 \ge d_2^\star$, Hopf bifurcation occurs at $d_2 = d_2^H(d_1,\tau,n)$; moreover, double Hopf bifurcation generated by mode-n and mode-m Hopf bifurcations occurs at $d_2 = d_2^H(d_1,\tau,n)$ provided that $d_2^H(d_1,\tau,n) = d_2^H(d_1,\tau,m)$ for some $n,m \in \mathbb{N}$ satisfying $n \neq m$.*

(III) If $A > 1$ and $B > -\frac{(A+1)^2}{4}$, then

- *(i) if $d_1 > d_{1,1}^*$, then u_* is unstable for $d_2 < d_2^T(d_1,1)$ or $d_2 > d_2^\star$; if further $0 < d_1 \le \widehat{d}_{1,1}$ or $d_1 \ge \widetilde{d}_{1,1}$, then mode-1 Turing bifurcation occurs at $d_2 = d_2^T(d_1,1)$, while if further $\widehat{d}_{1,1} < d_1 < \widetilde{d}_{1,1}$, then mode-1 Turing bifurcation occurs at $d_2 = d_2^T(d_1,1)$ provided that $\tau \neq \frac{\tau(d_1,1)}{2}$;*
- *(ii) if $d_{1,N}^* < d_1 < d_{1,N-1}^*$ for some positive integer $N \ge 2$, then u_* is unstable for $d_2 < d_2^T(d_1,N)$ or $d_2 > d_2^\star$; if further $0 < d_1 \le \widehat{d}_{1,N}$ or $d_1 \ge \widetilde{d}_{1,N}$, then mode-$N$ Turing bifurcation occurs at $d_2 = d_2^T(d_1,N)$, while if further $\widehat{d}_{1,N} < d_1 < \widetilde{d}_{1,N}$, mode-$N$ Turing bifurcation occurs at $d_2 = d_2^T(d_1,N)$ provided that $\tau \neq \frac{\tau(d_1,N)}{2}$;*
- *(iii) if $d_1 = d_{1,N}^*$ for some positive integer $N \ge 1$, then u_* is unstable for $d_2 < d_2^T(d_1,N)$ or $d_2 > d_2^\star$; if further $0 < d_1 \le \widehat{d}_{1,N+1}$ or $d_1 \ge \widetilde{d}_{1,N}$, then double Turing bifurcation generated by mode-N and mode-$N+1$ Turing bifurcations occurs at $d_2 = d_2^T(d_1,N)$, while if further $\widehat{d}_{1,N} < d_1 < \widetilde{d}_{1,N}$ (or $\widehat{d}_{1,N+1} < d_1 < \widetilde{d}_{1,N+1}$), then mode-$N$ (or mode-$N+1$) Turing bifurcation occurs at $d_2 = d_2^T(d_1,N)$ provided that $\tau \neq \frac{\tau(d_1,N)}{2}$ (or $\tau \neq \frac{\tau(d_1,N+1)}{2}$);*
- *(iv) when $d_2 \ge d_2^\star$, we have*
  - *(a) if further $0 < d_1 \le \widehat{d}_{1,n}$ or $d_1 \ge \widetilde{d}_{1,n}$ for some $n \ge 1$, then Hopf bifurcation occurs at $d_2 = d_2^H(d_1,\tau,n)$;*
  - *(b) if further $\widehat{d}_{1,n} < d_1 < \widetilde{d}_{1,n}$ for some $n \ge 1$, then Hopf bifurcation occurs at $d_2 = d_2^H(d_1,\tau,n)$ provided that $0 < \tau < \frac{\tau(d_1,n)}{2}$.*

Proof. In (2.15), $\sigma_n^2 > 0$ if and only if $\frac{1}{2} + \frac{\tau d_1 n^2}{\ell^2} - \tau A - \frac{\tau B \ell^2}{\ell^2 + d_1 n^2} > 0$. For any $n \ge 1$, Eq. (2.14) has a single eigenvalue at $d_2 = d_2^T(d_1,n)$ provided that

$\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}\neq 0$, while Eq. (2.4) has a pair of purely imaginary roots if and only if $d_2=d_2^H(d_1,\tau,n)$ and $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}>0$. We can use similar arguments in the proof of Lemma 2.2 to determine the sign of $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}$:

(A1) if one of (H1)–(H5) of Lemma 2.2 holds, then $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}>0$;

(A2) if one of (H6)–(H8) of Lemma 2.2 holds, then $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}=\frac{1}{2}>0$;

(A3) if (H9) of Lemma 2.2 holds, then $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}>0$ if and only if $\tau<\frac{\tau(d_1,n)}{2}$.

Regarding d_2 as a bifurcation parameter and noting that $\mu(d_2)=0$ for $d_2=d_2^T(d_1,n)$, it follows that for $\frac{1}{2}+\frac{\tau d_1 n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}\neq 0$,

$$\left.\frac{d\mu(d_2)}{dd_2}\right|_{d_2=d_2^T(d_1,n)}=-\frac{u_*n^2(\ell^2+d_1n^2)}{(\ell^2+d_1n^2)(\ell^2+2\tau d_1n^2-2\tau A\ell^2)-2\tau B\ell^4}\neq 0. \tag{2.18}$$

Regarding d_2 as a bifurcation parameter and noting that $\mu(d_2)=\pm i\sigma_n$ for $d_2=d_2^H(d_1,\tau,n)$, it follows that

$$\left.\frac{d\mathrm{Re}\mu(d_2)}{dd_2}\right|_{d_2=d_2^H(d_1,\tau,n)}=\frac{u_*n^2}{2\ell^2\left(\tau^2\sigma_n^2+\left(2+\tau\frac{d_1n^2}{\ell^2}-\tau A-\frac{\tau B\ell^2}{\ell^2+d_1n^2}\right)^2\right)}, \tag{2.19}$$

which is positive.

Using (iii) of (I) of Lemma 2.1, (II) of Lemma 2.1, Lemma 2.3 and the transversality condition (2.19) allows us to conclude (I). When $0<A\le 1$ or $A>1$, we utilize similar arguments in the proof of Theorem 2.1, the transversality conditions (2.18)–(2.19) and Lemma 2.3 to obtain the statements of (II) and (i)–(iii) of (III). We next concentrate on $A>1$ and $B>-\frac{(A+1)^2}{4}$. Recall that $\widehat{d}_{1,n}>\widehat{d}_{1,n+1}$ and $\widetilde{d}_{1,n}>\widetilde{d}_{1,n+1}$ for each $n\ge 1$. Then, the transversality condition (2.19), along with Lemma 2.3, demonstrates (iv) of (III). □

Remark 2.1. From the proof of Lemma 2.1, the following statements hold for either temporal kernel: if $0<A\le 1$ or $A>1$ and $B<-\frac{(A+1)^2}{4}$, then Turing bifurcation is possible to occur for $d_2<0$; if $A>1$ and $B=-\frac{(A+1)^2}{4}$, then Turing bifurcation can emerge for $d_2\le 0$; if $A>1$ and $B>-\frac{(A+1)^2}{4}$, then Turing bifurcation may appear for $d_2\in\mathbb{R}$.

3. Application

Based on the study of [1], we set $g(u,h)=u\left(1+\alpha u-(1+\alpha)h\right)$ with $\alpha>0$ and model (1.1) becomes

$$\begin{cases} u_t=d_1u_{xx}+d_2(uv_x)_x+u\left(1+\alpha u-(1+\alpha)h\right), 0<x<\ell\pi,\ t>0,\\ u_x(0,t)=u_x(\ell\pi,t)=0,\ t>0. \end{cases} \tag{3.1}$$

We focus on the positive steady state $u_\star=1$. Since $A=\alpha>0$ and $B=-(1+\alpha)<0$, it is obvious that $A+B=-1<0$. Then we set $\ell=2$ and apply finite difference method for numerical exploration.

Example 3.1. Suppose $\ell = 2$, $\alpha = 4$, $d_1 = 2$ and $v(x,t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(x,s)ds$. Then, $u_\star = 1$ is asymptotically stable for $d_2 > d_2^T(d_1, 1)$ and $0 < \tau < \tau(d_1, 2)$ and Turing-Hopf bifurcation generated by mode-1 Turing and mode-2 Hopf bifurcations occurs at $(\tau, d_2) = \big(\tau(d_1, 2), d_2^T(d_1, 1)\big)$.

Proof. Since $\ell = 2$, $\alpha = 4$ and $d_1 = 2$, we have $B > -\frac{(A+1)^2}{4}$, $d_1 > d_{1,1}^*$ and $\widehat{d}_{1,1} < d_1 < \widetilde{d}_{1,1}$, where $B = -5$, $-\frac{(A+1)^2}{4} = -6.25$, $d_{1,1}^* = 1.8042$, $\widehat{d}_{1,1} = 1.5279$ and $\widetilde{d}_{1,1} = 10.4721$. Moreover, we numerically calculate $d_2^T(d_1, 1) = 0.6667$, $d_2^T(d_1, 2) = 0.3333$, $\sqrt{\frac{\ell^2\left(A-1-\sqrt{(A+1)^2+4B}\right)}{2d_1}} = 0.8740$, $\sqrt{\frac{\ell^2\left(A-1+\sqrt{(A+1)^2+4B}\right)}{2d_1}} = 2.2882$, $\tau(d_1, 1) = 6$ and $\tau(d_1, 2) = 3$. Then we make use of Theorem 2.1 to complete the proof. $\square$

The statements of Example 3.1 are depicted in Figure 1 and it is displayed in Figure 2 that a stable constant steady state $u_\star = 1$ and a mode-2 periodic solution appear for P1(2.5, 1) and P2(3.45, 1), respectively.

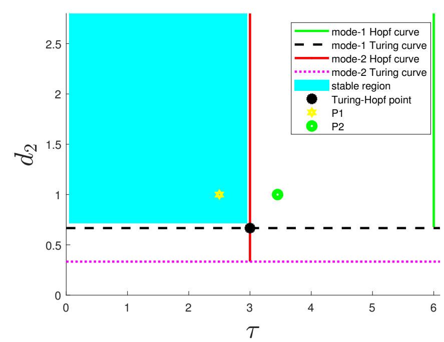

Figure 1. The stable region of $u_\star = 1$ for Example 3.1.

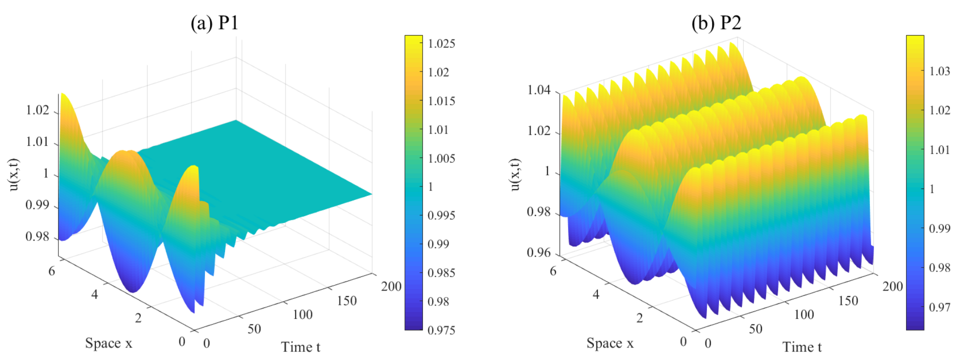

Figure 2. The solutions for P1(2.5, 1) and P2(3.45, 1), with the initial condition being $1 - 0.02 \cos x$.

Example 3.2. Suppose that $\ell = 2$, $\alpha = 1$ and $d_1 = 3$.

(I) For $v(x,t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(x,s)ds$, $u_\star = 1$ is asymptotically stable for $d_2 > d_2^T(d_1, 2)$ and is unstable for $d_2 < d_2^T(d_1, 2)$, and mode-2 Turing bifurcation occurs at $d_2 = d_2^T(d_1, 2)$.

(II) For $v(x,t) = \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} u(x,s)ds$ and fixed τ, $u_\star = 1$ is asymptotically stable for $d_2^T(d_1, 2) < d_2 < d_2^\star$ and is unstable for $d_2 < d_2^T(d_1, 2)$ or $d_2 > d_2^\star$, and mode-2 Turing and mode-n Hopf bifurcations occur at $d_2 = d_2^T(d_1, 2)$ and $d_2^H(d_1, \tau, n)$ with $n \geq 1$, respectively.

Proof. Using $\ell = 2$, $\alpha = 1$ and $d_1 = 3$, we have $A = 1$ and $d_{1,2}^* < d_1 < d_{1,1}^*$, where $d_{1,1}^* = 5.7016$, $d_{1,2}^* = 1.7051$. Besides, we numerically calculate $d_2^T(d_1, 2) = -2.5$. Then (I) and (II) follow from Theorems 2.1 and 2.2, respectively. $\square$

The statements of Example 3.2 are displayed in Figure 3, where it is shown that when $v(x,t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(x,s)ds$, $u_\star = 1$ is asymptotically stable for $d_2 \geq 0$ (see Figure 3 (a)), and when $v(x,t) = \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} u(x,s)ds$, there are infinitely many Hopf bifurcation curves for $d_2 > 0$, double Hopf bifurcation generated by mode-n and mode-m ($n \neq m$) Hopf bifurcations is obtained and stability switches emerge as τ varies for fixed $d_2 > d_2^H(d_1, -\tau(d_1, 2), 2)$ (see Figure 3 (b)). It is described in Figure 4 that system (3.1) has a mode-2 spatially nonhomogeneous steady state, a stable constant steady state $u_\star = 1$, a mode-1 periodic solution and a mode-2 periodic solution for P3$(2, -2.5)$, P4$(2, 26)$, P5$(2, 31.1)$ and P6$(1, 26)$, respectively.

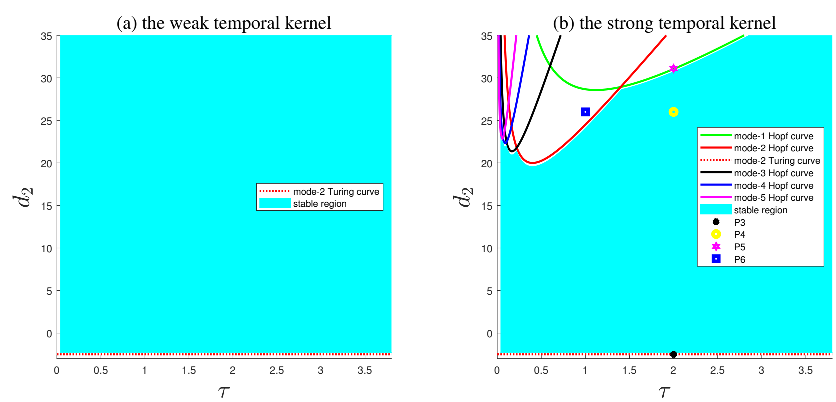

Figure 3. The stable region of $u_\star = 1$ for Example 3.2.

4. Discussion

In [6], a single population model with distributed memory-based diffusion is proposed. It is shown that the stability of the spatially homogeneous steady state are not affected by the weak temporal kernel and Hopf and double Hopf bifurcations emerge in the case of the strong temporal kernel. In this work, we propose a diffusive single-species model with nonlocality and distributed memory, in which the nonlocal term $h(x,t) = \int_0^{\ell\pi} K(x,y)u(y,t)dy$ represents nonlocal intraspecific competition of the species. Here, the kernel function $K(x,y)$ is the Green function of $-d_1 K_{xx} + I$ according to [2, 4, 7, 8]. As for the distributed memory, we take weak and strong temporal kernels into consideration.

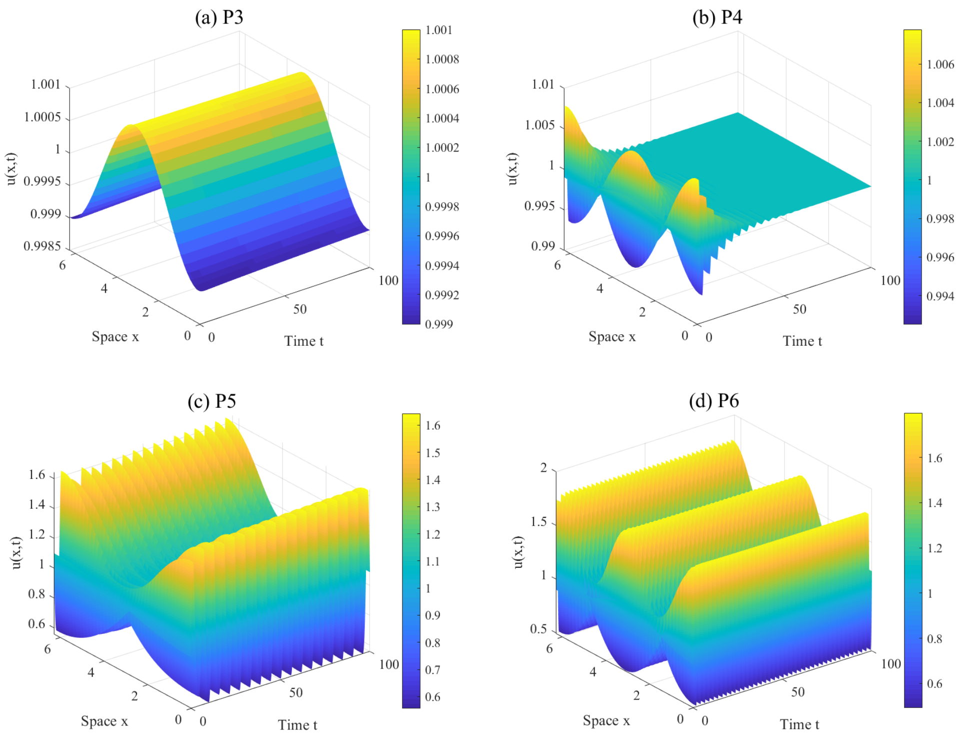

Figure 4. The solutions for P3$(2, -2.5)$, P4$(2, 26)$, P5$(2, 31.1)$ and P6$(1, 26)$, with the initial condition being $1 - 0.001 \cos x$, $1 - 0.001 \cos x$, $1 - 0.1 \cos 0.5x$ and $1 - 0.1 \cos x$, respectively.

When $A \leq 0$, the memory delay has no influence on the stability of the spatially homogeneous steady state for the weak temporal kernel, while large repulsive memory-based diffusion can induce Hopf and double Hopf bifurcations for the strong temporal kernel. When either $0 < A \leq 1$, or $A > 1$ and $B \leq -\frac{(A+1)^2}{4}$, the existence of Turing and double Turing bifurcations can be obtained by a consideration of large attractive memory-based diffusion for the weak temporal kernel, while for the strong temporal kernel, the existence of Turing and Hopf bifurcations can be obtained by considerations of large attractive and repulsive memory-based diffusion, respectively. When $A > 1$ and $B > -\frac{(A+1)^2}{4}$, Turing and Hopf bifurcations may be obtained by considering either kernel, and Hopf bifurcation is possible to emerge for small random diffusion or large repulsive memory-based diffusion by taking into account the weak or strong kernels, respectively.

A very interesting application to our theoretical findings is conducted in model (3.1), where the biological interpretation of the nonlocal term is presented in [1]. By considering the weak temporal kernel, Turing-Hopf bifurcation generated by mode-1 Turing and mode-2 Hopf bifurcations is obtained, while by considering the strong temporal kernel, stability switches and double Hopf bifurcation generated by mode-n and mode-m ($n \neq m$) Hopf bifurcations are observed.

The distributed memory-based diffusion model without nonlocality can not undergo Turing, Turing-Hopf and double Turing bifurcations [6]. In this paper, nonlocality and distributed memory may cause Hopf or Turing bifurcations, generating spatially nonhomogeneous periodic solutions or steady states, respectively. What's more, nonlocal spatial average can give rise to Turing bifurcation with the critical bifurcation curve being mode-1. In this work, the critical Turing bifurcation curve can be mode-n ($n \geq 2$) due to the nonlocal term $h(x,t)$ and distributed memory.

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References

[1] N. Britton, *Aggregation and the competitive exclusion principle*, J. Theor. Biol., 1989, 136(1), 57–66.

[2] S. Chen and J. Shi, *Global attractivity of equilibrium in gierer-meinhardt system with activator production saturation and gene expression time delays*, Nonlinear Anal.-Real World Appl., 2013, 14(4), 1871–1886.

[3] W. F. Fagan, M. A. Lewis, M. Auger-Méthé and T. Avgar, *Spatial memory and animal movement*, Ecology Letters, 2013, 16(10), 1316–1329.

[4] H. Hochstadt, *Integral Equations*, Wiley Interscience, New York, 1973.

[5] Q. Ji, R. Wu and T. Zhang, *Dynamics of a single population model with memory and nonlinear boundary condition*, Appl. Anal., 2024. DOI: 10.1080/00036811.2024.2426229.

[6] J. Lin and Y. Song, *Spatially inhomogeneous periodic patterns induced by distributed memory in the memory-based single population model*, Appl. Math. Lett., 2023, 137, 108490.

[7] W. Ni, J. Shi and M. Wang, *Global stability and pattern formation in a nonlocal diffusive Lotka-Volterra competition model*, J. Differ. Equ., 2018, 264(11), 6891–6932.

[8] F. Riez and B. Sz-Nagy, *Functional Analysis*, Ungar, New York, 1955.

[9] U. E. Schläegel and M. A. Lewis, *Detecting effects of spatial memory and dynamic information on animal movement decisions*, Methods Ecol. Evol., 2014, 5(11), 1236–1246.

[10] H. Shen and Y. Song, *Spatiotemporal patterns in a diffusive resource-consumer model with distributed memory and maturation delay*, Math. Comput. Simul., 2024, 221, 622–644.

[11] H. Shen, Y. Song and H. Wang, *Bifurcations in a diffusive resource-consumer model with distributed memory*, J. Differ. Equ., 2023, 347, 170–211.

[12] J. Shi and Q. Shi, *Spatial movement with temporally distributed memory and dirichlet boundary condition*, J. Differ. Equ., 2024, 389, 305–337.

[13] J. Shi, C. Wang and H. Wang, *Diffusive spatial movement with memory and maturation delays*, Nonlinearity, 2019, 32(9), 3188–3208.

[14] J. Shi, C. Wang, H. Wang and X. Yan, *Diffusive spatial movement with memory*, J. Dynam. Differential Equations, 2020, 32(2), 979–1002.

[15] Q. Shi, J. Shi and H. Wang, *Spatial movement with distributed memory*, J. Math. Biol., 2021, 82(4), 33.

[16] Y. Song, S. Wu and H. Wang, *Memory-based movement with spatiotemporal distributed delays in diffusion and reaction*, Appl. Math. Comput., 2021, 404, 126254.

[17] D. Wang and C. Wang, *Dynamics of a diffusive two-species interaction model with memory effect*, Discrete Contin. Dyn. Syst.-Ser. B, 2024. DOI: 10.3934/dcdsb.2024187.

[18] S. Wu and Y. Song, *Spatial movement with distributed memory and maturation delay*, Qual. Theor. Dyn. Syst., 2024, 23(3), 117.

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