Journal of Applied Analysis and Computation
Volume 15, Number 6, December 2025, 3769–3781
Website:http://www.jaac-online.com
DOI:10.11948/20250061

POSITIVE RADIAL SOLUTIONS FOR A SEMIPOSITONE PROBLEM OF ELLIPTIC KIRCHHOFF EQUATIONS WITH SUBLINEAR NONLINEARITIES*

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Abstract We study the semipositone problem of the elliptic Kirchhoff type equation

$$\begin{cases} -\Big(b\displaystyle\int_{\Omega_e} |\nabla u|^2 dx\Big)\Delta u = \lambda K(|x|)f(u), & x \in B_e, \\ u(x) = 0, & |x| = r_0, \\ u(x) \to 0, & |x| \to \infty, \end{cases} \tag{0.1}$$

where b is a positive constant, λ is a positive parameter, $B_e = \{x \in \mathbb{R}^N : |x| > r_0\}$, $N > 2$, $K : [r_0, +\infty) \to (0, +\infty)$ is continuous with $r^{N+\eta}K(r)$ bounded for some $\eta > 0$, $f : [0, +\infty) \to \mathbb{R}$ is continuous, $f(0) < 0$ and $\lim\limits_{u\to\infty} \frac{f(u)}{u^q} = \beta$ for some $q \in (0, 1]$. We show that there exists $\lambda^* > 0$, such that (0.1) has at least one positive radial solution if $\lambda > \lambda^*$. The proof of the main result is based upon bifurcation theory.

Keywords Kirchhoff equation, semipositone problem, positive radial solution, bifurcation.

MSC(2010) 34B10, 34B15, 34B18, 34C23, 35B09.

1. Introduction

In this paper, we are concerned with the existence of positive radial solutions to the Kirchhoff type problem

$$\begin{cases} -\Big(b\displaystyle\int_{\Omega_e} |\nabla u|^2 dx\Big)\Delta u = \lambda K(|x|)f(u), & x \in B_e, \\ u(x) = 0, & |x| = r_0, \\ u(x) \to 0, & |x| \to \infty, \end{cases} \tag{1.1}$$

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where b is a positive constant, λ is a positive parameter, $B_e = \{x \in \mathbb{R}^N : |x| > r_0\}$, $N > 2$, $K : [r_0, +\infty) \to (0, +\infty)$ is continuous with $r^{N+\eta}K(r)$ bounded for some $\eta > 0$, $f : [0, +\infty) \to \mathbb{R}$ is a continuous function.

Equation in (1.1) is the stationary case of a nonlinear wave equation

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda \hat{f}(x, u), \qquad x \in \Omega,$$

which was first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string, where u denotes the displacement, $\hat{f}$ is the external force, b represents the initial tension, and a is related to the intrinsic properties of the string.

Existence and multiplicity of positive solutions of the Kirchhoff type equation

$$\begin{cases} \Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda \hat{f}(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

and its more general problems with $\hat{f}(x, s) \geq 0$ for $(x, s) \in \Omega \times [0, \infty)$, have been extensively studied. When Ω is a bounded domain in $\mathbb{R}^N$, we refer to Perera and Zhang [20], Figueiredo etc [11], Liang, Li and Shi [15], Ambrosetti and Arcoya [3], Silva et al. [22], Shibata [21], Cao and Dai [6] and the references therein. When Ω is an exterior domain in $\mathbb{R}^3$, see Dai, Ou and Tang [8], Figueiredo and de Morais Filho [10], Wang, Yuan and Zhang [23], Ye, Yu and Tang [24] for references along this line.

For example, Figueiredo etc [11] considered the Kirchhoff type equation

$$\begin{cases} -\Big(\tilde{a}(x) + \tilde{b}(x)\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \lambda u^q, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < q < 1$, $\tilde{a}, \tilde{b} \in C^\gamma(\bar{\Omega}), \gamma \in (0, 1)$ and $\tilde{a}(x) \geq a_0 > 0$, $\tilde{b} \geq 0$. They obtained

Theorem A. (Theorem 5.1, [11]) *The value $\lambda = 0$ is the only bifurcation point from the trivial solution for* (1.3). *Moreover, there exists a continuum $\mathcal{C}_0$ of positive solutions of* (1.3) *unbounded in $\mathbb{R} \times C(\bar{\Omega})$ emanating from* $(0, 0)$.

All the results mentioned above are dependent on the nonlinearities are nonnegative. However, Lions [16] proposed that it is challenging to study the existence of solutions for semipositone problems (i.e., $\hat{f}(x, 0) < 0$ for $x \in \Omega$). In the case of $b = 0$ in (1.2), that is, (1.2) is a local problem, for which the local semipositone problem has been studied by serval authors, see Lions [16], Ali, Castro and Shivaji [1], Ambrosetti and Arcoya [4], Hai and Shivaji [14], Ma [18]. However, to our best knowledge, few results on the existence of solutions of nonlocal semipositone problems (i.e., $b > 0$ and $\hat{f}(x, 0) < 0$ for $x \in \Omega$ in (1.2)).

Recently, Graef etc [12] dealt with the nonlocal semipositone problems

$$\begin{cases} -\Big(a + b\int_{B_e} |\nabla u|^2 dx\Big)\Delta u = \lambda g(|x|, u), & x \in B_e, \\ u(x) = 0, & x \in \partial B_e. \end{cases} \tag{1.4}$$

They used variational method to prove that

Theorem B. (Theorem 1.5, [12]) *Assume the following*

$(F1)$ *there exist continuous functions* $A, B : [r_0, \infty) \to (0, \infty)$ *with* $q > 3$ *and* $\mu \in (0, N-2)$ *such that*

$$A(\xi)(t^q - 1) \leq g(\xi, t) \leq B(\xi)(t^q + 1) \quad \text{for all} \quad (\xi, t) \in [r_0, +\infty) \times [0, +\infty),$$

where $A(\xi), B(\xi) \leq \frac{1}{\xi^{N+\mu}}$ *for* $\xi \gg 1$.

$(F2)$ *for all* $\xi \in [r_0, +\infty)$, $g(\xi, 0) < 0$.

$(F3)$ *there exists* $\theta > 4$ *such that, for all sufficiently large* t,

$$tg(\xi, t) > \theta G(\xi, t) \quad \text{for all} \quad \xi \geq r_0,$$

where $G(\xi, t) = \int_0^t g(\xi, \sigma) d\sigma$.

Then there is λ_*, such that problem (1.4) has a positive weak solution for $\lambda \in (0, \lambda_*)$.

However, no any information about the global behavior of positive solutions of (1.4). Motivated by the above papers, we consider the existence of positive solutions when the nonnegative nonlinearity in Figueiredo etc [11] becomes semipositone and $q > 3$ in Graef etc [12, (F1)] is replaced by $0 < q \leq 1$.

It is the purpose to study the existence of positive radial solutions of (1.1) in semipositone case with sublinear growth nonlinearities (i.e., $0 < q \leq 1$) on exterior domains via bifurcation theory.

Assume that $K : [r_0, +\infty) \to (0, +\infty)$ and $f : [0, +\infty) \to \mathbb{R}$ are continuous and satisfy

(K) $r^{N+\eta} K(r)$ bounded for some $\eta \in (0, +\infty)$ satisfying $\frac{2-N+\eta}{N-2} \in (-1, 0)$.

(f_1) $f(0) < 0, \quad \forall\, x \in \bar{B}_e$.

(f_2) $\exists\, \beta \in (0, +\infty)$, such that

$$\lim_{u \to \infty} \frac{f(u)}{u^q} = \beta \qquad \text{uniformly in } x \in \bar{B}_e,$$

with $0 < q \leq 1$.

The main result of this paper is as follows.

Theorem 1.1. *Let* (K), (f_1) *and* (f_2). *Then there exists* $\lambda^* > 0$ *such that* (1.1) *has at least one positive radial solution for all* $\lambda \geq \lambda^*$. *More precisely, there exists a connected set of positive radial solutions of* (1.1) *bifurcating from infinity for* $\lambda_\infty = +\infty$.

Let $I = (0, 1)$. We denote by Y the Sobolev space $H_0^1(I)$ with the inner product $(u, v) = \int_0^1 u' \cdot v'$ and norm $||u||^2 = \int_0^1 |u'|^2$, by Y^* the duality space of Y, by $\rightharpoonup$ the weak convergence in Y, and by $\langle \cdot, \cdot \rangle$ the duality pairing between Y^* and Y. Let $P = \{u \in Y : u(x) \geq 0, \text{ a.e. } x \in I\}$ be the positive cone in Y and let $P^* = \{h \in Y^* : \langle h, u \rangle \geq 0, u \in P\}$ be its dual cone.

The rest of the paper is arranged as follows: In Section 2, we transfer (1.1) into a singular two-point boundary value problem by a radial transformation. Section 3 is devoted to study a nonlocal eigenvalue problem with singular weight which will be using in computing the fixed point index in Section 4. Finally in Section 4, we will prove our main results on the existence of positive radial solutions for the semipositone problems (1.1) with sublinear nonlinearity.

2. Changes (1.1) to a singular two-point boundary value problem

Let $r=|x|$, we have

$$\int_{\Omega_e}|\nabla u|^2dx = N\omega_N\int_{r_0}^{\infty} s^{N-1}|u'(s)|^2ds, \tag{2.1}$$

where $\omega_N=\frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$ is the valume of unit ball in $\mathbb{R}^N$. And then (1.1) becomes

$$\begin{cases} -\Big(bN\omega_N\displaystyle\int_{r_0}^{\infty} s^{N-1}|u'(s)|^2ds\Big)(u''+\dfrac{N-1}{r}u')=\lambda K(r)f(u(r),\ r\in(r_0,\infty),\\ u(r_0)=u(\infty)=0. \end{cases} \tag{2.2}$$

Let $t=\left(\frac{r}{r_0}\right)^{2-N}$ and $v(t)=u(r(t))$, that is $r=r_0t^{\frac{1}{2-N}}$, we have

$$\begin{aligned} u'(r)&=v'(t)\frac{dt}{dr}=\frac{2-N}{r_0}(\frac{r}{r_0})^{1-N}v'(t),\\ u''(r)&=(\frac{2-N}{r_0})^2(\frac{r}{r_0})^{2(1-N)}v''(t)+\frac{(2-N)(1-N)}{r_0^2}(\frac{r}{r_0})^{-N}v'(t). \end{aligned}$$

Then

$$u''(r)+\frac{N-1}{r}u'(r)=(\frac{2-N}{r_0})^2(\frac{r}{r_0})^{2(1-N)}v''(t),$$

and

$$\begin{aligned} &\int_{r_0}^{\infty} r^{N-1}|u'(r)|^2dr\\ =&\int_1^0 (r_0t^{\frac{1}{2-N}})^{N-1}|\frac{2-N}{r_0}(\frac{r}{r_0})^{1-N}v'(t)|^2d(r_0t^{\frac{1}{2-N}})\\ =&\int_1^0 r_0^{N-1}t^{\frac{N-1}{2-N}}\frac{(2-N)^2}{r_0^2}(\frac{r}{r_0})^{2(1-N)}|v'(t)|^2\cdot\frac{r_0}{2-N}t^{\frac{N-1}{2-N}}dt\\ =&\int_0^1 r_0^{N-1}t^{\frac{N-1}{2-N}}\frac{(2-N)^2}{r_0^2}t^{\frac{2-2N}{2-N}}|v'(t)|^2\cdot\frac{r_0}{N-2}t^{\frac{N-1}{2-N}}dt\\ =&r_0^{N-2}(N-2)\int_0^1|v'(t)|^2dt. \end{aligned}$$

Thus, (2.2) can be rewritten as

$$\begin{cases} -\Big(\alpha\displaystyle\int_0^1|v'(t)|^2dt\Big)(\dfrac{2-N}{r_0})^2t^{\frac{2-2N}{2-N}}v''=\lambda K(r_0t^{\frac{1}{2-N}})f(v(t)),\ t\in(0,1),\\ v(0)=v(1)=0, \end{cases} \tag{2.3}$$

i.e.,

$$\begin{cases} -\Big(\alpha\displaystyle\int_0^1|v'(t)|^2dt\Big)v''=\lambda(\dfrac{r_0}{N-2})^2t^{\frac{-2(N-1)}{N-2}}K(r_0t^{\frac{1}{2-N}})f(v(t)),\ t\in(0,1),\\ v(0)=v(1)=0, \end{cases} \tag{2.4}$$

where $\alpha = bN\omega_N r_0^{N-2}(N-2)$. Taking $h(t) = (\frac{r_0}{N-2})^2 t^{\frac{-2(N-1)}{N-2}} K(r_0 t^{\frac{1}{2-N}})$, it follows from (2.4) that

$$\begin{cases} -\Big(\alpha \displaystyle\int_0^1 |v'(t)|^2 dt\Big) v'' = \lambda h(t) f(v(t)),\ t \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{2.5}$$

Note that $h : (0,1) \to (0,\infty)$ is continuous and could be singular at 0 if $\eta \in (0, N-2)$. In addition, condition (K) guarantees $h \in L^1_{\text{loc}}(0,1)$.

Remark 2.1. If $N = 3$, then

$$N\omega_N = 3 \cdot \frac{2\pi^{\frac{3}{2}}}{3\Gamma(\frac{N}{2})} = 3 \cdot \frac{2\pi^{\frac{3}{2}}}{3\frac{\pi^{\frac{1}{2}}}{2}} = 4\pi.$$

Specifically, let $r_0 = 1$ and $K(r) = \frac{1}{r^{N+\eta}}$, then we have

$$h(t) = (\frac{1}{N-2})^2 t^{\frac{-2(N-1)}{N-2}} K(t^{\frac{1}{2-N}}) = (\frac{1}{N-2})^2 t^{\frac{2-N+\eta}{N-2}}. \tag{2.6}$$

Obviously,

$$\frac{2-N+\eta}{N-2} \in (-1,0) \iff \eta \in (0, N-2). \tag{2.7}$$

(2.7) is coincident with Graef etc [12, (F1)], which implies that there exists $p \in (1,\infty)$ such that

$$\frac{2-N+\eta}{N-2} p \ \in \ (-1,0)$$

as

$$\eta \in (0,1).$$

3. Nonlocal eigenvalue problem

3.1. Asakawa theory

Throughout this paper, X denotes the Banach space defined by

$$X = \Big\{\phi \in L^1_{\text{loc}}(0,1) \Big| \int_0^1 t(1-t)|\phi(t)|dt < +\infty\Big\} \tag{3.1}$$

equipped with the norm

$$||\phi||_X = \int_0^1 t(1-t)|\phi(t)|dt. \tag{3.2}$$

We will also denote by X_+ the subset $\{\phi \in X |\ \phi(t) \geq 0 \text{ for a.e. } t \in (0,1)\}$.

For $\phi \in X$, define the function $\mathcal{L}[\phi](\cdot)$ by

$$\mathcal{L}[\phi](t) = (1-t)\int_0^t s\phi(s)ds + t\int_t^1 (1-s)\phi(s)ds, \quad t \in [0,1].$$

By Asakawa [5, Lemma 2.1(ii)], $\mathcal{L}[\phi] \in C[0,1]$, so that $\mathcal{L}$ is a linear operator from X to $C[0,1]$. It is clear that

$$||\mathcal{L}[\phi]||_\infty \leq ||\phi||_X, \quad \forall\, \phi \in X. \tag{3.3}$$

Thus the operator $\mathcal{L}$ is bounded. Set

$$u = \mathcal{L}[\phi].$$

Then u is a solution of the boundary value problem

$$\begin{cases} u''(t) + \phi(t) = 0 \quad \text{a.e. } t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.4}$$

Lemma 3.1 (Lemma 2.2, [5]). *Suppose that $\phi \in X$ and that $u = \mathcal{L}[\phi]$. Then $u \in AC[0,1] \cap C^1(0,1)$, $u' \in AC_{loc}(0,1)$ and u is a solution of the boundary value problem (3.4). In addition if $\phi \in L^1(0,1)$, then $u \in C^1[0,1]$ and $u' \in AC[0,1]$.*

Lemma 3.2 (Lemma 2.3, [5]). *For every $\psi \in X_+$, the subset K defined by*

$$K = \mathcal{L}(\{\phi \in X \big|\; |\phi(t)| \leq \psi(t) \text{ a.e. } t \in (0,1)\})$$

is precompact in $C[0,1]$.

For $b \in X$ and $\mu > 0$, define $r(\cdot; b, \mu)$ and $\theta(\cdot; b, \mu)$ by

$$\begin{aligned} r(t; b, \mu) &= \sqrt{\mu^2 y_0(t;b)^2 + y_0'(t;b)^2}, \qquad 0 \leq t \leq 1, \\ \theta(t; b, \mu) &= \int_0^1 \frac{\mu(y_0'(s;b)^2 + b(s) y_0(s;b)^2)}{r(s;b,\mu)^2} ds, \qquad 0 \leq t \leq 1. \end{aligned} \tag{3.5}$$

Lemma 3.3 (Lemma 4.2, [5]). *Suppose that $\mu > 0$ and $b \in X$. Then*

(i) $r(t;b,\mu)$, $\theta(t;b,\mu) \in AC_{loc}[0,1)$ and $r(t;b,\mu) > 0$ for every $t \in [0,1)$ and

$$\begin{aligned} \mu y_0(t;b) &= r(t;b,\mu)\sin(\theta(t;b,\mu)), \\ y_0'(t;b) &= r(t;b,\mu)\cos(\theta(t;b,\mu)), \end{aligned} \tag{3.6}$$

for every $t \in [0,1)$. In addition if $b \in X_+$, then $r(t;b,\mu)$ is a nondecreasing function on $[0,1)$.

(ii) If $\int_0^1 t|b(t)|dt < \infty$, then $r(t;b,\mu)$, $\theta(t;b,\mu) \in AC_{loc}[0,1]$, $r(t;b,\mu) > 0$ for every $t \in [0,1]$ and equalities (3.6) hold for every $t \in [0,1]$. In addition if $b \in X_+$, then $r(t;b,\mu)$ is a nondecreasing function on $[0,1]$.

Lemma 3.4 (Lemma 4.3, [5]). *Let $\mu > 0$ and let $\{q_n\}_{n\in\mathbb{N}} \subset X$. Suppose that $q_n \to q$ in X as $n \to +\infty$. Then*

$$\begin{aligned} & y_0(\cdot; q_n) \to y_0(\cdot; q) \text{ in } C[0,1] \text{ as } n \to +\infty, \\ & y_0'(\cdot; q_n) \to y_0'(\cdot; q) \text{ in } C[0,\alpha] \text{ as } n \to +\infty, \\ & \theta(\cdot; q_n, \mu) \to \theta(\cdot; q, \mu) \text{ in } C[0,\alpha] \text{ as } n \to +\infty, \end{aligned}$$

for every $\alpha \in (0,1)$. In addition; if $\int_0^1 t|q_n(t)|dt < +\infty$ for every $n \in \mathbb{N}$ and if $\int_0^1 t|q_n(t) - q(t)|dt \to 0$ as $n \to \infty$, then

$$\begin{aligned} & y_0'(\cdot; q_n) \to y_0'(\cdot; q) \text{ in } C[0,1] \text{ as } n \to +\infty, \\ & \theta(\cdot; q_n, \mu) \to \theta(\cdot; q, \mu) \text{ in } C[0,1] \text{ as } n \to +\infty. \end{aligned}$$

3.2. Nonlocal eigenvalue problem with singular weight

Let us consider the nonlocal eigenvalue problem

$$\begin{cases} -\alpha||u||^2u''(x) = \lambda h(x)u(x), \quad x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.7}$$

In order to show our results, we need to introduce some notation. Given a subinterval $D = [a_1, b_1] \subset (0,1)$ and a positive constant $A > 0$, we denote by $\lambda_1(-A; D)$ the principal eigenvalue of the problem

$$\begin{cases} -A\varphi'' = \lambda h(x)\varphi, \quad x \in (a_1, b_1), \\ \varphi(a_1) = \varphi(b_1) = 0, \end{cases} \tag{3.8}$$

where $h \in X$.

Proposition 3.1. *Let B be a positive constant, consider the map*

$$\lambda_1(\mu) = \lambda_1(-\mu B, D), \quad \mu \geq 0.$$

Then, $\lambda_1(\mu)$ is a continuous and increasing function and

$$\lim_{\mu\to+\infty} \lambda_1(\mu) = \infty.$$

Proof. According to Figueiredo etc [11], it follows that for any positive constants $B_1, B_2 > 0$ that satisfy $B_1 \leq B_2$, then

$$\lambda_1(-B_1; D) \leq \lambda_1(-B_2; D).$$

So $\lambda_1(\mu)$ is increasing with respect to μ.

Next, we show that $\lambda_1(-\mu B)$ is continuous in μ.

For $q_n, q \in X$ with $q_n \to q$ in X, and for fixed $m \in \mathbb{N}$, let

$$q_n^{[m]}(t) = \min\{q_n(t), m\}, \qquad q^{[m]}(t) = \min\{q(t), m\}, \qquad t \in [0,1].$$

Then

$$q_n^{[m]}, q \in L^1(0,1) \text{ and } q_n^{[m]} \to q^{[m]} \text{ in } L^1(0,1), \tag{3.9}$$

and

$$\lambda_1(q_n^{[m]}) \to \lambda_1(q^{[m]}) \quad \text{ as } n \to \infty, \tag{3.10}$$

see Meng, Yan and Zhang [19, Theorem 1.1].

According to the same argument in Asakawa [5, proof of Lemma 4.4], the exists a unique $\lambda_1(q_n)$ and a unique $\lambda_1(q)$, such that

$$g(\lambda_1(q_n)) = \theta(1; \lambda_1(q_n)\, q_n, \mu) = \pi, \tag{3.11}$$

$$g(\lambda_1(q)) = \theta(1; \lambda_1(q)\, q, \mu) = \pi, \tag{3.12}$$

$$\lambda_1(q_n) = \inf_{m\in\mathbb{N}} \lambda_1(q_n^{[m]}), \tag{3.13}$$

$$\lambda_1(q) = \inf_{m\in\mathbb{N}} \lambda_1(q^{[m]}). \tag{3.14}$$

Combining (3.11)-(3.14) with the continuity of g and using the fact that $\lambda_1(q)$ is the isolated zero of

$$g(s) = \theta(1; sq, \mu) = \pi,$$

it may concludes that

$$\lambda_1(q_n) \to \lambda_1(q) \quad \text{as } n \to \infty. \tag{3.15}$$

Therefore, $\lambda_1(-\mu B)$ is continuous in μ. Then we conclude that $\lambda_1(\mu) \to \infty$ as $\mu \to \infty$. □

4. Existence of positive solutions for sublinear semipositone problems

In this section we deal with sublinear f, namely, $f : (0, \infty) \to \mathbb{R}$ is continuous that satisfy (f_1) and (f_2).

First, we prove the following theorem, which will lead directly to Theorem 1.1.

Theorem 4.1. *Let (K), (f_1) and (f_2). Then there exists $\lambda^* > 0$ such that (2.5) has at least one positive solution for all $\lambda \geq \lambda^*$. More precisely, there exists a connected set of positive solutions of (2.5) bifurcating from infinity for $\lambda_\infty = +\infty$.*

As before we set

$$F(v) = f(|v|), \tag{4.1}$$

and let

$$G(v) = F(v) - \beta|v|^q. \tag{4.2}$$

We may use the rescaling

$$w = \gamma v, \quad \lambda = \gamma^{q-3}, \quad \gamma > 0.$$

A direct calculation shows that v is a solution of (2.5) if and only if

$$\begin{cases} -\alpha||w||^2 w'' = \tilde{F}(\gamma, w), & t \in (0,1), \\ w(0) = w(1) = 0, \end{cases} \tag{4.3}$$

where

$$\tilde{F}(\gamma, w) := \beta h|w|^q + \gamma^q h G(\gamma^{-1} w). \tag{4.4}$$

We can extend $\tilde{F}$ to $\gamma = 0$ by setting

$$\tilde{F}(0, w) = \beta h|w|^q \tag{4.5}$$

and, by (f_2), such an extension is continuous.

Define nonlinear operators $A, L, K : Y \to Y^*$ by

$$\langle Au, v\rangle = \alpha||u||^2 \int_0^1 u' \cdot v', \ \langle L\tilde{F}(\gamma, u), v\rangle = \int_0^1 \tilde{F}(\gamma, u)v, \ \langle Ku, v\rangle = \int_0^1 uv,$$

for any $\gamma > 0$ and $u, v \in Y$. Let

$$\langle L_0 u, v\rangle := \langle L\tilde{F}(0, u), v\rangle = \int_0^1 \tilde{F}(0, u)v, \quad u, v \in Y.$$

Lemma 4.1 (Lemma 4.1, [7]). *For any given $f \in X^*$, the following auxiliary problem*

$$\begin{cases} -\alpha\Big(\int_0^1 |v'|^2 dx\Big)v'' = f(t), \quad t \in (0,1), \\ v(0) = v(1) = 0 \end{cases}$$

has a unique weak solution.

It follows Liu, Luo and Dai [17] that the operator $A^{-1} : Y^* \to Y$ is completely continous. In addition, by Liang, Li, and Shi [15], we know that $L_0 : Y \to Y^*$ is compact. Furthermore, we can easily see that L_0 maps P into P^*. Similarly, $K : Y \to Y^*$ is also compact and K maps P into P^*.

Next we show that $\lambda_\infty = +\infty$ is a bifurcation from infinity for

$$Av = \lambda L(hF(v)). \tag{4.6}$$

It is clear that v is a solution of (4.6) if and only if

$$Aw = L\tilde{F}(\gamma, w). \tag{4.7}$$

We set

$$S(\gamma, w) := w - A^{-1}L\tilde{F}(\gamma, w). \tag{4.8}$$

For $\gamma = 0$, solutions of

$$S_0(w) := S(0, w) = w - A^{-1}\tilde{F}(0, w) = w - A^{-1}L_0 w = 0 \tag{4.9}$$

are nothing but solutions of

$$\begin{cases} -\alpha\Big(\int_0^1 |w'|^2 dx\Big)w'' = \beta h|w|^q, \quad t \in (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{4.10}$$

Denote by P_r for $r > 0$ the bounded open subset $\{u \in P : ||u|| < r\}$ of P. If (4.9) has no solution on ∂P_r, that is, the completely continuous operator $A^{-1}L_0 : \bar{P}_r \subset P \to P$ has no fixed point on ∂P_r, then by Amann [2] the index of fixed point $i(A^{-1}L_0, P_r, P)$ is well defined. Hence, we can use the fixed point index theory to complete the proof of Theorem 4.1.

Proposition 4.1. *$i(A^{-1}L_0, P_r, P) = 0$ for small r.*

Proof. Given $0 \le \hat{h} \in C_0^\infty(\Omega)$ with $\hat{h} \neq 0$, define a completely continuous homotopy function $H : [0,1] \times Y \to Y^*$ by

$$H(t, w) = L_0 w + tK\hat{h}, \quad (t, w) \in [0,1] \times P.$$

We show that there exists $r_1 > 0$ such that the operator equation

$$Aw = H(t, w)$$

has no solutions on $[0,1] \times \partial P_r$ for $r \in (0, r_1)$. Suppose this is not true. Then there exist $t_1 \in [0,1]$ and $w_1 \in P$ with $0 < ||w_1|| < r_1$ such that

$$Aw_1 = H(t_1, w_1).$$

Thus for any $\bar{v} \in Y$, we have

$$\alpha||w_1||^2 \int_0^1 w_1' \cdot \bar{v}' = \int_0^1 h|w_1|^q \bar{v} + t_1 \int_0^1 \hat{h}\bar{v}.$$

That is, w_1 is a weak solution of the following problem

$$\begin{cases} -\Big(\alpha \displaystyle\int_0^1 |w'|^2\Big) w'' = h|w|^q + t\hat{h}, \quad t \in (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{4.11}$$

By the elliptic regularity theory and the strong maximum principle, we know that $w_1 \in C^2(I) \cap C_0^1(\bar{I})$ and $w_1 > 0$ in Ω. Hence, w_1 satisfies the following equation

$$-w'' = \lambda_1 w + \Big[\frac{h|w|^q/w}{b||w||^2} - \lambda_1\Big] w + \frac{t_1}{b||w||^2}\hat{h}, \qquad t \in (0,1). \tag{4.12}$$

For sufficiently small r_1, since $||w_1|| = r < r_1$ and $0 < q \leq 1$, then we have

$$\Big[\frac{h|w_1|^q/w_1}{b||w_1||^2} - \lambda_1\Big] w_1 + \frac{t_1}{b||w_1||^2}\hat{h} > 0, \qquad t \in (0,1).$$

This is impossible since (4.12) has no positive solution. Notice that this fact holds for the problem

$$\begin{cases} -\Big(\alpha \displaystyle\int_0^1 |w'|^2\Big) w'' = h|w|^q + \hat{h}, \quad t \in (0,1), \\ w(0) = w(1) = 0. \end{cases} \tag{4.13}$$

Indeed, the above problem has no solutions in P_r for $r \in (0, r_1)$. Consequently,

$$i(A^{-1}L_0, P_r, P) = i(A^{-1}H(0,\cdot), P_r, P) = i(A^{-1}H(1,\cdot), P_r, P) = 0, \quad r \in (0, r_1).$$

□

Proposition 4.2. *$i(A^{-1}L_0, P_R, P) = 1$ for large R.*

Proof. We define a completely continuous homotopy function $H : [0,1] \times Y \to Y^*$ by

$$H(t,w) = tL_0 w, \quad (t,w) \in [0,1] \times P.$$

We claim that there exists $R_1 > 0$ such that the operator equation

$$Aw = H(t,w) \tag{4.14}$$

has no solutions on $[0,1] \times \partial P_R$ for $R > R_1$. We prove by contradiction. Suppose that there exists a sequence $\{(t_n, w_n)\} \subset [0,1] \times P$ such that

$$t_n \to t_0 \in [0,1], \quad ||w_n|| \to \infty,$$

and (t_n, w_n) satisfies (4.14), that is,

$$\alpha||w_n||^2 \int_0^1 w_n' \cdot \bar{v}' = t_n \int_0^1 h|w_n|^q \bar{v}, \quad \bar{v} \in Y.$$

Let $\bar{w}_n = \frac{w_n}{||w_n||}$ for any n. Then we have, for any $\bar{v} \in Y$,

$$\frac{\alpha ||w_n||^2}{||w_n||^2} \int_0^1 \bar{w}_n' \cdot \bar{v}' = t_n \int_0^1 \frac{h|w_n|^q}{||w_n||^3} \bar{v} = t_n \int_0^1 \frac{h|\bar{w}_n|^q}{||w_n||^{3-q}} \bar{v}. \tag{4.15}$$

Since $\bar{w}_n$ is bounded in P, taking $\bar{v} = \bar{w}_n$ in (4.15) and letting $n \to \infty$, we have

$$\alpha = 0.$$

This is impossible since $\alpha > 0$. Taking $R > R_1$, we have

$$i(A^{-1}L_0, P_R, P) = i(A^{-1}H(1, \cdot), P_R, P) = i(A^{-1}H(0, \cdot), P_R, P) = 1, \quad R > R_1.$$

□

We can use Proposition 4.1 and Proposition 4.2 to infer the existence of $R > r > 0$ such that

$$S_0(w) \neq 0, \qquad \forall\, ||w|| \in \{r, R\} \tag{4.16}$$

and

$$\deg(S_0, P_R \setminus \bar{P}_r, 0) = \deg(I - A^{-1}L_0, P_R \setminus \bar{P}_r, 0) = i(A^{-1}L_0, P_R \setminus \bar{P}_r, P) = 1. \tag{4.17}$$

Remark 4.1. In the case that $P_R \setminus \bar{P}_r$ has an empty interior, it is not possible to define $\deg(S_0, P_R \setminus \bar{P}_r, 0)$ directly. Therefore, the retraction $\sigma : P_R \to P_R \setminus \bar{P}_r$ is used in Deimling [9], Guo and Lakshmikantham [13]. The degree is then defined as

$$\deg(I - A^{-1}L_0 \circ \sigma, \sigma^{-1}(P_R \setminus \bar{P}_r), 0) = i(A^{-1}L_0, P_R \setminus \bar{P}_r, P).$$

The definition of the retraction is as follows.

Definition 4.1. Let X be a topological space, $A \subset B \subset X$. If there is a continuous mapping $\sigma : B \to A$ such that when $x \in A$ there is $\sigma(x) = x$, then A is the retract of B, and σ is called the retraction.

Next we show

Lemma 4.2. *There exists $\gamma_0 > 0$ such that*

(i) *deg*$(S_\gamma, P_R \setminus \bar{P}_r, 0) = 1, \quad \forall\, 0 \leq \gamma \leq \gamma_0$*;*

(ii) *if* $S(\gamma, w) = 0$, $\gamma \in [0, \gamma_0]$, $r \leq ||w|| \leq R$, *then* $w > 0$ *in* $(0, 1)$.

Proof. Obviously, (i) follows if we show that $S(\gamma, w) \neq 0$ for all $||w|| \in \{r, R\}$ and all $0 \leq \gamma \leq \gamma_0$. Suppose, to the contrary, that there exists a sequence (γ_n, w_n) such that $\gamma_n \to 0$, $||w_n|| \in \{r, R\}$ and

$$w_n = A^{-1}Lw_n.$$

Since $A^{-1}L$ is compact, then, by passing to a subsequence if necessary, we have $w_n \to w$. Consequently,

$$S_0(w) = 0, \quad ||w|| \in \{r, R\},$$

which contradicts (4.10).

In order to prove (ii), we once again argue by contradiction. As in the previous argument, we find a sequence $w_n \in Y$, with $\{x \in I : w_n(x) \leq 0\} \neq \emptyset$, such that

$$w_n \to w, \quad ||w|| \in [r, R]$$

and $S_0(w) = 0$; namely, w solves (4.7). By the maximum principle $w > 0$ on $(0,1)$ and $w'(0) > 0,\ w'(1) < 0$. Moreover, the fact w_n is concave down on I implies that, without relabeling, $w_n \to w$ in $C^1(I)$. Therefore $w_n > 0$ on $(0,1)$ for n large, a contradiction. □

Proof of Theorem 4.1. By Lemma 4.2, problem (4.7) has a positive solution w_γ for all $0 \leq \gamma \leq \gamma_0$. As remarked before, for $\gamma > 0$, the rescaling $\lambda = \gamma^{q-3}$, $v = \frac{w}{\gamma}$ gives a solution (λ, v_λ) of (4.6) for all $\lambda \geq \lambda^*$ with

$$\lambda^* := \gamma_0^{q-3}.$$

Since $w_\gamma > 0$, (λ, v_λ) is a positive solution of (2.5). Finally $||v_\lambda|| \geq r$ for all $\gamma \in [0, \gamma_0]$ implies that

$$||v_\lambda|| = \frac{||w_\gamma||}{\gamma} \to \infty \ \text{ as } \ \gamma \to 0.$$

This completes the proof. □

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

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Received February 2025; Accepted June 2025; Available online July 2025.