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RELIABILITY ANALYSIS OF MASKED DATA FOR EXPONENTIALLY DISTRIBUTED COMPONENTS UNDER MULTIPLE TYPE-II CENSORING

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Abstract In the case of multiple type-II censoring, the maximum likelihood estimations of the parameters are proposed for the masked data of the series system of two components with exponential life distribution (same and different parameters), and the uniqueness of the likelihood equation root is proved. The Bayes point estimations and interval estimations of the parameters are also given under the assumption that the prior distribution is Gamma distribution. Besides, the likelihood function is deduced theoretically for the masked data of the parallel system with two components with exponential life distribution (same parameters), and the uniqueness of maximum likelihood estimation is proved. The Bayes point estimation and interval estimation of the parameter is proposed under the assumption of Gamma distribution. The application of the method is illustrated through simulation data in various situations.

Keywords Multiple type-II censoring, exponential distribution, masked data, series system, parallel system.

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1. Introduction

In reliability analysis, system lifetime data are often analyzed to estimate the unknown parameters of the lifetime distributions of individual system components. System lifetime test data generally consists of two aspects: failure times and failure causes. Ideally, system lifetime data should include the exact failure time of the system and the information on which specific component failure led to the overall system failure. However, in most cases, the exact component responsible for the system failure cannot be accurately identified. Instead, the cause of system failure is attributed to a subset of components, leaving the true failure cause masked. In real-world scenarios, the exact component failure information is often unavailable due to high costs associated with fault diagnosis and failure detection, particularly as modern systems increasingly adopt modular designs. This results in the

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inability to precisely determine the failed unit responsible for system failure. Similar masking problems arise in system reliability studies of computers, integrated circuits, and other complex systems. Various factors contribute to the occurrence of masking, including financial constraints, time limitations, recording errors, lack of diagnostic tools, and destructive consequences associated with certain component failures. Consequently, the statistical analysis of masked data has become a prominent research topic in recent years. Furthermore, as systems become more functionally sophisticated, their structures also grow increasingly complex. Systems such as aerospace power systems and radar systems are no longer simple series or parallel configurations but rather intricate multi-unit mixed configurations, often accompanied by the presence of masking phenomena.

Reliability lifetime test data samples can be classified into complete samples and censored samples. In practice, censored tests are widely used because they significantly save time, labor, and financial resources, making them a more commonly adopted testing method. The most frequently used censored data types include type-I censored samples, type-II censored samples, and multiple type-II censored samples (also referred to as missing data). Multiple type-II censored lifetime tests differ from conventional type-II censored lifetime tests. A standard type-II censored lifetime test involves subjecting n units to a lifetime test from time zero, with the test terminating upon the failure of the k^{th} unit (where k is a predetermined positive integer). The ordered observed failure times of these k units are recorded as $t_1, t_2, \cdots, t_k$. In contrast, a multiple (k -stage) type-II censored lifetime test also begins with n units under the test from time zero, but due to certain factors (e.g., lack of diligence by operators), some failure data are missing, leaving only k failure data available. The corresponding observed failure times are sequentially recorded as $t_{r_1}, t_{r_2}, \cdots, t_{r_k}$, where $1 \leq r_1 < r_2 < \cdots < r_k \leq n$. Notably, when $r_1 = 1, r_2 = 2, \cdots, r_k = k$, the multiple type-II censored sample reduces to a standard type-II censored sample. In other words, type-II censored samples are a special case of multiple type-II censored samples. Since multiple type-II censoring encompasses a broader range of real-world scenarios, research on this type of censored sample holds greater practical significance.

The estimation problem for parameters under masked data was first introduced by Usher and Hodgson in 1988 in [29]. They provided the maximum likelihood estimation (MLE) of parameters for systems composed of two or three series- connected units when masking occurs, assuming that the failure rate of each unit is constant. Statistical analysis of masked data has remained an active research area in recent years, with numerous scholars contributing valuable findings, resulting in a rich body of literature. For a comprehensive review, see references [1–36]. This paper presents a summary of research closely related to reliability statistical analysis under multiple type-II censoring with masked data.

For the multiple type-II censoring case, Zhang Meng, Lu Shan, and others in [36] studied the reliability estimation of components in series systems under masked system lifetime data. They first derived the likelihood function of the sample using a probabilistic element analysis approach and then provided approximate MLE and Bayesian estimates of the parameters under the assumption that component lifetimes follow an exponential distribution with possibly unequal parameters. It is worth noting that they [36] did not prove the uniqueness of the MLE and only provided an approximate MLE. Additionally, the study did not consider the case where the parameters of the exponential distribution are equal.

This paper first examines a system composed of two series-connected components with exponentially distributed lifetimes (considering both equal and unequal parameter cases). Under multiple type-II censoring with masking, we derive the MLE of the parameters and prove its uniqueness. When the prior distribution of the parameters follows a gamma distribution, Bayesian estimation and interval estimation are also provided. Next, we consider a system composed of two parallel-connected components with identical exponential distribution parameters. Under multiple type-II censoring with masking, we derive the likelihood function and obtain the MLE of the parameters, proving its uniqueness. Bayesian estimation and interval estimation are also provided under the assumption that the prior distribution of the parameters follows a gamma distribution.

Let the lifetime of the system i be denoted as T_i, while the lifetimes of the two components 1 and 2 forming the system i, denoted as T_{i1}, T_{i2}, are independent and follow exponential distributions $\text{Exp}(\lambda_1), \text{Exp}(\lambda_2)$ with parameters λ_1, λ_2, respectively, where the parameters may be either equal or different. Denote the cumulative distribution function (CDF) be $F_j(t)$, the reliability function be $\overline{F}_j(t)$, and the hazard function be $h_j(t)$ of $T_{ij}, j = 1, 2$, which are given by:

$$F_j(t) = 1 - e^{-\lambda_j t}, \overline{F}_j(t) = e^{-\lambda_j t}, h_j(t) = \lambda_j, j = 1, 2\,.$$

Remark 1.1. The conclusions of this paper are fully applicable to the case where the failure rate of lifetime T_{ij} for components $j = 1, 2$ passes through the origin. In fact, suppose the hazard function $h_j(t)$ of the component's lifetime T_{ij} passes through the origin, i.e., $h_j(t) = \beta_j t$. In this case, its distribution function is given by $F_j(t) = 1 - e^{-\beta_j t^2/2}$. If we set $Y_{ij} = T_{ij}^2$, then Y_{ij} follows an exponential distribution with parameter $\beta_j/2$.

Remark 1.2. The conclusions of this paper are also fully applicable to the case where the component $j = 1, 2$ lifetime T_{ij} follows an inverse exponential distribution. In fact, suppose the component lifetime T_{ij} follows an inverse exponential distribution with parameter β_j, whose distribution function is $F_j(t) = e^{-\beta_j/t}$. If we set $Y_{ij} = T_{ij}^{-1}$, then Y_{ij} follows an exponential distribution with parameter β_j.

2. System composed of two series-connected components with exponentially distributed lifetimes (identical parameters)

2.1. Maximum likelihood estimation of the parameter

First, consider a system composed of J components in series, where the system $i, i = 1, 2, \cdots, n$ lifetime is denoted as T_i, and the lifetime of the component $l, l = 1, 2, \cdots, J$ forming the i^{th} system is denoted as T_{il}. The cumulative distribution function (CDF) and probability density function (PDF) of the component lifetimes are denoted as $F_l(t), f_l(t)$, respectively. Under multiple type-II censored cases with masked data, the likelihood function has been given in reference [36], as the following theorem.

Theorem 2.1. *Assume that a system consists of J components in series. A lifetime experiment starts at time 0, and by the end of the test, only the failure time*

$t_{r_1}, t_{r_2}, \cdots, t_{r_k}$ of the $r_1, r_2, \cdots, r_k$ failed system is observed, where $t_{r_1} < t_{r_2} < \cdots < t_{r_k}$. Additionally, the possible failure causes of these k systems are observed. The censored sample with masked failure causes is given as $(t_{r_1}, s_{r_1}), (t_{r_2}, s_{r_2}), \cdots, (t_{r_k}, s_{r_k})$, where $s_{r_i}, i = 1, 2, \cdots, k$ represents the possible failure cause of the system. Then, the likelihood function is given by (where C^+ is a positive constant):

$$L = C^+\left[1 - \prod_{l=1}^{J} \overline{F}_l(t_{r_1})\right]^{r_1-1} \cdot \prod_{j=1}^{k}\left[\sum_{m\in s_{r_j}} h_m(t_{r_j})\right] \cdot \left[\prod_{j=1}^{k}\prod_{l=1}^{J} \overline{F}_l(t_{r_j})\right]$$
$$\times \prod_{j=1}^{k-1}\left[\prod_{l=1}^{J} \overline{F}_l(t_{r_j}) - \prod_{l=1}^{J} \overline{F}_l(t_{r_{j+1}})\right]^{r_{j+1}-r_j-1} \cdot \left[\prod_{l=1}^{J} \overline{F}_l(t_{r_k})\right]^{n-r_k}.$$

Consider a system i composed of two series-connected components (i.e. $J = 2$), where the lifetimes T_{i1}, T_{i2} of components 1 and 2 are independent and identically distributed according to an exponential distribution with parameter λ. That is:

$$F_1(t) = F_2(t) = 1 - e^{-\lambda t},\ \overline{F}_1(t) = \overline{F}_2(t) = e^{-\lambda t},\ h_1(t) = h_2(t) = \lambda\,.$$

At $t = 0$, a lifetime test is conducted on n systems. By the end of the test, only k failure data points are retained from the failed systems. The corresponding failure times of these k systems are denoted as $t_{r_1}, t_{r_2}, \cdots, t_{r_k}$ (where $1 \le r_1 < r_2 < \cdots < r_k \le n$), and $t_{r_1} < t_{r_2} < \cdots < t_{r_k}$, the possible failure causes of these k failed systems are observed and denoted as $s_{r_i}, i = 1, 2, \cdots, k$. Under this scenario, the system failure sample data with masking can be represented as $(t_{r_1}, s_{r_1}), (t_{r_2}, s_{r_2}), \cdots, (t_{r_k}, s_{r_k})$, where $s_{r_i}, i = 1, 2, \cdots, k$ takes values in $\{1\}$, $\{2\}$, $\{1, 2\}$. Specifically, $\{1\}$ indicates that the system failure is caused by component 1, and a total of k_1 system failures are attributed to component 1. $\{2\}$ indicates that the system failure is caused by component 2, and a total of k_2 system failures are attributed to component 2. $\{1, 2\}$ indicates that the specific cause of failure is unknown, meaning a masking effect occurs, and a total of k_3 system failures have an uncertain cause of failure. It is easy to see that: $k = k_1 + k_2 + k_3$.

According to Theorem 2.1, the likelihood function is given by (where $C^+ > 0$ is the multinomial coefficient due to ordering failure times and independent of the parameters):

$$\begin{aligned}
L(\lambda) = {} & C^+\left[1 - \prod_{l=1}^{2} \overline{F}_l(t_{r_1})\right]^{r_1-1} \cdot \prod_{j=1}^{k}\left[\sum_{m\in s_{r_j}} h_m(t_{r_j})\right] \cdot \left[\prod_{j=1}^{k}\prod_{l=1}^{2} \overline{F}_l(t_{r_j})\right] \\
& \times \prod_{j=1}^{k-1}\left[\prod_{l=1}^{2} \overline{F}_l(t_{r_j}) - \prod_{l=1}^{2} \overline{F}_l(t_{r_{j+1}})\right]^{r_{j+1}-r_j-1} \cdot \left[\prod_{l=1}^{2} \overline{F}_l(t_{r_k})\right]^{n-r_k} \\
= {} & C^+ 2^{k_3}\ \lambda^k (1 - e^{-2\lambda t_1})^{r_1-1} e^{-2\lambda\left[\sum_{j=1}^{k} t_{r_j} + (n-r_k)t_{r_k}\right]} \\
& \times \prod_{j=1}^{k-1} (e^{-2\lambda t_{r_j}} - e^{-2\lambda t_{r_{j+1}}})^{r_{j+1}-r_j-1},
\end{aligned}$$

$$\begin{aligned}
\ln L(\lambda) = {} & \ln C^+ + k_3 \ln 2 + k \ln\lambda \\
& + (r_1 - 1)\ln(1 - e^{-2\lambda t_{r_1}}) - 2\lambda\left[\sum_{j=1}^{k} t_{r_j} + (n - r_k)t_{r_k}\right]
\end{aligned}$$

$$+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\ln(e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}),$$

$$\frac{d\ln L(\lambda)}{d\lambda}=\frac{k}{\lambda}+2(r_1-1)\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}-2\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right]$$
$$+2\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}}}{e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}}.$$

Let $\frac{d\ln L(\lambda)}{d\lambda}=0$, then the following equation is obtained

$$\frac{1}{2\lambda}+\frac{r_1-1}{k}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}+\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}}}{e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}}$$
$$=\frac{1}{k}\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right]. \tag{2.1}$$

The root of Equation (2.1) is the maximum likelihood estimate (MLE) $\hat{\lambda}$ of the parameter λ.

Lemma 2.1. *The equation* (2.1) *has a unique positive real root for* λ.

Proof. Define the function $g(\lambda)$ as

$$g(\lambda)=\frac{1}{2\lambda}+\frac{r_1-1}{k}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}+\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}}}{e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}}.$$

Since $\lim_{\lambda\to 0}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}=\lim_{\lambda\to 0}\frac{t_{r_1}}{e^{2\lambda t_{r_1}}-1}=+\infty$, $\lim_{\lambda\to+\infty}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}=\lim_{\lambda\to+\infty}\frac{t_{r_1}}{e^{2\lambda t_{r_1}}-1}=0$,

$$\lim_{\lambda\to 0}\frac{-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}}}{e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}}=\lim_{\lambda\to 0}\frac{t_{r_{j+1}}-t_{r_j}e^{2\lambda(t_{r_{j+1}}-t_{r_j})}}{e^{2\lambda(t_{r_{j+1}}-t_{r_j})}-1}=+\infty,$$
$$\lim_{\lambda\to+\infty}\frac{-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}}}{e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}}=\lim_{\lambda\to+\infty}\frac{t_{r_{j+1}}-t_{r_j}e^{2\lambda(t_{r_{j+1}}-t_{r_j})}}{e^{2\lambda(t_{r_{j+1}}-t_{r_j})}-1}=-t_{r_j},$$

we have $\lim_{\lambda\to 0}g(\lambda)=+\infty$, $\lim_{\lambda\to+\infty}g(\lambda)=-\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)t_{r_j}$,

$$g'(\lambda)=-\frac{1}{2\lambda^2}-2\frac{r_1-1}{k}\frac{t_{r_1}^2e^{-2\lambda t_{r_1}}}{(1-e^{-2\lambda t_{r_1}})^2}+\frac{1}{k}\sum_{j=1}^{k-1}\frac{r_{j+1}-r_j-1}{(e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}})^2}$$
$$\times\Big[2(t_{r_j}^2e^{-2\lambda t_{r_j}}-t_{r_{j+1}}^2e^{-2\lambda t_{r_{j+1}}})\cdot(e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}})$$
$$-2(-t_{r_j}e^{-2\lambda t_{r_j}}+t_{r_{j+1}}e^{-2\lambda t_{r_{j+1}}})^2\Big]$$
$$=-\frac{1}{2\lambda^2}-2\frac{r_1-1}{k}\frac{t_{r_1}^2e^{-2\lambda t_{r_1}}}{\left(1-e^{-2\lambda t_{r_1}}\right)^2}$$

$$- \frac{2}{k}\sum_{j=1}^{k-1}\frac{(r_{j+1}-r_j-1)(t_{r_{j+1}}-t_{r_j})^2}{\left(e^{-2\lambda t_{r_j}}-e^{-2\lambda t_{r_{j+1}}}\right)^2}e^{-2\lambda(t_{r_{j+1}}+t_{r_j})}$$
$$<0\,.$$

Therefore, the equation (2.1) has a unique positive real root. □

2.2. Bayesian point estimation and interval estimation of the parameter

First,we expand the likelihood function as follows:

$$\begin{aligned}L(\lambda) =& C^{+}2^{k_3}\,\lambda^k\sum_{s_1=0}^{r_1-1}(-1)^{r_1-1-s_1}C_{r_1-1}^{s_1}e^{-2\lambda(r_1-s_1-1)t_{r_1}}\cdot e^{-2\lambda\left[\sum\limits_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right]}\\ &\times\prod_{j=1}^{k-1}\sum_{l_j=0}^{r_{j+1}-r_j-1}(-1)^{r_{j+1}-r_j-1-l_j}C_{r_{j+1}-r_j-1}^{l_j}e^{-2\lambda[l_jt_{r_j}+(r_{j+1}-r_j-1-l_j)t_{r_{j+1}}]}\\ =& C^{+}2^{k_3}\sum_{s_1=0}^{r_1-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_1-\sum\limits_{i=1}^{k-1}l_i}\\ &\times C_{r_1-1}^{s_1}C_{r_2-r_1-1}^{l_1}C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}}\lambda^k\cdot e^{-2\lambda u_1}.\end{aligned}$$

Here $u_1=(r_1+l_1-s_1)t_{r_1}+\sum\limits_{j=1}^{k-2}(r_{j+1}-r_j-l_j+l_{j+1})t_{r_{j+1}}+(n-r_{k-1}-l_{k-1})t_{r_k}$. Following the approach in reference [36], assume the prior distribution of parameter λ follows a Gamma distribution:

$$g(\lambda)=\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda},\ \alpha>0,\beta>0\,.$$

Then, the posterior density function of λ is given by $h(\lambda\mid\text{data})=\frac{L(\lambda)g(\lambda)}{\int_0^{+\infty}L(\lambda)g(\lambda)d\lambda}$.

From this, a Bayesian estimate, the posterior mean of λ is $\hat{\lambda}=\dfrac{\int_0^{+\infty}\lambda L(\lambda)g(\lambda)d\lambda}{\int_0^{+\infty}L(\lambda)g(\lambda)d\lambda}$.

To compute this for $m=0,1$, evaluate the integral:

$$\begin{aligned}M^{(m)} =& \int_0^{+\infty}\lambda^mL(\lambda)g(\lambda)d\lambda\\ =& C^{+}2^{k_3}\frac{\beta^\alpha}{\Gamma(\alpha)}\sum_{s_1=0}^{r_1-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_1-\sum_{i=1}^{k-1}l_i}\\ &\times C_{r_1-1}^{s_1}C_{r_2-r_1-1}^{l_1}C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}}\\ &\times\int_0^{+\infty}\lambda^{\alpha-1+k+m}e^{-\lambda(2u_1+\beta)}d\lambda\,.\end{aligned}$$

Note the following integral: $\int_0^{+\infty} \lambda^a e^{-\lambda b} d\lambda = \frac{\Gamma(a+1)}{b^{a+1}}$.

$$\begin{aligned} M^{(m)} =& C^{+} 2^{k_3} \frac{\Gamma(\alpha+k+m)}{\Gamma(\alpha)} \beta^{\alpha} \sum_{s_1=0}^{r_1-1} \sum_{l_1=0}^{r_2-r_1-1} \sum_{l_2=0}^{r_3-r_2-1} \cdots \\ & \times \sum_{l_{k-1}=0}^{r_k-r_{k-1}-1} (-1)^{r_k-k-s_1-\sum_{i=1}^{k-1} l_i} C_{r_1-1}^{s_1} C_{r_2-r_1-1}^{l_1} \\ & \times C_{r_3-r_2-1}^{l_2} \cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} \cdot (2u_1+\beta)^{-(\alpha+k+m)}. \end{aligned}$$

Thus, the Bayesian estimate of λ can be expressed as: $\hat{\lambda} = \frac{M^{(1)}}{M^{(0)}}$.

The Bayesian interval estimation for the parameter λ is derived as follows.

Since the posterior density function of λ is given by $h(\lambda\,|\,\text{data}) = \dfrac{L(\lambda)g(\lambda)}{\int_0^{+\infty} L(\lambda)g(\lambda)d\lambda}$.

Then the Bayesian interval estimation of λ at the confidence level $1-\alpha'$ is given by $(\hat{\lambda}_1, \hat{\lambda}_2)$, where $\hat{\lambda}_1, \hat{\lambda}_2$ satisfy the following equations:

$$\int_0^{\hat{\lambda}_1} h(\lambda \mid \text{data}) d\lambda = \frac{\alpha'}{2}, \int_{\hat{\lambda}_2}^{+\infty} h(\lambda \mid \text{data}) d\lambda = \frac{\alpha'}{2}.$$

Example 2.1. Let $n = 30$, $k = 24$, $r_1, r_2, \cdots, r_{24}$ be given as 1, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 25, 26, 27, 28, 30, $k_1 = 8, k_2 = 8, k_3 = 8$, respectively. The failure data of a system consisting of two components,both following the distribution Exp(0.8) generated by Monte-Carlo simulation, are as follows.

The failure data belonging to $\{1\}$ are 0.0219, 0.0412, 0.1097, 0.2199, 0.5936, 0.6716, 0.8847, 0.9609.

The failure data belonging to $\{2\}$ are 0.0965, 0.1347, 0.1540, 0.3035, 0.6244, 0.8192, 1.0644, 1.9287.

The failure data belonging to $\{1,2\}$ are 0.0882, 0.1256, 0.2192, 0.7332, 0.8715, 1.0229, 1.2131, 1.2701.

Under the k-multiple type-II censored life test, the maximum likelihood estimate (MLE) of the parameter λ is obtained as $\hat{\lambda} = 0.8467$. If the prior distribution is assumed to follow a Gamma distribution with parameters $\alpha = 1$, $\beta = 2$, the Bayes point estimate of λ is computed as $\hat{\lambda} = 0.8282$, and the 0.95 Bayesian confidence interval is given by $[0.4152, 1.6818]$.

3. System composed of two series-connected components with exponentially distributed lifetimes (different parameters)

3.1. Maximum likelihood estimations of parameters

Consider the system i composed of two series-connected components 1 and 2, where the lifetimes T_{i1}, T_{i2} of components 1 and 2 are independent and follow exponential distributions with parameters λ_1, λ_2, respectively. The test scheme is the same as that in Section 2.1.

According to Theorem 2.1, the likelihood function is given by (where C^+ is a positive constant):

$$
\begin{aligned}
&L(\lambda_1,\lambda_2)\\
=&C^+\left[1-\prod_{l=1}^{2}\overline{F}_l(t_{r_1})\right]^{r_1-1}\cdot\prod_{j=1}^{k}\left[\sum_{m\in s_{r_j}}h_m(t_{r_j})\right]\cdot\left[\prod_{j=1}^{k}\prod_{l=1}^{2}\overline{F}_l(t_{r_j})\right]\\
&\times\prod_{j=1}^{k-1}\left[\prod_{l=1}^{2}\overline{F}_l(t_{r_j})-\prod_{l=1}^{2}\overline{F}_l(t_{r_{j+1}})\right]^{r_{j+1}-r_j-1}\cdot\left[\prod_{l=1}^{2}\overline{F}_l(t_{r_k})\right]^{n-r_k}\\
=&C^+\lambda_1^{k_1}\lambda_2^{k_2}\left(\lambda_1+\lambda_2\right)^{k_3}\left[1-e^{-(\lambda_1+\lambda_2)t_{r_1}}\right]^{r_1-1}e^{-(\lambda_1+\lambda_2)\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right]}\\
&\times\prod_{j=1}^{k-1}\left[e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}\right]^{r_{j+1}-r_j-1},\\
&\ln L(\lambda_1,\lambda_2)\\
=&\ln C^+ + k_1\ln\lambda_1+k_2\ln\lambda_2+k_3\ln(\lambda_1+\lambda_2)+(r_1-1)\ln\left[1-e^{-(\lambda_1+\lambda_2)t_{r_1}}\right]\\
&+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\ln\left[e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}\right]\\
&-(\lambda_1+\lambda_2)\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right],\\
&\frac{\partial\ln L(\lambda_1,\lambda_2)}{\partial\lambda_1}\\
=&\frac{k_1}{\lambda_1}+\frac{k_3}{\lambda_1+\lambda_2}+(r_1-1)\frac{t_{r_1}e^{-(\lambda_1+\lambda_2)t_{r_1}}}{1-e^{-(\lambda_1+\lambda_2)t_{r_1}}}\\
&+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-(\lambda_1+\lambda_2)t_{r_j}}+t_{r_{j+1}}e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}{e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}\\
&-\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right],\\
&\frac{\partial\ln L(\lambda_1,\lambda_2)}{\partial\lambda_2}\\
=&\frac{k_2}{\lambda_2}+\frac{k_3}{\lambda_1+\lambda_2}+(r_1-1)\frac{t_{r_1}e^{-(\lambda_1+\lambda_2)t_{r_1}}}{1-e^{-(\lambda_1+\lambda_2)t_{r_1}}}\\
&+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-(\lambda_1+\lambda_2)t_{r_j}}+t_{r_{j+1}}e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}{e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}\\
&-\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right].
\end{aligned}
$$

Let $\frac{\partial \ln L(\lambda_1,\lambda_2)}{\partial \lambda_1}=0, \frac{\partial \ln L(\lambda_1,\lambda_2)}{\partial \lambda_2}=0$, yield the following two equations:

$$\begin{aligned}
&\frac{k_1}{\lambda_1}+\frac{k_3}{\lambda_1+\lambda_2}+(r_1-1)\frac{t_{r_1}e^{-(\lambda_1+\lambda_2)t_{r_1}}}{1-e^{-(\lambda_1+\lambda_2)t_{r_1}}}\\
&\quad+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-(\lambda_1+\lambda_2)t_{r_j}}+t_{r_{j+1}}e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}{e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}\\
&=\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k},\\
&\frac{k_2}{\lambda_2}+\frac{k_3}{\lambda_1+\lambda_2}+(r_1-1)\frac{t_{r_1}e^{-(\lambda_1+\lambda_2)t_{r_1}}}{1-e^{-(\lambda_1+\lambda_2)t_{r_1}}}\\
&\quad+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-(\lambda_1+\lambda_2)t_{r_j}}+t_{r_{j+1}}e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}{e^{-(\lambda_1+\lambda_2)t_{r_j}}-e^{-(\lambda_1+\lambda_2)t_{r_{j+1}}}}\\
&=\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}.
\end{aligned}$$

From $\frac{k_1}{\lambda_1}=\frac{k_2}{\lambda_2}$, we obtain $\lambda_2=\frac{k_2}{k_1}\lambda_1$. Substituting this into the first equation,

$$\begin{aligned}
&\frac{k_1}{\lambda_1}+\frac{k_3}{\lambda_1(1+k_2/k_1)}+(r_1-1)\frac{t_{r_1}e^{-\lambda_1(1+k_2/k_1)t_{r_1}}}{1-e^{-\lambda_1(1+k_2/k_1)t_{r_1}}}\\
&\quad+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-\lambda_1(1+\frac{k_2}{k_1})t_{r_j}}+t_{r_{j+1}}e^{-\lambda_1(1+\frac{k_2}{k_1})t_{r_{j+1}}}}{e^{-\lambda_1(1+\frac{k_2}{k_1})t_{r_j}}-e^{-\lambda_1(1+\frac{k_2}{k_1})t_{r_{j+1}}}}\\
&=\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}.
\end{aligned}$$

The equation is then transformed into:

$$\begin{aligned}
&\frac{1}{1+k_2/k_1}\frac{1}{\lambda_1}+\frac{r_1-1}{k}\frac{t_{r_1}e^{-\lambda_1(1+k_2/k_1)t_{r_1}}}{1-e^{-\lambda_1(1+k_2/k_1)t_{r_1}}}\\
&\quad+\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-\lambda_1(1+k_2/k_1)t_{r_j}}+t_{r_{j+1}}e^{-\lambda_1(1+k_2/k_1)t_{r_{j+1}}}}{e^{-\lambda_1(1+k_2/k_1)t_{r_j}}-e^{-\lambda_1(1+k_2/k_1)t_{r_{j+1}}}}\\
&=\frac{1}{k}\left[\sum_{j=1}^{k}t_{r_j}+(n-r_k)t_{r_k}\right].
\end{aligned}\tag{3.1}$$

The root of equation (3.1) is the maximum likelihood estimate (MLE) $\hat{\lambda}_1$ of the parameter λ_1, leading to the MLE of $\hat{\lambda}_2$ as follows: $\hat{\lambda}_2=\frac{k_2}{k_1}\hat{\lambda}_1$.

Lemma 3.1. *Equation* (3.1) *has a unique positive real root with respect to* λ_1.

Proof. Define $\lambda_1'=\left(1+\frac{k_2}{k_1}\right)\lambda_1$, then equation (3.1) can be rewritten as

$$\frac{1}{\lambda_1'}+\frac{r_1-1}{k}\frac{t_{r_1}e^{-\lambda_1' t_{r_1}}}{1-e^{-\lambda_1' t_{r_1}}}+\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{-t_{r_j}e^{-\lambda_1' t_{r_j}}+t_{r_{j+1}}e^{-\lambda_1' t_{r_{j+1}}}}{e^{-\lambda_1' t_{r_j}}-e^{-\lambda_1' t_{r_{j+1}}}}$$

$$=\frac{1}{k}\left[\sum_{j=1}^{k} t_{r_j}+(n-r_k)t_{r_k}\right]. \tag{3.2}$$

From Lemma 2.1, it follows that equation (3.2) has a unique positive real root, which implies that equation (3.1) also has a unique positive real root. □

It is worth noting that reference [36] does not provide proof of the uniqueness of the maximum likelihood estimate (MLE) of the parameter λ_1, λ_2; instead, it only presents its approximate maximum likelihood estimate as follows.

Define $r_0 = 0, m_0 = r_1 - 1, m_j = r_{j+1} - r_j - 1, j = 0, 1, \cdots, k-1$.

Furthermore, for $j = 1, 2, \cdots, k$, denote

$$p_j = \frac{r_j}{n+1}, q_j = 1 - p_j\ ,\ \delta_j = \frac{q_j}{q_j - q_{j+1}} - \frac{q_j q_{j+1}}{(q_j - q_{j+1})^2}\ln\frac{q_j}{q_{j+1}}\ ,$$
$$\gamma_j = \frac{q_{j+1}\ln q_{j+1} - q_j \ln q_j}{q_j - q_{j+1}} + \delta_j \ln q_j - (1-\delta_j)\ln q_{j+1}.$$

Thus, the approximate maximum likelihood estimates of the parameter λ_1, λ_2 are given by:

$$\hat{\lambda}_1 = \frac{k_1}{k_1+k_2}\left(k - \sum_{j=0}^{k-1} m_j\gamma_j\right) \times \left\{\sum_{j=0}^{k-1} m_j\left[\delta_j t_{r_j} + (1-\delta_j)t_{r_{j+1}}\right] + \sum_{j=1}^{k} t_{r_j} + (n-r_k)t_{r_k}\right\}^{-1},$$
$$\hat{\lambda}_2 = \frac{k_2}{k_1+k_2}\left(k - \sum_{j=0}^{k-1} m_j\gamma_j\right) \times \left\{\sum_{j=0}^{k-1} m_j\left[\delta_j t_{r_j} + (1-\delta_j)t_{r_{j+1}}\right] + \sum_{j=1}^{k} t_{r_j} + (n-r_k)t_{r_k}\right\}^{-1}.$$

Assume that $n = 25$, $k = 20$, $r_1, r_2, \cdots, r_{20}$ takes the values 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, $k_1 = 5, k_2 = 10, k_3 = 5$. By conducting 1000 Monte Carlo simulations, failure data are generated for a system composed of two components, each following distribution $\text{Exp}(\lambda_1)$ and $\text{Exp}(\lambda_2)$, respectively. Under the k-multiple type-II censored life test, the mean and mean squared error (MSE) of the maximum likelihood estimate (MLE) and the approximate maximum likelihood estimates (AMLE) of parameters λ_1, λ_2 are computed. The results are presented in Table 1, from which it can be observed that there is no significant advantage or disadvantage between the MLE and AMLE for the two parameters.

3.2. Bayesian point estimations and interval estimations of parameters

First, the likelihood function $L(\lambda_1, \lambda_2)$ is expanded as follows:

$$L(\lambda_1,\lambda_2) = C^{+}\lambda_1^{k_1}\lambda_2^{k_2}\cdot\sum_{s_1=0}^{k_3} C_{k_3}^{s_1}\lambda_1^{s_1}\lambda_2^{k_3-s_1}\cdot\sum_{s_2=0}^{r_1-1}(-1)^{r_1-1-s_2}C_{r_1-1}^{s_2}$$

Table 1. Simulation results of the MLE and AMLE of parameter λ_1, λ_2.

λ_1	λ_2	MLE $\hat{\lambda}_1$		MLE $\hat{\lambda}_2$		AMLE $\hat{\lambda}_1$		AMLE $\hat{\lambda}_2$	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
0.5	0.8	0.4464	0.0124	0.8929	0.0468	0.3878	0.0198	0.7756	0.0294
0.5	1	0.5205	0.0121	1.0411	0.0485	0.4522	0.0111	0.9044	0.0444
0.5	1.5	0.6991	0.0630	1.3981	0.1040	0.6073	0.0292	1.2146	0.1521
1	1.5	0.8697	0.0523	1.7394	0.1986	0.7555	0.0864	1.5110	0.1068
1	2	1.0504	0.0537	2.1009	0.2147	0.9125	0.0462	1.8250	0.1850
1	2.5	1.2210	0.1146	2.4419	0.2665	1.0607	0.0533	2.1213	0.3421
1.5	2.5	1.4146	0.0988	2.8292	0.4744	1.2289	0.1426	2.4577	0.2780
1.5	3	1.5680	0.1234	3.1360	0.4935	1.3621	0.1086	2.7242	0.4344
1.5	3.5	1.7341	0.1829	3.4682	0.5133	1.5064	0.0967	3.0128	0.6240
2	3.5	1.9010	0.1600	3.8020	0.6918	1.6514	0.2348	3.3028	0.4920
2	4	2.1186	0.2071	4.2373	0.8282	1.8405	0.1711	3.6810	0.6844
2	4.5	2.2949	0.3416	4.5898	1.0267	1.9936	0.1922	3.9871	1.0316

$$\begin{aligned}
&\times e^{-(\lambda_1+\lambda_2)(r_1-1-s_2)t_{r_1}} e^{-(\lambda_1+\lambda_2)\left[\sum_{j=1}^{k} t_{r_j}+(n-r_k)t_{r_k}\right]} \\
&\times \prod_{j=1}^{k-1} \sum_{l_j=0}^{r_{j+1}-r_j-1} (-1)^{r_{j+1}-r_j-1-l_j} C_{r_{j+1}-r_j-1}^{l_j} \\
&\times e^{-(\lambda_1+\lambda_2)l_j t_{r_j}} e^{-(\lambda_1+\lambda_2)(r_{j+1}-r_j-1-l_j)t_{r_{j+1}}} \\
=&C^+ \sum_{s_1=0}^{k_3} \sum_{s_2=0}^{r_1-1} \sum_{l_1=0}^{r_2-r_1-1} \sum_{l_2=0}^{r_3-r_2-1} \cdots \sum_{l_{k-1}=0}^{r_k-r_{k-1}-1} (-1)^{r_k-k-s_2-\sum_{i=1}^{k-1} l_i} \\
&\times C_{k_3}^{s_1} C_{r_1-1}^{s_2} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2} \cdots \\
&\times C_{r_k-r_{k-1}-1}^{l_{k-1}} \lambda_1^{k_1+s_1} \lambda_2^{k_2+k_3-s_1} \cdot e^{-(\lambda_1+\lambda_2)u_2}.
\end{aligned}$$

Here $u_2 = (r_1 + l_1 - s_2)t_{r_1} + \sum_{j=1}^{k-2} (r_{j+1} - r_j - l_j + l_{j+1})t_{r_{j+1}} + (n - r_{k-1} - l_{k-1})t_{r_k}$. Like reference [36], assume that the parameters $\lambda_j, j = 1, 2$ are independent, with their prior distributions following Gamma distributions:

$$g_j(\lambda_j) = \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_j^{\alpha_j-1} e^{-\beta_j\lambda_j}, \alpha_j > 0, \beta_j > 0, j = 1, 2\,.$$

The joint prior distribution of parameters λ_1, λ_2 is then given by:

$$g(\lambda_1, \lambda_2) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \lambda_1^{\alpha_1-1} \lambda_2^{\alpha_2-1} e^{-\beta_1\lambda_1} e^{-\beta_2\lambda_2}\,.$$

Thus, the joint posterior distribution of λ_1, λ_2 is given by:

$$h(\lambda_1, \lambda_2 \mid \text{data}) = \frac{L(\lambda_1, \lambda_2)g(\lambda_1, \lambda_2)}{\int_0^{+\infty} \int_0^{+\infty} L(\lambda_1, \lambda_2)g(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2}\,.$$

From this, the Bayesian estimates of parameters λ_1, λ_2 are obtained as:

$$\hat{\lambda}_1 = \frac{\int_0^{+\infty}\int_0^{+\infty} \lambda_1 L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}{\int_0^{+\infty}\int_0^{+\infty} L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2},$$
$$\hat{\lambda}_2 = \frac{\int_0^{+\infty}\int_0^{+\infty} \lambda_2 L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}{\int_0^{+\infty}\int_0^{+\infty} L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2}.$$

The following integral is computed for $m = 0, 1$:

$$\begin{aligned}
M_1^{(m)} =& \int_0^{+\infty}\int_0^{+\infty} \lambda_1^m L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
=& C^+ \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{s_1=0}^{k_3}\sum_{s_2=0}^{r_2-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_2-\sum_{i=1}^{k-1} l_i} \\
&\times C_{k_3}^{s_1} C_{r_1-1}^{s_2} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} \\
&\times \int_0^{+\infty} \lambda_1^{m+v_1-1} e^{-\lambda_1(u_2+\beta_1)} d\lambda_1 \cdot \int_0^{+\infty} \lambda_2^{v_2-1} e^{-\lambda_2(u_2+\beta_2)} d\lambda_2 \\
=& C^+ \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{s_1=0}^{k_3}\sum_{s_2=0}^{r_1-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_2-\sum_{i=1}^{k-1} l_i} \\
&\times C_{k_3}^{s_1} C_{r_1-1}^{s_2} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} \\
&\times \frac{\Gamma(m+v_1)}{(u_2+\beta_1)^{m+v_1}} \cdot \frac{\Gamma(v_2)}{(u_2+\beta_2)^{v_2}}.
\end{aligned}$$

Here $v_1 = k_1 + s_1 + \alpha_1, v_2 = k_2 + k_3 - s_1 + \alpha_2$.

Similarly, following the same approach, we obtain:

$$\begin{aligned}
M_2^{(m)} =& \int_0^{+\infty}\int_0^{+\infty} \lambda_2^m L(\lambda_1,\lambda_2)g(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2 \\
=& C^+ \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{s_1=0}^{k_3}\sum_{s_2=0}^{r_2-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_2-\sum_{i=1}^{k-1} l_i} \\
&\times C_{k_3}^{s_1} C_{r_1-1}^{s_2} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} \\
&\times \int_0^{+\infty} \lambda_1^{v_1-1} e^{-\lambda_1(u_2+\beta_1)} d\lambda_1 \cdot \int_0^{+\infty} \lambda_2^{m+v_2-1} e^{-\lambda_2(u_2+\beta_2)} d\lambda_2 \\
=& C^+ \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \sum_{s_1=0}^{k_3}\sum_{s_2=0}^{r_1-1}\sum_{l_1=0}^{r_2-r_1-1}\sum_{l_2=0}^{r_3-r_2-1}\cdots\sum_{l_{k-1}=0}^{r_k-r_{k-1}-1}(-1)^{r_k-k-s_2-\sum_{i=1}^{k-1} l_i} \\
&\times C_{k_3}^{s_1} C_{r_1-1}^{s_2} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2}\cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} \\
&\times \frac{\Gamma(v_1)}{(u_2+\beta_1)^{v_1}} \cdot \frac{\Gamma(m+v_2)}{(u_2+\beta_2)^{m+v_2}}.
\end{aligned}$$

Finally, the Bayesian estimates of parameter λ_1, λ_2 can be expressed as:

$$\hat{\lambda}_1 = \frac{M_1^{(1)}}{M_1^{(0)}},\ \hat{\lambda}_2 = \frac{M_2^{(1)}}{M_2^{(0)}},$$

where $M_1^{(0)} = M_2^{(0)}$.

The Bayesian confidence interval estimation for parameter λ_1, λ_2 is derived as follows.

The posterior density function of λ_1 is given by:

$$\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_2 = \frac{\int_0^{+\infty} L(\lambda_1, \lambda_2) g(\lambda_1, \lambda_2) d\lambda_2}{\int_0^{+\infty}\int_0^{+\infty} L(\lambda_1, \lambda_2) g(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2}.$$

Thus, the Bayesian confidence interval estimate for parameter λ_1 at a confidence level $1-\alpha'$ of is given by $(\hat{\lambda}_{11}, \hat{\lambda}_{12})$, where $\hat{\lambda}_{11}, \hat{\lambda}_{12}$ satisfy the following equations:

$$\int_0^{\hat{\lambda}_{11}}\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_2 d\lambda_1 = \frac{\alpha'}{2}, \int_{\hat{\lambda}_{12}}^{+\infty}\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_2 d\lambda_1 = \frac{\alpha'}{2}.$$

Similarly, the posterior density function of λ_2 is given by:

$$\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_1 = \frac{\int_0^{+\infty} L(\lambda_1, \lambda_2) g(\lambda_1, \lambda_2) d\lambda_1}{\int_0^{+\infty}\int_0^{+\infty} L(\lambda_1, \lambda_2) g(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2}.$$

Thus, the Bayesian confidence interval for parameter λ_2 at a confidence level $1-\alpha'$ is given by $(\hat{\lambda}_{21}, \hat{\lambda}_{22})$, where $\hat{\lambda}_{21}, \hat{\lambda}_{22}$ satisfy the following equations:

$$\int_0^{\hat{\lambda}_{21}}\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_1 d\lambda_2 = \frac{\alpha'}{2}, \int_{\hat{\lambda}_{22}}^{+\infty}\int_0^{+\infty} h(\lambda_1, \lambda_2 \mid \text{data}) d\lambda_1 d\lambda_2 = \frac{\alpha'}{2}.$$

Example 3.1. Let $n = 30$, $k = 25$, $r_1, r_2, \cdots, r_{25}$ be given as 1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, $k_1 = 5, k_2 = 10, k_3 = 10$. The failure data of a system composed of two components in series, where the components following distributions Exp(1) and Exp(2), respectively, are generated through Monte Carlo simulation as follows.

The failure data belonging to {1} are 0.0105, 0.0399, 0.0701, 0.1753, 0.7504.

The failure data belonging to {2} are 0.0391, 0.0897, 0.0948, 0.1873, 0.3256, 0.3344, 0.4234, 0.5307, 0.5692, 1.0179.

The failure data belonging to {1,2} are 0.0156, 0.0358, 0.0496, 0.1238, 0.1923, 0.2094, 0.3265, 0.5683, 0.7179, 0.8811.

Under the k-multiple type-II censored life test, the maximum likelihood estimates (MLE) of parameters λ_1, λ_2 are obtained as: $\hat{\lambda}_1 = 1.0712$ and $\hat{\lambda}_2 = 2.1423$. The approximate maximum likelihood estimates (AMLE) are given by: $\hat{\lambda}_1 = 0.9059$ and $\hat{\lambda}_2 = 1.8118$. If the prior distribution of λ_1 follows a Gamma distribution with parameters $\alpha_1 = 1$, $\beta_1 = 2$, and the prior distribution of λ_2 follows a Gamma distribution with parameters $\alpha_2 = 2$, $\beta_2 = 3$, then the Bayesian point estimates of the parameters λ_1, λ_2 are calculated as $\hat{\lambda}_1 = 1.0148$ and $\hat{\lambda}_2 = 1.7423$. The 0.95 Bayesian credible intervals for the parameters are given by $[0.4058, 1.8937], [0.9904, 2.9108]$.

4. System composed of two parallel components with identical exponential lifetime parameters

4.1. Maximum likelihood estimation of the parameter

First, consider a system composed of J components in parallel, where the system $i, i = 1, 2, \cdots, n$ lifetime is denoted as T_i, and the lifetime of the component $l, l =$

$1, 2, \cdots, J$ forming the i^{th} system is denoted as T_{il}. The cumulative distribution function (CDF) and probability density function (PDF) of the component lifetimes are denoted as $F_l(t), f_l(t)$, respectively. Under multiple type-II censored cases with masked data, the likelihood function can be derived from the following theorem.

Theorem 4.1. *Assume that a system consists of J components in parallel. A lifetime experiment starts at time 0, and by the end of the test, only the failure time $t_{r_1}, t_{r_2}, \cdots, t_{r_k}$ of the $r_1, r_2, \cdots, r_k$ failed system is observed, where $t_{r_1} < t_{r_2} < \cdots < t_{r_k}$. Additionally, the possible failure causes of these k systems are observed. The censored sample with masked failure causes is given as $(t_{r_1}, s_{r_1}), (t_{r_2}, s_{r_2}), \cdots, (t_{r_k}, s_{r_k})$, where $s_{r_i}, i = 1, 2, \cdots, k$ represents the possible failure cause of the system. Then, the likelihood function is given by (where C^+ is a positive constant):*

$$\begin{aligned} L =& C^+ \left[\prod_{l=1}^{J} F_l(t_{r_1})\right]^{r_1-1} \cdot \prod_{j=1}^{k}\left[\sum_{m\in s_{r_j}} \frac{f_m(t_{r_j})}{F_m(t_{r_j})}\right] \cdot \left[\prod_{j=1}^{k}\prod_{l=1}^{J} F_l(t_{r_j})\right] \\ &\times \prod_{j=1}^{k-1}\left[\prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j})\right]^{r_{j+1}-r_j-1} \cdot \left[1 - \prod_{l=1}^{J} F_l(t_{r_k})\right]^{n-r_k}. \end{aligned}$$

Proof. The likelihood function is derived using the probability element method. Given that

$$\begin{aligned} (0, +\infty) = (0, t_{r_1}) \bigcup [t_{r_1}, t_{r_1} + dt_{r_1}) \bigcup [t_{r_1} + dt_{r_1}, t_{r_2}) \\ \bigcup \cdots \bigcup [t_{r_k}, t_{r_k} + dt_{r_k}) \bigcup [t_{r_k} + dt_{r_k}, +\infty), \end{aligned}$$

we can decompose $(0, +\infty)$ into the following four parts:

Part I: $(0, t_{r_1})$, **Part II:** $\bigcup_{j=1}^{k} [t_{r_j}, t_{r_j} + dt_{r_j})$,

Part III: $\bigcup_{j=1}^{k-1} [t_{r_j} + dt_{r_j}, t_{r_{j+1}})$, **Part IV:** $[t_{r_k} + dt_{r_k}, +\infty)$.

For Part I: Within the interval $(0, t_{r_1})$, there are $r_1 - 1$ system failures. The probability that a particular system i fails within $(0, t_{r_1})$ is given by:

$$P(T_i < t_{r_1}) = P(T_{i1} < t_{r_1}, T_{i2} < t_{r_1}, \cdots, T_{iJ} < t_{r_1}) = \prod_{l=1}^{J} F_l(t_{r_1}).$$

Thus, the probability that exactly $r_1 - 1$ systems fail within the interval $(0, t_{r_1})$ is given by (where C_1^+ is a positive constant): $C_1^+ \left[\prod_{l=1}^{J} F_l(t_{r_1})\right]^{r_1-1}$.

For Part II: The $r_1^{th}, r_2^{th}, \cdots, r_k^{th}$ system failure occurs within the interval $[t_{r_1}, t_{r_1} + dt_{r_1}), [t_{r_2}, t_{r_2} + dt_{r_2}), \cdots, [t_{r_k}, t_{r_k} + dt_{r_k})$. The failure time of the r_j^{th}, $j = 1, 2, \cdots, k$ failed system is recorded as t_{r_j}, thus:

$$\begin{aligned} &P(t_{r_j} \le T_{r_j} < t_{r_j} + dt_{r_j}) \\ =& \sum_{m\in s_{r_j}} P(T_{r_j 1} < t_{r_j}, \cdots, T_{r_j(m-1)} < t_{r_j}, t_{r_j} \le T_{r_j} < t_{r_j} + dt_{r_j}, \end{aligned}$$

$$
\begin{aligned}
& T_{r_j(m+1)} < t_{r_j}, \cdots, T_{r_jJ} < t_{r_j}) \\
&= \sum_{m \in s_{r_j}} f_m(t_{r_j}) dt_{r_j} \prod_{l=1}^{J_m} F_l(t_{r_j}) \\
&= \sum_{m \in s_{r_j}} \frac{f_m(t_{r_j})}{F_m(t_{r_j})} \cdot \prod_{l=1}^{J} F_l(t_{r_j}) dt_{r_j}
\end{aligned}
$$

where $J_m = \{1, \cdots, m-1, m+1, \cdots, n\}$.

Therefore, the density function corresponding to the k failed systems in Part II is given by (where C_2^+ is a positive constant):

$$
C_2^+ \prod_{j=1}^{k} \left[\sum_{m \in s_{r_j}} \frac{f_m(t_{r_j})}{F_m(t_{r_j})} \cdot \prod_{l=1}^{J} F_l(t_{r_j}) dt_{r_j} \right] = C_2^+ \prod_{j=1}^{k} \left[\sum_{m \in s_{r_j}} \frac{f_m(t_{r_j})}{F_m(t_{r_j})} \right] \cdot \left[\prod_{j=1}^{k} \prod_{l=1}^{J} F_l(t_{r_j}) \right].
$$

For Part III: The number of system failures occurring in each interval $[t_{r_1} + dt_{r_1}, t_{r_2}), [t_{r_2} + dt_{r_2}, t_{r_3}), \cdots, [t_{r_{k-1}} + dt_{r_{k-1}}, t_{r_k})$ is given by $r_2 - r_1 - 1, r_3 - r_2 - 1, \cdots, r_k - r_{k-1} - 1$. Thus, for a particular system i, the probability of failure within the interval $[t_{r_j} + dt_{r_j}, t_{r_{j+1}})$ for $j = 1, 2, \cdots k-1$ is:

$$
\begin{aligned}
& P(t_{r_j} + dt_{r_j} \leq T_i < t_{r_{j+1}}) \\
=& P(T_i < t_{r_{j+1}}) - P(T_i < t_{r_j} + dt_{r_j}) \\
=& P(T_{i1} < t_{r_{j+1}}, \cdots, T_{iJ} < t_{r_{j+1}}) - P(T_{i1} < t_{r_j} + dt_{r_j}, \cdots, T_{iJ} < t_{r_j} + dt_{r_j}) \\
=& \prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j} + dt_{r_j}).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\lim_{dt_{r_j} \to 0} P(t_{r_j} + dt_{r_j} \leq T_i < t_{r_{j+1}}) &= \lim_{dt_{r_j} \to 0} \left[\prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j} + dt_{r_j}) \right] \\
&= \prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j}).
\end{aligned}
$$

The density function corresponding to the $r_{j+1} - r_j - 1$ failed systems within the interval $[t_{r_j} + dt_{r_j}, t_{r_{j+1}}), j = 1, 2, \cdots, k-1$ is given by (where C_{3j}^+ is a positive integer): $C_{3j}^+ \left[\prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j}) \right]^{r_{j+1} - r_j - 1}$.

Thus, the density function corresponding to all failed systems in Part III is given by (where C_3^+ is a positive constant): $C_3^+ \prod_{j=1}^{k-1} \left[\prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j}) \right]^{r_{j+1} - r_j - 1}$.

For Part IV: There are $n - r_k$ systems that fail within the interval $[t_{r_k} + dt_{r_k}, +\infty)$. The probability that a given system i fails within the interval $[t_{r_k} + dt_{r_k}, +\infty)$ is:

$$
\begin{aligned}
P(T_i \geq t_{r_k} + dt_{r_k}) &= 1 - P(T_i < t_{r_k} + dt_{r_k}) \\
&= 1 - P(T_{i1} < t_{r_k} + dt_{r_k}, \cdots, T_{iJ} < t_{r_k} + dt_{r_k})
\end{aligned}
$$

$$= 1 - \prod_{l=1}^{J} F_l(t_{r_k} + dt_{r_k}),$$

$$\lim_{dt_{r_k} \to 0} P(T_i \geq t_{r_k} + dt_{r_k}) = 1 - \prod_{l=1}^{J} F_l(t_{r_k}).$$

Therefore, the probability density function corresponding to the $n - r_k$ failed systems in Part IV is (where C_4^+ is a positive constant): $C_4^+ \left[1 - \prod_{l=1}^{J} F_l(t_{r_k})\right]^{n-r_k}$. Combining all parts, the likelihood function is given by:

$$L = C^+ \left[\prod_{l=1}^{J} F_l(t_{r_1})\right]^{r_1 - 1} \cdot \prod_{j=1}^{k} \left[\sum_{m \in s_{r_j}} \frac{f_m(t_{r_j})}{F_m(t_{r_j})}\right] \cdot \left[\prod_{j=1}^{k} \prod_{l=1}^{J} F_l(t_{r_j})\right]$$
$$\times \prod_{j=1}^{k-1} \left[\prod_{l=1}^{J} F_l(t_{r_{j+1}}) - \prod_{l=1}^{J} F_l(t_{r_j})\right]^{r_{j+1} - r_j - 1} \cdot \left[1 - \prod_{l=1}^{J} F_l(t_{r_k})\right]^{n - r_k}.$$

□

Let system i be composed of component 1 and component 2 in a parallel configuration (i.e. $J = 2$), where the lifetimes T_{i1}, T_{i2} of component 1 and component 2 are independent and both follow an exponential distribution with parameter λ. In this case:

$$F_1(t) = F_2(t) = 1 - e^{-\lambda t}\,,\ \overline{F}_1(t) = \overline{F}_2(t) = e^{-\lambda t}\,,\ h_1(t) = h_2(t) = \lambda\,.$$

At time 0, n systems are subjected to a lifetime test. By the end of the test, only k failure data points are retained from the failed systems. The corresponding failure times of these k systems are denoted sequentially as $t_{r_1}, t_{r_2}, \cdots, t_{r_k}$, where $1 \leq r_1 < r_2 < \cdots < r_k \leq n$. Additionally, $t_{r_1} < t_{r_2} < \cdots < t_{r_k}$. The potential failure causes of these k failed systems are observed and recorded as $s_{r_i}, i = 1, 2, \cdots, k$. Under this scenario, the system failure sample data with censoring is given by $(t_{r_1}, s_{r_1}), (t_{r_2}, s_{r_2}), \cdots, (t_{r_k}, s_{r_k})$, where $s_{r_i}, i = 1, 2, \cdots, k$ represents the failure cause for the corresponding system $\{1\}, \{2\}, \{1, 2\}$ with the following interpretations: $\{1\}$ indicates that the system failure is caused by component 1. $\{2\}$ indicates that the system failure is caused by component 2. $\{1, 2\}$ indicates that the exact failure cause is unknown, implying that a censoring effect has occurred.

Without loss of generality, rearrange $t_{r_1}, t_{r_2}, \cdots, t_{r_k}$ according to the failure causes: $y_1, y_2, \cdots, y_{k_1}$ correspond to system failures caused by component 1. y_{k_1+1}, $y_{k_1+2}, \cdots, y_{k_1+k_2}$ correspond to system failures caused by component 2. $y_{k_1+k_2+1}$, $y_{k_1+k_2+2}, \cdots, y_k$ correspond to censored failures. At this point, we have: $k = k_1 + k_2 + k_3$.

From the above theorem, the likelihood function is given by (where C^+ is a positive constant):

$$L(\lambda)$$
$$= C^+ (1 - e^{-\lambda t_{r_1}})^{2(r_1 - 1)} \cdot \prod_{j=1}^{k_1} \frac{\lambda e^{-\lambda y_j}}{1 - e^{-\lambda y_j}} \cdot \prod_{j=1}^{k_2} \frac{\lambda e^{-\lambda y_{k_1+j}}}{1 - e^{-\lambda y_{k_1+j}}}$$
$$\times \prod_{j=1}^{k_3} \frac{2\lambda e^{-\lambda y_{k_1+k_2+j}}}{1 - e^{-\lambda y_{k_1+k_2+j}}} \cdot \prod_{j=1}^{k} (1 - e^{-\lambda t_{r_j}})^2$$

$$
\begin{aligned}
&\times \prod_{j=1}^{k-1}\left[(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2\right]^{r_{j+1}-r_j-1}\cdot\left[1-(1-e^{-\lambda t_{r_k}})^2\right]^{n-r_k}\\
=&C^{+}2^{k_3}\lambda^k(1-e^{-\lambda t_{r_1}})^{2(r_1-1)}\prod_{j=1}^{k}(1-e^{-\lambda t_{r_j}})\cdot e^{-\lambda\sum\limits_{j=1}^{k}t_{r_j}}\\
&\times \prod_{j=1}^{k-1}\left[(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2\right]^{r_{j+1}-r_j-1}\cdot\left[1-(1-e^{-\lambda t_{r_k}})^2\right]^{n-r_k},\\
&\ln L(\lambda)\\
=&\ln C^{+}+k_3\ln 2+k\ln\lambda+2(r_1-1)\ln(1-e^{-\lambda t_{r_1}})+\sum_{j=1}^{k}\ln(1-e^{-\lambda t_{r_j}})\\
&-\lambda\sum_{j=1}^{k}t_{r_j}+\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\ln\left[(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2\right]\\
&+(n-r_k)\ln\left[1-(1-e^{-\lambda t_{r_k}})^2\right],\\
&\frac{d\ln L(\lambda)}{d\lambda}\\
=&\frac{k}{\lambda}+2(r_1-1)\frac{t_{r_1}e^{-\lambda t_{r_1}}}{1-e^{-\lambda t_{r_1}}}+\sum_{j=1}^{k}\frac{t_{r_j}e^{-\lambda t_{r_j}}}{1-e^{-\lambda t_{r_j}}}-\sum_{j=1}^{k}t_{r_j}\\
&+2\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\,\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}\left(1-e^{-\lambda t_{r_{j+1}}}\right)-t_{r_j}e^{-\lambda t_{r_j}}\left(1-e^{-\lambda t_{r_j}}\right)}{\left(1-e^{-\lambda t_{r_{j+1}}}\right)^2-\left(1-e^{-\lambda t_{r_j}}\right)^2}\\
&+2(n-r_k)\frac{-t_{r_k}e^{-\lambda t_{r_k}}\left(1-e^{-\lambda t_{r_k}}\right)}{1-\left(1-e^{-\lambda t_{r_k}}\right)^2}.
\end{aligned}
$$

Let $\frac{d\ln L(\lambda)}{d\lambda}=0$, and we obtain the following equation:

$$
\begin{aligned}
&\frac{1}{\lambda}+2\frac{r_1-1}{k}\frac{t_{r_1}e^{-\lambda t_{r_1}}}{1-e^{-\lambda t_{r_1}}}+\frac{1}{k}\sum_{j=1}^{k}\frac{t_{r_j}e^{-\lambda t_{r_j}}}{1-e^{-\lambda t_{r_j}}}+2\frac{n-r_k}{k}\frac{-t_{r_k}e^{-\lambda t_{r_k}}(1-e^{-\lambda t_{r_k}})}{1-(1-e^{-\lambda t_{r_k}})^2}\\
&+\frac{2}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}(1-e^{-\lambda t_{r_{j+1}}})-t_{r_j}e^{-\lambda t_{r_j}}(1-e^{-\lambda t_{r_j}})}{(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2}\\
=&\frac{1}{k}\sum_{j=1}^{k}t_{r_j}. \qquad (4.1)
\end{aligned}
$$

The root of Equation (4.1) provides the maximum likelihood estimate (MLE) $\hat{\lambda}$ of the parameter λ.

Lemma 4.1. *Equation* (4.1) *with respect to* λ *has a unique positive real root.*

Proof. Define the function

$$
g(\lambda)=\frac{1}{\lambda}+2\frac{r_1-1}{k}\frac{t_{r_1}e^{-\lambda t_{r_1}}}{1-e^{-\lambda t_{r_1}}}+\frac{1}{k}\sum_{j=1}^{k}\frac{t_{r_j}e^{-\lambda t_{r_j}}}{1-e^{-\lambda t_{r_j}}}+2\frac{n-r_k}{k}\frac{-t_{r_k}e^{-\lambda t_{r_k}}(1-e^{-\lambda t_{r_k}})}{1-(1-e^{-\lambda t_{r_k}})^2}
$$

$$+\frac{2}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}(1-e^{-\lambda t_{r_{j+1}}})-t_{r_j}e^{-\lambda t_{r_j}}(1-e^{-\lambda t_{r_j}})}{(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2}.$$

Since

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}=\lim_{\lambda\to 0}\frac{t_{r_1}}{e^{2\lambda t_{r_1}}-1}=+\infty,\\
&\lim_{\lambda\to +\infty}\frac{t_{r_1}e^{-2\lambda t_{r_1}}}{1-e^{-2\lambda t_{r_1}}}=\lim_{\lambda\to +\infty}\frac{t_{r_1}}{e^{2\lambda t_{r_1}}-1}=0\,,\\
&\lim_{\lambda\to 0}\frac{-t_{r_k}e^{-\lambda t_{r_k}}\left(1-e^{-\lambda t_{r_k}}\right)}{1-\left(1-e^{-\lambda t_{r_k}}\right)^2}=0\,,\\
&\quad\lim_{\lambda\to +\infty}\frac{-t_{r_k}\,e^{-\lambda t_{r_k}}\,(1-e^{-\lambda t_{r_k}})}{1-(1-e^{-\lambda t_{r_k}})^2}\\
&=\lim_{\lambda\to +\infty}\frac{-t_{r_k}\,(e^{\lambda t_{r_k}}-1)}{e^{2\lambda t_{r_k}}-(e^{\lambda t_{r_k}}-1)^2}\\
&=\lim_{\lambda\to +\infty}\frac{-t_{r_k}\,(e^{\lambda t_{r_k}}-1)}{2e^{\lambda t_{r_k}}-1}\\
&=-\frac{t_{r_k}}{2}\,,
\end{aligned}$$

and $1-e^{-\lambda t_{r_{j+1}}}\sim\lambda t_{r_{j+1}}$, $1-e^{-\lambda t_{r_j}}\sim\lambda t_{r_j}$ as $\lambda\to 0$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}\left(1-e^{-\lambda t_{r_{j+1}}}\right)-t_{r_j}e^{-\lambda t_{r_j}}\left(1-e^{-\lambda t_{r_j}}\right)}{\left(1-e^{-\lambda t_{r_{j+1}}}\right)^2-\left(1-e^{-\lambda t_{r_j}}\right)^2}\\
&=\lim_{\lambda\to 0}\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}\cdot\lambda t_{r_{j+1}}-t_{r_j}e^{-\lambda t_{r_j}}\cdot\lambda t_{r_j}}{\left(\lambda t_{r_{j+1}}\right)^2-\left(\lambda t_{r_j}\right)^2}\\
&=\lim_{\lambda\to 0}\frac{t_{r_{j+1}}^2e^{-\lambda t_{r_{j+1}}}-t_{r_j}^2e^{-\lambda t_{r_j}}}{\lambda(t_{r_{j+1}}^2-t_{r_j}^2)}\\
&=+\infty\,,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\lambda\to +\infty}\frac{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}(1-e^{-\lambda t_{r_{j+1}}})-t_{r_j}e^{-\lambda t_{r_j}}(1-e^{-\lambda t_{r_j}})}{(1-e^{-\lambda t_{r_{j+1}}})^2-(1-e^{-\lambda t_{r_j}})^2}\\
&=\frac{1}{2}\lim_{\lambda\to +\infty}\frac{-t_{r_{j+1}}^2e^{-\lambda t_{r_{j+1}}}\left(1-e^{-\lambda t_{r_{j+1}}}\right)+t_{r_{j+1}}^2e^{-2\lambda t_{r_{j+1}}}+t_{r_j}^2e^{-\lambda t_{r_j}}\left(1-e^{-\lambda t_{r_j}}\right)-t_{r_j}^2e^{-2\lambda t_{r_j}}}{t_{r_{j+1}}e^{-\lambda t_{r_{j+1}}}\left(1-e^{-\lambda t_{r_{j+1}}}\right)-t_{r_j}e^{-\lambda t_{r_j}}\left(1-e^{-\lambda t_{r_j}}\right)}\\
&=\frac{1}{2}\lim_{\lambda\to +\infty}\frac{-t_{r_{j+1}}^2e^{-\lambda(t_{r_{j+1}}-t_{r_j})}(1-e^{-\lambda t_{r_{j+1}}})+t_{r_{j+1}}^2e^{-\lambda(2t_{r_{j+1}}-t_{r_j})}+t_{r_j}^2(1-e^{-\lambda t_{r_j}})-t_{r_j}^2e^{-\lambda t_{r_j}}}{t_{r_{j+1}}e^{-\lambda(t_{r_{j+1}}-t_{r_j})}(1-e^{-\lambda t_{r_{j+1}}})-t_{r_j}(1-e^{-\lambda t_{r_j}})}\\
&=-\frac{t_{r_j}^2}{2t_{r_j}}\\
&=-\frac{t_{r_j}}{2}\,.
\end{aligned}$$

Thus, $\lim_{\lambda\to 0}g(\lambda)=+\infty$, $\lim_{\lambda\to +\infty}g(\lambda)=-\frac{1}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)t_{r_j}-\frac{n-k}{k}t_{r_k}$.

Denote $x = e^{-\lambda}$,

$$
\begin{aligned}
&A_j \\
=&t_{r_{j+1}}^2 x^{3t_{r_{j+1}}} + 2(t_{r_{j+1}} - t_{r_j})^2\, x^{t_{r_j}+t_{r_{j+1}}} - (t_{r_{j+1}} - 2t_{r_j})^2\, x^{2t_{r_j}+t_{r_{j+1}}} \\
&- (2t_{r_{j+1}} - t_{r_j})^2\, x^{t_{r_j}+2t_{r_{j+1}}} + 2(t_{r_{j+1}} - t_{r_j})^2 x^{2t_{r_j}+2t_{r_{j+1}}} + t_{r_j}^2 x^{3t_{r_j}}\ , \\
&g'(\lambda) \\
=&-\frac{1}{\lambda^2} - 2\frac{r_1-1}{k}\frac{t_{r_1}^2 e^{-\lambda t_{r_1}}}{\left(1-e^{-\lambda t_{r_1}}\right)^2} - \frac{1}{k}\sum_{j=1}^{k}\frac{t_{r_j}^2 e^{-\lambda t_{r_j}}}{\left(1-e^{-\lambda t_{r_j}}\right)^2} + 2\frac{n-r_k}{k} t_{r_k}\left(\frac{e^{-\lambda t_{r_k}}-1}{2-e^{-\lambda t_{r_k}}}\right)' \\
&+ \frac{2}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\left(\frac{t_{r_{j+1}} x^{t_{r_{j+1}}} - t_{r_{j+1}} x^{2t_{r_{j+1}}} - t_{r_j} x^{t_{r_j}} + t_{r_j} x^{2t_{r_j}}}{-2x^{t_{r_{j+1}}} + x^{2t_{r_{j+1}}} + 2x^{t_{r_j}} - x^{2t_{r_j}}}\right)_x' \frac{dx}{d\lambda} \\
=&-\frac{1}{\lambda^2} - 2\frac{r_1-1}{k}\frac{t_{r_1}^2 e^{-\lambda t_{r_1}}}{\left(1-e^{-\lambda t_{r_1}}\right)^2} - \frac{1}{k}\sum_{j=1}^{k}\frac{t_{r_j}^2 e^{-\lambda t_{r_j}}}{\left(1-e^{-\lambda t_{r_j}}\right)^2} - 2\frac{n-r_k}{k}\frac{t_{r_k}^2 e^{-\lambda t_{r_k}}}{\left(2-e^{-\lambda t_{r_k}}\right)^2} \\
&- \frac{2}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1)\frac{A_j}{(-2x^{t_{r_{j+1}}} + x^{2t_{r_{j+1}}} + 2x^{t_{r_j}} - x^{2t_{r_j}})^2} \\
=&-\frac{1}{\lambda^2} - 2\frac{r_1-1}{k}\frac{t_{r_1}^2 e^{-\lambda t_{r_1}}}{\left(1-e^{-\lambda t_{r_1}}\right)^2} - \frac{1}{k}\sum_{j=1}^{k}\frac{t_{r_j}^2 e^{-\lambda t_{r_j}}}{\left(1-e^{-\lambda t_{r_j}}\right)^2} \\
&- 2\frac{n-r_k}{k}\frac{t_{r_k}^2 e^{-\lambda t_{r_k}}}{\left(2-e^{-\lambda t_{r_k}}\right)^2} - \frac{2}{k}\sum_{j=1}^{k-1}(r_{j+1}-r_j-1) \\
&\times \frac{(t_{r_{j+1}}^2 x^{t_{r_{j+1}}} + t_{r_j}^2 x^{t_{r_j}})(x^{t_{r_{j+1}}} - x^{t_{r_j}})^2 + 2(t_{r_{j+1}} - t_{r_j})^2 x^{t_{r_{j+1}}+t_{r_j}}(1-x^{t_{r_j}})(1-x^{t_{r_{j+1}}})}{(-2x^{t_{r_{j+1}}} + x^{2t_{r_{j+1}}} + 2x^{t_{r_j}} - x^{2t_{r_j}})^2} \\
<&0.
\end{aligned}
$$

Thus, the equation has a unique positive real root $g(\lambda) = \frac{1}{k}\sum_{j=1}^{k} t_{r_j}$. □

4.2. Bayesian point estimation and interval estimation of the parameter

First, the likelihood function $L(\lambda)$ is expanded as follows:

$$
\begin{aligned}
L(\lambda) =& C^{+} 2^{k_3}\, \lambda^k \cdot \sum_{k_1=0}^{2(r_1-1)} (-1)^{2(r_1-1)-k_1} C_{2(r_1-1)}^{k_1} e^{-[2(r_1-1)-k_1]\lambda t_{r_1}} \\
&\times \prod_{j=1}^{k}\left[\sum_{s_j=0}^{1}(-1)^{1-s_j} e^{-(1-s_j)\lambda t_{r_j}}\right] \cdot e^{-\lambda\sum_{j=1}^{k} t_{r_j}} \\
&\times \prod_{j=1}^{k-1}\left[\sum_{l_j=0}^{r_{j+1}-r_j-1} (-1)^{r_{j+1}-r_j-1-l_j} C_{r_{j+1}-r_j-1}^{l_j}\right.
\end{aligned}
$$

$$\times (1-e^{-\lambda t_{r_{j+1}}})^{2l_j}(1-e^{-\lambda t_{r_j}})^{2(r_{j+1}-r_j-1-l_j)}\Bigg]$$

$$\times e^{-(n-r_k)\lambda t_{r_k}} \sum_{k_2=0}^{n-r_k} (-1)^{n-r_k-k_2} 2^{k_2} C_{n-r_k}^{k_2} e^{-(n-r_k-k_2)\lambda t_{r_k}}$$

$$= C^{+} 2^{k_3} \lambda^{k} \cdot \sum_{k_1=0}^{2(r_1-1)} (-1)^{2(r_1-1)-k_1} C_{2(r_1-1)}^{k_1} e^{-[2(r_1-1)-k_1]\lambda t_{r_1}}$$

$$\times \prod_{j=1}^{k}\left[\sum_{s_j=0}^{1}(-1)^{1-s_j} e^{-(1-s_j)\lambda t_{r_j}}\right] \cdot e^{-\lambda \sum\limits_{j=1}^{k} t_{r_j}}$$

$$\times \prod_{j=1}^{k-1}\Bigg[\sum_{l_j=0}^{r_{j+1}-r_j-1} \sum_{a_j=0}^{2l_j} \sum_{b_j=0}^{2(r_{j+1}-r_j-1-l_j)} (-1)^{3(r_{j+1}-r_j-1)-l_j-a_j-b_j}$$

$$\times C_{r_{j+1}-r_j-1}^{l_j} C_{2l_j}^{a_j} C_{2(r_{j+1}-r_j-1-l_j)}^{b_j} e^{-\lambda\left\{(2l_j-a_j)t_{r_{j+1}}+[2(r_{j+1}-r_j-1-l_j)-b_j]t_j\right\}}\Bigg]$$

$$\times e^{-(n-r_k)\lambda t_{r_k}} \sum_{k_2=0}^{n-r_k} (-1)^{n-r_k-k_2} 2^{k_2} C_{n-r_k}^{k_2} e^{-(n-r_k-k_2)\lambda t_{r_k}}$$

$$= C^{+} 2^{k_3} \lambda^{k} \sum_{k_1=0}^{2(r_1-1)} \sum_{s_1=0}^{1} \sum_{s_2=0}^{1} \cdots \sum_{s_k=0}^{1} \sum_{l_1=0}^{r_2-r_1-1} \sum_{a_1=0}^{2l_1} \sum_{b_1=0}^{2(r_2-r_1-1-l_1)}$$

$$\times \sum_{l_2=0}^{r_3-r_2-1} \sum_{a_2=0}^{2l_2} \sum_{b_2=0}^{2(r_3-r_2-1-l_2)} \cdots \sum_{l_{k-1}=0}^{r_k-r_{k-1}-1} \sum_{a_{k-1}=0}^{2l_{k-1}} \sum_{b_{k-1}=0}^{2(r_k-r_{k-1}-1-l_{k-1})} \sum_{k_2=0}^{n-r_k}$$

$$\times 2^{k_2} (-1)^{2r_1-2-k_1+k-\sum\limits_{j=1}^{k} s_j+\sum\limits_{j=1}^{k-1}[3(r_{j+1}-r_j-1)-l_j-a_j-b_j]+n-r_k-k_2}$$

$$\times C_{2(r_1-1)}^{k_1} C_{r_2-r_1-1}^{l_1} C_{r_3-r_2-1}^{l_2} \cdots$$

$$\times C_{r_k-r_{k-1}-1}^{l_{k-1}} C_{2l_j}^{a_j} C_{2(r_{j+1}-r_j-1-l_j)}^{b_j} C_{n-r_k}^{r_k} \cdot e^{-\lambda w}.$$

Here

$$w = (2(r_1-1)-k_1)t_{r_1} + \sum_{j=1}^{k} t_{r_j}$$
$$+ \sum_{j=1}^{k-1}\left\{(2l_j-a_j)t_{r_{j+1}} + [2(r_{j+1}-r_j-1-l_j)-b_j]\, t_j\right\}$$
$$+ (2(n-r_k)-k_2)t_{r_k}.$$

Assume that the prior distribution of the parameter λ is a Gamma distribution:

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}, \alpha > 0, \beta > 0\,.$$

Then, the posterior density function of λ is given by: $h(\lambda \mid \text{data}) = \frac{L(\lambda)g(\lambda)}{\int_0^{+\infty} L(\lambda)g(\lambda)d\lambda}$.

Thus, the Bayesian estimate of the parameter λ is: $\hat{\lambda} = \frac{\int_0^{+\infty} \lambda L(\lambda)g(\lambda)d\lambda}{\int_0^{+\infty} L(\lambda)g(\lambda)d\lambda}$.

Compute the integral for $m = 0, 1$:

$$
\begin{aligned}
&M^{(m)} \\
&= \int_0^{+\infty} \lambda^m L(\lambda) g(\lambda) d\lambda \\
&= C^{+} 2^{k_3} \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{k_1=0}^{2(r_1-1)} \sum_{s_1=0}^{1} \sum_{s_2=0}^{1} \cdots \sum_{s_k=0}^{1} \sum_{l_1=0}^{r_2-r_1-1} \sum_{a_1=0}^{2l_1} \sum_{b_1=0}^{2(r_2-r_1-1-l_1)} \sum_{l_2=0}^{r_3-r_2-1} \sum_{a_2=0}^{2l_2} \\
&\quad \times \sum_{b_2=0}^{2(r_3-r_2-1-l_2)} \cdots \sum_{l_{k-1}=0}^{r_k-r_{k-1}-1} \sum_{a_{k-1}=0}^{2l_{k-1}} \sum_{b_{k-1}=0}^{2(r_k-r_{k-1}-1-l_{k-1})} \sum_{k_2=0}^{n-r_k} 2^{k_2} \\
&\quad \times (-1)^{2r_1-2-k_1+k-\sum_{j=1}^{k} s_j + \sum_{j=1}^{k-1}[3(r_{j+1}-r_j-1)-l_j-a_j-b_j]+n-r_k-k_2} C_{2(r_1-1)}^{k_1} C_{r_2-r_1-1}^{l_1} \\
&\quad \times C_{r_3-r_2-1}^{l_2} \cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} C_{2l_j}^{a_j} C_{2(r_{j+1}-r_j-1-l_j)}^{b_j} C_{n-r_k}^{r_k} \\
&\quad \times \int_0^{+\infty} \lambda^{k+m+\alpha-1} e^{-\lambda(w+\beta)} d\lambda \\
&= C^{+} 2^{k_3} \frac{\beta^\alpha}{\Gamma(\alpha)} \sum_{k_1=0}^{2(r_1-1)} \sum_{s_1=0}^{1} \sum_{s_2=0}^{1} \cdots \sum_{s_k=0}^{1} \sum_{l_1=0}^{r_2-r_1-1} \sum_{a_1=0}^{2l_1} \sum_{b_1=0}^{2(r_2-r_1-1-l_1)} \sum_{l_2=0}^{r_3-r_2-1} \\
&\quad \times \sum_{a_2=0}^{2l_2} \sum_{b_2=0}^{2(r_3-r_2-1-l_2)} \cdots \sum_{l_{k-1}=0}^{r_k-r_{k-1}-1} \sum_{a_{k-1}=0}^{2l_{k-1}} \sum_{b_{k-1}=0}^{2(r_k-r_{k-1}-1-l_{k-1})} \sum_{k_2=0}^{n-r_k} \\
&\quad \times 2^{k_2} (-1)^{2r_1-2-k_1+k-\sum_{j=1}^{k} s_j + \sum_{j=1}^{k-1}[3(r_{j+1}-r_j-1)-l_j-a_j-b_j]+n-r_k-k_2} C_{2(r_1-1)}^{k_1} C_{r_2-r_1-1}^{l_1} \\
&\quad \times C_{r_3-r_2-1}^{l_2} \cdots C_{r_k-r_{k-1}-1}^{l_{k-1}} C_{2l_j}^{a_j} C_{2(r_{j+1}-r_j-1-l_j)}^{b_j} C_{n-r_k}^{r_k} \cdot \frac{\Gamma(k+m+\alpha)}{(w+\beta)^{k+m+\alpha}}.
\end{aligned}
$$

Thus, the Bayesian estimate of the parameter λ can be expressed as: $\hat{\lambda} = \frac{M^{(1)}}{M^{(0)}}$. Next, we derive the Bayesian interval estimation for the parameter λ as follows. Since the posterior density function of λ is given by:

$$
h(\lambda \mid \text{data}) = \frac{L(\lambda) g(\lambda)}{\int_0^{+\infty} L(\lambda) g(\lambda) d\lambda} .
$$

The Bayesian interval estimation of λ at the confidence level $1 - \alpha'$ is given by $(\hat{\lambda}_1, \hat{\lambda}_2)$, where $\hat{\lambda}_1, \hat{\lambda}_2$ satisfy the following equations:

$$
\int_0^{\hat{\lambda}_1} h(\lambda \mid \text{data}) d\lambda = \frac{\alpha'}{2} , \int_{\hat{\lambda}_2}^{+\infty} h(\lambda \mid \text{data}) d\lambda = \frac{\alpha'}{2} .
$$

Example 4.1. Let $n = 25$, $k = 20$, $r_1, r_2, \cdots, r_{20}$ be given as: 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, $k_1 = 8, k_2 = 8, k_3 = 4$, respective. Using Monte Carlo simulation, the failure data of a system composed of two components in parallel, each following an exponential distribution Exp(2), are generated as follows.

The failure data belonging to $\{1\}$ are 0.3222, 0.3839, 0.3950, 0.5075, 0.5077, 0.8455, 1.0428, 1.1193.

The failure data belonging to $\{2\}$ are 0.1327, 0.3670, 0.4689, 0.5017, 0.5767, 0.8140, 1.2409, 1.3270.

The failure data belonging to $\{1, 2\}$ are 0.1417, 0.3270, 0.8992, 0.9238.

Under a k-multiple Type-II censoring lifetime test, the maximum likelihood estimate (MLE) of the parameter λ is calculated as $\hat{\lambda} = 2.1612$. If the prior distribution follows a Gamma distribution with parameters $\alpha = 2$, $\beta = 3$, the Bayesian point estimate of λ is computed as $\hat{\lambda} = 1.9740$. The 0.95 Bayesian confidence interval estimate for λ is given by $[1.4518, 2.5809]$.

5. Conclusion

In the case of multiple type-II censoring, the maximum likelihood estimations, Bayesian point estimations and interval estimations of parameters are studied for the masked data of the series system of two components with exponential life distribution (same and different parameters). For the masked data of parallel system $(J = 2)$, only the case with the same parameter is given. For the case with different parameters, due to the complexity of the likelihood function, the likelihood equation involves the solution of the two dimensional transcendental equation, which needs further study.

As for the case of multiple type-II censoring $J \geq 3$, the expression of likelihood function is more complicated due to the complexity of masked causes, which also needs further study.

Moreover, this study assumes that masking and failure causes are independent, and if this assumption is relaxed, it will also be one of the contents of future researches.

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