Journal of Applied Analysis and Computation
Volume 16, Number 2, April 2026, 860–894

Website:http://www.jaac-online.com
DOI:10.11948/20230330

BIFURCATION AND SPATIOTEMPORAL PATTERNS IN A HOMOGENEOUS DIFFUSIVE GIERER-MEINHARDT SYSTEM*

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Abstract This paper investigates an activator-inhibitor system with diffusion under homogeneous Neumann boundary conditions. Firstly, we consider the Hopf bifurcation at the positive equilibrium, and obtain the conditions for determining the bifurcation direction and stability of the bifurcating periodic solutions. Secondly, we demonstrate that the system undergoes a Turing-Hopf bifurcation with codimension-two. By calculating the normal form on the center manifold, we show that the system has the complex spatiotemporal dynamics near the Turing-Hopf bifurcation point. Moreover, the Turing instability of the positive equilibrium is discussed. By the bifurcation theory, we establish the local structure of the steady state bifurcations, and describe some conditions for determining the direction of bifurcations. Finally, some numerical simulations are carried out to explain and supplement the results of various bifurcation analyses.

Keywords Gierer-Meinhardt system, Hopf bifurcation, Turing-Hopf bifurcation, steady state bifurcation.

MSC(2010) 35K57.

1. Introduction

A chemical reaction diffusion system is said to be an activation inhibition reaction if the reactant has a blocking effect on the reaction. The mechanism of the reaction is the bireactant interaction, which has been widely studied and applied to a wide range of fields of developmental biology, with varying degrees of rationality [17].

To study relatively simple molecular mechanisms based on auto and cross catalysis, Gierer and Meinhardt in 1972 [6] proposed the following Gierer-Meinhardt reaction-diffusion activator-

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*This authors were supported by the National Natural Science Foundation of China (11901370, 12301215), Natural Science Basic Research Plan in Shannxi Province (2019JQ516), Natural Science Foundation of Shaanxi Provincial Department of Education grant (19JK0142), Natural Science Foundation of China (2019M653578), Shaanxi Provincial Association for Science and Technology, China (20200508), and Zhejiang Provincial Natural Science Foundation of China (LQ23A010009).
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inhibitor system:

$$\begin{cases} \dfrac{\partial U}{\partial t} - D_U \Delta U = \alpha_0 \alpha + \kappa \alpha \dfrac{U^m}{V^n} - \gamma U, & x \in \Omega, t > 0, \\ \dfrac{\partial V}{\partial t} - D_V \Delta V = c' \rho \dfrac{U^l}{V^k} - rV, & x \in \Omega, t > 0, \\ \partial_\nu U = \partial_\nu V = 0, & x \in \partial\Omega, t > 0, \\ U(x,0) = U_0(x) \geq 0,\ V(x,0) = V_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, U and V represent the concentration of two reactants with the corresponding diffusion rate $D_U > 0$ and $D_V > 0$, respectively. Here α_0 represents the reaction rate of self activating substances, α indicates the degree of inhibition of self activating substances by inhibitory substances, ρ represents the source concentration, r, γ denote the decomposition rates, and κ, c' are coefficients associated with the generation of two reactants. U^m, U^l, V^n and V^k are the powers of U and V, which are used to measure the generation and destruction of activator and inhibitor caused by autocatalysis and cross catalysis, respectively.

The Gierer-Meinhardt activator-inhibitor system (1.1) has been widely studied. For example, for the case of $(m,n,l,k) = (2,1,2,0)$, Gonpot and Colet [7] investigated the effects of diffusion and the Turing instability on pattern formation in activation inhibition system through nonlinear bifurcation analysis. Yang et al. [29] not only studied the Turing-Hopf bifurcation of the system, proved the occurrence of spatial resonance phenomenon, but also proposed the normal form calculation method for two-dimensional spatial resonance bifurcation in order to better understand the dynamic behavior of the system near Turing-Hopf bifurcation point. Wu et al. [27] considered the Turing instability of the positive constant equilibrium and spatially homogeneous periodic solutions by employing normal form and center manifold reduction. Sun et al. [24] used the bifurcation theorem to study the Hopf bifurcation, and the existence of codimension-two Turing-Hopf bifurcation was further studied, and the amplitude equation was derived using multi-time scale analysis. Moreover, Chen et al. [3] considered the Hopf bifurcation and the steady state bifurcation of saturated Gierer-Meinhardt system, and obtained the global bifurcation diagram of spatial nonhomogeneous periodic solutions and the steady state solutions of critical parameters. Additionally, more bifurcation analyses of the Gierer-Meinhardt system with different saturation terms can be found in the references [2, 13, 15, 16, 20, 26, 30]. Readers interested in the study of equations and bifurcation analysis can also refer to some relevant content in the references [1, 5, 11, 12, 18, 21, 28, 32–39]. In addition, Liu et al. [14] also studied the Hopf bifurcation and steady state bifurcation of the Gierer-Meinhardt system under non-homogeneous Dirichlet boundary conditions, and the results indicated the existence of spatial non-homogeneous periodic solutions and nonconstant positive solutions. For the case of $(m,n,l,k) = (2,2,1,0)$, Wang et al. [25] studied the Turing instability and the Hopf bifurcation of such system at the homogeneous Neumann boundary condition.

Based on the above statements, extensive research has been conducted on the Gierer-Meinhardt activator-inhibitor system (1.1), but research on the simplest version of the Gierer-Meinhardt reaction diffusion system is still very rare [23]. It is natural to consider this in order to better understand the dynamic behavior of the system (1.1). Therefore, in this paper, we will mainly focus on the simplified version of the Gierer-Meinhardt reaction diffusion system as the follows system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{u^2}{v} - bu, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - d\Delta v = u^2 - v, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where the parameters b, d are positive constants.

For the system (1.2), Suleiman [23] investigated the stability of positive constant equilibrium and proved that diffusion can destabilize the unique positive constant equilibrium of the system (1.2). The direction and stability of the Hopf bifurcation of system (1.2) and the dynamic behavior near the Turing-Hopf bifurcation point with codimension-two are not analyzed in detail in the references. At the same time, few authors have studied the local structure of steady state bifurcation and bifurcation direction [3, 7, 14, 25, 27]. In this paper, we will focus on the Hopf bifurcation, the Turing-Hopf bifurcation and the steady-state bifurcation.

We first consider that the system (1.2) undergoes a Hopf bifurcation and calculate its direction, while using numerical simulations to verify that the system (1.2) has spatially homogeneous and nonhomogeneous periodic solutions. Secondly, we investigate that the steady state solution is a codimension-two Turing-Hopf bifurcation point, and we obtain that the parameter space near the Turing-Hopf bifurcation point can be divided into six regions, the system (1.2) has different dynamic phenomena. Finally, we mainly discuss local bifurcation and establish conditions for determining the direction of bifurcation. In particular, we use some parameter examples to illustrate the existence of such bifurcation diagrams for the system (1.2).

The rest of the paper is structured as follows. In Section 2, we consider Hopf bifurcation, including the existence, direction of bifurcation and the stability of bifurcating periodic solutions. In Section 3, we investigate the existence of Turing-Hopf bifurcation. Finally, we discuss the Turing instability of positive solutions and establish local bifurcation and its direction in Section 4. Meanwhile, we supplement and illustrate some theoretical results of various bifurcations with numerical simulations. Throughout the paper, we denote by $\mathbb{N}$ the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Hopf bifurcation

In this section, we consider that the Hopf bifurcation for Gierer-Meinhardt system with diffusion subject to Neumann boundary condition on the spatial domain $\Omega = (0, l\pi)$, with $l \in \mathbb{R}^+$. Then system (1.2) takes the following form:

$$\begin{cases} u_t - u_{xx} = \dfrac{u^2}{v} - bu, & x \in (0, l\pi),\ t > 0, \\ v_t - dv_{xx} = u^2 - v, & x \in (0, l\pi),\ t > 0, \\ u_x(x,t) = v_x(x,t) = 0, & x = 0, l\pi,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{2.1}$$

It is clear that the system (2.1) has a unique positive constant equilibrium solution $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ for $b > 0$. Thus, we always assume that $b > 0$ and choose b as the bifurcation parameter.

Firstly, we make the translation $\hat{u} = u - \frac{1}{b}$, $\hat{v} = v - \frac{1}{b^2}$, and we still let u, v denote $\hat{u}$, $\hat{v}$ respectively. Then, we can rewrite (2.1) as

$$\begin{cases} u_t - u_{xx} = \dfrac{\left(u + \frac{1}{b}\right)^2}{v + \frac{1}{b^2}} - b\left(u + \dfrac{1}{b}\right), & x \in (0, l\pi), t > 0, \\ v_t - dv_{xx} = \left(u + \dfrac{1}{b}\right)^2 - \left(v + \dfrac{1}{b^2}\right), & x \in (0, l\pi), t > 0, \\ u_x(x,t) = v_x(x,t) = 0, & x = 0, l\pi, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{2.2}$$

The linearized operator of system (2.2) near $(0, 0)$ takes the form

$$L(b) = \begin{pmatrix} \dfrac{\partial^2}{\partial x^2} + b & -b^2 \\ \dfrac{2}{b} & d\dfrac{\partial^2}{\partial x^2} - 1 \end{pmatrix}.$$

It is well known that the eigenvalue problem

$$-\psi'' = \mu\psi,\ x \in (0, l\pi),\ \psi'(0) = \psi'(l\pi) = 0,$$

has eigenvalues $\mu_n = \frac{n^2}{l^2}, (n = 0, 1, 2, \dots)$ and the corresponding eigenfunctions $\psi_n(x) = \cos \frac{n}{l} x$.

For each $n \in \mathbb{N}_0$, we define a 2×2 matrix

$$L_n(b) := \begin{pmatrix} b - \dfrac{n^2}{l^2} & -b^2 \\ \dfrac{2}{b} & -1 - \dfrac{dn^2}{l^2} \end{pmatrix}.$$

It follows from [31] that the eigenvalues of $L(b)$ are given by the eigenvalues of $L_n(b)$ for $n = 0, 1, 2, \cdots$. The characteristic equation of $L_n(b)$ is

$$\lambda^2 - T_n(b)\lambda + D_n(b) = 0,\ n = 0, 1, \cdots, \tag{2.3}$$

where

$$T_n(b) = b - 1 - \frac{(d+1)n^2}{l^2},\ D_n(b) = b - (bd - 1)\frac{n^2}{l^2} + \frac{dn^4}{l^4}, \tag{2.4}$$

and its eigenvalues are

$$\lambda_{1,2}^{(n)} = \frac{T_n(b) \pm \sqrt{T_n^2(b) - 4D_n(b)}}{2},\ n = 0, 1, 2, \cdots.$$

Here, we identify the Hopf bifurcation value b that satisfies the Hopf bifurcation condition, there exists a number $n \in \mathbb{N}_0$, such that

$$T_n(b) = 0,\ D_n(b) > 0 \text{ and } T_m(b) \neq 0,\ D_m(b) \neq 0 \text{ for } m \neq n, \tag{2.5}$$

and for the unique complex eigenvalues pairs $\alpha(b) \pm i\omega(b)$ near the imaginary axis that satisfies transversality condition

$$\alpha'(b) \neq 0. \tag{2.6}$$

For later calculations, we define

$$b = b_n^H := (1+d)p + 1, \tag{2.7}$$

where $p = \frac{n^2}{l^2}$. Clearly, when $n = 0$, we have $D_0(b_0^H) = b > 0$.

Define

$$n^* = \max\left\{k \in \mathbb{N}_0 | D_n(b_n^H) > 0, b_n^H > 0 \quad \text{for} \quad n = 0, 1, \cdots, k-1\right\}. \tag{2.8}$$

Then, it follows from (2.4) and (2.7) that

$$T_m(b_n^H) = b_n^H - 1 - \frac{(1+d)m^2}{l^2} = \frac{(1+d)(n^2 - m^2)}{l^2}.$$

It is obvious that $T_n(b_n^H) = 0$, $T_m(b_n^H) \neq 0$ for $m \neq n$. Now, we only need to prove that whether $D_m(b_n^H) \neq 0$ for all $1 \leq n, m \leq n^* - 1$, and in particular, we have $D_n(b_n^H) > 0$. Here we will give some conditions so that $D_m(b_n^H) > 0$, for each $1 \leq m \leq n^* - 1$.

Lemma 2.1. *Suppose that $3 - 2\sqrt{2} < bd < 3 + 2\sqrt{2}$, and let b_n^H be well defined as in (2.7). Then $D_m(b_n^H) > 0$ for each $1 \leq n, m \leq n^* - 1$.*

Proof. Since $D_n(b) = b - (bd-1)\frac{n^2}{l^2} + \frac{dn^4}{l^4}$, then $\Delta = d^2b^2 - 6db + 1$. If for certain ranges of b, d, $\Delta < 0$, then $D_m(b_n^H) > 0$ for all $1 \leq n, m \leq n^* - 1$. Thus, it is important for us to know in what case $\Delta < 0$. Rewrite Δ as $\Delta(y) = y^2 - 6y + 1$, where $y = bd$. Clearly, $\Delta(y) < 0$ in $(3 - 2\sqrt{2}, 3 + 2\sqrt{2})$. Then for any $bd \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$, $\Delta < 0$.

So far, we have proved that $D_m(b_n^H) > 0$ for any $bd \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2})$, $1 \leq n, m \leq n^* - 1$. The proof is completed. □

Finally, for any Hopf bifurcation point $b = b_n^H, n = 0, 1, \cdots, n^* - 1$, we have $T_n(b_n^H) = 0$, $D_n(b_n^H) > 0$, implying that (2.3) has purely imaginary eigenvalues. Let

$$\lambda_n(b) = \alpha_n(b) \pm i\omega_n(b),\ n = 0, 1, \cdots, n^* - 1$$

be the roots of (2.3) satisfying $\alpha_n(b_n^H) = 0,\ \omega_n(b_n^H) = \sqrt{D_n(b_n^H)}$. Then, when b is near b_n^H

$$\alpha_n(b) \pm i\omega_n(b) = \frac{T_n(b) \pm \sqrt{T_n^2(b) - 4D_n(b)}}{2},$$

and

$$\alpha_n(b) = \frac{T_n(b)}{2},\ \omega_n(b) = \sqrt{D_n(b) - \frac{T_n^2(b)}{4}}.$$

In (2.4), we obtain

$$\alpha_n'(b_n^H) = \frac{1}{2} > 0. \tag{2.9}$$

This means that the transversal conditions are satisfied at each $b_n^H, n = 0, 1, \cdots, n^* - 1$. Hence, the existence of the spatial homogeneous and the nonhomogeneous periodic solutions of the Hopf bifurcation can be established in the following theorem. In particular, we have

Theorem 2.1. *For $0 \leq n \leq n^* - 1, l \in \mathbb{N}_0$, at each $b = b_n^H$, then the system (2.1) undergoes a Hopf bifurcation, and the bifurcating periodic solutions near $(b, u, v) = \left(b_n^H, \frac{1}{b_n^H}, \frac{1}{(b_n^H)^2}\right)$ can be* *parameterized as* $(b(s), u(s), v(s))$ *so that* $b(s) \in C^\infty$ *in the form of* $b(s) = b_n^H + o(s)$ *for* $s \in (0, \delta)$ *for some small* $\delta > 0$*, and*

$$\begin{cases} u(s)(x,t) = \dfrac{1}{b_n^H} + s(a_n e^{2\pi i t/T(s)} + \bar{a_n} e^{-2\pi i t/T(s)}) \cos \dfrac{n}{l} x + o(s), \\ v(s)(x,t) = \dfrac{1}{(b_n^H)^2} + s(b_n e^{2\pi i t/T(s)} + \bar{b_n} e^{-2\pi i t/T(s)}) \cos \dfrac{n}{l} x + o(s), \end{cases} \tag{2.10}$$

where (a_n, b_n) *is the corresponding eigenvector, and* $T(s) = 2\pi/\sqrt{D_n(b_n^H)} + o(s)$*, and* D_n *is defined by (2.4). Moreover,*

(1) the bifurcating periodic solutions from $b = b_0^H$ *are spatially homogeneous, which coincide with the periodic solutions of the corresponding ODE system;*

(2) the bifurcating periodic solutions from $b = b_n^H$ *are spatially nonhomogeneous.*

Proof. Since $l \in \mathbb{N}_0$ and $b_n^H (n = 0, 1, \cdots, n^* - 1)$ are well defined and we know that $T_n(b_n^H) = 0$, $T_m(b_n^H) \neq 0$ for $m \neq n$, and $D_m(b_n^H) > 0$ for each $1 \leq n, m \leq n^* - 1$. And by (2.9), the transverse conditions holds, hence, we can apply the Hopf bifurcation theorem [10] to get the desired results. The proof is completed. □

Next, we consider the bifurcation direction and stability of the spatially homogeneous bifurcating periodic solutions from $b = b_0^H$ according to [31].

Theorem 2.2. *For the system* (2.1)*, we have*

(1) if $Re(c_1(b_0^H)) < 0$*, the Hopf bifurcation at* $b = b_0^H$ *is supercritical, and bifurcating periodic solutions are locally asymptotically stable;*

(2) if $Re(c_1(b_0^H)) > 0$*, the Hopf bifurcation at* $b = b_0^H$ *is subcritical, and the bifurcating periodic solutions are unstable.*

Proof. Here we follow the notations and calculations in [38], we set

$$q := (a_0, b_0)^T = (1, \frac{(b_0^H)^2}{b_0^H - i\omega_0}), \ q^* := (a_0^*, b_0^*)^T = (\frac{\omega_0 + i b_0^H}{2l\pi\omega_0}, -\frac{-i(b_0^H)^2}{2l\pi\omega_0}), \tag{2.11}$$

where $\omega_0 = \sqrt{b_0^H}$, and denote

$$f(u,v) = \frac{(u + \frac{1}{b})^2}{v + \frac{1}{b^2}} - b\left(u + \frac{1}{b}\right), \ g(u,v) = \left(u + \frac{1}{b}\right)^2 - \left(v + \frac{1}{b^2}\right).$$

By direct calculation, we have

$$\begin{aligned} & f_{uu}\left(b_0^H, 0, 0\right) = 2\left(b_0^H\right)^2, \ f_{uv}\left(b_0^H, 0, 0\right) = -2\left(b_0^H\right)^3, \\ & f_{vv}\left(b_0^H, 0, 0\right) = 2\left(b_0^H\right)^4, \ f_{uuv}\left(b_0^H, 0, 0\right) = -2\left(b_0^H\right)^4, \\ & f_{uvv}\left(b_0^H, 0, 0\right) = 4\left(b_0^H\right)^5, \ f_{vvv}\left(b_0^H, 0, 0\right) = -6\left(b_0^H\right)^6, \ g_{uu}(b_0^H, 0, 0) = 2, \\ & f_{uuu}(b_0^H, 0, 0) = g_{uv}(b_0^H, 0, 0) = g_{vv}(b_0^H, 0, 0) = g_{uuu}(b_0^H, 0, 0) = g_{uuv}(b_0^H, 0, 0) \\ & \qquad = g_{uvv}(b_0^H, 0, 0) = g_{vvv}(b_0^H, 0, 0) = 0. \end{aligned}$$

Then

$$c_0=-\omega_0(3ib_0^H+\omega_0),\ d_0=2,\ e_0=2(b_0^H)^2-4(b_0^H)^3+2(b_0^H-i\omega_0)^2,\ f_0=2,$$
$$g_0=4(b_0^H)^3-12i\omega_0(b_0^H)^2+2\omega_0^2b_0^H-6i\omega_0^3,\ h_0=0,$$

and

$$Q_{qq}=\begin{pmatrix}c_0\\d_0\end{pmatrix},\ Q_{q\bar{q}}\begin{pmatrix}e_0\\f_0\end{pmatrix},\ Q_{qq\bar{q}}\begin{pmatrix}g_0\\h_0\end{pmatrix},$$

hence

$$\begin{aligned}
\langle q^*,Q_{qq}\rangle&=\frac{(ib_0^H-\omega_0)(3ib_0^H+\omega_0)}{2}+\frac{i(b_0^H)^2}{\omega_0},\\
\langle \bar{q}^*,Q_{qq}\rangle&=-\frac{(ib_0^H+\omega_0)(3ib_0^H+\omega_0)}{2}-\frac{i(b_0^H)^2}{\omega_0},\\
\langle q^*,Q_{q\bar{q}}\rangle&=\frac{(\omega_0-ib_0^H)[(b_0^H)^2-2(b_0^H)^3+(b_0^H-\omega_0)^2]+i(b_0^H)^2}{\omega_0},\\
\langle \bar{q}^*,Q_{q\bar{q}}\rangle&=\frac{(\omega_0+ib_0^H)[(b_0^H)^2-2(b_0^H)^3+(b_0^H-\omega_0)^2]-i(b_0^H)^2}{\omega_0},\\
\langle q^*,Q_{qq\bar{q}}\rangle&=\frac{(\omega_0-ib_0^H)(4(b_0^H)^3-12i\omega_0(b_0^H)^2+2\omega_0^2b_0^H-i\omega_0^3b_0^H)}{\omega_0}.
\end{aligned}$$

Thus, we have

$$H_{20}=\begin{pmatrix}c_0\\d_0\end{pmatrix}-\langle q^*,Q_{qq}\rangle\begin{pmatrix}a_0\\b_0\end{pmatrix}-\langle \bar{q}^*,Q_{qq}\rangle\begin{pmatrix}\bar{a_0}\\\bar{b_0}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$
$$H_{11}=\begin{pmatrix}e_0\\f_0\end{pmatrix}-\langle q^*,Q_{q\bar{q}}\rangle\begin{pmatrix}a_0\\b_0\end{pmatrix}-\langle \bar{q}^*,Q_{q\bar{q}}\rangle\begin{pmatrix}\bar{a_0}\\\bar{b_0}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

which derives that $\omega_{20}=\omega_{11}=0$, and then we obtain $\langle q^*,Q_{\omega_{20}q}\rangle=\langle q^*,Q_{\omega_{11}q}\rangle=0$. Since

$$c_1(b_0^H)=\frac{i}{2\omega_0}\langle q^*,Q_{qq}\rangle\cdot\langle q^*,Q_{q\bar{q}}\rangle+\langle q^*,Q_{\omega_{11}q}\rangle+\frac{1}{2}\langle q^*,Q_{\omega_{20}q}\rangle+\frac{1}{2}\langle q^*,Q_{qq\bar{q}}\rangle,$$

hence we have

$$Re(c_1(b_0^H))=Re\left\{\frac{i}{2\omega_0}\langle q^*,Q_{qq}\rangle\cdot\langle q^*,Q_{q\bar{q}}\rangle+\frac{1}{2}\langle q^*,Q_{qq\bar{q}}\rangle\right\}.$$

From previous calculations, it follows that $\alpha'(b_0^H)>0$. Therefore, we obtain the conclusion on the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions according to [38]. The proof is completed. □

To visualize the cascade of the Hopf bifurcations described the Lemma 2.1, in particular, we consider a numerical example. In the following, we use the spatial discrete finite difference method to solve the initial boundary value problem of system (2.1) at $\Omega=(0,\pi)$.

Fix $d=1.5$, we choose $b=0.5<b_0^H=1$, the solution of the system (2.1) tends to the positive constant solution $(2,4)$(see Figure 1); $b=b_0^H=1$, the solution tends to a spatially

homogeneous time periodic orbit (see Figure 2); and $b = 0.5 < b_0^H = 1$, the solution tends to the spatially non-homogeneous time periodic orbit (see Figure 3).

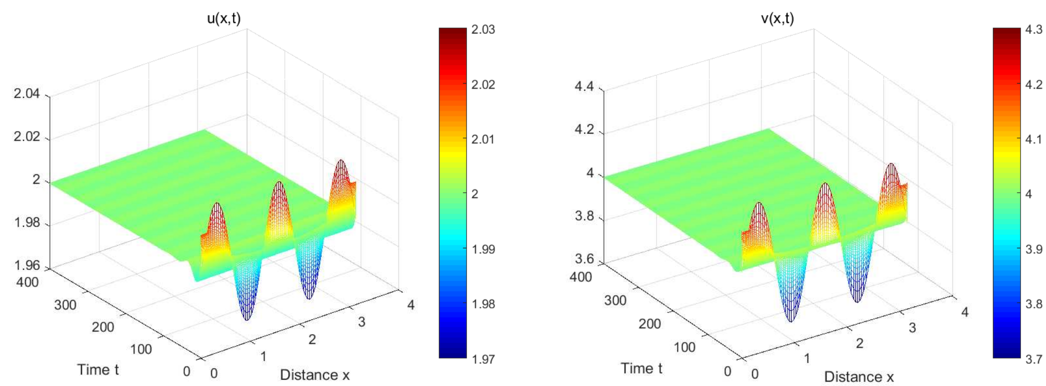


Figure 1. Here $d = 1.5$, choose $b = 0.5 < b_0^H = 1$, the initial values $u_0(x) = 2+0.5\cos(5x)$, $v_0(x) = 4+0.5\cos(5x)$, and the system (2.1) tends to the positive constant solution $(2, 4)$.

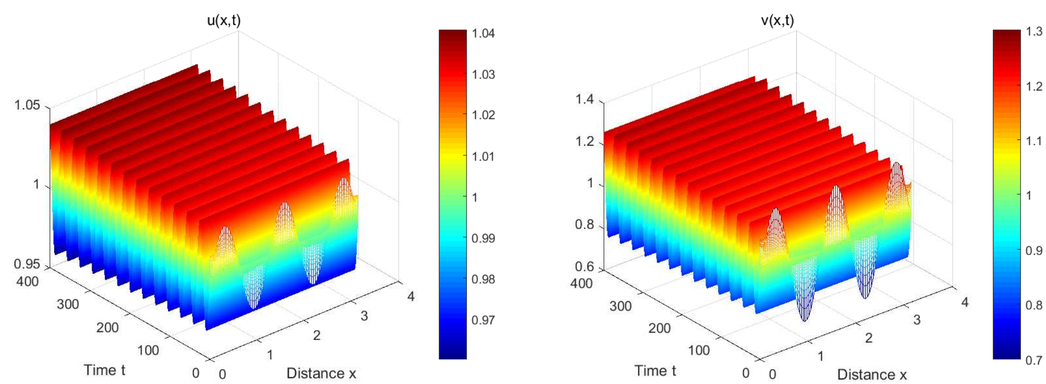


Figure 2. Here $d = 1.5$, choose $b = b_0^H = 1$, the initial values $u_0(x) = 1 + 5\sin(4x/5)$, $v_0(x) = 1 + 5\sin(4x/5)$, and the solution tends to a spatially homogeneous time periodic orbit.

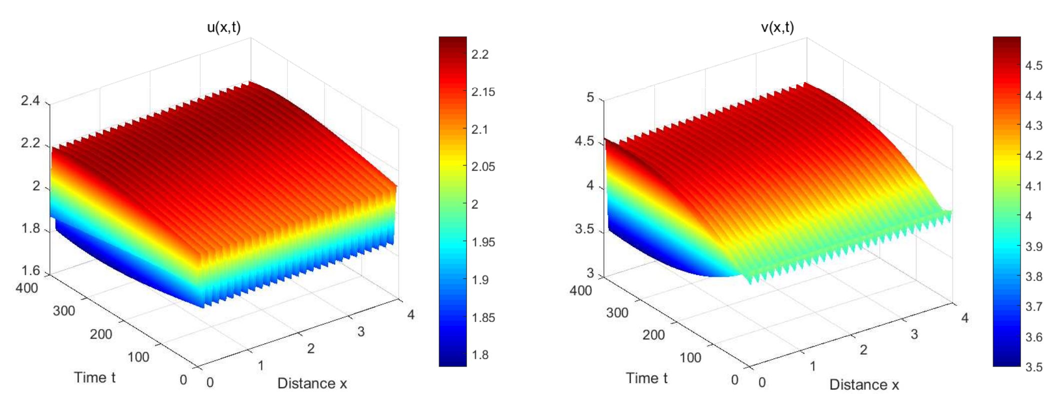


Figure 3. Here $d = 1.5$, choose $b = 0.5 < b_0^H = 1$, the initial values $u_0(x) = 2+5\sin(6x/5)$, $v_0(x) = 4+5\sin(6x/5)$, and the solution tends to the spatially non-homogeneous time periodic orbit.

3. Turing-Hopf bifurcation

In this setion, we will analyze the Turing-Hopf bifurcation for the following three parts. In section 2, we know that the positive equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ undergoes the Hopf bifurcation. And we find that the positive equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ can be a codimension-2 Turing-Hopf bifurcation point for some suitable conditions. To better understand the dynamics of spatiotemporal near the Turing-Hopf bifurcation point, we examine the following contents. In patricular, in subsection 3.1, we consider the existence of the Turing-Hopf bifurcation; in subsection 3.2, we calculate the normal form on the center manifold of the Turing-Hopf bifurcation, with the corresponding normal form, we get that the parameter space near the Turing-Hopf bifurcation point can be divided into six regions, and for each region, the system (2.1) can exhibit different dynamic phenomena; finally, some numerical simulations of the theoretical results is considered in subsection 3.3.

3.1. Existence of Turing-Hopf bifurcation

In this subsection, we will consider the existence of the Turing-Hopf bifurcation, and from the Theorem 2.2, we know that the system (2.1) undergoes the Hopf bifurcation, and the periodic solution is spatially homogeneous for $b = 1$. Since the Turing-Hopf bifurcation is spatially codimension-2 bifurcation, then if system (2.1) undergoes Turing-Hopf bifurcation, the following conditions need to be satisfied:

(1) for $n = 0$, system (2.1) has a pair of simple purely imaginary roots $\pm i\omega$;

(2) for $n > 0$, system (2.1) has a simple zero root $\lambda = 0$.

Define

$$d_k = \frac{\mu_k + b}{\mu_k(b - \mu_k)}, \quad S = \{k \in \mathbb{N}_0 : b - \mu_k > 0\},$$

then there exist $k_* \in \mathbb{N}_0$ such that when $k = k_*$, we can get the following Turing bifurcation curves:

$$d^*_{k_*} = \frac{\mu_{k_*} + b^*}{\mu_{k_*}(b^* - \mu_{k_*})} = \min_{k \in S} \frac{\mu_k + b^*}{\mu_k(b^* - \mu_k)}.$$

From the Theorem 2.2, we know that when $b = 1$, the system (2.1) undergoes Hopf bifurcation. Hence, when $b = b^*$, we have the Hopf bifurcation curve $b^* = 1$.

Here, denote that the Turing bifurcation curve is d_k, and the Hopf bifurcation curve is b, when $k = k_*$, we find the first intersection point in the first quadrant as the Turing-Hopf bifurcation point [9, 22]. In particular, we have the following theorem for the Turing-Hopf bifurcation.

Theorem 3.1. *For the system* (2.1)*, the following statements hold,*

(1) if $S = \emptyset$, the system (2.1) *does not undergo Turing-Hopf bifurcation;*

(2) if $S \neq \emptyset$, the system (2.1) *undergoes Turing-Hopf bifurcation at the point* $(b, d) = (b^*, d^*_{k_*})$. *Furthermore, when* $(b, d) \in \{(b, d) | 0 < b < b^*, 0 < d < \frac{\mu_{k_*} + b^*}{\mu_{k_*}(b^* - \mu_{k_*})}\}$*, the equilibrium* $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ *is locally asymptotically stable.*

Proof. In $b - d$ plane, Hopf bifurcation curve is

$$\mathcal{H}_0 : b = b^*, \tag{3.1}$$

and Turing bifurcation curves are

$$\mathcal{L}_k : d = \frac{\mu_k + b}{\mu_k(b - \mu_k)}, \tag{3.2}$$

where $k \in S$ is a series of curves through the origin.

1. If $S = \emptyset$, then Turing bifurcation curves $\mathcal{L}_k$ and Hopf bifurcation curve $\mathcal{H}_0$ have no intersection in the first quadrant, which means that system (2.1) can not undergo Turing-Hopf bifurcation.

2. If $S \neq \emptyset$, when $(b, d) \in \{(b, d)|0 < b < b^*, 0 < d < d^*_{k_*}\}$, it is clear that $T_n < 0$ and $D_n > 0$ for $n \in \mathbb{N}_0$, so the system (2.1) has a equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$, which is locally asymptotically stable, and indicates that Turing bifurcation curves $\mathcal{L}_k$ intersects with Hopf bifurcation curve $\mathcal{H}_0$ at Turing-Hopf bifurcation point $(b^*, d^*_{k_*})$. In this case, the all other eigenvalues of (2.3) $(n \neq 0, k_*)$ have negative real parts. Moreover, assume that $\lambda_1(b) = \alpha_1(b) + i\beta_1(b)$ with $\alpha_1(b^*) = 0, \beta_1(b^*) = \omega > 0$, and $\lambda_2(b) = \alpha_2(b) + i\beta_2(b)$ with $\alpha_2(b^*) = 0, \beta_2(b^*) = 0$, then the transversality conditions are given as follows:

$$\frac{\mathrm{d}Re(\lambda_1(b))}{\mathrm{d}b}\Big|_{b=1,\mathcal{H}_0} = \frac{1}{2} > 0, \quad \frac{\mathrm{d}Re(\lambda_2(b))}{\mathrm{d}b}\Big|_{b=1,\mathcal{L}_k} = \frac{-d^*_{k_*}\mu_{k_*} + 1}{T_{k_*}} < 0.$$

Since $\mu_{k_*} = \frac{{k_*}^2}{l^2}$, when l is large enough, there is $-d^*_{k_*}\mu_{k_*} + 1 > 0$. The proof is completed. □

3.2. Normal form of the Turing-Hopf bifurcation

In this subsection, we consider that the Turing-Hopf bifurcation for the system (2.1) at the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$, and we choose b and d as bifurcation parameters, $\varepsilon_1, \varepsilon_2$ as perturbation parameters and set $b = b^* + \varepsilon_1$, $d = d^* + \varepsilon_2$. Then the system (2.1) is rewritten as

$$\begin{cases} u_t = \Delta u + \dfrac{u^2}{v} - (b^* + \varepsilon_1)u, \\ v_t = (d^* + \varepsilon_2)\Delta v + u^2 - v. \end{cases} \tag{3.3}$$

Here, the system (3.3) has the unique positive equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$. Transfering $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ to the origin by $\bar{u} = u - \frac{1}{b}$, $\bar{v} = v - \frac{1}{b^2}$, after removing the horizontal bar, then the system (3.3) becomes

$$\begin{cases} u_t = \Delta u + \dfrac{(u + \frac{1}{b})^2}{v + \frac{1}{b^2}} - (b^* + \varepsilon_1)(u + \dfrac{1}{b}), \\ v_t = (d^* + \varepsilon_2)\Delta v + (u + \dfrac{1}{b})^2 - (v + \dfrac{1}{b^2}). \end{cases} \tag{3.4}$$

From [8], fix $m \in \mathbb{N}$ and $r > 0$, define $\mathcal{C} = C([-r, 0]; X^m)(r > 0)$ to be the Banach space of continuous maps from $[-r, 0]$ to X^m with the sup norm, we consider system (3.4) with parameters in the phase space $\mathcal{C}$ defined as

$$\dot{u}(t) = D(\varepsilon)\Delta u(t) + L(\varepsilon)u_t + G(u_t, \varepsilon), \tag{3.5}$$

where $u_t \in \mathcal{C}$, $D(\varepsilon) = \mathrm{diag}(d_1(\varepsilon), d_2(\varepsilon), \cdots, d_m(\varepsilon))$ with $d_i(0) > 0$ for $1 \le i \le m$, the domain of Δ is defined by $dom(\Delta) = Y^m \subseteq X^m$ where Y is defined as

$$Y = \left\{ u \in W^{2,2}(\Omega) : \frac{\partial u}{\partial n} = 0, x \in \partial\Omega \right\},$$

the parameter vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is in a neighborhood $V \subset \mathbb{R}^2$ of $(0, 0)$, and $L : V \to L(\mathcal{C}, \mathbb{R}^m)$, $G : \mathcal{C} \times V \to \mathbb{R}^m$. Then from the system (3.4), we have

$$D(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & d^* + \varepsilon_2 \end{pmatrix}, \quad L(\varepsilon) = \begin{pmatrix} b^* + \varepsilon_1 & -(b^* + \varepsilon_1)^2 \\ \dfrac{2}{b^* + \varepsilon_1} & -1 \end{pmatrix},$$

$$G(\phi,\varepsilon)=\begin{pmatrix}\dfrac{(\phi_1+u^*)^2}{\phi_2+v^*}-(b^*+\varepsilon_1)(\phi_1+u^*)-(b^*+\varepsilon_1)\phi_1+(b^*+\varepsilon_1)^2\phi_2\\ (\phi_1+u^*)^2-(\phi_2+v^*)-\dfrac{2}{b^*+\varepsilon_1}\phi_1+\phi_2\end{pmatrix},$$

where $\phi=(\phi_1,\phi_2)^T\in\mathcal{C}$.

Let $L_0=L(0)$, $D_0=D(0)$, then the linearized equation for the zero equilibrium of system (3.5) can be written as

$$\dot{u}(t)=D_0\Delta u(t)+L_0u_t. \tag{3.6}$$

We ignore the dependency on higher-order (≥ 2) terms of small parameters $\varepsilon_1,\varepsilon_2$ in the normal form of the system (3.5). By the Taylor expansion formally for the operator $L(\varepsilon)$ and diagonal matrix $D(\varepsilon)$ at $\varepsilon=0$, we have

$$L(\varepsilon)\phi=L(0)\phi+\frac{1}{2}L_1(\varepsilon)\phi+\cdots,\ \text{for } \phi\in\mathcal{C},\ D(\varepsilon)=\frac{1}{2}D_1(\varepsilon)+\cdots, \tag{3.7}$$

where $L_1:V\to L(\mathcal{C},X^m)$, and $D_1:V\to\mathbb{R}^m\times\mathbb{R}^m$ are linear. By [32], we write G for

$$G(\phi,0)=\frac{1}{2}Q(\phi,\phi)+\frac{1}{3}C(\phi,\phi,\phi)+O(|\ \phi\ |^4),\ \phi\in\mathcal{C}, \tag{3.8}$$

where Q,C are symmetric multilinear forms. For convenience, we also write $Q(\phi,\psi)$ as $Q_{\phi\psi}$, $Q_\phi\psi$ or $Q_\psi\phi$, and $C(\phi,\psi,\upsilon)$ as $C_{\phi\psi\upsilon}$. So we have

$$D(0)=\begin{pmatrix}1&0\\0&d^*\end{pmatrix},D_1(\varepsilon)=\begin{pmatrix}0&0\\0&2\varepsilon_2\end{pmatrix},L(0)=\begin{pmatrix}b^*&-b^{*2}\\ \dfrac{2}{b^*}&-1\end{pmatrix},L_1(\varepsilon)=\begin{pmatrix}2\varepsilon_1&-4(b^*+\varepsilon_1)\varepsilon_1\\ -\dfrac{4\varepsilon_1}{(b+\varepsilon_1)^2}&0\end{pmatrix},$$

and

$$Q(\phi,\psi)=\begin{pmatrix}\alpha_{11}\phi_1\psi_1+\alpha_{12}(\phi_1\psi_2+\phi_2\psi_1)+\alpha_{13}\phi_2\psi_2\\ \alpha_{21}\phi_1\psi_1+\alpha_{22}(\phi_1\psi_2+\phi_2\psi_1)+\alpha_{23}\phi_2\psi_2\end{pmatrix},$$

$$C(\phi,\psi,\upsilon)$$
$$=\begin{pmatrix}\beta_{11}\phi_1\psi_1\upsilon_1+\beta_{12}(\phi_1\psi_1\upsilon_2+\phi_1\psi_2\upsilon_1+\phi_2\psi_1\upsilon_1)+\beta_{13}(\phi_1\psi_2\upsilon_2+\phi_2\psi_1\upsilon_2+\phi_2\psi_2\upsilon_1)+\beta_{14}\phi_2\psi_2\upsilon_2\\ \beta_{21}\phi_1\psi_1\upsilon_1+\beta_{22}(\phi_1\psi_1\upsilon_2+\phi_1\psi_2\upsilon_1+\phi_2\psi_1\upsilon_1)+\beta_{23}(\phi_1\psi_2\upsilon_2+\phi_2\psi_1\upsilon_2+\phi_2\psi_2\upsilon_1)+\beta_{24}\phi_2\psi_2\upsilon_2\end{pmatrix},$$

where

$$\alpha_{11}=2b^{*2},\ \alpha_{12}=-2b^{*3},\ \alpha_{13}=2b^{*4},\ \alpha_{21}=2,\ \beta_{12}=-2b^{*4},\ \beta_{13}=4b^{*5},\ \beta_{14}=-6b^{*6},$$
$$\alpha_{22}=\alpha_{23}=\beta_{11}=\beta_{21}=\beta_{22}=\beta_{23}=\beta_{24}=0,$$

for $\phi=(\phi_1,\phi_2)^T,\psi=(\psi_1,\psi_2)^T,\upsilon=(\upsilon_1,\upsilon_2)^T\in\mathcal{C}$.

The corresponding feature matrix of the system (3.4) is

$$D_k(\lambda)=\begin{pmatrix}\lambda+\mu_k-b^*&b^{*2}\\ \dfrac{2}{b^*}&\lambda+d\mu_k+1\end{pmatrix},\ k\in\mathbb{N}.$$

According to the Theorem 3.1, $\lambda = i\omega$, $\omega = \sqrt{b}$ are eigenvalues of $D_0(\lambda)$, $\lambda = 0$ is a simple eigenvalue for $D_{k_*}(\lambda)$ and the real parts of the other eigenvalues are negative. Through straightforward calculations, we obtain

$$\phi_1 = \begin{pmatrix} 1 \\ \dfrac{2}{b^*(d_{k_*}\mu_{k_*}+1)} \end{pmatrix}, \psi_1 = \begin{pmatrix} \dfrac{b^{*2}(d_{k_*}\mu_{k_*}+1)^2}{4+b^{*2}(d_{k_*}\mu_{k_*}+1)^2} \\ \dfrac{2b^{*2}(d_{k_*}\mu_{k_*}+1)^2}{4+b^{*2}(d_{k_*}\mu_{k_*}+1)^2}. \end{pmatrix}, \phi_2 = \begin{pmatrix} 1 \\ \dfrac{b^*+i\omega}{b^{*2}} \end{pmatrix}, \psi_2 = \begin{pmatrix} \dfrac{1}{2} + \dfrac{ib^*}{2\omega} \\ -\dfrac{ib^{*2}}{2\omega} \end{pmatrix},$$

and $\Phi = (\phi_1, \phi_2, \bar{\phi_2})$, $\Psi = (\psi_1, \psi_2, \bar{\psi_2})^T$ satisfying $\Phi\Psi = I_2$, and I_2 is a 2×2 identity matrix.

Remark 3.1. If the effect of higher-order terms of the perturbation parameters on the system (2.1) is ignored, then the three-truncated normal form of the system (2.1) confined to the central manifold near the Turing-Hopf bifurcation point $b = b^*$ is as follows:

$$\begin{cases} \dot{z_1} = a_1(\varepsilon)z_1 + a_{200}z_1^2 + a_{011}z_2\bar{z}_2 + a_{300}z_1^3 + a_{111}z_1z_2\bar{z}_2 + \cdots, \\ \dot{z_2} = i\omega z_2 + b_2(\varepsilon)z_2 + b_{110}z_1z_2 + b_{210}z_1^2z_2 + b_{021}z_2^2\bar{z}_2 + \cdots, \\ \dot{\bar{z}}_2 = -i\omega\bar{z}_2 + \bar{b}_2(\varepsilon)\bar{z}_2 + \bar{b}_{110}z_1\bar{z}_2 + \bar{b}_{210}z_1^2\bar{z}_2 + \bar{b}_{021}z_2^2\bar{z}_2{}^2 + \cdots. \end{cases} \tag{3.9}$$

Let $z_1 = r, z_2 = \rho\cos\theta - i\rho\sin\theta$, we transform system (3.9) to cylindrical form:

$$\begin{cases} \dot{r} = a_1(\varepsilon)r + a_{300}r^3 + a_{111}r\rho^3, \\ \dot{\rho_2} = Re(b_2(\varepsilon))\rho + Re(b_{210})\rho r^3 + Re(b_{021})\rho^3. \end{cases} \tag{3.10}$$

Here, the parameter calculation is as follows (see [8]):

$$\begin{aligned}
a_1(\varepsilon) &= \frac{1}{2}\psi_1(L_1(\varepsilon)\phi_1 - \mu_{k_*}D_1(\varepsilon)\phi_1), a_{200} = a_{011} = b_{110} = 0, \\
b_2(\varepsilon) &= \frac{1}{2}\psi_2(L_1(\varepsilon)\phi_2 - 0D_1(\varepsilon)\phi_2), \\
a_{300} &= \frac{1}{4}\psi_1 C_{\phi_1\phi_1\phi_1} + \frac{1}{\omega}\psi_1 \mathrm{Re}[iQ_{\phi_1\phi_2}\psi_2]Q_{\phi_1\phi_1} + \psi_1 Q_{\phi_1(h^0_{200} + \frac{1}{\sqrt{2}}h^{2k_*}_{200})}, \\
a_{111} &= \psi_1 C_{\phi_1\phi_2\bar{\phi_2}} + \frac{2}{\omega}\psi_1 Re[iQ_{\phi_1\phi_2}\psi_2]Q_{\phi_1\bar{\phi_2}} + \psi_1(Q_{\phi_1(h^0_{011} + \frac{1}{\sqrt{2}}h^{2k_*}_{011})} + Q_{\phi_2 h^{k_*}_{101}} + Q_{\bar{\phi_2}h^{k_*}_{110}}), \\
b_{210} &= \frac{1}{2}\psi_2 C_{\phi_1\phi_1\phi_2} + \frac{1}{2i\omega}\psi_2(2Q_{\phi_1\phi_1}\psi_1 Q_{\phi_1\phi_2} + (-Q_{\phi_2\phi_2}\psi_2 + Q_{\phi_2\bar{\phi_2}}\bar{\psi_2})Q_{\phi_1\phi_1}) \\
&\quad + \psi_2(Q_{\phi_1 h^{k_*}_{110}} + Q_{\phi_2 h^0_{200}}), \\
b_{021} &= \frac{1}{2}\psi_2 C_{\phi_2\phi_2\bar{\phi_2}} + \frac{1}{4i\omega}\psi_2\left(\frac{2}{3}Q_{\bar{\phi_2}\bar{\phi_2}}\bar{\psi_2}Q_{\phi_2\phi_2} + (-2Q_{\phi_2\phi_2}\psi_2 + 4Q_{\phi_2\bar{\phi_2}}\bar{\psi_2})Q_{\phi_2\bar{\phi_2}}\right) \\
&\quad + \psi_2(Q_{\phi_2 h^0_{011}} + Q_{\bar{\phi_2}h^0_{020}}),
\end{aligned}$$

where

$$\begin{aligned}
h^0_{200} &= -\frac{1}{2}L^{-1}(0)Q_{\phi_1\phi_1} + \frac{1}{2i\omega}(\phi_2\psi_2 - \bar{\phi_2}\bar{\psi_2})Q_{\phi_1\phi_1}, \\
h^{2k_*}_{200} &= -\frac{1}{2\sqrt{2}}[L(0) + \mathrm{diag}(-4\mu_{k_*}, -4d_{k_*}\mu_{k_*})]^{-1}Q_{\phi_1\phi_1}, \\
h^0_{011} &= -L^{-1}(0)Q_{\phi_2\bar{\phi_2}} + \frac{1}{i\omega}(\phi_2\psi_2 - \bar{\phi_2}\bar{\psi_2})Q_{\phi_2\bar{\phi_2}},
\end{aligned}$$

$$h_{020}^{0}=\frac{1}{2}[2i\omega I-L(0)]^{-1}Q_{\phi_2\phi_2}+\frac{1}{2i\omega}(\phi_2\psi_2-\frac{1}{3}\bar{\phi_2}\bar{\psi_2})Q_{\phi_2\phi_2},$$
$$h_{110}^{k_*}=[i\omega I-(L(0)-\mathrm{diag}(-\mu_{k_*},-d_{k_*}\mu_{k_*}))]^{-1}Q_{\phi_1\phi_2}-\frac{1}{i\omega}\phi_1\psi_1Q_{\phi_1\phi_2},$$
$$h_{002}^{0}=h_{020}^{\bar{0}},\ h_{101}^{k_*}=h_{110}^{\bar{k}_*},\ h_{011}^{2k_*}=0.$$

3.3. Numerical simulations

In this subsection, we use the spatial discrete finite difference method for numerical verification,

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\Delta u+\dfrac{u^2}{v}-bu,\ x\in(0,\pi),t>0,\\ \dfrac{\partial v(x,t)}{\partial t}=d\Delta v+u^2-v,\ x\in(0,\pi),t>0,\end{cases}\tag{3.11}$$

then $\left(\frac{1}{b},\frac{1}{b^2}\right)=(1,1)$ is a unique equilibrium. In addition, $S=\{12\},b^*=1,\ k_*=2,\ d_{k_*}^*=5.85$. The Hopf bifurcation curve in $b-d$ plane is $\mathcal{H}_0:b=b^*$. The Turing bifurcation curves are

$$\mathcal{L}_k:d=\frac{\mu_k+b}{\mu_k(b-\mu_k)},\quad k\in S.$$

For the system (3.11), the normal form restricted on center manifold at the Turing-Hopf singularity is

$$\begin{cases}\dot{z_1}=\left(0.1049\varepsilon_1-0.7344\varepsilon_1^2-\dfrac{3.3574\varepsilon_1}{(1+\varepsilon_1)^2}-0.4131\varepsilon_2\right)z_1+6.6099z_1^3-0.2505z_1z_2\bar{z}_2+\cdots,\\ \dot{z_2}=iz_2+\left(\dfrac{1+i}{4}(2-4(1+\varepsilon_1)(1+i))+\dfrac{i}{(1+\varepsilon_1)^2}\right)\varepsilon_1z_2+(1.3445+1.9439i)z_1^2z_2\\ \qquad+(12.7949+0.8732i)z_2^2\bar{z}_2+\cdots,\\ \dot{\bar{z}}_2=-i\bar{z}_2+\left(\dfrac{1-i}{4}(2-4(1+\varepsilon_1)(1-i))-\dfrac{i}{(1+\varepsilon_1)^2}\right)\varepsilon_1\bar{z}_2+(1.3445-1.9439i)z_1^2\bar{z}_2\\ \qquad+(12.7949-0.8732i)z_2\bar{z}_2{}^2+\cdots.\end{cases}\tag{3.12}$$

Transformed to cylindrical coordinate form:

$$\begin{cases}\dot{r}=\left(0.1049\varepsilon_1-0.7344\varepsilon_1^2-\dfrac{3.3574\varepsilon_1}{(1+\varepsilon_1)^2}-0.4131\varepsilon_2\right)r+6.6099r^3-0.2505r\rho^2,\\ \dot{\rho}=0.5000\varepsilon_1\rho+1.3445\rho r^2+12.7949\rho^3.\end{cases}\tag{3.13}$$

Notice that $\rho>0,r\in\mathbb{N}_0$, from [22], system (3.13) has coexistence equilibrium E_0; spatially nonhomogeneous steady states $E_1^{\pm}$; spatially homogeneous periodic solution E_2; spatially nonhomogeneous periodic solutions $E_3^{\pm}$ as follows:

$$\begin{aligned}&E_0=(0,0),E_1^{\pm}=(\pm\sqrt{c_1},0),\ \text{for}\ c_1>0,\\ &E_2=(0,\sqrt{-0.0391\varepsilon_1}),\ \text{for}\ \varepsilon_1<0,\\ &E_3^{\pm}=(\pm\sqrt{c_2},\sqrt{c_3}),\ \text{for}\ c_2>0,c_3>0,\end{aligned}\tag{3.14}$$

where

$$c_1 = -0.0395\varepsilon_1 + 0.1111\varepsilon_1^2 + \frac{0.5079\varepsilon_1}{(1+\varepsilon_1)^2} + 0.0625\varepsilon_2,$$
$$c_2 = -0.0173\varepsilon_1 + 0.1107\varepsilon_1^2 + \frac{0.5059\varepsilon_1}{(1+\varepsilon_1)^2} + 0.0623\varepsilon_2,$$
$$c_3 = -0.0373\varepsilon_1 - 0.0116\varepsilon_1^2 - \frac{0.0532\varepsilon_1}{(1+\varepsilon_1)^2} - 0.0065\varepsilon_2.$$

Since we intend to investigate the local stability of the equilibrium of system (3.13). Then,

$$\tilde{c_1} = -0.0395\varepsilon_1 + 0.0625\varepsilon_2,\ \tilde{c_2} = -0.0173\varepsilon_1 + 0.0623\varepsilon_2,\ \tilde{c_3} = -0.0373\varepsilon_1 - 0.0065\varepsilon_2.$$

Let $\tilde{c_1} > 0, \tilde{c_2} > 0, \tilde{c_3} > 0$. By direct calculation, it follows that

$$\varepsilon_2 > 0.6320\varepsilon_1,\ \varepsilon_2 > 0.2777\varepsilon_1,\ \varepsilon_2 < -5.7231\varepsilon_1.$$

Then we can obtain the following critical bifurcation lines:

$$\mathcal{H}_0 : \varepsilon_1 = 0,\ \mathcal{T} : \varepsilon_2 = 0.6320\varepsilon_1, \mathcal{T}_1 : \varepsilon_2 = 0.2777\varepsilon_1,\ \varepsilon_1 < 0, \mathcal{T}_2 : \varepsilon_2 = -5.7231\varepsilon_1,\ \varepsilon_1 < 0. \quad (3.15)$$

According to the above bifurcation curves. From Figure 4 (Left), it can be seen that the Turing bifurcation curve and the Hopf bifurcation curve have multiple intersections, we choose the first intersection point (b, d) of the Turing curve $\mathcal{L}_k$ and the Hopf curve $\mathcal{H}_0$ as the Turing-Hopf bifurcation point $(b, d) = (b^*, d^*_{k_*})$. The system (2.1) undergoes Turing-Hopf bifurcation at the point $(b^*, d^*_{k_*}) \approx (1, 5.85)$ for $k_* = 2$. The parameter space in Figure 4 (Right) is divided into six regions, for each region, the system (2.1) can show different dynamic phenomena.

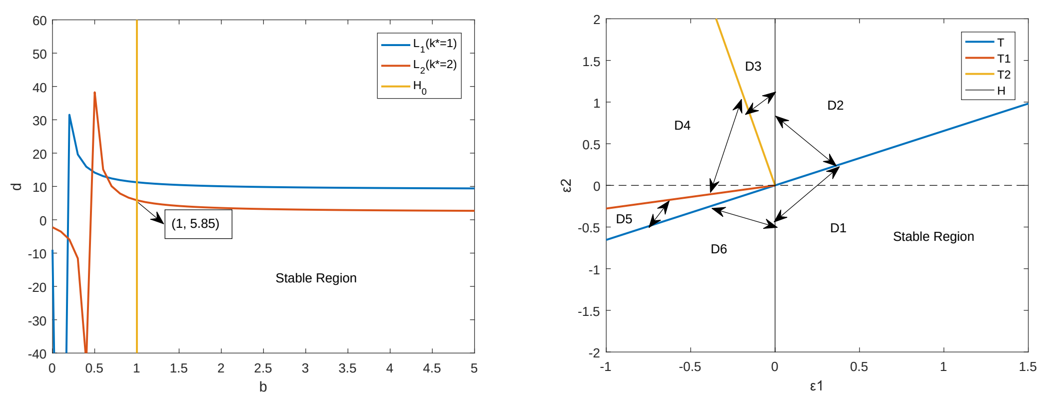


Figure 4. Left: Turing-Hopf bifurcation point $(b^*, d^*_{k_*})$ in b-d plane. Right: Local bifurcation sets.

For each region, we obtain the following conclusions.

Proposition 3.1. *In the $\varepsilon_1 - \varepsilon_2$ parameter plane, these four bifurcation curves $\mathcal{H}_0, \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ divide a small neighborhood of the origin into six regions. For each region, the system* (2.1) *has the different dynamic phenomena, we can get the following results:*

1. when $(\varepsilon_1, \varepsilon_2) \in D1$, the positive constant solution for the system (2.1) is locally asymptotically stable (see Figure 5), otherwise, when $(\varepsilon_1, \varepsilon_2) \notin D1$, the equilibrium becomes unstable;

2. when $(\varepsilon_1, \varepsilon_2) \in D2$, the system (2.1) has an unstable positive constant solution and a pair of stable spatially inhomogeneous steady states (see Figure 6);

3. when $(\varepsilon_1, \varepsilon_2) \in D3$, the system (2.1) *has an unstable positive constant solution, an unstable spatially homogeneous periodic solution and a pair of stable spatially inhomogeneous steady states (see Figure 7);*

4. when $(\varepsilon_1, \varepsilon_2) \in D4$, the system (2.1) *has an unstable positive constant solution, a pair of unstable spatially inhomogeneous periodic solutions, a stable spatially homogeneous periodic solution and a pair of stable spatially inhomogeneous steady states (see Figure 8);*

5. when $(\varepsilon_1, \varepsilon_2) \in D5$, the system (2.1) has an unstable positive constant solution, a pair of unstable spatially inhomogeneous steady states and a stable spatially homogeneous periodic solution (see Figure 9);

6. when $(\varepsilon_1, \varepsilon_2) \in D6$, the system (2.1) *has an unstable positive constant solution and a stable spatially homogeneous periodic solution (see Figure 10).*

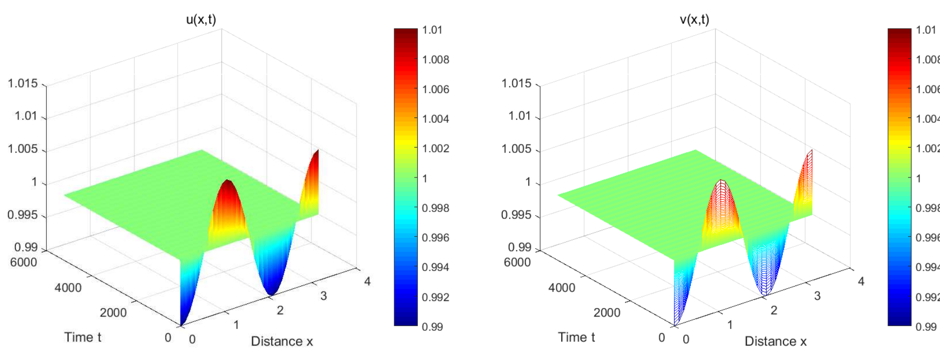
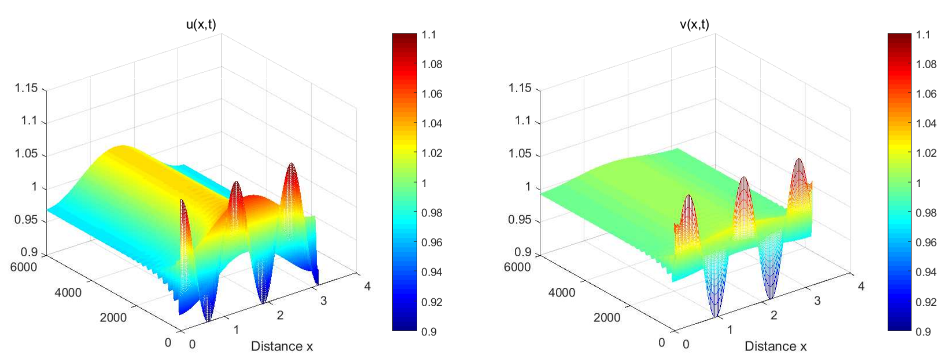
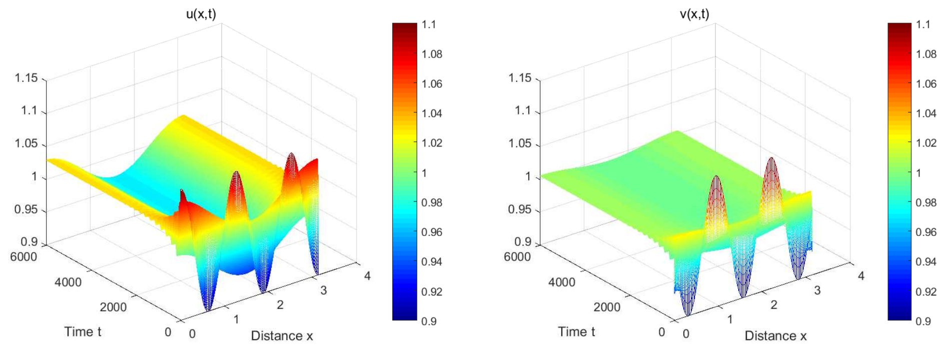


Figure 5. $(\varepsilon_1, \varepsilon_2) = (0.1, -0.1)$, and the initial values $u(x, 0) = u_0 + 2\cos 5x, v(x, 0) = v_0 - 2\cos 5x$. E_0 is stable.

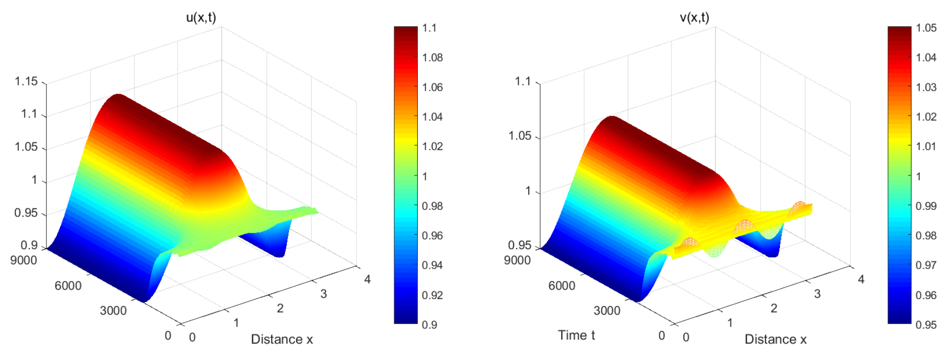


(a) $u(x, 0) = u_0 + 0.1\cos 2x, v(x, 0) = v_0 - 0.1\cos 2x$.

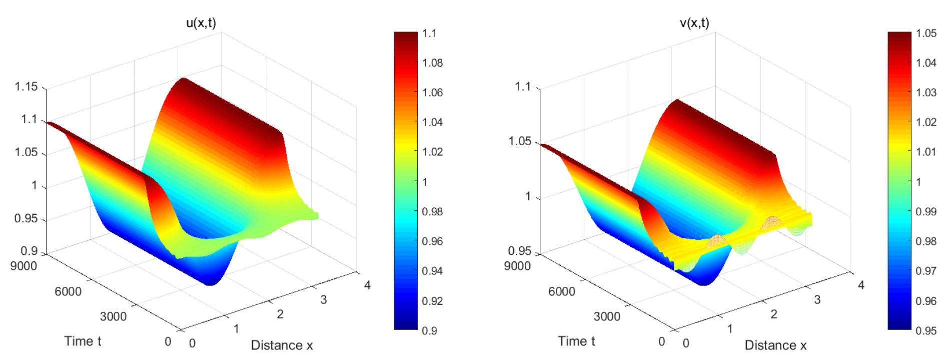


(b) $u(x,0) = u_0 - 0.1\cos 2x, v(x,0) = v_0 + 0.1\cos 2x$.

Figure 6. $(\varepsilon_1, \varepsilon_2) = (0.1, 0.07)$, E_0 is unstable, $E_1^{\pm}$ is stable.

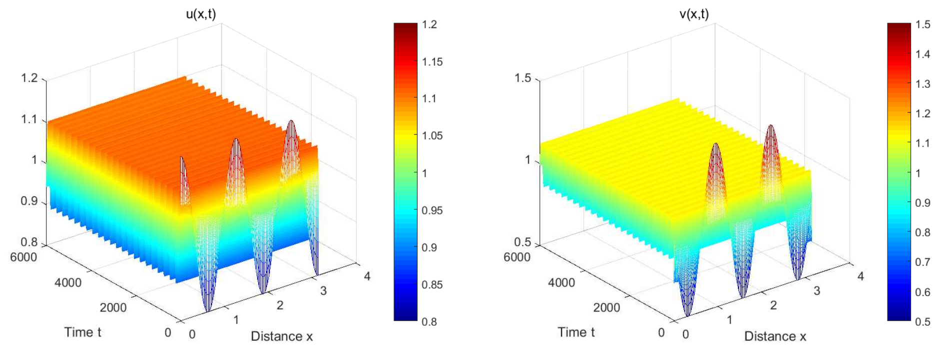


(a) $u(x,0) = u_0 + 0.15\cos 2x, v(x,0) = v_0 - 0.15\cos 2x$.

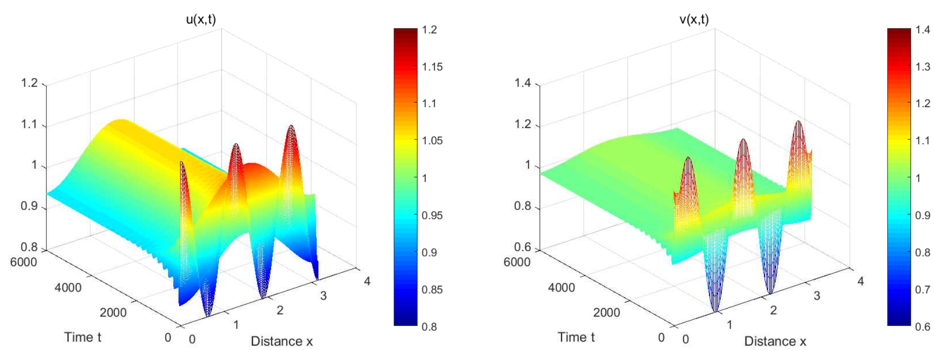


(b) $u(x,0) = u_0 - 0.15\cos 2x, v(x,0) = v_0 + 0.15\cos 2x$.

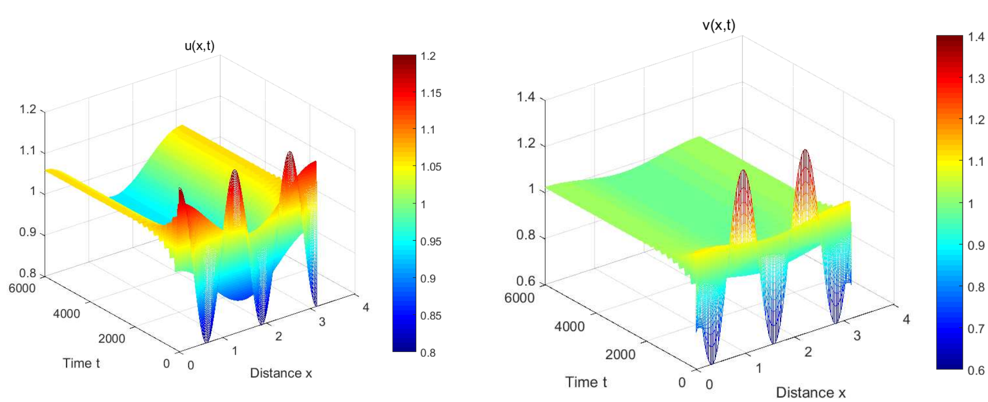
Figure 7. When $(\varepsilon_1, \varepsilon_2) = (-0.1, 0.6)$, E_0, E_2 are unstable, $E_1^{\pm}$ is stable.



(a) $u(x,0) = u_0 + 0.2\cos 2x, v(x,0) = v_0 - 0.2\cos 2x$.



(b) $u(x,0) = u_0 + 0.2\cos 2x, v(x,0) = v_0 - 0.1\cos 2x$.



(c) $u(x,0) = u_0 - 0.2\cos 2x, v(x,0) = v_0 + 0.1\cos 2x$.

Figure 8. When $(\varepsilon_1, \varepsilon_2) = (-0.1, 0.02)$, $E_0, E_3^{\pm}$ are unstable equilibria, $E_2, E_1^{\pm}$ are asymptotically stable equilibria.

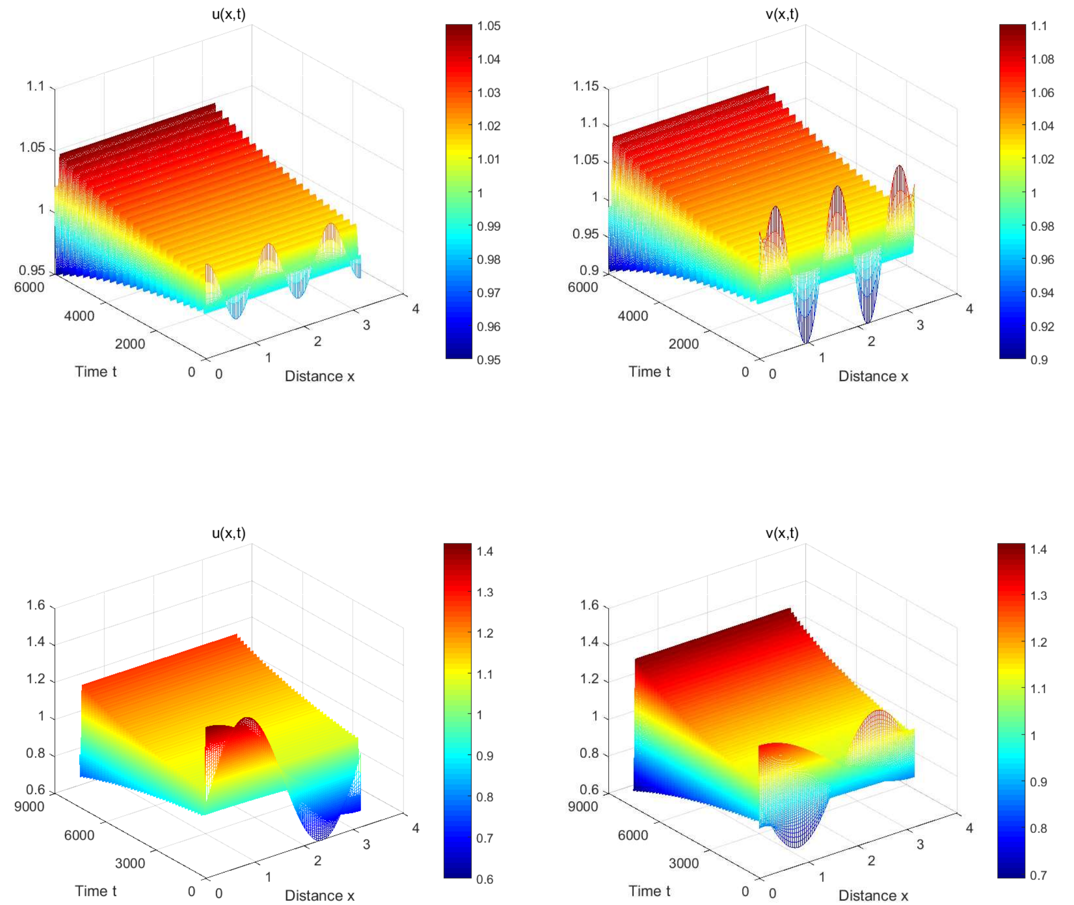


Figure 9. When $(\varepsilon_1, \varepsilon_2) = (-0.1, -0.03)$, the initial values $u(x, 0) = u_0 + 0.01 \cos 2x, v(x, 0) = v_0 - 0.01 \cos 2x$, and $E_0, E_1^{\pm}$ are unstable equilibria, E_2 is asymptotically stable equilibria.

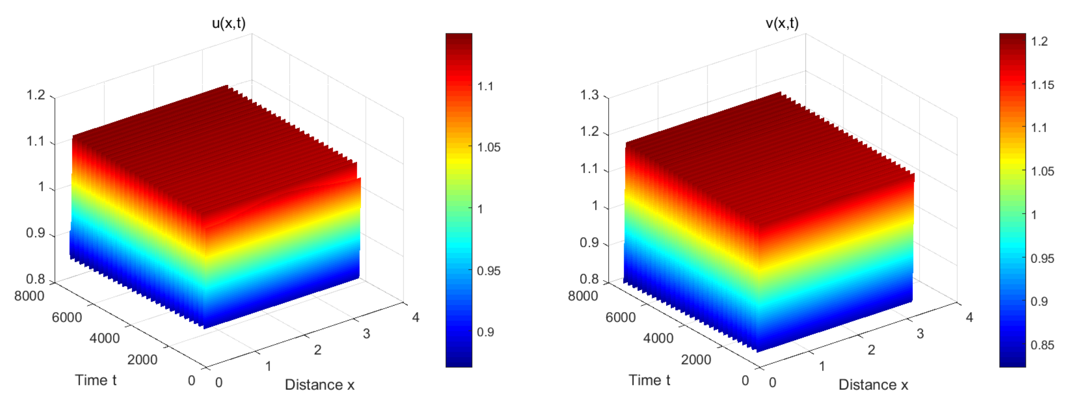


Figure 10. When $(\varepsilon_1, \varepsilon_2) = (-0.1, -0.08)$, the initial values $u(x, 0) = u_0 + 0.1 \cos 2x, v(x, 0) = v_0 + 0.1 \cos 2x$, and E_0 is unstable equilibria, E_2 is asymptotically stable equilibria.

4. Steady state bifurcation

In this section, we will study the steady state bifurcations in following four parts. In particular, in subsection 4.1, we consider the stability of the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ of the system (2.1); in subsection 4.2, we study the local bifurcation for the system (4.7), for the case of simple eigenvalues, we apply the Crandall-Rabinowitz bifurcation theorem, for the case of double eigenvalues, we complete the proof by spatial decomposition and implicit function theorem; in subsection 4.3, we calculate the direction of steady state bifurcation at simple eigenvalues; finally, by virtue of numerical simulation, the theoretical results are supplemented and explained in subsection 4.4.

4.1. Turing instability

In this subsection, we mainly consider the stability of $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ of the system (2.1) and some the Turing instability results will be obtained.

For the system (2.1), the ordinary differential equations take the following form:

$$\begin{cases} u_t = \dfrac{u^2}{v} - bu, & t > 0, \\ v_t = u^2 - v, & t > 0. \end{cases} \tag{4.1}$$

The system (4.1) has a unique positive constant equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$, and we can obtain the Jacobian matrix at $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ of the system (4.1)

$$J = \begin{pmatrix} b & -b^2 \\ \dfrac{2}{b} & -1 \end{pmatrix},$$

then the characteristic equation is

$$\mu^2 - \mathrm{Tr}(J)\mu + \mathrm{Det}(J) = 0, \tag{4.2}$$

where

$$\mathrm{Tr}(J) = b - 1,\ \mathrm{Det}(J) = b.$$

It is easy to see that if $0 < b < 1$, the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ of the system (4.1) is locally asymptotically stable.

Let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$$

be the eigenvalues for the operator $-\Delta$ subject to the Neumann boundary condition on Ω, where λ_i has multiplicity $m_i \geq 1$. Set $\phi_{ij}(1 \leq j \leq m_i)$ be the normalized eigenfunctions corresponding to λ_i, and then $\phi_{ij}, i \geq 0, 1 \leq j \leq m_i$, forms a complete orthogonal basis in $L^2(\Omega)$.

If $\lambda_1 < b$, then there exists a maximum number i_0 such that

$$\lambda_i < b,\ 1 \leq i \leq i_0. \tag{4.3}$$

In this case, we set

$$\tilde{d} = \tilde{d}(b, \Omega) = \min_{i \leq i \leq i_0} d_i, \text{ where } \quad d_i = \frac{\lambda_i + b}{\lambda_i(b - \lambda_i)}. \tag{4.4}$$

Naturally we can give the stability of the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ of the system (2.1) as follows.

Theorem 4.1. *For the system* (2.1), *we have*

(i) if $\lambda_1 \geq b$ *or* $\lambda_1 < b$ *and* $0 < d < \tilde{d}$, *then the equilibrium* $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ *is locally asymptotically stable;*

(ii) if $\lambda_1 < b$ *and* $d > \tilde{d}$, *then the equilibrium* $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ *is unstable.*

Proof. The linearized system of (2.1) at $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ has the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} b+\Delta & -b^2 \\ \dfrac{2}{b} & -1+d\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Let (Φ, Ψ) be an eigenfunction of L, corresponding to the eigenvalue μ. Then we have

$$\begin{pmatrix} b+\Delta & -b^2 \\ \dfrac{2}{b} & -1+d\Delta \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \mu \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}.$$

Set

$$\Phi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i}^{\infty} a_{ij}\phi_{ij}, \quad \Psi = \sum_{0\leq i\leq\infty, 1\leq j\leq m_i}^{\infty} b_{ij}\phi_{ij},$$

we have

$$\sum_{0\leq i\leq\infty, 1\leq j\leq m_i}^{\infty} \begin{pmatrix} b-\lambda_i-\mu & -b^2 \\ \dfrac{2}{b} & -1-d\lambda_i-\mu \end{pmatrix} \begin{pmatrix} a_{ij} \\ b_{ij} \end{pmatrix} \phi_{ij} = 0.$$

Notice that μ is an eigenvalue of L if and only if

$$\mu^2 - P_i\mu + Q_i = 0, \ i = 0, 1, 2, \cdots, \tag{4.5}$$

where

$$P_i = b - 1 - (1+d)\lambda_i, \ Q_i = d\lambda_i^2 + (1-bd)\lambda_i + b.$$

Since $0 < b < 1$, we have $P_i < 0$ for all $i \geq 0$. Clearly, $Q_0 > 0$. Furthermore, if $\lambda_1 \geq b$, we have $Q_i = d\lambda_i(\lambda_i - b) + \lambda_i + b > 0$ for $i \geq 1$, which implies that $\mathrm{Re}(\mu) < 0$ for all eigenvalues μ, therefore, the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ is locally asymptotically stable; if $\lambda_1 < b$ and $0 < d < \tilde{d}$, we see $\lambda_i < b$ and $d < \tilde{d}$ for $1 \leq i \leq i_0$, which leads to $Q_i > 0$ for $1 \leq i \leq i_0$, for $i > i_0$, we see $\lambda_i \geq b$ and then we obtain $Q_i > 0$, hence, we have $Q_i > 0$ for $i \geq 1$. This shows that $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ is locally asymptotically stable; if $\lambda_1 < b$ and $d > \tilde{d}$, then we can assume that the minimum value is at $1 \leq j \leq i_0$, thus $d > d_j$, which implies $Q_j = d\lambda_j(\lambda_j - b) + \lambda_j + b < 0$, hence, the equilibrium $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ is unstable. The proof is completed. □

4.2. Local steady state bifurcation

In this subsection, we consider the steady state bifurcation for the Gierer-Meinhardt system (2.1), and we consider the following semilinear elliptic system:

$$\begin{cases} -u'' = \dfrac{u^2}{v} - bu, x \in (0, l\pi), \\ -dv'' = u^2 - v, \ x \in (0, l\pi), \\ u' = v' = 0, \qquad x = 0, l\pi. \end{cases} \tag{4.6}$$

Here, we denote $d_1 = 1, d_2 = d$, and we choose d as a bifurcation parameter.

For the sake of simplicity, we translate $\left(\frac{1}{b}, \frac{1}{b^2}\right)$ to the origin by the translation $(\hat{u}, \hat{v}) = (u - \frac{1}{b}, v - \frac{1}{b^2})$ and still denote $\hat{u}, \hat{v}$ by u, v, respectively, so we have the following system:

$$\begin{cases} -u'' = \dfrac{(u+\frac{1}{b})^2}{v+\frac{1}{b^2}} - b(u+\dfrac{1}{b}), & x \in (0, l\pi), \\ -dv'' = \left(u+\dfrac{1}{b}\right)^2 - \left(v+\dfrac{1}{b^2}\right), & x \in (0, l\pi), \\ u' = v' = 0, & x = 0, l\pi. \end{cases} \tag{4.7}$$

Then, we consider the local bifurcation for system (4.7), which includes the cases of linearization operators with simple and double eigenvalues, particular, for the case of simple eigenvalues, we apply the Crandall-Rabinowitz bifurcation theorem; for the case of double eigenvalues, we complete the proof by spatial decomposition and implicit function theorem.

Let $X = \{(u, v) \in W^{2,2}(0, l\pi) \times W^{2,2}(0, l\pi) : u' = v' = 0, x = 0, l\pi\}$ and $Y = L^2(0, l\pi) \times L^2(0, l\pi)$. Define the map $F : R^+ \times X \to Y$ by

$$F(d, U) = \begin{pmatrix} u'' + \dfrac{(u+\frac{1}{b})^2}{v+\frac{1}{b^2}} - b(u+\dfrac{1}{b}) \\ dv'' + (u+\dfrac{1}{b})^2 - (v+\dfrac{1}{b^2}) \end{pmatrix}, \ U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is easy to see that we only need to consider that the zero solution of the mapping F, and $(0, 0)$ is the unique constant solution of (4.7), so we have $F(d, (0, 0)) = 0$. By simple calculations, the Frchet derivative of F with respect to U at the point $(0, 0)$ can be written as

$$L(d) = F_U(d, (0, 0)) = \begin{pmatrix} \Delta + b & -b^2 \\ \dfrac{2}{b} & d\Delta - 1 \end{pmatrix},$$

where $\Delta = \frac{d^2}{dx^2}$.

Throughout this subsection, we always assume that $\lambda_1 < b$. Then there exists a largest positive integer i_0 such that $\lambda_i < b$ for $1 \le i \le i_0$. Letting $\mu = 0$ in (4.5) and then we have

$$(H) : d = d_i := \frac{\lambda_i + b}{\lambda_i(b - \lambda_i)}, \text{ where } \quad \lambda_i = i^2, \ 1 \le i \le i_0.$$

In the following, we prove the existence of non-constant positive solution of $F(d, (u, v)) = 0$ near $F(d, (0, 0)) = 0 (1 \le i \le i_0)$. Note that d_i may be equal or not equal to d_j when $i \ne j$. Thus, we will prove two different cases, namely, $d_i \ne d_j$ and $d_i = d_j$ for $i \ne j$.

Theorem 4.2. *Suppose that (H) holds,*

(i) if $d_i \ne d_j$ whenever $i \ne j$, $i, j \in [1, i_0]$, then $(d_i, (0, 0))$ is a bifurcation point of $F(d, U) = 0$. Moreover, there is a curve of non-constant solutions $(d(s), u(s), v(s))$ of $F(d, U) = 0$ for $|s|$ sufficiently small, satisfying $d(0) = d_i, (u(0), v(0)) = (0, 0)$, $u(s) = s\phi_i + o(s)$, $v(s) = se_i\phi_i + o(s)$, where $d(s), u(s), v(s)$ are continuously differential functions with respect to s, and $\phi_i = \sqrt{\frac{2}{\pi}} \cos ix$, $e_i = \frac{b-\lambda_i}{b^2}$;

(ii) suppose that there exists a positive integer $i(\neq j)$ such that $d_i = d_j = \hat{d}$. Let

$$e_i = \frac{b-\lambda_i}{b^2},\ e_i^* = \frac{b(\lambda_i - b)}{2},\ \Phi_i = \begin{pmatrix} 1 \\ e_i \end{pmatrix} \phi_i, \tag{4.8}$$

$$\begin{aligned} &A_1 = b^2 - 2b^3 e_i + b^4 e_i^2 \neq 0, `A_2 = 2b^2 - 2b^3(e_i + e_j) + 2b^4 e_i e_j \neq 0, \\ &A_3 = b^2 - 2b^3 e_j + b^4 e_j^2 \neq 0, \end{aligned} \tag{4.9}$$

$$X_2 := \{(y,z)^T \in X : \int_0^\pi (y + e_i z)\phi_i dx = \int_0^\pi (y + e_j z)\phi_j dx = 0\}. \tag{4.10}$$

If $1 + e_i e_i^* \neq 0, 1 + e_j e_j^* \neq 0$ and $j = 2i(\text{resp.} i = 2j)$, then $(\hat{d}, (0,0))$ is a bifurcation point of $F(d,U) = 0$. Moreover, there is a curve of nonconstant solutions $(d(\omega), s(\omega)(\cos\omega\Phi_i + \sin\omega\Phi_j + W(\omega)))$ of $F(d,U) = 0$ for $|\omega - \omega_0|$ sufficiently small, where $(d(\omega), s(\omega), W(\omega))$ are continuously differentiable functions with respect to ω and satisfy $d(\omega_0) = \hat{d}$, $s(\omega_0) = 0$, $W(\omega_0) = 0$, here ω_0 is any constant satisfying

$$\cos\omega_0 \neq 0 \text{ and } A_2 c_2 c_4 e_j \lambda_j \sin^2\omega_0 \neq A_1 c_1 c_3 e_i \lambda_i \cos^2\omega_0, \tag{4.11}$$

$$\left(\textit{resp. } \sin\omega_0 \neq 0 \text{ and } A_3 c_2 c_5 e_j \lambda_j \sin^2\omega_0 \neq A_2 c_1 c_6 e_i \lambda_i \cos^2\omega_0\right), \tag{4.12}$$

where

$$c_1 = \frac{e_i^*}{1 + e_i e_i^*},\ c_2 = \frac{e_j^*}{1 + e_j e_j^*},\ c_3 = \sqrt{\frac{1}{2\pi} \frac{1 + \frac{e_j^*}{A_1}}{1 + e_j e_j^*}},$$

$$c_4 = \sqrt{\frac{1}{2\pi} \frac{1 + \frac{2e_i^*}{A_2}}{1 + e_i e_i^*}},\ c_5 = \sqrt{\frac{1}{2\pi} \frac{1 + \frac{e_i^*}{A_3}}{1 + e_i e_i^*}},\ c_6 = \sqrt{\frac{1}{2\pi} \frac{1 + \frac{2e_j^*}{A_2}}{1 + e_j e_j^*}}.$$

Proof. We first prove the statement (i). By applying the Crandall-Rabinowitz bifurcation theorem about simple eigenvalues in [4], we obtain that $(d_i, (0,0))$ is a bifurcation point if the following conditions are satisfied:

(1) the partial derivatives F_d, F_U and F_{dU} exist and are continuous;

(2) dim ker $F_U(d_i, (0,0)) = \text{codim } R(F_U(d_i, (0,0))) = 1$;

(3) let ker $F_U(d_i, (0,0)) = \text{span}\{\Phi_i\}$, then $F_{dU}(d_i, (0,0))\Phi_i \notin R(F_U(d_i, (0,0)))$.

When $d_i \neq d_j$, note that the operator

$$L(d_i) = F_U(d_i, (0,0)) = \begin{pmatrix} \Delta + b & -b^2 \\ \frac{2}{b} & d_i\Delta - 1 \end{pmatrix},\ \Delta = \frac{d^2}{dx^2}.$$

It is obvious that the linear operators F_d, F_u and F_{dU} are continuous. By making simple calculations, we have

$$\ker L(d_i) = \text{span}\{\Phi_i\},\ \Phi_i = \begin{pmatrix} 1 \\ e_i \end{pmatrix} \phi_i,$$

where $e_i = \frac{b-\lambda_i}{b^2} > 0$. Hence, dim ker $L(d_i) = 1$.

The adjoint operator is defined by

$$L^*(d_i) = \begin{pmatrix} \Delta + b & \dfrac{2}{b} \\ -b^2 & d_i\Delta - 1 \end{pmatrix}.$$

In the same way, we get

$$\ker L^*(d_i) = \text{span}\{\Phi_i^*\},\ \Phi_i^* = \begin{pmatrix} 1 \\ e_i^* \end{pmatrix} \phi_i,$$

where $e_i^* = \frac{b(\lambda_i - b)}{2} < 0$. Since $R(L) = (\ker L^*)^T$, then codim $R(L(d_i)) = \dim \ker L^*(d_i) = 1$.

Finally, since

$$F_{dU}(d_i, (0,0))\Phi_i = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \Phi_i = \begin{pmatrix} 0 \\ -\lambda_i e_i \phi_i \end{pmatrix},$$

and

$$\langle F_{dU}(d_i, (0,0))\Phi_i, \Phi_i^* \rangle = -\lambda_i e_i e_i^* \int_0^\pi \phi_i^2 dx = -\lambda_i e_i e_i^* \neq 0, \tag{4.13}$$

we find $F_{dU}(d_i, (0,0))\Phi_i \notin R(L(d_i))$. Hence, the proof of (i) is completed.

(ii) Suppose that there exists a positive integer $i (\neq j)$ such that $d_i = d_j = \hat{d}$. Clearly, $F(\hat{d}, (0,0)) = 0$. Let $L(\hat{d}) = F_U(\hat{d}, (0,0))$. Then

$$\begin{aligned} &\ker L(\hat{d}) = \text{span}\{\Phi_i, \Phi_j\},\ \ker L^*(\hat{d}) = \text{span}\{\Phi_i^*, \Phi_j^*\}, \\ &R(L(\hat{d})) = \{(u,v)^T \in Y : \int_0^\pi (u + e_i^* v)\phi_i dx = \int_0^\pi (u + e_j^* v)\phi_j dx = 0\}, \end{aligned}$$

which leads to $\dim \ker L(\hat{d}) = \text{codim } R(L(\hat{d})) = 2$ Obviously, the Crandall-Rabinowitz bifurcation theorem does not work in the situation. Now, we solve this situation by techniques of spatial decomposition and implicit function theorem.

To complete our idea, we first make the following decomposition. Let $X_1 = \ker L(\hat{d})$ and define the operator P on Y by

$$P\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{1 + e_i e_i^*}\left[\int_0^\pi (u + e_i^* v)\phi_i dx\right]\Phi_i + \frac{1}{1 + e_j e_j^*}\left[\int_0^\pi (u + e_j^* v)\phi_j dx\right]\Phi_j.$$

Then, $R(P) = \text{span}\{\Phi_i, \Phi_j\} = X_1 \subset Y$ and $P^2 = P$. Hence, P is the projection from Y to $X_1 \subset Y$. Decompose Y as $Y = Y_1 \oplus Y_2$ with $Y_1 = R(P) = X_1$, $Y_2 = \ker\,(P) = R(L(\hat{d}))$. Meantime, we decompose $X = X_1 \oplus X_2$, where $X_1 = \text{span}\{\Phi_i, \Phi_j\}$ and X_2 is defined by (4.10). Next, We will look for solutions of $F(d, U) = 0$ in the form

$$U = s(\cos\omega\Phi_i + \sin\omega\Phi_j + W),\ W = (\omega_1, \omega_2)^T \in X_2,$$

where $s, \omega \in R$ are parameters.

Then, we rewrite that map $F : R^+ \times X \to Y$ by

$$F(d,U) = \begin{pmatrix} u'' + \dfrac{(u+\frac{1}{b})^2}{v+\frac{1}{b^2}} - b(u+\dfrac{1}{b}) \\ dv'' + (u+\dfrac{1}{b})^2 - (v+\dfrac{1}{b^2}) \end{pmatrix} = L(d)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F^1(u,v) \\ F^2(u,v) \end{pmatrix},$$

where

$$F^1(u,v) = b^2u^2 - 2b^3uv + b^4v^2 - b^4u^2v + 2b^5uv^2 - b^6v^3 + O(|u|^4, |v|^3|u|), \tag{4.14}$$

$$F^2(u,v) = u^2 + O(|u|^4, |v|^3|u|). \tag{4.15}$$

It is easy to see that (u,v) is a solution of (4.6) if and only if (u,v) satisfies $F(d,U) = 0$. This leads us to take the next step to find the existence of nonconstant pairs (u,v).

Fix $\omega_0 \in R$ for the time being, and we define a nonlinear mapping $K(d,s,W;\omega) : R^+ \times R \times X_2 \times (\omega_0 - \delta, \omega_0 + \delta) \to Y$ by

$$\begin{aligned} K(d,s,W;\omega) &= s^{-1}F(d, s(\cos\omega\Phi_i + \sin\omega\Phi_j + W)) \\ &= L(d)(\cos\omega\Phi_i + \sin\omega\Phi_j + W) + s\begin{pmatrix} \tilde{F^1} \\ \tilde{F^2} \end{pmatrix}, \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} \tilde{F^1} &= b^2(\cos\omega\phi_i + \sin\omega\phi_j + \omega_1)^2 + b^4(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + \omega_2)^2 \\ &\quad - 2b^3(\cos\omega\phi_i + \sin\omega\phi_j + \omega_1)(e_i\cos\omega\phi_i + e_j\sin\omega\phi_j + \omega_2) + O(|s|), \\ \tilde{F^2} &= (\cos\omega\phi_i + \sin\omega\phi_j + \omega_1)^2 + O(|s|). \end{aligned}$$

It is obvious that $K(\hat{d},0,0;\omega_0) = 0$. So we can obtain the Fréchet derivative of $K(d,s,W;\omega)$ with respect to (d,s,W) at $(\hat{d},0,0;\omega_0)$ for the linear mapping

$$\begin{aligned} K_{d,s,W}(\hat{d},0,0;\omega_0)(d,s,W) =& L(\hat{d})W - de_i\lambda_i\cos\omega_0\begin{pmatrix} 0 \\ \phi_i \end{pmatrix} - de_j\lambda_j\sin\omega_0\begin{pmatrix} 0 \\ \phi_j \end{pmatrix} \\ &+ sA_1\cos^2\omega_0\begin{pmatrix} \phi_i^2 \\ \dfrac{1}{A_1}\phi_i^2 \end{pmatrix} + sA_2\cos\omega_0\sin\omega_0\begin{pmatrix} \phi_i\phi_j \\ \dfrac{2}{A_2}\phi_i\phi_j \end{pmatrix} \\ &+ sA_3\sin^2\omega_0\begin{pmatrix} \phi_j^2 \\ \dfrac{1}{A_3}\phi_j^2 \end{pmatrix}, \end{aligned}$$

where A_1, A_2, A_3 are given by (4.9).

Next we will prove that $K_{d,s,W}(\hat{d},0,0;\omega_0) : R^+ \times R \times X_2 \to Y$ is an isomorphism. Then, we decompose

$$\begin{pmatrix} 0 \\ \phi_i \end{pmatrix} = c_1\Phi_i + \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = c_2\Phi_j + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

where

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -c_1 \\ 1 - c_1 e_i \end{pmatrix} \phi_i, \ c_1 = \frac{e_i^*}{1 + e_i e_i^*} \neq 0,$$

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -c_2 \\ 1 - c_2 e_j \end{pmatrix} \phi_j, \ c_2 = \frac{e_j^*}{1 + e_j e_j^*} \neq 0,$$

and we can clearly check $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in Y_2$.

Now, we divide our discussion into two cases $j = 2i$ and $i = 2j$.

Case I. $j = 2i$.

By simple calculations, we have

$$\int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}}, \ \int_0^\pi \phi_i \phi_j^2 dx = 0, \ \int_0^\pi \phi_j^3 dx = 0,$$

then, it is obvious that $\begin{pmatrix} \phi_j^2 \\ \frac{1}{A_3}\phi_j^2 \end{pmatrix} \in Y_2$ and we decompose further

$$\begin{pmatrix} \phi_i^2 \\ \dfrac{1}{A_1}\phi_i^2 \end{pmatrix} = c_3 \Phi_j + \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}, \ \begin{pmatrix} \phi_i \phi_j \\ \dfrac{2}{A_2}\phi_i\phi_j \end{pmatrix} = c_4 \Phi_i + \begin{pmatrix} u_4 \\ v_4 \end{pmatrix},$$

where

$$c_3 = \frac{1 + \frac{e_j^*}{A_1}}{1 + e_j e_j^*} \int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}} \frac{1 + \frac{e_j^*}{A_1}}{1 + e_j e_j^*} \neq 0, \quad \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} \phi_i^2 - c_3 \phi_j \\ \dfrac{1}{A_1}\phi_i^2 - c_3 e_j \phi_j \end{pmatrix} \in Y_2,$$

$$c_4 = \frac{1 + \frac{2e_i^*}{A_2}}{1 + e_i e_i^*} \int_0^\pi \phi_i^2 \phi_j dx = \sqrt{\frac{1}{2\pi}} \frac{1 + \frac{2e_i^*}{A_2}}{1 + e_i e_i^*} \neq 0, \quad \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} \phi_i \phi_j - c_4 \phi_i \\ \dfrac{2}{A_2}\phi_i \phi_j - c_4 e_i \phi_i \end{pmatrix} \in Y_2.$$

At this time, we have

$$K_{d,s,W}(\hat{d}, 0, 0; \omega_0)(d, s, W) = Z_1 + Z_2,$$

where

$$\begin{aligned} Z_1 =& (-dc_1 e_i \lambda_i \cos\omega_0 + sc_4 A_2 \cos\omega_0 \sin\omega_0)\Phi_i + (-dc_2 e_j \lambda_j \sin\omega_0 + sc_3 A_1 \cos^2\omega_0)\Phi_j \in Y_1, \\ Z_2 =& L(\hat{d})W - de_i\lambda_i \cos\omega_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - de_j \lambda_j \sin\omega_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + sA_1 \cos^2\omega_0 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \\ &+ sA_2 \cos\omega_0 \sin\omega_0 \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} + sA_3 \sin^2\omega_0 \begin{pmatrix} \phi_j^2 \\ \dfrac{1}{A_3}\phi_j^2 \end{pmatrix} \in Y_2. \end{aligned}$$

Let

$$K_{d,s,W}(\hat{d},0,0;\omega_0)(d,s,W)=0. \tag{4.17}$$

Since $L(\hat{d})$ is an isomorphism from X_2 to Y_2, (4.17) is equivalent to

$$\begin{cases}(-dc_1e_i\lambda_i\cos\omega_0+sc_4A_2\cos\omega_0\sin\omega_0)\Phi_i+(-dc_2e_j\lambda_j\sin\omega_0+sc_3A_1\cos^2\omega_0)\Phi_j=0,\\ L(\hat{d})W-de_i\lambda_i\cos\omega_0\begin{pmatrix}u_1\\ v_1\end{pmatrix}-de_j\lambda_j\sin\omega_0\begin{pmatrix}u_2\\ v_2\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}u_3\\ v_3\end{pmatrix}\\ \quad+sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}u_4\\ v_4\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}\phi_j^2\\ \dfrac{1}{A_3}\phi_j^2\end{pmatrix}=0.\end{cases} \tag{4.18}$$

By (4.11), we obtain $d=0,s=0$ from the first equation of (4.18). Embedding them into the second equation we obtain $W=0$. This implies that $K_{d,s,W}(\hat{d},0,0;\omega_0)$ is injective.

Now we prove that $K_{d,s,W}(\hat{d},0,0;\omega_0)$ is surjective. For an arbitrary element $\begin{pmatrix}u\\ v\end{pmatrix}\in Y$, we shall find $K_{d,s,W}(\hat{d},0,0;\omega_0)(d,s,W)$ such that

$$K_{d,s,W}(\hat{d},0,0;\omega_0)(d,s,W)=\begin{pmatrix}u\\ v\end{pmatrix}. \tag{4.19}$$

By the decomposition of Y, then exist $\alpha,\beta\in R$ and $\begin{pmatrix}u_0\\ v_0\end{pmatrix}\in Y_2$ such that

$$\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}u_0\\ v_0\end{pmatrix}+\alpha\Phi_i+\beta\Phi_j.$$

Substituting it into (4.6) yields

$$\begin{cases}-dc_1e_i\lambda_i\cos\omega_0+sc_4A_2\cos\omega_0\sin\omega_0=\alpha,\\ -dc_2e_j\lambda_j\sin\omega_0+sc_3A_1\cos^2\omega_0=\beta,\\ L(\hat{d})W-de_i\lambda_i\cos\omega_0\begin{pmatrix}u_1\\ v_1\end{pmatrix}-de_j\lambda_j\sin\omega_0\begin{pmatrix}u_2\\ v_2\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}u_3\\ v_3\end{pmatrix}\\ \quad+sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}u_4\\ v_4\end{pmatrix}+sA_3\sin^2\omega_0\begin{pmatrix}\phi_j^2\\ \dfrac{1}{A_3}\phi_j^2\end{pmatrix}=\begin{pmatrix}u_0\\ v_0\end{pmatrix}.\end{cases} \tag{4.20}$$

By (4.11), we obtain

$$d=\bar{d}:=\frac{\alpha c_3A_1\cos\omega_0-\beta c_4A_2\sin\omega_0}{c_2c_4e_j\lambda_jA_2\sin^2\omega_0-c_1c_3e_i\lambda_iA_1\cos^2\omega_0},$$

$$s=\bar{s}:=\frac{\alpha c_2 e_j\lambda_j\sin\omega_0-\beta c_1 e_i\lambda_i\cos\omega_0}{c_2c_4e_j\lambda_jA_2\cos\omega_0\sin^2\omega_0-c_1c_3e_i\lambda_iA_1\cos^3\omega_0}.$$

By embedding $\bar{d},\bar{s}$ into the third equation of (4.20) to get

$$\begin{aligned}L(\hat{d})W&=\begin{pmatrix}u_0\\v_0\end{pmatrix}+\bar{d}e_i\lambda_i\cos\omega_0\begin{pmatrix}u_1\\v_1\end{pmatrix}+\bar{d}e_j\lambda_j\sin\omega_0\begin{pmatrix}u_2\\v_2\end{pmatrix}-\bar{s}A_1\cos^2\omega_0\begin{pmatrix}u_3\\v_3\end{pmatrix}\\&\quad-\bar{s}A_2\cos\omega_0\sin\omega_0\begin{pmatrix}u_4\\v_4\end{pmatrix}-\bar{s}A_3\sin^2\omega_0\begin{pmatrix}\phi_j^2\\ \dfrac{1}{A_3}\phi_j^2\end{pmatrix}\\&:=\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\in Y_2,\end{aligned}$$

which implies $W=L^{-1}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\in Y_2$.

Then we find that $(d,s,W)=\left(\bar{d},\bar{s},L^{-1}\begin{pmatrix}\bar{u}\\ \bar{v}\end{pmatrix}\right)$ is the solution of (4.19), which shows $K_{d,s,W}(\hat{d},0,0;\omega_0)$ is surjective.

Hence, $K_{d,s,W}(\hat{d},0,0;\omega_0)$ is an isomorphism from $R^+\times R\times X_2$ to Y. Applying the implicit function theorem to

$$K(d,0,0;\omega_0)=0, \tag{4.21}$$

we get that there is a curve of non-constant solution $(d(\omega),s(\omega),W(\omega))$ for $|\omega-\omega_0|$ sufficiently small, where ω_0 satisfies (4.11). Moreover, $d(\omega),s(\omega),W(\omega)$ are continuously differentiable functions and satisfy $d(\omega_0)=\hat{d},s(\omega_0)=0,W(\omega_0)=0$.

Case II. $i=2j$.

By simple calculations, $\int_0^\pi\phi_i^2\phi_jdx=0,\ \int_0^\pi\phi_i\phi_j^2dx=\sqrt{\frac{1}{2\pi}}\neq0$. Then $\begin{pmatrix}\phi_i^2\\ \frac{1}{A_1}\phi_i^2\end{pmatrix}\in Y_2$ and we decompose

$$\begin{pmatrix}\phi_j^2\\ \dfrac{1}{A_3}\phi_j^2\end{pmatrix}=c_5\Phi_i+\begin{pmatrix}u_5\\v_5\end{pmatrix},\quad\begin{pmatrix}\phi_i\phi_j\\ \dfrac{2}{A_2}\phi_i\phi_j\end{pmatrix}=c_6\Phi_j+\begin{pmatrix}u_6\\v_6\end{pmatrix},$$

where

$$\begin{pmatrix}u_5\\v_5\end{pmatrix}=\begin{pmatrix}\phi_j^2-c_5\phi_i\\ \dfrac{1}{A_3}\phi_j^2-c_5e_i\phi_i\end{pmatrix},\quad c_5=\frac{1+\frac{e_i^*}{A_3}}{1+e_ie_i^*}\int_0^\pi\phi_i\phi_j^2dx=\sqrt{\frac{1}{2\pi}}\frac{1+\frac{e_i^*}{A_3}}{1+e_ie_i^*},$$

$$\begin{pmatrix}u_6\\v_6\end{pmatrix}=\begin{pmatrix}\phi_i\phi_j-c_6\phi_j\\ \dfrac{2}{A_2}\phi_i\phi_j-c_6e_j\phi_j\end{pmatrix},\quad c_6=\frac{1+\frac{2e_j^*}{A_2}}{1+e_je_j^*}\int_0^\pi\phi_i\phi_j^2dx=\sqrt{\frac{1}{2\pi}}\frac{1+\frac{2e_j^*}{A_2}}{1+e_je_j^*}.$$

Hence, we have

$$\begin{aligned}
&K_{d,s,W}(\hat{d},0,0;\omega_0)(d,s,W)\\
=&L(\hat{d})W+(-dc_1e_i\lambda_i\cos\omega_0+sc_5A_3\sin^2\omega_0)\Phi_i+(-dc_2e_j\lambda_j\sin\omega_0\\
&+sc_6A_2\cos\omega_0\sin\omega)\Phi_j-de_i\lambda_i\cos\omega_0\begin{pmatrix}u_1\\ v_1\end{pmatrix}-de_j\lambda_j\sin\omega_0\begin{pmatrix}u_2\\ v_2\end{pmatrix}\\
&+sA_3\sin^2\omega_0\begin{pmatrix}u_5\\ v_5\end{pmatrix}+sA_2\cos\omega_0\sin\omega_0\begin{pmatrix}u_6\\ v_6\end{pmatrix}+sA_1\cos^2\omega_0\begin{pmatrix}\phi_i^2\\ \dfrac{1}{A_1}\phi_i^2\end{pmatrix},
\end{aligned}$$

where

$$\begin{pmatrix}u_1\\ v_1\end{pmatrix},\begin{pmatrix}u_2\\ v_2\end{pmatrix},\begin{pmatrix}u_5\\ v_5\end{pmatrix},\begin{pmatrix}u_6\\ v_6\end{pmatrix},\begin{pmatrix}\phi_i^2\\ \dfrac{1}{A_1}\phi_i^2\end{pmatrix}\in Y_2.$$

As in **case I**, we can prove that the map $K_{d,s,W}(\hat{d},0,0;\omega_0)$ is an isomorphism from $R^+\times R\times X_2$ to Y if ω_0 satisfies (4.12). According to the implicit function theorem, we have completed the proof of this case. □

4.3. Direction of the bifurcation

In this subsection, we consider the direction of the steady state bifurcation from simple eigenvalues derived the Theorem 4.2(i).

We rewrite the map $F:\mathbb{R}^+\times X\to Y$ by

$$F(d,U)=\begin{pmatrix}u''+\tilde{f}(u,v)\\ dv''+\tilde{g}(u,v)\end{pmatrix},$$

where

$$\tilde{f}(u,v)=\frac{(u+\frac{1}{b})^2}{v+\frac{1}{b^2}}-b(u+\frac{1}{b}),\ \tilde{g}(u,v)=(u+\frac{1}{b})^2-(v+\frac{1}{b^2}).$$

Then a straightforward calculation gives

$$\begin{cases}\tilde{f}_u(0,0)=b,\ \tilde{f}_v(0,0)=-b^2,\ \tilde{f}_{uu}(0,0)=2b^2,\ \tilde{f}_{uv}(0,0)=-2b^3,\\ \tilde{f}_{vv}(0,0)=2b^4,\ \tilde{f}_{uuv}(0,0)=-2b^4,\ \tilde{f}_{uvv}(0,0)=4b^5,\ \tilde{f}_{vvv}(0,0)=-6b^6,\\ \tilde{g}_u(0,0)=\dfrac{2}{b},\ \tilde{g}_v(0,0)=-1,\ \tilde{g}_{uu}(0,0)=2,\\ \tilde{f}_{uuu}(0,0)=\tilde{g}_{uv}(0,0)=\tilde{g}_{vv}(0,0)=\tilde{g}_{uuu}(0,0)=\tilde{g}_{uuv}(0,0)=\tilde{g}_{uvv}(0,0)=\tilde{g}_{vvv}(0,0)=0.\end{cases}\tag{4.22}$$

By the Theorem 4.2(i), we can see that $\dim\ker F_U(d_i,(0,0))=\operatorname{codim} R(d_i,(0,0))=1$ and $\ker F_U(d_i,(0,0))=\operatorname{span}\{\Phi_i\}$. Hence, X and Y can be decomposed as

$$X=\ker F_U(d_i,(0,0))\oplus Z\text{ and }Y=R(F_U(d_i,(0,0)))\oplus Z',$$

where Z and Z' are the complements of $\ker F_U(d_i,(0,0))$ in X and $R(F_U(d_i,(0,0)))$ in Y, respectively. By (4.13), we find that

$$\langle F_{dU}(d_i,(0,0))\Phi_i, \Phi_i^*\rangle = -\lambda_i e_i e_i^* = -i^2 e_i e_i^* \neq 0.$$

According to the [19], we get $d'(0) = -\frac{\langle F_{UU}(d_i,(0,0))\Phi_i^2,\Phi_i^*\rangle}{2\langle F_{dU}(d_i,(0,0))\Phi_i,\Phi_i^*\rangle}$.

By some calculations, we have

$$\langle F_{UU}(d_i,(0,0))\Phi_i^2, \Phi_i^*\rangle = (k_i + l_i e_i^*)\int_0^\pi \phi_i^3 dx,$$

where

$$k_i = \tilde{f}_{uu}(0,0) + 2\tilde{f}_{uv}(0,0)e_i + \tilde{f}_{vv}(0,0)e_i^2 = 2(b^2 e_i - b)^2 = 2\lambda_i^2, \tag{4.23}$$

$$l_i = \tilde{g}_{uu}(0,0) + 2\tilde{g}_{uv}(0,0)e_i + \tilde{g}_{vv}(0,0)e_i^2 = 2. \tag{4.24}$$

Since $\phi_i = \sqrt{\frac{2}{b}}\cos ix$, we have $\int_0^\pi \phi_i^3 dx = 0$, this leads to $\langle F_{UU}(d_i,(0,0))\Phi_i^2,\Phi_i^*\rangle = 0$. Thus, $d'(0) = 0$.

From [19], we can obtain that the direction of steady state bifurcation is supercritical (resp. subcritical) if

$$d''(0) = -\frac{\langle F_{UUU}(d_i,(0,0))\Phi_i^3,\Phi_i^*\rangle + 3\langle F_{UU}(d_i,(0,0))\Phi_i\theta,\Phi_i^*\rangle}{3\langle F_{dU}(d_i,(0,0))\Phi_i^2,\Phi_i^*\rangle} > (<)0,$$

where θ is the solution of the following problem

$$F_{UU}(d_i,(0,0))\Phi_i^2 + F_U(d_i,(0,0))\theta = 0.$$

By direct calculation, we have

$$\langle F_{UUU}(d_i,(0,0))\Phi_i^3,\Phi_i^*\rangle = \frac{4}{\pi^2}(m_i + n_i e_i^*)\int_0^\pi \cos^4(ix)dx = \frac{3}{2\pi}(m_i + n_i e_i^*),$$

where

$$m_i = \tilde{f}_{uuu}(0,0) + 3\tilde{f}_{uuv}(0,0)e_i + 3\tilde{f}_{uvv}(0,0)e_i^2 + \tilde{f}_{vvv}(0,0)e_i^3 = -6b^4 e_i + 12b^5 e_i^2 - 6b^6 e_i^3, \tag{4.25}$$

$$n_i = \tilde{g}_{uuu}(0,0) + 3\tilde{g}_{uuv}(0,0)e_i + 3\tilde{g}_{uvv}(0,0)e_i^2 + \tilde{g}_{vvv}(0,0)e_i^3 = 0. \tag{4.26}$$

Let $\theta = (\theta_1,\theta_2)$, and it satisfies

$$\begin{cases} \theta_1'' + \tilde{f}_u(0,0)\theta_1 + \tilde{f}_v(0,0)\theta_2 = -k_i\phi_i^2, & x \in (0,\pi), \\ d_i\theta_2'' + \tilde{g}_u(0,0)\theta_1 + \tilde{g}_v(0,0)\theta_2 = -l_i\phi_i^2, & x \in (0,\pi), \\ \theta_i'(0) = \theta_i'(\pi) = 0, & i = 1,2. \end{cases} \tag{4.27}$$

Integrating (4.27) by part, we derive

$$\int_0^\pi \theta_1 dx = \frac{l_i\tilde{f}_v(0,0) - k_i\tilde{g}_v(0,0)}{\tilde{f}_u(0,0)\tilde{g}_v(0,0) - \tilde{f}_v(0,0)\tilde{g}_u(0,0)} = \frac{2(\lambda_i^2 - b^2)}{b}, \tag{4.28}$$

$$\int_0^\pi \theta_2 dx = \frac{k_i\tilde{g}_u(0,0) - l_i\tilde{f}_u(0,0)}{\tilde{f}_u(0,0)\tilde{g}_v(0,0) - \tilde{f}_v(0,0)\tilde{g}_u(0,0)} = \frac{2(2\lambda_i^2 - b^2)}{b^2}. \tag{4.29}$$

A direct calculation shows that

$$\langle F_{UU}(d_i,(0,0))\Phi_i\theta,\Phi_i^*\rangle = C_1\int_0^\pi \theta_1\phi_i^2 dx + C_2\int_0^\pi \theta_2\phi_i^2 dx,$$

where

$$C_1 = \tilde{f}_{uu}(0,0) + 2\tilde{f}_{uv}(0,0)e_i + (\tilde{g}_{uu}(0,0) + 2\tilde{g}_{uv}(0,0)e_i)e_i^*,$$
$$C_2 = \tilde{f}_{uv}(0,0) + 2\tilde{f}_{vv}(0,0)e_i + (\tilde{g}_{uv}(0,0) + 2\tilde{g}_{vv}(0,0)e_i)e_i^*.$$

By (4.22) and the expressions of e_i, e_i^*, then

$$C_1 = 2b^2 - 2b^3 e_i + 2e_i^* = b(3\lambda_i - b), \tag{4.30}$$
$$C_2 = -2b^3 + 2b^4 e_i = -2b^2\lambda_i. \tag{4.31}$$

For the later calculations, we now compute $\int_0^\pi \theta_1\phi_i^2 \mathrm{d}x$ and $\int_0^\pi \theta_2\phi_i^2 \mathrm{d}x$. Multiplying (4.28) by ϕ_i^2 and integrating by parts, we get

$$\int_0^\pi \phi_i^4 dx = \frac{4}{\pi^2}\int_0^\pi \cos^4(ix)dx = \frac{3}{2\pi},$$

then

$$\int_0^\pi \theta_1''\phi_i^2 dx + \tilde{f}_u(0,0)\int_0^\pi \theta_1\phi_i^2 dx + \tilde{f}_v(0,0)\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}k_i, \tag{4.32}$$

and

$$d_i\int_0^\pi \theta_2''\phi_i^2 dx + \tilde{g}_u(0,0)\int_0^\pi \theta_1\phi_i^2 dx + \tilde{g}_v(0,0)\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}l_i, \tag{4.33}$$

where

$$\int_0^\pi \theta_j''\phi_i^2 dx = \frac{4i^2}{\pi}\int_0^\pi \theta_j(1-\pi\phi_i^2)dx, \quad j=1,2. \tag{4.34}$$

Substituting (4.28), (4.29) and (4.34) to (4.32) and (4.33) yields

$$(\tilde{f}_u(0,0) - 4i^2)\int_0^\pi \theta_1\phi_i^2 dx + \tilde{f}_v(0,0)\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}k_i - \frac{8i^2(\lambda_i^2 - b^2)}{\pi b},$$

and

$$\tilde{g}_u(0,0)\int_0^\pi \theta_1\phi_i^2 dx + (\tilde{g}_v(0,0) - 4i^2 d_i)\int_0^\pi \theta_2\phi_i^2 dx = -\frac{3}{2\pi}l_i - \frac{8i^2(2\lambda_i^2 - b^2)}{\pi b^2}d_i.$$

Thus,

$$\beta_1 \triangleq \int_0^\pi \theta_1\phi_i^2 dx = \frac{(1+4\lambda_i d_i)(3b^2 k_i + 16b\lambda_i(\lambda_i - b^2)) - b^2(3b^2 l_i + 16\lambda_i d_i(2\lambda_i^2 - b^2))}{b - 4b\lambda_i d_i + 4\lambda_i + 16\lambda_i^2 d_i},$$
$$\beta_2 \triangleq \int_0^\pi \theta_2\phi_i^2 dx = \frac{(4\lambda_i - b)(3b^2 l_i + 16\lambda_i d_i(2\lambda_i^2 - b^2)) + (6bk_i + 32\lambda_i(\lambda_i^2 - b^2))}{b - 4b\lambda_i d_i + 4\lambda_i + 16\lambda_i^2 d_i}.$$

In totally, we have

$$d''(0) = \frac{(m_i + n_i e_i^*) + 2\pi(C_1\beta_1 + C_2\beta_2)}{2\pi e_i e_i^*\lambda_i}. \tag{4.35}$$

Here, by (4.22), (4.27), (4.30), (4.31) and the expressions of e_i, e_i^*, d_i, we calculate that

$$m_i + n_i e_i^* = -6\lambda_i^2(b - \lambda_i), \tag{4.36}$$

and

$$\begin{aligned} 2\pi(C_1\beta_1 + C_2\beta_2) &= 2\pi b((3\lambda_i - b)\beta_1 - 2b\lambda_i\beta_2) \\ &= \frac{(6b\lambda_i^2 + 16\lambda_i(\lambda_i^2 - b^2))(4\lambda_i + (b - 3\lambda_i)(1 + 4\lambda_i d_i))}{b} \\ &= \frac{[6b\lambda_i^2 + 16\lambda_i(\lambda_i^2 - b^2)][(5b + 3\lambda_i)(b - 3\lambda_i) + 4\lambda_i(b - \lambda_i)]}{b(b - \lambda_i)}. \end{aligned} \tag{4.37}$$

Substituting (4.36), (4.37) into (4.35), we obtain

$$d''(0) = \frac{b[6b\lambda_i^2 + 16\lambda_i(\lambda_i^2 - b^2)][(5b + 3\lambda_i)(b - 3\lambda_i) + 4\lambda_i(b - \lambda_i)] - 6b^2\lambda_i^2(b - \lambda_i)^2}{\pi b\lambda_i(\lambda_i - b)^3}. \tag{4.38}$$

Hence, according to the above analysis, we can determine the local bifurcation direction of the following theorem by considering the sign of $d''(0)$. Note that $\lambda_i = i^2$ throughout this subsection.

Theorem 4.3. *For the bifurcation of $(\hat{d}, (0,0))$ obtained in Theorem 4.2(i), it is supercritical if $d''(0) > 0$, and it is subcritical if $d''(0) < 0$, where $i \in [1, i_0]$, and i_0 is defined by* (4.3).

4.4. Numerical simulations

In this subsection, we will use the spatial discrete finite difference method to further numerically show different the dynamic phenomena of the system (2.1) with $\Omega = (0, \pi)$.

(1) The neutral curves d about $i \in R$ are shown in Figure 11. Clearly, when $d_1 \neq d_2$ is on the left, the steady state bifurcation is located at the simple eigenvalues and when $d_1 = d_2$ is on the right, the steady state bifurcation is located at a double eigenvalue.

(2) Let $b = 6$. It follows from (4.4) or the condition (H) that $d_1 = 1.400$ and $d_2 = 1.250$. According to Theorem 4.2 (i), we can see that the steady state bifurcation occurs at $d = d_i, i = 1, 2$. See Figure 12(a) for $d = 1.400$ and See Figure 12(b) for $d = 1.250$.

(3) Let $b = 5.7$. It follows from (4.4) or the condition (H) that $d_1 = d_2 = 1.425$. According to the Theorem 4.2 (ii), we can see that the steady state bifurcation occurs at $d = \hat{d}$. See Figure 13 for $d = 1.425$, and we must point out that the bifurcation at this point is a double eigenvalue.

(4) If $d \geq 1$, the system (2.1) has positive periodic solutions on the spatial domain $\Omega = (0, \pi)$. See Figure 13 for $d = 1.1$.

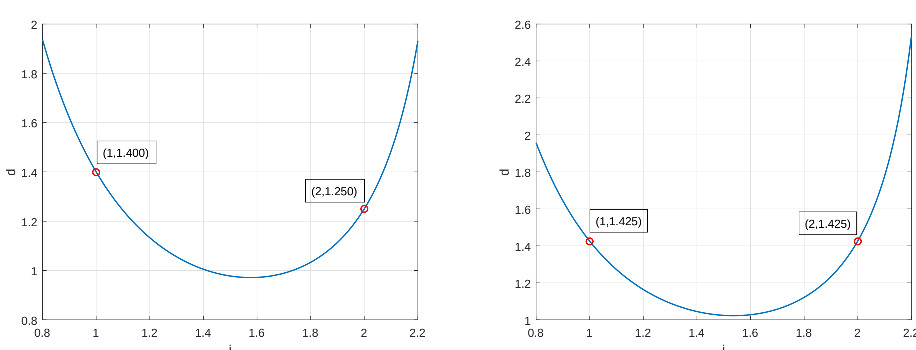
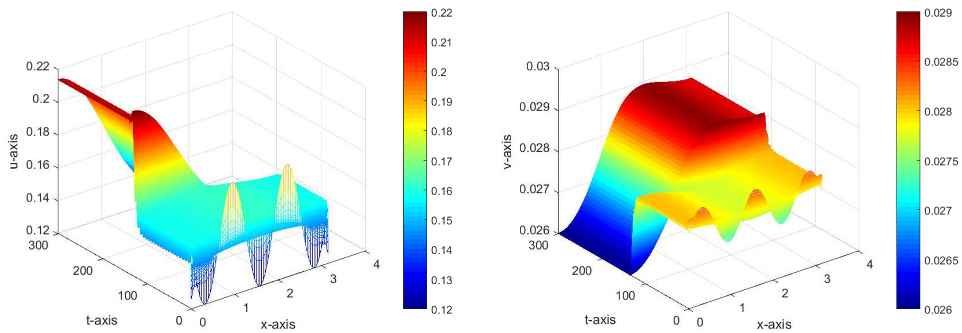
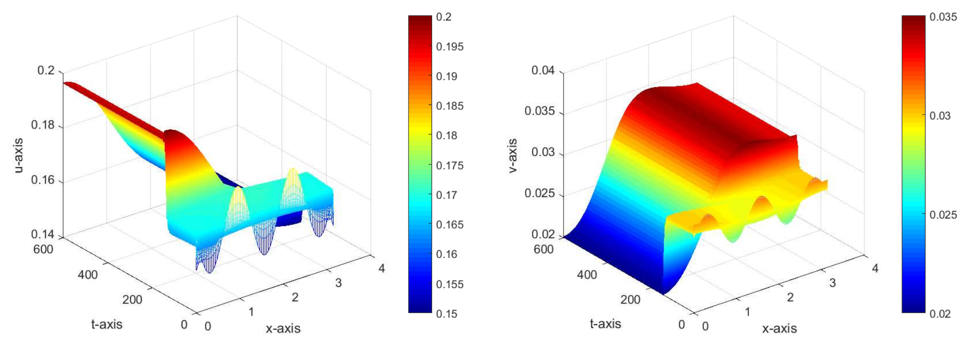


Figure 11. The neutral curves d about $i \in R$. Left: $b = 6$; right: $b = 5.7$.



(a) $d = 1.4$ (a) $d = 1.4$



(b) $d = 1.25$ (b) $d = 1.25$

Figure 12. Steady state bifurcation solutions at the simple eigenvalue for $b = 6$. Here, (a) $d = 1.4$; (b) $d = 1.25$. The solutions tends to a spatially nonhomogeneous steady state. Left: u; right: v.

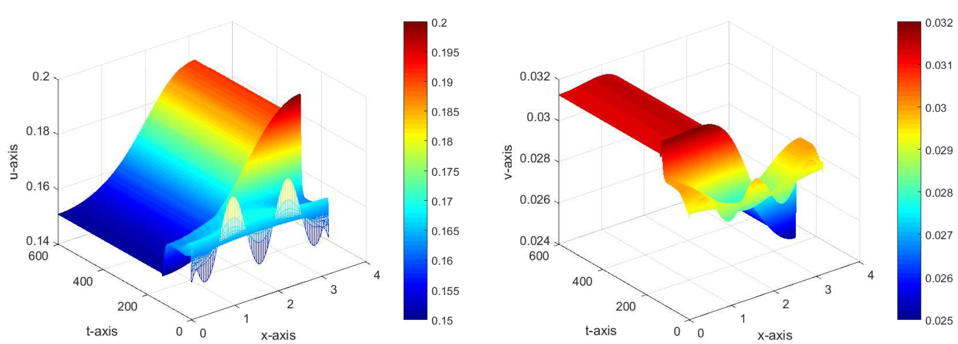


Figure 13. Steady state bifurcation solution at the double eigenvalue for $b = 5.7$. Here, $d = 1.425$. The solution tends to a spatially nonhomogeneous steady state. Left: u; right: v.

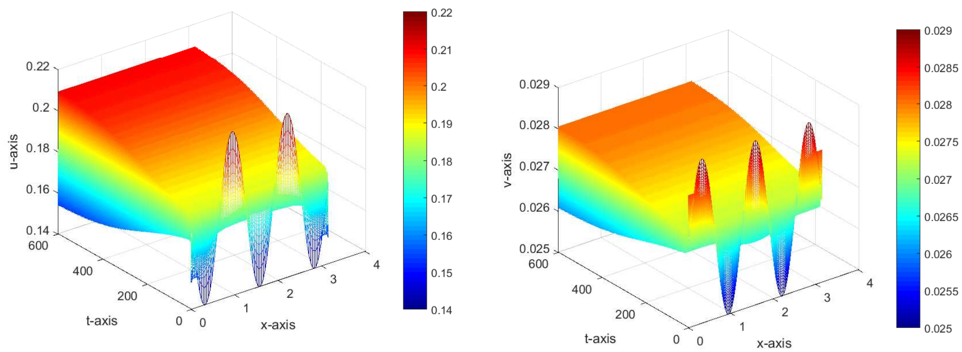


Figure 14. Positive periodic solution of (1.2) for $b = 6$. Here, $d = 1.1$ and the solution tends to a spatially homogeneous time periodic orbit. Left: u; right: v.

5. Conclusion

This paper investigates the bifurcation problem of a class of Gierer-Meinhardt model. Firstly, we use the Hopf bifurcation theorem to prove the existence of the Hopf bifurcation and obtain conditions for determining the direction and stability of the bifurcating periodic solutions; secondly, the existence of the Turing-Hopf bifurcation is also established, and we conclude that there exist complex spatiotemporal dynamics near the Turing-Hopf bifurcation point; finally, by the Crandall-Rabinowitz bifurcation theorem, the spatial decomposition and the implicit function theorem, we establish the local structure of steady state bifurcation and provide conditions for determining the direction of bifurcation.

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Received September 2023; Accepted March 2024; Available online October 2025.