Journal of Applied Analysis and Computation
Volume 16, Number 2, April 2026, 895–915

Website:http://www.jaac-online.com
DOI:10.11948/20250132

# MITTAG-LEFFLER EULER-MARUYAMA METHOD FOR LINEAR STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNELS*

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**Abstract** This paper reconsiders the linear stochastic Volterra integral equations with weakly singular kernels. For the equivalent form of the underlying equation, we propose the improved version of the existing mean-square asymptotical stability result for the exact solution. Moreover, some new or improved results are obtained for the Mittag-Leffler Euler–Maruyama method for the equations, and it is shown that the method shares sharp strong convergence with order $1/2-\beta$ and carries preferable stability. The numerical examples are performed to show the accuracy and effectiveness of the numerical scheme and verify the correctness of the theoretical analysis.

**Keywords** Stochastic Volterra integral equations, weakly singular kernels, Mittag-Leffler Euler-Maruyama method, mean-square asymptotic stability, strong convergence.

**MSC(2010)** 60H20, 60H35, 65C30.

## 1. Introduction

The stochastic Volterra integral equation (SVIE) can be regarded as a generalization of the stochastic differential equation (SDE), and its general form can be expressed as

$$x(t) = \phi(t) + \int_0^t f(t,s,x(s))ds + \int_0^t g(t,s,x(s))dW_s, \quad t \geq 0.$$

It can be observed that the solution of the SVIEs is related to all the historical information of the equation, so the memory effect can be described. Since the coefficients of the SVIEs depend on the variable $t$, the stochastic integral in the equation usually does not have martingale properties and the solution to the equation is a non-Marov process. Therefore, compared with SDEs, the explicit solutions to SVIEs are more difficult to obtain and the corresponding numerical studies are more challenging. Consider the linear SVIE

$$x(t) = x(0) + \int_0^t (t-s)^\alpha x(s)ds + \int_0^t (t-s)^\alpha x(s)dW_s,$$

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*The authors were supported by National Natural Science Foundation of China (Nos. 12471391, 12401547).
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when $\alpha = 1$, the Euler–Maruyama (EM) method for this equation can achieve strong convergence with order 1 [20]. This result is interesting. In fact, when $\alpha = 0$, the equation degenerates into an SDE, and the EM method in this case has only strong convergence with order 0.5 [15,22]. Therefore, it can be seen that the appearance of kernel functions may make the convergence order of numerical methods better in some extent.

According to singularity of the kernel functions of the SVIEs, they can be divided into two main categories: 1) SVIEs with non-singular kernels; 2) SVIEs with singular kernels. For the case of non-singular kernels, some important research progresses have been made in theoretical analysis and numerical computation. For example, [10] and [14] generalized SVIEs to SVIEs with delay and SVIEs with jump, respectively. In addition, theta type method [5], Split-step $\theta$ method [29](integrator [24]), split-step collocation methods [28] and operational matrix method [12] have also been applied to solve numerically SVIEs with non-singular kernels. For the other type, we consider SVIEs with weakly singular kernels

$$x(t) = x_0 + \int_0^t (t-s)^{-\alpha} f(x(s))ds + \int_0^t (t-s)^{-\beta} g(x(s))dW_s, \tag{1.1}$$

where $0 < \alpha < 1$, $0 < \beta < 1/2$, $W$ denotes an $m$-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. Let $x_0 \in \mathbb{R}^d$, the coefficients $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable. The equations (1.1) have been allplied in anamalous diffusion [18] and rough volatility [13], as well as the EM method and the exponential EM method for the equations were proposed in [30] and [8], respectively. Meanwhile, a stochastic collocation method was carried out and its effectiveness was demonstrated in [4]. Moreover, the fast EM method was constructed to improve computing efficiency in [6]. In addition, Li et al. [19] proposed the $\theta$-EM method and Milstein method and showed that their strong convergence rates are $\min\{1-\alpha, 1/2-\beta\}$ and $\min\{1-\alpha, 1-2\beta\}$, respectively. Alfonsi et al. [2] developed a new multi-factor Euler method, which significantly reduces the computational cost in an asymptotic manner compared to the traditional Euler method.

For now, the studies on the above numerical methods for singular SVIEs mainly focuses on the strong convergence. But the analytical and numerical stability properties of SVIEs with singular kernel were discussed very limitedly. For the case $\alpha = \beta \in (0, 1/2)$, the mean-square stability of the analytical solution has been studied in [26], and the numerical stability of the exponential EM method was discussed in [8]. We want to relax the requirement $0 < \alpha < 1/2$ of the kernel of the drift term to the case $0 < \alpha < 1$ and further investigate the stability of the exact solutions and numerical solutions.

Consider a bilinear scalar SVIE with weakly singular kernels in this paper

$$x(t) = x_0 + \int_0^t (t-s)^{-\alpha} \lambda x(s)ds + \int_0^t (t-s)^{-\beta} \mu x(s)dW_s, \tag{1.2}$$

where $\lambda$, $\mu$ are any given constant, and $0 < \alpha < 1$, $0 < \beta < \frac{1}{2}$. Firstly, our aim is to obtain the equivalent form with the Mittag-Leffler (ML) kernel function of (1.2) and prove analytical stability. For $\alpha, \beta \in (0, 1)$, the Mittag-Leffler functions $E_{\alpha,\beta}$, $E_\alpha : \mathbb{R} \to \mathbb{R}$ are defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where $\Gamma$ is the Gamma function, i.e., $\Gamma(x) := \int_0^\infty s^{x-1} e^{-s} ds$. The ML function appeared frequently in the research of fractional differential equations and different types of integral equations [1,16,23]. Secondly, the stability study of the numerical solution of the stochastic Volterra equation with two singular kernels is reflected only in the exponential EM method [8]. Based on it, we construct the Mittag-Leffler Euler–Maruyama (MLEM) method on the form of the equivalent equation. Regarding this, we obtain some improved or new results on analytical and numerical mean-square stability as well as strong convergence in comparison to the corresponding results in [26], [27] and [19].

The paper is organized as follows. In Section 2, we obtain the equivalent equation form with the ML kernel function of (1.2) and prove the improved result of the asymptotic stabilty of solutions in mean square sense. In Section 3 and Section 4, the MLEM method for the equation (1.2) is established, and its mean square stability and strong convergence are analyzed anew. In Section 5, we will implement the MLEM method through concrete examples and verify the obtained theoretical results.

## 2. Equivalent form of SVIEs with ML kernel function and its stability

Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a given complete probability space. The $\sigma$-algebraic flow $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the conditions of being monotonically increasing and right-continuous. Meanwhile, $\mathbb{E}$ denote the expectation corresponding to probability measure $\mathbb{P}$. This section obtains the equivalent equation form with the ML kernel function of (1.2) and proves the analytical stability. In order to get the equivalent form of the equation and estimates of the ML function, we first give the following lemma.

**Lemma 2.1.** (Lemma 8.1 in [27]) *According to a variation of constants formula* [3]*, we can get the following equivalent equation form of* (1.2) *with ML kernel function*

$$\begin{aligned} X(t) \ &= E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})X_0 \\ &+ \mu\Gamma(1-\beta)\int_0^t (t-s)^{-\beta}E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})X(s)dW_s. \end{aligned} \tag{2.1}$$

**Proof.** The proof will be shown in the Appendix. □

**Lemma 2.2.** (Lemma 8.2 in [27]) *Suppose that $\lambda \neq 0$, $-1 < \alpha < 1$ and $\beta$ is a real number. Then there exist positive constants $M(\alpha, \beta)$ and $\bar{M}(\alpha)$ depending on $\alpha$ and $\beta$ such that for any $t \geq 0$, it holds when $\beta \neq \alpha$,*

$$|E_{(1-\alpha),(1-\beta)}(\lambda t^{1-\alpha})| \leq \frac{M(\alpha,\beta)}{|\lambda|\max\{1, t^{1-\alpha}\}}.$$

*Moreover, when $\beta = \alpha$, we can obtain*

$$|E_{(1-\alpha),(1-\alpha)}(\lambda t^{1-\alpha})| \leq \frac{\bar{M}(\alpha)}{\lambda^2\max\{1, t^{2(1-\alpha)}\}}.$$

Next, we prove the stability of the equivalent equation form (2.1).

**Theorem 2.1.** *Assume that the conditions*

$$\lambda < 0 \ \text{and} \ \mu^2\Gamma^2(1-\beta)\int_0^\infty s^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2 ds < 1$$

*hold. Then for* $1+2\beta-2\alpha>0$ *and any* $\delta\in(0,1)$, *we have*

$$\sup_{t\ge 0} t^{\delta}\mathbb{E}|X(t)|^2<\infty. \tag{2.2}$$

**Proof.** Using Itô's isometry for (2.1), for all $t\in[0,\infty)$, we see that

$$\begin{aligned}\mathbb{E}|X(t)|^2 &= \left(E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})\right)^2\mathbb{E}|X_0|^2\\ &\quad + \mu^2\Gamma^2(1-\beta)\int_0^t (t-s)^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2\mathbb{E}|X(s)|^2ds.\end{aligned}$$

Put $y(t)=\mathbb{E}|X(t)|^2$ and $y_0=\mathbb{E}|X_0|^2$. On the space of bounded and continuous real-valued functions $C_b([0,\infty),\mathbb{R})$, we establish a function as below. For any $\xi\in C_b([0,\infty),\mathbb{R})$, we define

$$\|\xi\|_w := \sup_{t\in[0,\infty)}\alpha(t)|\xi(t)|,$$

where

$$\alpha(t):=\begin{cases}T^{\delta}, & t\in[0,T],\\ t^{\delta}, & t\ge T.\end{cases}$$

Here, $T$ is a positive coefficient and chosen later. It is obvious that the set $C_w([0,\infty),\mathbb{R}):=\{\xi\in C_b([0,\infty),\mathbb{R}):\|\xi\|_w<\infty\}$ is a Banach space with the norm $\|\cdot\|_w$. Now on $C_w([0,\infty),\mathbb{R})$ we introduce the operator $\mathcal{T}_{y_0}$ as follows. For any $\xi\in C_w([0,\infty),\mathbb{R})$, we define

$$\begin{aligned}\mathcal{T}_{y_0}\xi(t) &= \left(E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})\right)^2 y_0\\ &\quad + \mu^2\Gamma^2(1-\beta)\int_0^t (t-s)^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2\xi(s)ds.\end{aligned}$$

This operator is contractive. Indeed, for any $\xi$, $\hat{\xi}\in C_w([0,\infty),\mathbb{R})$, we have

$$\begin{aligned}&\mathcal{T}_{y_0}\xi(t)-\mathcal{T}_{y_0}\hat{\xi}(t)\\ =&\mu^2\Gamma^2(1-\beta)\int_0^t (t-s)^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2(\xi(s)-\hat{\xi}(s))ds\end{aligned}$$

for all $t\in[0,\infty)$. Consider the case where $t\in[0,T]$. In this case,

$$\begin{aligned}&\alpha(t)|\mathcal{T}_{y_0}\xi(t)-\mathcal{T}_{y_0}\hat{\xi}(t)|\\ \le& T^{\delta}\mu^2\Gamma^2(1-\beta)\int_0^t (t-s)^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2|\xi(s)-\hat{\xi}(s)|ds\\ \le& \mu^2\Gamma^2(1-\beta)\int_0^t \alpha(s)(t-s)^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2|\xi(s)-\hat{\xi}(s)|ds\\ \le& \mu^2\Gamma^2(1-\beta)\int_0^{\infty} u^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)u^{1-\alpha})\right)^2du\|\xi-\hat{\xi}\|_w.\end{aligned} \tag{2.3}$$

Next, for $t>T$, we have

$$\alpha(t)|\mathcal{T}_{y_0}\xi(t)-\mathcal{T}_{y_0}\hat{\xi}(t)|$$

$$\leq t^\delta \mu^2 \Gamma^2(1-\beta) \int_0^t (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2 |\xi(s) - \hat{\xi}(s)| ds$$
$$\leq t^\delta \mu^2 \Gamma^2(1-\beta) \int_0^t (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2 s^{-\delta} ds \|\xi - \hat{\xi}\|_w. \tag{2.4}$$

Note the following three facts. On the interval $[0, t/2]$, it holds that

$$\begin{aligned} & t^\delta \mu^2 \Gamma^2(1-\beta) \int_0^{t/2} (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2 s^{-\delta} ds \\ \leq\; & t^\delta \mu^2 \Gamma^2(1-\beta) \int_0^{t/2} \frac{C}{(t-s)^{2-2\alpha+2\beta}} s^{-\delta} ds \\ \leq\; & \frac{C t^\delta \mu^2 \Gamma^2(1-\beta)}{(t/2)^{2-2\alpha+2\beta}} \int_0^{t/2} s^{-\delta} ds \\ \leq\; & \frac{C 2^{1-2\alpha+2\beta+\delta} \mu^2 \Gamma^2(1-\beta)}{(1-\delta) t^{1-2\alpha+2\beta}} \\ \leq\; & \frac{C 2^{1-2\alpha+2\beta+\delta} \mu^2 \Gamma^2(1-\beta)}{(1-\delta) T^{1-2\alpha+2\beta}}. \end{aligned} \tag{2.5}$$

On the interval $[t/2, t-M]$,

$$\begin{aligned} & t^\delta \mu^2 \Gamma^2(1-\beta) \int_{t/2}^{t-M} (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2 s^{-\delta} ds \\ \leq\; & \frac{t^\delta}{(t/2)^\delta} \mu^2 \Gamma^2(1-\beta) \int_{t/2}^{t-M} \frac{C}{(t-s)^{2-2\alpha+2\beta}} ds \\ \leq\; & \frac{C 2^\delta \mu^2 \Gamma^2(1-\beta)}{(1-2\alpha+2\beta) M^{1+2\beta-2\alpha}}. \end{aligned} \tag{2.6}$$

And on $[t-M, t]$,

$$\begin{aligned} & t^\delta \mu^2 \Gamma^2(1-\beta) \int_{t-M}^{t} (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha}\right)^2 s^{-\delta} ds \\ \leq\; & \frac{t^\delta \mu^2 \Gamma^2(1-\beta)}{(t-M)^\delta} \int_{t-M}^{t} (t-s)^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2 ds \\ \leq\; & \frac{t^\delta \mu^2 \Gamma^2(1-\beta)}{(t-M)^\delta} \int_0^\infty u^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha) u^{1-\alpha})\right)^2 du. \end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7), $M > 0$, $T > 2M$ and $1 + 2\beta - 2\alpha > 0$, it holds that for any $t > T$,

$$\begin{aligned} & \frac{C\mu^2 \Gamma^2(1-\beta) 2^{2\beta+\delta+1-2\alpha}}{(1-\delta) T^{1+2\beta-2\alpha}} + \frac{C\mu^2 \Gamma^2(1-\beta) 2^\delta}{(1+2\beta-2\alpha) M^{1+2\beta-2\alpha}} \\ & + \frac{\mu^2 \Gamma^2(1-\beta) t^\delta}{(t-M)^\delta} \int_0^\infty u^{-2\beta} \left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha) u^{1-\alpha})\right)^2 du < 1. \end{aligned}$$

This combines with (2.3) and (2.4) to show that $\mathcal{T}_{y_0}$ is contractive on the space $C_w([0,\infty),\mathbb{R})$. On the other hand, it is easy to see that $\mathcal{T}_{y_0}$ is bounded in $C_w([0,\infty),\mathbb{R})$. Hence, by Banach fixed point theorem, there exists a unique fixed point $\xi^*$ in $C_w([0,\infty),\mathbb{R})$, which is also the fixed point of this operator in $C_b([0,\infty),\mathbb{R})$. This implies that the estimate (2.2) holds. The proof is completed. $\square$

**Remark 2.1.** The above theorem shows that the exact solution of SVIEs can achieve the mean-square asymptotical stability under certain conditions, where $1+2\beta-2\alpha>0$ is essential in the proof process. When $\alpha=\beta$, the result is similar to Proposition 11 in [26]. However, if the above conditions hold true, it will include $1+2\beta-2\alpha>0$. By applying Lemma 2.2 and simple calculations yields

$$\mu^2\Gamma^2(1-\beta))\int_0^\infty s^{-2\beta}(E_{(1-\alpha),(1-\beta)}(\lambda\Gamma(1-\alpha)s^{(1-\alpha)}))^2 ds \le \tilde{M}(\alpha,\beta)\frac{\mu^2}{\lambda^2},$$

where $\tilde{M}(\alpha,\beta)=\frac{2M^2(\alpha,\beta)(1-\alpha)\Gamma^2(1-\beta)}{\Gamma^2(1-\alpha)(1-2\beta)(1+2\beta-2\alpha)}$ is a positive constant depending on $\alpha$ and $\beta$ when $1+2\beta-2\alpha>0$. For given $\alpha$ and $\beta$, there will always exist $\lambda$ and $\mu$ such that $\tilde{M}(\alpha,\beta)\frac{\mu^2}{\lambda^2}<1$.

**Remark 2.2.** We expand the range of $\delta$ to $(0,1)$, therefore, the result in Theorem 2.1 improves the corresponding result in Theorem 4.1 in [27].

## 3. Stability of numerical solutions for SVIEs

In this section, we present the MLEM method for (2.1) and state its stability result.

For a fixed step-size $h>0$, the MLEM method for (2.1) is given by

$$\begin{aligned}\widehat{X}_h(t) =& E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})X_0 \\ &+\mu\Gamma(1-\beta)\int_0^t (t-\tau_h(s))^{-\beta}E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\widehat{X}_h(\tau_h(s))dW_s,\end{aligned} \tag{3.1}$$

where $\tau_h:(0,\infty)\to[0,\infty)$ is defined by

$$\tau_h(s)=kh \quad \text{for } s\in(kh,(k+1)h], \quad k=0,1,2,\dots.$$

Before going to the proof of stability analysis, we recall the following result about the property of the ML function.

**Lemma 3.1.** (Section 4.10.2 in [11]) $E_{\alpha,\beta}(-x)$, $x>0$ *is a completely monotone function if and only if* $0<\alpha\le 1$ *and* $\beta\ge\alpha$.

Next, we present the following lemma.

**Lemma 3.2.** *Suppose that*

$$\mu^2\Gamma^2(1-\beta)\int_0^\infty s^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2 ds<1.$$

*Let* $\delta\in(0,\beta-\alpha+\frac{1}{2})$ *be arbitrary. Then*

$$\limsup_{t\to\infty}\mu^2\Gamma^2(1-\beta)\int_0^t (t-s)^{-2\beta}\cdot\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2\frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}ds<1.$$

**Proof.** Since $\mu^2\Gamma^2(1-\beta)\int_0^\infty s^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2 ds<1$, there exists $\eta\in(0,1)$ such that

$$\frac{\mu^2\Gamma^2(1-\beta)}{\eta^{2\delta}}\int_0^\infty\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2}{s^{2\beta}}ds<1.$$

Choose and fix such $\eta$ satisfying the preceding inequality. Then

$$\begin{aligned}
&\lim_{t\to\infty}\sup \mu^2\Gamma^2(1-\beta)\int_{\eta t}^{t}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2}{(t-s)^{2\beta}}\cdot\frac{\max\left\{1,t^{2\delta}\right\}}{\max\left\{1,s^{2\delta}\right\}}ds\\
\leq&\lim_{t\to\infty}\sup\frac{\mu^2\Gamma^2\left(1-\beta\right)}{\eta^{2\delta}}\int_{\eta t}^{t}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2}{(t-s)^{2\beta}}ds\\
<&\frac{\mu^2\Gamma^2\left(1-\beta\right)}{\eta^{2\delta}}\int_0^{\infty}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)u^{1-\alpha})\right)^2}{u^{2\beta}}ds\\
<&1.
\end{aligned}\tag{3.2}$$

On the other hand, by virtue of Lemma 2.2 we have

$$\begin{aligned}
&\int_0^{\eta t}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2}{(t-s)^{2\beta}}\cdot\frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}ds\\
\leq&\frac{M^2(\alpha,\beta)}{|\lambda\Gamma(1-\alpha)|^2}\int_0^{\eta t}\frac{\max\{1,t^{2\delta}\}}{(t-s)^{2-2\alpha+2\beta}}ds.
\end{aligned}$$

A direct estimation yields that

$$\lim_{t\to\infty}\sup t^{2\delta}\int_0^{\eta t}\frac{1}{(t-s)^{2-2\alpha+2\beta}}ds\leq\lim_{t\to\infty}\sup t^{2\delta}\frac{\eta t}{(t-\eta t)^{2-2\alpha+2\beta}}=0,$$

which implies that

$$\lim_{t\to\infty}\sup\int_0^{\eta t}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right)^2}{(t-s)^{2\beta}}\cdot\frac{\max\{1,t^{2\delta}\}}{\max\{1,s^{2\delta}\}}ds=0.$$

This together with (3.2) completes the proof. □

Finally, we are now in a position to prove the stability of the MLEM method.

**Theorem 3.1.** *Assume that the conditions*

$$\lambda<0 \text{ and } \mu^2\Gamma^2(1-\beta)\int_0^{\infty}s^{-2\beta}\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2ds<1\tag{3.3}$$

*hold. Then for any step-size $h>0$ and $\alpha\geq\beta$, there exists $K>0$ such that the solution $\widehat{X}_h$ of* (3.1) *satisfies*

$$\mathbb{E}|\widehat{X}_h(t)|^2\leq K\mathbb{E}|X(0)|^2\quad\text{for all t}\geq 0,$$

*and furthermore, for any $\delta\in(0,\beta-\alpha+\frac{1}{2})$, we have* $\lim_{t\to\infty}t^{2\delta}\mathbb{E}|\widehat{X}_h(t)|^2=0$.

**Proof.** By (3.1) and using Itô's isometry, we arrive at

$$\begin{aligned}
\mathbb{E}|\widehat{X}_h(t)|^2=&\left(E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})\right)^2\mathbb{E}|X(0)|^2\\
&+\mu^2(\Gamma(1-\beta))^2\int_0^t\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\right)^2}{(t-\tau_h(s))^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2ds.
\end{aligned}$$

Note that $\tau_h(s) \le s$ and using Lemma 3.1, we obtain that

$$\mathbb{E}|\widehat{X}_h(t)|^2 \le \big(E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})\big)^2\,\mathbb{E}|X(0)|^2 + \mu^2(\Gamma(1-\beta))^2 \int_0^t \frac{\big(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\big)^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds.$$

By virtue of Lemma 2.2, there exists $M(\alpha,\beta) > 0$ such that for any $X(0) \neq 0$,

$$\frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} \le \frac{M(\alpha,\beta)}{|\lambda\Gamma(1-\alpha)|^2\max\{1,t^{2-2\alpha}\}} + \mu^2(\Gamma(1-\beta))^2 \int_0^t \frac{\big(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\big)^2}{(t-s)^{2\beta}}\frac{\mathbb{E}|\widehat{X}_h(\tau_h(s))|}{\mathbb{E}|X(0)|^2}ds. \tag{3.4}$$

Now, let

$$K := \frac{M(\alpha,\beta)}{1-\mu^2(\Gamma(1-\beta))^2\int_0^\infty \frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)s^{1-\alpha})\right)^2}{s^{2\beta}}ds}. \tag{3.5}$$

Thanks to (3.3), $K > 0$ and we are now proving $\sup_{t\ge 0}\frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} < K$ by contradiction. So assume that there exists $T > 0$ being the first time for which $\frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} \ge K$, i.e.,

$$\frac{\mathbb{E}|\widehat{X}_h(T)|^2}{\mathbb{E}|X(0)|^2} = K, \quad \frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} < K \quad \text{for } t\in[0,T).$$

Thus, replacing $t = T$ in (3.4) yields that

$$\begin{aligned} K &\le M(\alpha,\beta) + \mu^2(\Gamma(1-\beta))^2K\int_0^T \frac{\big(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(T-s)^{1-\alpha})\big)^2}{(T-s)^{2\beta}}ds \\ &< M(\alpha,\beta) + \mu^2(\Gamma(1-\beta))^2K\int_0^\infty \frac{\big(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)u^{1-\alpha})\big)^2}{u^{2\beta}}du, \end{aligned}$$

which contradicts to the definition of $K$ as in (3.5), and the desired result holds.

Let $\delta \in (0, \beta-\alpha+\frac{1}{2})$ be arbitrary and then to complete the proof, we need to show that $\lim_{t\to\infty} t^{2\delta}\mathbb{E}|\widehat{X}_h(t)|^2 = 0$. In fact, choose and fix an arbitrary constant $\hat{\delta} \in (\delta, \beta-\alpha+\frac{1}{2})$ so that it is sufficient to show that

$$\limsup_{t\to\infty} t^{2\hat{\delta}}\frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} < \infty.$$

Suppose that the contrary holds. Then, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ tending to $\infty$ such that $\gamma_n := \max\{1, t_n^{2\hat{\delta}}\}\frac{\mathbb{E}|\widehat{X}_h(t_n)|^2}{\mathbb{E}|X(0)|^2}$ satisfies

$$\gamma_n = \max\left\{\max\{1,t^{2\hat{\delta}}\}\frac{\mathbb{E}|\widehat{X}_h(t)|^2}{\mathbb{E}|X(0)|^2} : t\in[0,t_n]\right\} \quad \text{for } n\in\mathbb{N} \tag{3.6}$$

and $\lim_{n\to\infty}\gamma_n=\infty$. Replacing $t=t_n$ in (3.4) yields that

$$\begin{aligned}\frac{\mathbb{E}|\widehat{X}_h(t_n)|^2}{\mathbb{E}|X(0)|^2} \leq& \frac{M(\alpha,\beta)}{|\lambda\Gamma(1-\alpha)|^2\max\{1,t_n^{2-2\alpha}\}}\\ &+\mu^2(\Gamma(1-\beta))^2\int_0^{t_n}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t_n-s)^{1-\alpha})\right)^2}{(t_n-s)^{2\beta}}\frac{\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2}{\mathbb{E}|X(0)|^2}ds,\end{aligned}$$

which together with (3.6) implies that

$$\begin{aligned}\gamma_n \leq& \frac{M(\alpha,\beta)\max\{1,t_n^{2\hat{\delta}}\}}{|\lambda\Gamma(1-\alpha)|^2\max\{1,t_n^{2-2\alpha}\}}\\ &+\gamma_n\mu^2(\Gamma(1-\beta))^2\int_0^{t_n}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t_n-s)^{1-\alpha})\right)^2}{(t_n-s)^{2\beta}}\frac{\max\{1,t_n^{2\hat{\delta}}\}}{\max\{1,s^{2\hat{\delta}}\}}ds.\end{aligned}$$

Among them,

$$\begin{aligned}\frac{\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2}{\mathbb{E}|X(0)|^2} &\leq \frac{\max\{\mathbb{E}|\widehat{X}_h(s)|^2\}}{\mathbb{E}|X(0)|^2}\\ &=\max\{\frac{1}{\max\{1,s^{2\hat{\delta}}\}}\max\{1,s^{2\hat{\delta}}\}\frac{\mathbb{E}|\widehat{X}_h(s)|^2}{\mathbb{E}|X(0)|^2}\}\\ &\leq\frac{1}{\max\{1,s^{2\hat{\delta}}\}}\max\{\max\{1,s^{2\hat{\delta}}\}\frac{\mathbb{E}|\widehat{X}_h(s)|^2}{\mathbb{E}|X(0)|^2}\}\\ &=\frac{\gamma_n}{\max\{1,s^{2\hat{\delta}}\}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\gamma_n\left(1-\mu^2(\Gamma(1-\beta))^2\int_0^{t_n}\frac{\left(\mathrm{E}_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t_n-s)^{1-\alpha})\right)^2}{(t_n-s)^{2\beta}}\frac{\max\{1,t_n^{2\hat{\delta}}\}}{\max\{1,s^{2\hat{\delta}}\}}ds\right)\\ \leq&\frac{M(\alpha,\beta)\max\{1,t_n^{2\hat{\delta}}\}}{|\lambda\Gamma(1-\alpha)|^2\max\{1,t_n^{2-2\alpha}\}}.\end{aligned}$$

Since $2\hat{\delta}<2\beta-2\alpha+1<2(1-\alpha)$, it follows that

$$\lim_{n\to\infty}\sup\frac{M(\alpha,\beta)\max\{1,t_n^{2\hat{\delta}}\}}{|\lambda\Gamma(1-\alpha)|^2\max\{1,t_n^{2-2\alpha}\}}=0.$$

However, by virtue of Lemma 3.2 and $\lim_{n\to\infty}\gamma_n=\infty$, we have

$$\lim_{n\to\infty}\sup\gamma_n\left(1-\mu^2(\Gamma(1-\beta))^2\int_0^{t_n}\frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t_n-s)^{1-\alpha})\right)^2}{(t_n-s)^{2\beta}}\frac{\max\{1,t_n^{2\hat{\delta}}\}}{\max\{1,s^{2\hat{\delta}}\}}ds\right)=\infty,$$

which leads to a contradiction. The proof is completed. □

**Remark 3.1.** The proof of the above theorem uses certain conditions in Theorem 2.1, and the difference is that the condition $\alpha\geq\beta$ is added. As for $\alpha<\beta$, it remains to know whether the MLEM method is stable or not. In the above parameter region, we did some numerical simulations for $\alpha=0.2$ and $\beta=0.3$ in Figure 3 of Section 5, it is shown that the scheme is still stable.

**Remark 3.2.** Based on relaxing the requirement $0 < \alpha < 1/2$ of the kernel of the drift term to the case $0 < \alpha < 1$, we prove the stability of the numerical solutions. The result in Theorem 3.1 extends the corresponding result in [8].

## 4. Strong convergence of numerical solution for SVIEs

Firstly, we give the following lemma for the bound on $\sup_{0\le t\le T} \mathbb{E}|\widehat{X}_h(t)|$.

**Lemma 4.1.** *Let*

$$C_1 = M_1 E_{1-2\beta}\left(\mu^2(\Gamma(1-\beta))^2 M_2 \Gamma(1-2\beta) T^{1-2\beta}\right)$$

*with*

$$M_1 := \max_{0\le t\le T} (E_{1-\alpha}(\lambda\Gamma(1-\alpha)T^{1-\alpha}))^2,$$
$$M_2 := \max_{0\le t\le T} (E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)T^{\alpha}))^2.$$

*Then, for all $h > 0$ we have*

$$\sup_{0\le t\le T} \mathbb{E}|\widehat{X}_h(t)|^2 \le C_1 \mathbb{E}|X(0)|^2.$$

**Proof.** By (3.1), we arrive at

$$(\widehat{X}_h(t))^2 = \Bigg\{ (E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha}))X(0) + \mu\Gamma(1-\beta)\int_0^t \frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(t-\tau_h(s))^{2\beta}} \widehat{X}_h(\tau_h(s))dW_s \Bigg\}^2.$$

Taking the expectation of the both sides of the above equality and using Itô's isometry, we obtain that

$$\mathbb{E}|\widehat{X}_h(t)|^2 = (E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha}))^2 \mathbb{E}|X(0)|^2 + \mu^2(\Gamma(1-\beta))^2 \int_0^t \frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\right)^2}{(t-\tau_h(s))^{2\beta}} \mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds.$$

Noting that $\tau_h(s) \le s$, we derive

$$\begin{aligned}\mathbb{E}|\widehat{X}_h(t)|^2 \le& \left(E_{1-\alpha}(\lambda\Gamma(1-\beta)t^{1-\alpha})\right)^2 \mathbb{E}|X(0)|^2 \\ &+ \mu^2(\Gamma(1-\beta))^2 \int_0^t \frac{\left(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\right)^2}{(t-s)^{2\beta}} \mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds \\ \le& M_1 \mathbb{E}|X(0)|^2 + \mu^2(\Gamma(1-\beta))^2 M_2 \int_0^t \frac{\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2}{(t-s)^{2\beta}} ds.\end{aligned}$$

Let $m_t := \sup_{0\le s\le t} \mathbb{E}|\widehat{X}_h(s)|^2$. Then

$$m_t \le M_1 \mathbb{E}|X(0)|^2 + \mu^2(\Gamma(1-\beta))^2 M_2 \int_0^t \frac{m_s}{(t-s)^{2\beta}}\, ds.$$

Applying Gronwall's inequality (e.g., Lemma 6.19 in [7]), we arrive at

$$\sup_{0\le s\le t} \mathbb{E}|\widehat{X}_h(s)|^2 \le M_1 E_{1-2\beta}\left(\mu^2(\Gamma(1-\beta))^2 M_2\Gamma(1-2\beta)t^{1-2\beta}\right)\mathbb{E}|X(0)|^2,$$

which completes the proof. □

Secondly, we obtain the following preparatory lemma to realize scaling better.

**Lemma 4.2.** *Let $1-\alpha\le \frac{1}{2}-\beta$ and $t,s\in[0,T]$, $\tilde{\alpha}=1-\alpha$, $\tilde{\beta}=\frac{1}{2}-\beta$. Then*

$$|t^{\tilde{\alpha}}-s^{\tilde{\alpha}}|\le s^{\tilde{\alpha}-\tilde{\beta}}|t-s|^{\tilde{\beta}}$$

*for all $t\ge s$.*

**Proof.** Since $\delta\in(0,1)$, using the inequality $|t^{\delta}-s^{\delta}|\le|t-s|^{\delta}$, we derive that

$$|t^{\tilde{\alpha}}-s^{\tilde{\alpha}}|=|t^{\tilde{\alpha}-\tilde{\beta}}\cdot t^{\tilde{\beta}}-s^{\tilde{\alpha}-\tilde{\beta}}\cdot s^{\tilde{\beta}}|\le|s^{\tilde{\alpha}-\tilde{\beta}}t^{\tilde{\beta}}-s^{\tilde{\alpha}-\tilde{\beta}}\cdot s^{\tilde{\beta}}|=s^{\tilde{\alpha}-\tilde{\beta}}|t-s|^{\tilde{\beta}}.$$

The proof is complete. □

Thirdly, we establish an upper bound on the difference $\mathbb{E}|\widehat{X}_h(t)-\widehat{X}_h(\tilde{t})|^2$ in terms of $\max\{1,\tilde{t}^{1+2\beta-2\alpha}\}|t-\tilde{t}|^{1-2\beta}$.

**Lemma 4.3.** *Let*

$$M_3:=\left(\max_{x\in[0,T^{1-\alpha}]}\partial_x E_{1-\alpha}(\lambda\Gamma(1-\alpha)x)\right)^2, M_4:=\left(\max_{x\in[0,T^{1-\alpha}]}\partial_x E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)x)\right)^2,$$

*and*

$$\begin{aligned}C_2=&\Bigg\{4M_3\mathbb{E}|X(0)|^2\max\{1,T^{1+2\beta-2\alpha}\}\\&+\frac{8\mu^2(\Gamma(1-\beta))^2M_2C_1\max\{1,T^{2\alpha-1-2\beta}\}+4\mu^2(\Gamma(1-\beta))^2M_4C_1T^{1-2\beta}}{\min\{2-2\alpha,1-2\beta\}}\Bigg\},\end{aligned}$$

*where $C_1$ is given in Lemma 4.1. Then, for all $h>0$ and $t$, $\tilde{t}\in(0,T]$, we have*

$$\mathbb{E}|\widehat{X}_h(t)-\widehat{X}_h(\tilde{t})|^2\le C_2\max\{1,\tilde{t}^{1+2\beta-2\alpha}\}|t-\tilde{t}|^{1-2\beta}.$$

**Proof.** Choose and fix $t,\tilde{t}\in(0,T]$ with $t>\tilde{t}$. By (3.1), we have

$$\begin{aligned}&\widehat{X}_h(t)-\widehat{X}_h(\tilde{t})\\&=\left(E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha})-E_{1-\alpha}(\lambda\Gamma(1-\alpha)\tilde{t}^{1-\alpha})\right)X_0\\&\quad+\mu\Gamma(1-\beta)\Bigg\{\int_{\tilde{t}}^{t}\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(t-\tau_h(s))^{\beta}}\widehat{X}_h(\tau_h(s))dW_s\\&\quad+\int_0^{\tilde{t}}\left(\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(t-\tau_h(s))^{\beta}}-\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}}\right)\\&\quad\times\widehat{X}_h(\tau_h(s))dW_s\\&\quad+\int_0^{\tilde{t}}\left(\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}}-\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(\tilde{t}-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}}\right)\end{aligned}$$

$$\times \widehat{X}_h(\tau_h(s))dW_s\Bigg\}.$$

Using the inequality $|x+y+z+w|^2 \leq 4(|x|^2+|y|^2+|z|^2+|w|^2)$ for all $x, y, z, w \in \mathbb{R}^d$ and Itô's isometry, we derive that

$$\begin{aligned}
&\mathbb{E}|\widehat{X}_h(t) - \widehat{X}_h(\tilde{t})|^2 \\
\leq &4|E_{1-\alpha}(\lambda\Gamma(1-\alpha)t^{1-\alpha}) - E_{1-\alpha}(\lambda\Gamma(1-\alpha)\tilde{t}^{1-\alpha})|^2\mathbb{E}|X(0)|^2 \\
&+ 4\mu^2(\Gamma(1-\beta))^2\Bigg\{\int_{\tilde{t}}^t \frac{\big(E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\big)^2}{(t-\tau_h(s))^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds \\
&+ \int_0^{\tilde{t}} \Big(\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(t-\tau_h(s))^{\beta}} - \frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}}\Big)^2 \\
&\times \mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds \\
&+ \int_0^{\tilde{t}} \Big(\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}} - \frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(\tilde{t}-\tau_h(s))^{1-\alpha})}{(\tilde{t}-\tau_h(s))^{\beta}}\Big)^2 \\
&\times \mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds\Bigg\}.
\end{aligned}$$

By Mean Value Theorem, Lemma 4.1 and the fact that $\left(\frac{1}{(t-\tau_n(s))^\beta} - \frac{1}{(\tilde{t}-\tau_n(s))^\beta}\right)^2 \leq \frac{1}{(\tilde{t}-s)^{2\beta}} - \frac{1}{(t-s)^{2\beta}}$, we arrive at

$$\begin{aligned}
\mathbb{E}|\widehat{X}_h(t) - \widehat{X}_h(\tilde{t})|^2 \leq &4M_3|t^{1-\alpha} - \tilde{t}^{1-\alpha}|^2\mathbb{E}|X(0)|^2 + \int_{\tilde{t}}^t \frac{4\mu^2(\Gamma(1-\beta))^2 M_2 C_1}{(t-s)^{2\beta}} ds \\
&+ 4\mu^2(\Gamma(1-\beta))^2 M_2 C_1 \int_0^{\tilde{t}} \left(\frac{1}{(\tilde{t}-s)^{2\beta}} - \frac{1}{(t-s)^{2\beta}}\right) ds \\
&+ 4\mu^2(\Gamma(1-\beta))^2 C_1 \int_0^{\tilde{t}} \frac{M_4\left((t-\tau_h(s))^{1-\alpha} - (\tilde{t}-\tau_h(s))^{1-\alpha}\right)^2}{(\tilde{t}-s)^{2\beta}} ds.
\end{aligned}$$

Since $\delta \in (0,1)$, it follows that $|t^\delta - s^\delta| \leq |t-s|^\delta$ for $t, s \geq 0$. When $1-\alpha > \frac{1}{2} - \beta$, we can obtain that

$$\begin{aligned}
&\mathbb{E}|\widehat{X}_h(t) - \widehat{X}_h(\tilde{t})|^2 \\
&\leq 4M_3(t-\tilde{t})^{2-2\alpha}\mathbb{E}|X(0)|^2 + \frac{4\mu^2(\Gamma(1-\beta))^2 M_2 C_1}{1-2\beta}(t-\tilde{t})^{1-2\beta} \\
&\quad + \frac{4\mu^2(\Gamma(1-\beta))^2 M_2 C_1}{1-2\beta}(t-\tilde{t})^{1-2\beta} + \frac{4\mu^2(\Gamma(1-\beta))^2 M_4 C_1 (\tilde{t})^{1-2\beta}}{1-2\beta}(t-\tilde{t})^{1-2\beta} \\
&\leq \left(4M_3\mathbb{E}|X(0)|^2 T^{1+2\beta-2\alpha} + \frac{8\mu^2(\Gamma(1-\beta))^2 M_2 C_1 + 4\mu^2(\Gamma(1-\beta))^2 M_4 C_1 T^{1-2\beta}}{1-2\beta}\right)(t-\tilde{t})^{1-2\beta}.
\end{aligned}$$

When $1-\alpha \leq \frac{1}{2} - \beta$, by virtue of Lemma 4.2, we can also obtain that

$$\mathbb{E}|\widehat{X}_h(t) - \widehat{X}_h(\tilde{t})|^2$$

$$
\begin{aligned}
&\le 4M_3\tilde{t}^{1+2\beta-2\alpha}(t-\tilde{t})^{1-2\beta}\mathbb{E}|X(0)|^2+\frac{4\mu^2(\Gamma(1-\beta))^2M_2C_1}{1-2\beta}(t-\tilde{t})^{1-2\beta}\\
&\quad+\frac{4\mu^2(\Gamma(1-\beta))^2M_2C_1}{1-2\beta}(t-\tilde{t})^{1-2\beta}+\frac{4\mu^2(\Gamma(1-\beta))^2M_4C_1\tilde{t}^{2-2\alpha}}{2-2\alpha}(t-\tilde{t})^{1-2\beta}\\
&\le\left(4M_3\tilde{t}^{1+2\beta-2\alpha}\mathbb{E}|X(0)|^2+\frac{8\mu^2(\Gamma(1-\beta))^2M_2C_1+4\mu^2(\Gamma(1-\beta))^2M_4C_1\tilde{t}^{2-2\alpha}}{\min\{2-2\alpha,1-2\beta\}}\right)(t-\tilde{t})^{1-2\beta}\\
&\le\left(4M_3\mathbb{E}|X(0)|^2+\frac{8\mu^2(\Gamma(1-\beta))^2M_2C_1T^{2\alpha-2\beta-1}+4\mu^2(\Gamma(1-\beta))^2M_4C_1T^{1-2\beta}}{\min\{2-2\alpha,1-2\beta\}}\right)\\
&\quad\times\tilde{t}^{1+2\beta-2\alpha}(t-\tilde{t})^{1-2\beta}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}|\widehat{X}_h(t)-\widehat{X}_h(\tilde{t})|^2\\
&\le\Bigg\{\frac{8\mu^2(\Gamma(1-\beta))^2M_2C_1\max\{1,T^{2\alpha-1-2\beta}\}+4\mu^2(\Gamma(1-\beta))^2M_4C_1T^{1-2\beta}}{\min\{2-2\alpha,1-2\beta\}}\\
&\quad+4M_3\mathbb{E}|X(0)|^2\max\{1,T^{1+2\beta-2\alpha}\}\Bigg\}\times\max\{1,\tilde{t}^{1+2\beta-2\alpha}\}(t-\tilde{t})^{1-2\beta}.
\end{aligned}
$$

Then the proof is completed. □

We are now in a position to prove the strong convergence of the MLEM method.

**Theorem 4.1.** *For any $T>0$, there exists a constant $C$ depending on $T$, $\lambda$ and $\mu$ such that*

$$\sup_{0\le t\le T}\mathbb{E}|\widehat{X}_h(t)-X(t)|^2\le Ch^{1-2\beta}.$$

**Proof.** From (2.1) and (3.1), we have

$$
\begin{aligned}
&\widehat{X}_h(t)-X(t)\\
=&\Bigg\{\int_0^t\left(\frac{1}{(t-\tau_h(s))^\beta}-\frac{1}{(t-s)^\beta}\right)E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})\widehat{X}_h(\tau_h(s))dW_s\\
&+\int_0^t\Big(\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})}{(t-s)^\beta}-\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})}{(t-s)^\beta}\Big)\widehat{X}_h(\tau_h(s))dW_s\\
&+\int_0^t\frac{E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})}{(t-s)^\beta}(\widehat{X}_h(\tau_h(s))-X(s))dW_s\Bigg\}\mu\Gamma(1-\beta).
\end{aligned}
$$

Using the inequality $|x+y+z|^2\le 3(|x|^2+|y|^2+|z|^2)$ for all $x,y,z\in\mathbb{R}^d$ and Itô's isometry, we derive

$$
\begin{aligned}
&\mathbb{E}|\widehat{X}_h(t)-X(t)|^2\\
\le&\Bigg\{\int_0^t\Big(\frac{1}{(t-s)^\beta}-\frac{1}{(t-\tau_h(s))^\beta}\Big)^2|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})|^2\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2ds\\
&+\int_0^t\frac{\big|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})-E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\big|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2ds
\end{aligned}
$$

$$+\int_0^t \frac{|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))-X(s)|^2 ds\Bigg\}3\mu^2(\Gamma(1-\beta))^2.$$

Moreover, using the inequality $\left(\frac{1}{(t-\tau_n(s))^\beta}-\frac{1}{(t-s)^\beta}\right)^2 \le \frac{1}{(t-s)^{2\beta}}-\frac{1}{(t-\tau_n(s))^{2\beta}}$ and $t-\tau_n(s)\le t+h-s$, we obtain that

$$\begin{aligned}\int_0^t \Big(\frac{1}{(t-\tau_n(s))^\beta}-\frac{1}{(t-s)^\beta}\Big)^2 ds &\le \int_0^t \Big(\frac{1}{(t-s)^{2\beta}}-\frac{1}{(h+t-s)^{2\beta}}\Big)ds\\ &\le \frac{h^{1-2\beta}}{1-2\beta}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^t \Big(\frac{1}{(t-s)^\beta}-\frac{1}{(t-\tau_h(s))^\beta}\Big)^2|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})|^2\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds\\ \le&\frac{M_2C_1}{1-2\beta}h^{1-2\beta}.\end{aligned}\tag{4.1}$$

By the Mean Value Theorem and Lemma 4.1, when $1-\alpha>\frac{1}{2}-\beta$, we arrive at

$$\begin{aligned}&\int_0^t \frac{\big|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})-E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\big|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds\\ \le& M_4C_1\int_0^t \frac{|(t-\tau_h(s))^{1-\alpha}-(t-s)^{1-\alpha}|^2}{(t-s)^{2\beta}}ds\\ \le& M_4C_1\int_0^t \frac{|s-\tau_h(s)|^{2-2\alpha}}{(t-s)^{2\beta}}ds\\ \le& M_4C_1h^{2-2\alpha}\int_0^t \frac{1}{(t-s)^{2\beta}}ds\\ \le&\frac{M_4C_1T^{1-2\beta}h^{1+2\beta-2\alpha}}{1-2\beta}h^{1-2\beta}\\ \le&\frac{M_4C_1C_2T^{1-2\beta}}{1-2\beta}h^{1-2\beta},\end{aligned}$$

where $h^{1+2\beta-2\alpha}<C_2$ and $C_2$ is a constant. When $1-\alpha\le\frac{1}{2}-\beta$, by virtue of Lemma 4.2, we also arrive at

$$\begin{aligned}&\int_0^t \frac{\big|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha})-E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\big|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds\\ \le& M_4C_1\int_0^t \frac{|(t-\tau_h(s))^{1-\alpha}-(t-s)^{1-\alpha}|^2}{(t-s)^{2\beta}}ds\\ \le& M_4C_1\int_0^t \frac{|s-\tau_h(s)|^{1-2\beta}}{(t-s)^{2\alpha-1}}ds\\ \le&\frac{M_4C_1T^{2-2\alpha}}{2-2\alpha}h^{1-2\beta}.\end{aligned}$$

Therefore, we can get

$$\int_0^t \frac{\left|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-\tau_h(s))^{1-\alpha}) - E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})\right|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s))|^2 ds$$
$$\le \frac{M_4 C_1 T^{\max\{2-2\alpha,1-2\beta\}}}{\min\{2-2\alpha,1-2\beta\}} h^{1-2\beta}. \tag{4.2}$$

Moreover, in light of Lemma 4.3,

$$\begin{aligned}
&\mathbb{E}|\widehat{X}_h(\tau_h(s)) - X(s)|^2 \\
\le& 2\mathbb{E}|\widehat{X}_h(\tau_h(s)) - \widehat{X}_h(s)|^2 + 2\mathbb{E}|\widehat{X}_h(s) - X(s)|^2 \\
\le& 2C_2 \max\{1, s^{1+2\beta-2\alpha}\}|\tau_h(s) - s|^{1-2\beta} + 2\mathbb{E}|\widehat{X}_h(s) - X(s)|^2 \\
\le& 2C_2 \max\{1, s^{1+2\beta-2\alpha}\} h^{1-2\beta} + 2\mathbb{E}|\widehat{X}_h(s) - X(s)|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&3\mu^2(\Gamma(1-\beta))^2 \int_0^t \frac{|E_{1-\alpha,1-\beta}(\lambda\Gamma(1-\alpha)(t-s)^{1-\alpha})|^2}{(t-s)^{2\beta}}\mathbb{E}|\widehat{X}_h(\tau_h(s)) - X(s)|^2 ds \\
\le& 3\mu^2(\Gamma(1-\beta))^2 M_2 \left(\int_0^t \frac{2C_2\max\{1, s^{1+2\beta-2\alpha}\}h^{1-2\beta}}{(t-s)^{2\beta}}\, ds + \int_0^t \frac{2\mathbb{E}|\widehat{X}_h(s) - X(s)|^2}{(t-s)^{2\beta}}\, ds\right) \\
\le& 6\mu^2(\Gamma(1-\beta))^2 M_2 \Bigg\{C_2 \max\{T^{2-2\alpha}B(2+2\beta-2\alpha, 1-2\beta), \frac{1}{1-2\beta}T^{1-2\beta}\}h^{1-2\beta} \\
&+ \int_0^t \frac{\mathbb{E}|\widehat{X}_h(s) - X(s)|^2}{(t-s)^{2\beta}}\, ds\Bigg\},
\end{aligned}$$

where $\int_0^t s^{1+2\beta-2\alpha}(t-s)^{-2\beta}ds = t^{2-2\alpha}B(2+2\beta-2\alpha, 1-2\beta)$ and $B(\cdot,\cdot)$ is the Beta function, i.e. $B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}dx$. This together with (4.1) and (4.2) implies that

$$\begin{aligned}
&\mathbb{E}|\widehat{X}_h(t) - X(t)|^2 \\
\le& 6\mu^2(\Gamma(1-\beta))^2 M_2 \int_0^t \frac{\mathbb{E}|\widehat{X}_h(s) - X(s)|^2}{(t-s)^{2\beta}}\, ds + 6h^{1-2\beta}\mu^2(\Gamma(1-\beta))^2 \Bigg\{\frac{M_2C_1}{1-2\beta} \\
&+ \frac{M_4 C_1 T^{\max\{2-2\alpha,1-2\beta\}}}{\min\{2-2\alpha,1-2\beta\}} + M_2C_2\max\{T^{2-2\alpha}B(2+2\beta-2\alpha, 1-2\beta), \frac{1}{1-2\beta}T^{1-2\beta}\}\Bigg\}.
\end{aligned}$$

Applying the Gronwall's inequality (e.g., Lemma 6.19 in [7]), we arrive at

$$\sup_{0\le t\le T} \mathbb{E}|\widehat{X}_h(t) - X(t)|^2 \le Ch^{1-2\beta},$$

where

$$\begin{aligned}
C :=& E_{1-2\beta}(6\mu^2(\Gamma(1-\beta))^2 M_2\Gamma(1-2\beta)T^{1-2\beta})\Bigg\{\frac{M_2C_1}{1-2\beta} + \frac{M_4C_1T^{\max\{2-2\alpha,1-2\beta\}}}{\min\{2-2\alpha,1-2\beta\}} \\
&+ M_2C_2\max\{T^{2-2\alpha}B(2+2\beta-2\alpha,1-2\beta), \frac{1}{1-2\beta}T^{1-2\beta}\}\Bigg\} \times 6\mu^2(\Gamma(1-\beta))^2.
\end{aligned}$$

The proof is completed. □

**Remark 4.1.** The above theorem shows that the MLEM solution can achieve sharp strong convergence with order $1/2-\beta$, which is superior to the $\theta$-EM method for SVIEs with weakly singular kernels because the latter only up to $\min\{1-\alpha, 1/2-\beta\}$ in [19].

## 5. Numerical examples

In the section, we will verify the strong convergence order and stability of the MLEM method for the bilinear SVIE (1.2).

**Example 5.1.** Consider a simple bilinear scalar SVIE with the initial value $x_0 = 2$

$$x(t) = x_0 - \int_0^t (t-s)^{-\alpha} x(s)ds - \int_0^t (t-s)^{-\beta} x(s)dW_s, \qquad t \in [0, \frac{1}{2}].$$

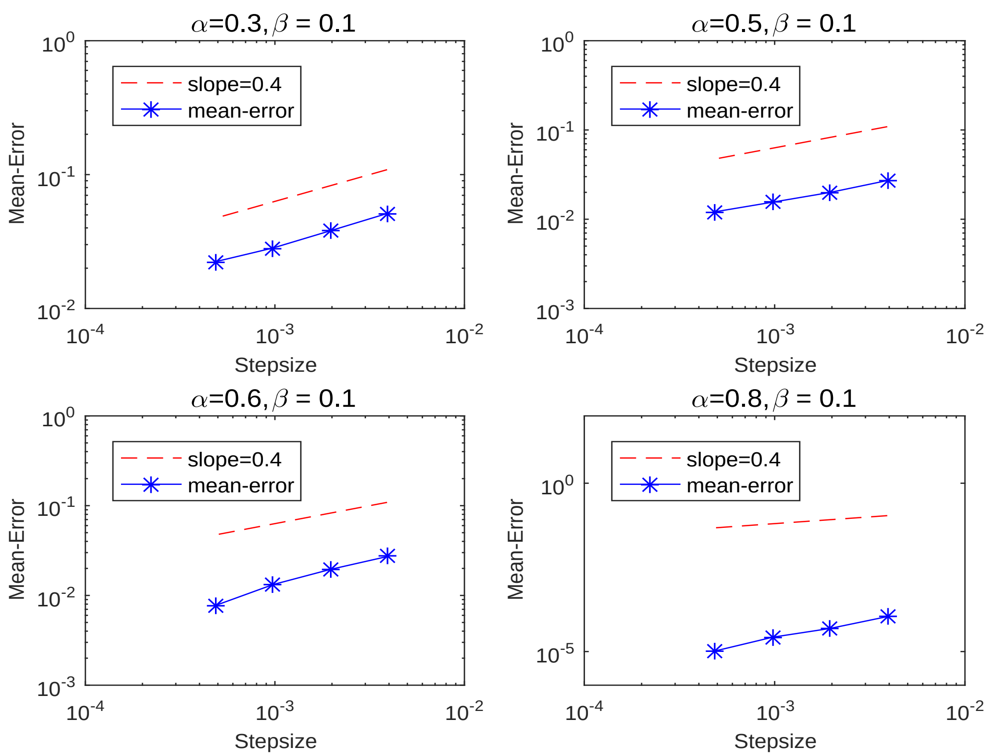


**Figure 1.** Mean-square error of the numerical solution of the MLEM method (3.1).

In a similar way, we use the sample average to approximate the expectation. More precisely, we measure the mean-square errors at the terminal time $t_N$ and the computing order by

$$\Lambda_{h,N} = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000} \mid X^{(i)}(t_N) - X_N^{(i)} \mid^2}, \quad Order = \frac{\log(\Lambda_{h,N}/\Lambda_{\frac{h}{2},N})}{\log(2)},$$

where $X^{(i)}(t_N)$ and $X_N^{(i)}$ indicate the exact solution and the numerical solution on the $i$th sample path, respectively.

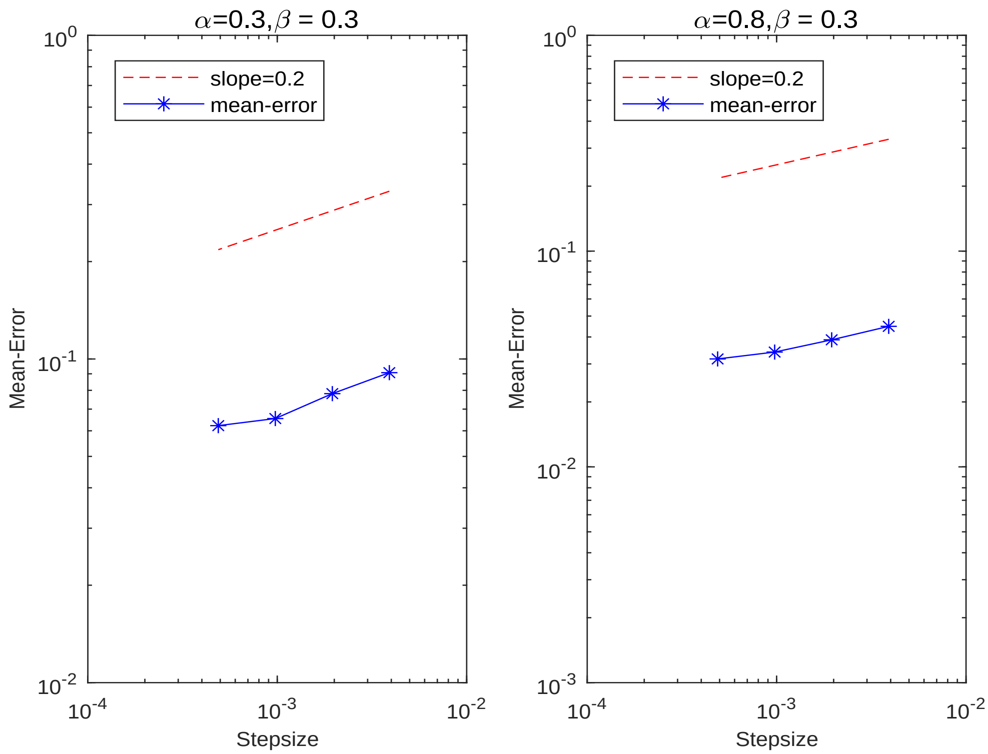


**Figure 2.** Mean-square error of the numerical solution of the MLEM method (3.1).

Considering the calculation efficiency of ML functions, the numerical solutions by the MLEM method with a small stepsize $h = 2^{-12}$ is used as a replacement of the unknown exact solution. The numerical solutions of the MLEM method will be obtained by using four different stepsizes $h = 2^{-8},\ 2^{-9},\ 2^{-10},\ 2^{-11}$ on the same Brownian path when $\alpha = 0.3,\ 0.5,\ 0.6,\ 0.8$. Figures 1 and 2 reveal that the strong convergence order of this method at time $T = \frac{1}{2}$ has nothing to do with the drift term parameter $\alpha$, and they are all $1/2 - \beta$ for various values of $\alpha$.

To test the mean-square asymptotic stability with $h = 2^{-1}$ for (3.1), we give the following example.

**Example 5.2.** Consider the bilinear scalar SVIE with the initial value $x_0 = 10$

$$x(t) = x_0 + \int_0^t (t-s)^{-\alpha} \lambda x(s) ds + \int_0^t (t-s)^{-\beta} \mu x(s) dW_s, \qquad t \in [0, 20],$$

where $\alpha = 0.3$, $\beta = 0.2$, satisfying $\alpha > \beta$.

Referring to Remark 4.1 in [27], we can find that the condition (3.3) in Lemma 3.2 holds, which means that there is indeed the corresponding $\lambda,\ \mu$ to make the numerical method stable. As for the following condition in Lemma 3.2

$$\mu^2 \Gamma^2(1-\beta) \int_0^\infty s^{-2\beta} \left( E_{1-\alpha,1-\beta}(\lambda \Gamma(1-\alpha) s^{1-\alpha}) \right)^2 ds < 1,$$

fixing the value of $\lambda$, we can observe that the left side of the inequality increases as $\mu$ increases, then there exists $\mu_0$ such that the inequality holds when $\mu < \mu_0$. Take $\mu = 5,\ 1,\ 1/5,\ 1/10$ and

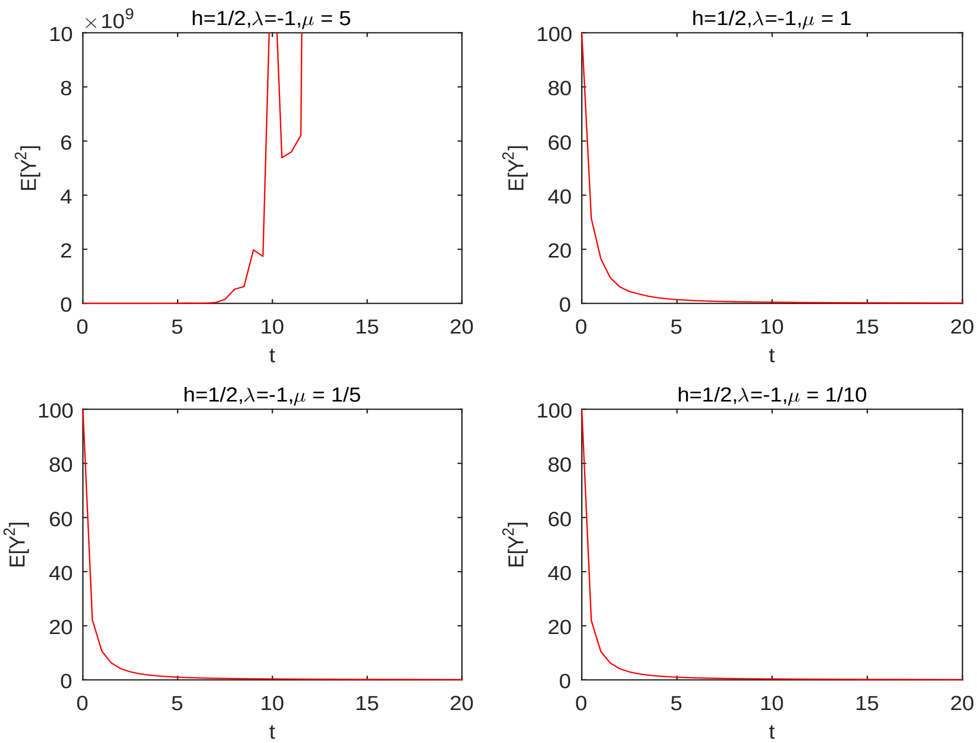


**Figure 3.** Second moment of the numerical solution of the MLEM method (3.1).

$\lambda = -1$ unchanged and generating 2000 numerical sample paths over [0,20], we plot the mean square of the numerical solution in Figure 3, which verifies our result.

On the other hand, we currently are not able to use our methods to deal with the case $\beta > \alpha$. In this case, we simulate the Example 5.2 with $\alpha = 0.2, \beta = 0.3$ by using the MLEM method (3.1), the numerical results in Figure 4 show that the method is still stable for some $\beta > \alpha$.

## Acknowledgements

We declare that all authors conceive of the study and carry out the proof. All authors of this paper read and approve the final manuscript. Deep thanks go to the referees for their many constructive comments and suggestions to improve this article.

## Appendix

To prove Theorem 2.1, we first present the result for the deterministic equation in the following form

$$x(t) = x_0 + \lambda \int_0^t (t-s)^{-\alpha} x(s) \mathrm{d}s + \mu \int_0^t (t-s)^{-\beta} f(s) \mathrm{d}s,$$

where $\lambda$ and $\mu$ are constants, $f(s)$ is measurable and bounded, and $\alpha \in (0,1), \beta \in (0,1)$.

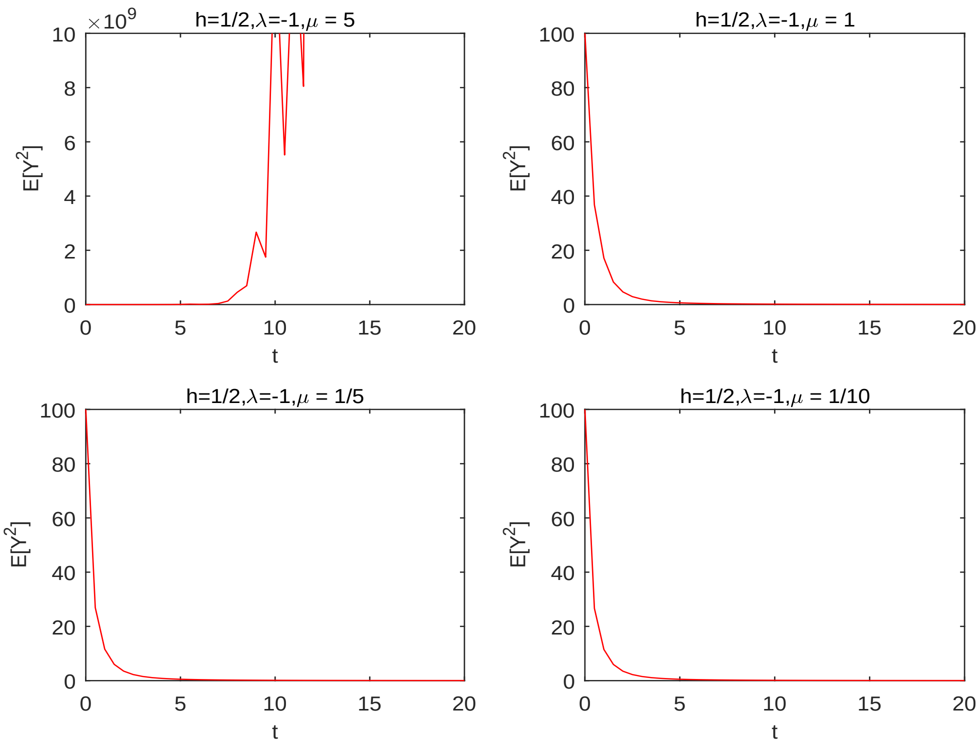


**Figure 4.** Second moment of the numerical solution of the MLEM method (3.1).

**Theorem 5.1.** *For any $T > 0$, there exists a constant $C$ depending on $T$, $\lambda$ and $\mu$ such that*

$$\begin{aligned}x(t) =&E_{(1-\alpha)}(\lambda\Gamma(1-\alpha)t^{(1-\alpha)})x_0\\ &+\mu\Gamma(1-\beta)\int_0^t (t-s)^{-\beta}E_{(1-\alpha),(1-\beta)}(\lambda\Gamma(1-\alpha)(t-s)^{(1-\alpha)})f(s)ds.\end{aligned}$$

**Proof.** Using the similar way in the [17], we can get that $x(t)$ is exponentially bounded, which means its Laplace transforms exist. Then, taking Laplace transform with respect to $t$ in both sides of (1), we obtain

$$Y(s) = \frac{x_0}{s} + \lambda L(t^{-\alpha})Y(s) + \mu L(t^{-\beta})F(s),$$

where $Y(s)$ and $F(s)$ denotes the Laplace transform of $x(s)$ and $f(s)$, respectively. The inverse Laplace transform using (8) yields

$$\begin{aligned}x(t) =&E_{(1-\alpha)}(\lambda\Gamma(1-\alpha)t^{(1-\alpha)})x_0\\ &+\mu\Gamma(1-\beta)\int_0^t (t-s)^{-\beta}E_{(1-\alpha),(1-\beta)}(\lambda\Gamma(1-\alpha)(t-s)^{(1-\alpha)})f(s)\mathrm{d}s.\end{aligned}$$

□

As an application of the preceding theorem, we obtain the Eq. (2.1) is an explicit representation of the solution of the bilinear scalar SVIE with weakly singular kernels (1.2), and the proof is similar with the Section 3.2 in the [3].

**Remark 5.1.** In order to prove Theorem 2.3 it is sufficient to show that $x(t) = X(t)$. Using the Itô representation theorem, it is sufficient to show that the following statement

$$\left\langle x(t), C + \int_0^T \Xi(\tau)\mathrm{d}W_\tau \right\rangle = \left\langle X(t), C + \int_0^T \Xi(\tau)\mathrm{d}W_\tau \right\rangle \tag{5.1}$$

holds for all $C \in \mathbb{R}^d$ and $\Xi \in \mathbb{H}^2([0,d], \mathbb{R}^d)$. Using the similar proof as in Lemma 3.4, Proposition 3.5 and Theorem 2.3 in [3], the proof of the Lemma 2.1 is completed.

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Received April 2025; Accepted October 2025; Available online October 2025.