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LONG-TIME BEHAVIOR OF AVIAN INFLUENZA MODEL WITH NONLOCAL DIFFUSION*

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Abstract The aim of this study is to investigate the long-time behavior of a model of avian influenza incorporating nonlocal diffusion. We establish the existence, uniqueness, positivity and boundedness of the solution by constructing a Lyapunov function and utilizing the eigenvalue problem of the nonlocal diffusion term. The basic reproduction number is determined through the generation matrix method. By constructing Lyapunov function and using the comparison principle, we demonstrate the global stability and uniform persistence of the system. Numerical simulations are performed to validate our theoretical findings, indicating that diffusion has a pronounced impact on the disease. Our findings reveal that slight changes in the diffusion coefficient lead to significant changes in both susceptible and infected groups. Therefore, to control the development and spread of the disease, it is essential to cull avian populations and limit human movement during outbreaks.

Keywords Avian influenza model, nonlocal diffusion, basic reproduction number, global stability, uniform persistence.

MSC(2010) 35E20, 37N35, 93E03.

1. Introduction

Avian influenza is an acute respiratory infectious disease caused by certain subtypes of avian influenza A virus, which can have a severe impact on public health. Since 2023, highly pathogenic avian influenza cases have been reported in several South American countries, including Chile, Argentina, Uruguay, Bolivia, and Japan. Bolivia has reported 22 confirmed cases of the disease as of 14 March, resulting in the culling of more than 270,000 poultry nationwide. Argentina has identified 59 confirmed cases and more than 300 suspected cases in 11 provinces as of 18 March, with over 700,000 poultry culled in six affected breeding grounds. On March 28, about 560,000 chickens were culled at a farm in Hokkaido, Japan, due to an outbreak of avian influenza. which had significant impacts on human health and the economy. As evident from the above data, avian influenza outbreaks cause substantial losses to the local economy. Thus, it is crucial to study avian influenza.

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Mathematical modeling is considered an effective tool for analyzing the transmission dynamics of avian influenza. For instance, Tang et al. [21] studied an avian-human influenza epidemic model with diffusion and non-local delay, which describes the transmission of avian influenza between birds and humans, including the infectivity of asymptomatic persons during the incubation period. Tadmon et al. [20] researched an avian-human influenza epidemic model with diffusion, non-local delay, and a spatially homogeneous environment, which describes the spread of avian influenza among birds, humans, and the environment. The best strategy to prevent the outbreak is to eliminate asymptomatic birds, and to reduce the contact rate between susceptible populations and poultry environments. Zheng et al. [27] proposed a time-periodic reaction-diffusion model for avian influenza with spatial heterogeneity, and considered the impact of temperature and bird-poultry diffusion on the spread of the avian influenza virus, suggesting that preventing virus-carrying birds from contacting poultry and enhancing environmental disinfection are effective control measures. A study [18] presented near-optimal control of avian influenza models with complex networks and spatial diffusion, and obtained necessary and sufficient conditions for approximate optimality. More avian influenza models can be found in the literature [1, 2, 5, 8, 12, 14, 17, 24].

We noted that some researchers assume local diffusion for avian and human populations as measured by the classical Laplacian operator [18, 20, 21, 27]. However, in reference [15], Murray pointed out that local diffusion flux proportional to gradients is not sufficient to accurately describe certain biological phenomena, while emphasizing the importance and intuitive necessity of long-term effects in biology. Therefore, the classical Laplacian operator cannot simulate diffusion accurately. To better describe the disease spread, in this paper, the diffusion process is described by integral operators($\int_\Omega J(x-y)\varphi(y)dy - \varphi(x)$) [13] to represent the movement between non-adjacent locations in space. Meanwhile, non-local diffusion equations are used in different fields, such as epidemiology [4, 10, 23], population ecology [7, 9], etc. Of which, Yang et al. [23] considered the impact of disease persistence and extinction through the relationship between transmission rate and recovery rate. Kao and Lou [9] compared the strengths and weaknesses of random diffusion and nonlocal diffusion. For other recent studies on the nonlocal diffusion equation, see [3, 6]. Due to the strong coupling of the avian influenza model and the difficulty of calculation, there is no work in this field at present. Through the research of this paper, it provides a reference for the research of related models. The main work of this paper is as follows: (1) we build a nonlocal diffusion avian influenza model. (2) We prove the existence, uniqueness, positivity and boundedness of the solution, and demonstrated the global stability and uniform persistence of the system. (3) The results of the theorem are verified by numerical simulations.

The article is arranged as follows. In Section 2, the existence, uniqueness, positivity and boundedness of the solution are proved by constructing Lyapunov function and applying the eigenvalue problem of nonlocal diffusion term. In Section 3, using Lyapunov function and comparison principle, we proved the global stability and uniform persistence of the system. Numerical simulations are shown in Section 4. The article ends in Section 5 with some conclusions.

2. Model and well-posedness of the solution

In reference [11], Liu et al. built the following model:

$$\frac{dS_a}{dt} = g(S_a) - \mu_a S_a - \beta_a S_a I_a,$$

$$\frac{dI_a}{dt} = \beta_a S_a I_a - (\mu_a + \delta_a) I_a,$$
$$\frac{dS_h}{dt} = \Lambda_h - \mu_h S_h - \beta_h S_h I_a,$$
$$\frac{dI_h}{dt} = \beta_h S_h I_a - (\mu_h + \delta_h + \gamma) I_h,$$
$$\frac{dR_h}{dt} = \gamma I_h - \mu_h R_h. \tag{2.1}$$

By virtue of the model (2.1), based on the previous analysis, and consider the long distance transport and migration of avian and the ease of human movement across regions, hence, in order to better reflect the dynamic behavior of avian influenza disease, this paper will construct a nonlocal diffusion model of avian influenza. In equation (2.1), the fifth equation and other equations without coupling, in this case, we don't consider the effect of R_h for the system. Where, $g(S_a)$ denotes net growth rate of the susceptible avian, let $g(S_a) = \Lambda_a$. Other parameter meanings are shown in Table 1.

$$\begin{cases} \dfrac{\partial S_a}{\partial t} = d_1 \displaystyle\int_\Omega \mathbb{J}(x-y) S_a(y,t)dy - d_1 S_a(x,t) + \Lambda_a(x) - \mu_a(x) S_a(x,t) \\ \qquad -\beta_a(x) S_a(x,t) I_a(x,t), \\ \dfrac{\partial I_a}{\partial t} = d_1 \displaystyle\int_\Omega \mathbb{J}(x-y) I_a(y,t)dy - d_1 I_a(x,t) + \beta_a(x) S_a(x,t) I_a(x,t) \\ \qquad - (\mu_a(x) + \delta_a(x)) I_a(x,t), \\ \dfrac{\partial S_h}{\partial t} = d_2 \displaystyle\int_\Omega \mathbb{J}(x-y) S_h(y,t)dy - d_2 S_h(x,t) + \Lambda_h(x) - \mu_h(x) S_h(x,t) \\ \qquad - \beta_h(x) S_h(x,t) I_a(x,t), \\ \dfrac{\partial I_h}{\partial t} = d_2 \displaystyle\int_\Omega \mathbb{J}(x-y) I_h(y,t)dy - d_2 I_h(x,t) + \beta_h(x) S_h(x,t) I_a(x,t) \\ \qquad - (\mu_h(x) + \delta_h(x) + \gamma(x)) I_h(x,t), \\ S_a(x,0) = S_{a,0}(x), I_a(x,0) = I_{a,0}(x), S_h(x,0) = S_{h,0}(x) I_h(x,0) = I_{h,0}(x), \\ \qquad x \in \Omega, \quad t > 0, \end{cases} \tag{2.2}$$

with boundary condition

$$\frac{\partial S_a}{\partial \nu} = \frac{\partial I_a}{\partial \nu} = \frac{\partial S_h}{\partial \nu} = \frac{\partial I_h}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \tag{2.3}$$

and initial condition

$$S_a(x,0) = S_{a,0}(x), I_a(x,0) = I_{a,0}(x), S_h(x,0) = S_{h,0}(x), I_h(x,0) = I_{h,0}(x), x \in \Omega. \tag{2.4}$$

$d_1 > 0$, $d_2 > 0$. $\Lambda_a(x)$, $\Lambda_h(x)$, $\mu_a(x)$, $\mu_h(x)$, $\beta_a(x)$, $b(x)$, $\beta_h(x)$, $\delta_a(x)$, $\delta_h(x)$ and $\gamma(x)$ are positive continuous bounded functions on $\overline{\Omega}$. The dispersal kernel function $\mathbb{J}$ is continuous and satisfies the following properties:

$$\mathbb{J}(0) > 0, \ \int_R \mathbb{J}(x)dx = 1, \ \mathbb{J}(x) > 0 \text{ on } \overline{\Omega}, \ \mathbb{J}(x) = \mathbb{J}(-x) \geq 0 \text{ on } R. \tag{2.5}$$

Let us consider the following function spaces and positive cones,

$$X := C(\overline{\Omega}), \ X_+ := C_+(\overline{\Omega}).$$

Table 1. Description of parameter.

Parameter	Description
β_a	the transmission rate from infective avian to susceptible avian
δ_a	the disease-related death rate of the infected avian
δ_h	the disease-related death rate of the infected human
μ_a	the natural death rate of the avian
μ_h	the natural death rate of the human
γ	the recovery rate of the infective human
β_h	the transmission rate from infective avian to susceptible human
Λ_h	the net growth rate of the susceptible human
d_1	the diffusion rate of avian
d_2	the diffusion rate of human

The norms in X be defined as follows:

$$\|\hbar\|_X := \sup_{x\in\overline{\Omega}} |\hbar(x)|, \ \hbar \in X.$$

Next, we define the linear operators on X,

$$\begin{aligned}
&\Xi_1\hbar_1(x) := d_1 \int_\Omega \mathbb{J}(x-y)\hbar_1(y)dy - d_1\hbar_1(x) - \mu_a(x)\hbar_1(x),\\
&\Xi_2\hbar_2(x) := d_1 \int_\Omega \mathbb{J}(x-y)\hbar_2(y)dy - d_1\hbar_2(x) - (\mu_a(x)+\delta_a(x))\hbar_2(x),\\
&\Xi_3\hbar_3(x) := d_2 \int_\Omega \mathbb{J}(x-y)\hbar_3(y)dy - d_2\hbar_3(x) - \mu_h(x)\hbar_3(x),\\
&\Xi_4\hbar_4(x) := d_2 \int_\Omega \mathbb{J}(x-y)\hbar_4(y)dy - d_2\hbar_4(x) - (\mu_h(x)+\delta_h(x)+\gamma(x))\hbar_4(x).
\end{aligned} \tag{2.6}$$

By virtue of [16, Theorem 1.2], we obtain that $\Xi_1(t)_{t\geq 0}$, $\Xi_2(t)_{t\geq 0}$, $\Xi_3(t)_{t\geq 0}$ and $\Xi_4(t)_{t\geq 0}$ are uniformly continuous semigroups on X. Furthermore, according to [9, Section 2.1.1], the semigroups $\Xi_1(t)_{t\geq 0}$, $\Xi_2(t)_{t\geq 0}$, $\Xi_3(t)_{t\geq 0}$ and $\Xi_4(t)_{t\geq 0}$ are positive.

Notation

$$\overline{g} = \sup_{t\to\infty} g(t), \underline{g} = \inf_{t\to\infty} g(t),$$

here, $g(t)$is a continuous bounded function. Next, we will prove the existence and uniqueness of the solution for system (2.2).

Lemma 2.1. *The solution* $(S_a(x,t), I_a(x,t), S_h(x,t), I_h(x,t))$ *of system* (2.2) *satisfy that*

$$\lim_{t\to\infty} \int_\Omega [S_a(x,t)+I_a(x,t)]dx \leq \frac{\bar{\Lambda}_a|\Omega|}{\underline{\mu}_a}, \quad \lim_{t\to\infty} \int_\Omega [S_h(x,t)+I_h(x,t)]dx \leq \frac{\bar{\Lambda}_h|\Omega|}{\underline{\mu}_h},$$

where $|\Omega|$ *denotes the volume of* Ω.

Proof. By (2.2), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega [S_a(x,t)+I_a(x,t)]dx\\
=&\,d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)S_a(y,t)dydx-\int_\Omega d_1S_a(x,t)dx+\int_\Omega \Lambda_a(x)dx-\int_\Omega \mu_a(x)S_a(x,t)dx\\
&-\int_\Omega \beta_a(x)S_a(x,t)I_a(x,t)dx+d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)I_a(y,t)dydx-\int_\Omega d_1I_a(x,t)dx\\
&+\int_\Omega \beta_a(x)S_a(x,t)I_a(x,t)dx-\int_\Omega(\mu_a(x)+\delta_a(x))I_a(x,t)dx\\
=&\,d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)S_a(y,t)dydx-\int_\Omega d_1S_a(x,t)dx+\int_\Omega \Lambda_a(x)dx-\int_\Omega \mu_a(x)S_a(x,t)dx\\
&+d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)I_a(y,t)dydx-\int_\Omega d_1I_a(x,t)dx-\int_\Omega(\mu_a(x)+\delta_a(x))I_a(x,t)dx,\\
&\frac{d}{dt}\int_\Omega [S_h(x,t)+I_h(x,t)]dx\\
=&\,d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)S_h(y,t)dydx-\int_\Omega d_2S_h(x,t)dx+\int_\Omega \Lambda_h(x)dx-\int_\Omega \mu_h(x)S_h(x,t)dx\\
&-\int_\Omega \beta_h(x)S_h(x,t)I_a(x,t)dx+d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)I_h(y,t)dydx-\int_\Omega d_2I_h(x,t)dx\\
&+\int_\Omega \beta_h(x)S_h(x,t)I_a(x,t)dx-\int_\Omega(\mu_h(x)+\delta_h(x)+\gamma(x))I_h(x,t)dx\\
=&\,d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)S_h(y,t)dydx-\int_\Omega d_2S_h(x,t)dx+\int_\Omega \Lambda_h(x)dx-\int_\Omega \mu_h(x)S_h(x,t)dx\\
&+d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)I_h(y,t)dydx-\int_\Omega d_2I_h(x,t)dx-\int_\Omega(\mu_h(x)+\delta_h(x)+\gamma(x))I_h(x,t)dx.
\end{aligned}
$$

Moreover, according to (2.5), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega [S_a(x,t)+I_a(x,t)]dx\\
\le&\, d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)dyS_a(y,t)dx+d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)dyI_a(y,t)dx\\
&-\int_\Omega d_1(S_a(x,t)+I_a(x,t))dx-\int_\Omega \underline{\mu}_a(S_a(x,t)+I_a(x,t))dx+\int_\Omega \Lambda_a(x)dx\\
\le&\, \bar{\Lambda}_a|\Omega|-\underline{\mu}_a\int_\Omega (S_a(x,t)+I_a(x,t))dx,\\
&\frac{d}{dt}\int_\Omega [S_h(x,t)+I_h(x,t)]dx\\
\le&\, d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)dyS_h(y,t)dx+d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)dyI_h(y,t)dx\\
&-\int_\Omega d_2(S_h(x,t)+I_h(x,t))dx-\underline{\mu}_h\int_\Omega (S_h(x,t)+I_h(x,t))dx+\int_\Omega \Lambda_h(x)dx\\
\le&\, \bar{\Lambda}_h|\Omega|-\underline{\mu}_h\int_\Omega (S_h(x,t)+I_h(x,t))dx.
\end{aligned}
$$

By calculating, we have

$$\lim_{t\to\infty}\int_\Omega [S_a(x,t)+I_a(x,t)]dx \le \frac{\bar{\Lambda}_a|\Omega|}{\underline{\mu}_a},\quad \lim_{t\to\infty}\int_\Omega [S_h(x,t)+I_h(x,t)]dx \le \frac{\bar{\Lambda}_h|\Omega|}{\underline{\mu}_h}.$$

Hence, the result holds. □

To prove the following theorem, we introduce the following eigenvalue problem [13],

$$\begin{cases}\displaystyle\int_{R^N}\mathbb{J}(x-y)(\varphi(y)-\varphi(x))dy = -\lambda_e\varphi(x), & \text{in } \Omega,\\ \varphi(x)=0, & \text{on } R^N\setminus\Omega.\end{cases} \tag{2.7}$$

Lemma 2.2. *For system* (2.7)*, there exists a unique principal eigenvalue* λ_1 *correspond to eigenfunction* $\varphi(x)$*. Furthermore,* $0<\lambda_1<1$ *and*

$$\lambda_1 = \inf_{\psi\in L^2(\Omega),\psi\neq 0}\frac{\int_\Omega \varphi^2(x)dx - \int_\Omega\int_\Omega \mathbb{J}(x-y)\varphi(y)\varphi(x)dydx}{\int_\Omega \varphi^2(x)}.$$

Theorem 2.1. *For any initial data* $(S_{a,0},I_{a,0},S_{h,0},I_{h,0})>0$*, there exists a unique positive solution* $(S_a(x,t),I_a(x,t),S_h(x,t),I_h(x,t))$ *of system* (2.2) *for* $t>0$ *on* Ω*.*

Proof. For the proof of positivity of the solution, refer to [10, Proposition 2.2], which we do not prove again here. Next, the uniqueness of the solution will be proved. We know that the exist a unique local solution for system (2.2) on $t\in[0,\tau_e)$(Due to the coefficients of system (2.2) satisfy the local Lipschitz condition), where τ_e is the explosion time. Let $k_0>1$ be sufficiently large for

$$\frac{1}{k_0}\le \min_{0<t<\tau_e}|\mathcal{N}(x,t)| \le \max_{0<t<\tau_e}|\mathcal{N}(x,t)|\le k_0,$$

where $\mathcal{N}(x,t)=S_a(x,t)+I_a(x,t)+S_h(x,t)+I_h(x,t)$. For each integer $k>k_0$, define the stopping time

$$\begin{aligned}\tau_k =& \inf\{t\in[0,\tau_e]:\min(S_a(x,t),I_a(x,t),S_h(x,t),I_h(x,t))\le\frac{1}{k}\\ &\text{or}\quad \max(S_a(x,t),I_a(x,t),S_h(x,t),I_h(x,t))\ge k\}.\end{aligned}$$

For $k\to\infty$, τ_k is increasing and $\tau_\infty=\lim_{k\to\infty}\tau_k$, then $\tau_\infty<\tau_e$ a.s.. Next, we need to show $\tau_\infty=\infty$ a.s..

$$\begin{aligned}&\frac{d}{dt}(\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2)\\ =&2\langle S_a(x,t), d_1\int_\Omega \mathbb{J}(x-y)S_a(y,t)dy - d_1S_a(x,t)+\Lambda_a(x)-\mu_a(x)S_a(x,t)\\ &-\beta_a(x)S_a(x,t)I_a(x,t)\rangle\\ &+2\langle I_a(x,t), d_1\int_\Omega \mathbb{J}(x-y)I_a(y,t)dy - d_1I_a(x,t)+\beta_a(x)S_a(x,t)I_a(x,t)\\ &-(\mu_a(x)+\delta_a(x))I_a(x,t)\rangle\\ &+2\langle S_h(x,t), d_2\int_\Omega \mathbb{J}(x-y)S_h(y,t)dy - d_2S_h(x,t)+\Lambda_h(x)-\mu_h(x)S_h(x,t)\\ &-\beta_h(x)S_h(x,t)I_a(x,t)\rangle\end{aligned}$$

$$
\begin{aligned}
&+ 2\langle I_h(x,t), d_2\int_\Omega \mathbb{J}(x-y)I_h(y,t)dy - d_2 I_h(x,t) + \beta_h(x)S_h(x,t)I_a(x,t)\\
&- (\mu_h(x)+\delta_h(x)+\gamma(x))I_h(x,t)\rangle\\
\leq & 2\{d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)S_a(y,t)S_a(x,t)dydx - d_1\int_\Omega S_a^2(x,t)dx + \int_\Omega \Lambda_a(x)S_a(x,t)dx\\
&+ d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)I_a(y,t)I_a(x,t)dydx - d_1\int_\Omega I_a^2(x,t)dx + \int_\Omega \beta_a(x)S_a(x,t)I_a^2(x,t)dx\\
&+ d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)S_h(y,t)S_h(x,t)dydx - d_2\int_\Omega S_h^2(x,t)dx + \int_\Omega \Lambda_h(x)S_h(x,t)dx\\
&+ d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)I_h(y,t)I_h(x,t)dydx - d_2\int_\Omega I_h^2(x,t)dx\\
&+ \int_\Omega \beta_h(x)S_h(x,t)I_a(x,t)I_h(x,t)dx\}\\
:= & f_1 + f_2.
\end{aligned}
\tag{2.8}
$$

Further, we calculate

$$
\begin{aligned}
f_1 = & 2\{d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)S_a(y,t)S_a(x,t)dydx - d_1\int_\Omega S_a^2(x,t)dx\\
&+ d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)I_a(y,t)I_a(x,t)dydx - d_1\int_\Omega I_a^2(x,t)dx\\
&+ d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)S_h(y,t)S_h(x,t)dydx - d_2\int_\Omega S_h^2(x,t)dx\\
&+ d_2\int_\Omega\int_\Omega \mathbb{J}(x-y)I_h(y,t)I_h(x,t)dydx - d_2\int_\Omega I_h^2(x,t)dx\},
\end{aligned}
$$

and

$$
\begin{aligned}
f_2 = & 2\{\int_\Omega \Lambda_a(x)S_a(x,t)dx + \int_\Omega \beta_a(x)S_a(x,t)I_a^2(x,t)dx\\
&+ \int_\Omega \Lambda_h(x)S_h(x,t)dx + \int_\Omega \beta_h(x)S_h(x,t)I_a(x,t)I_h(x,t)dx\}.
\end{aligned}
$$

According to Lemma 2.2, we have

$$
\begin{aligned}
f_1 &\leq -2d_1\lambda_1(\int_\Omega S_a^2(x,t) + \int_\Omega I_a^2(x,t)dx + \int_\Omega S_h^2(x,t)dx + \int_\Omega I_h^2(x,t)dx)\\
&\leq -2d_1\lambda_1(\|S_a(x,t)\|^2 + \|I_a(x,t)\|^2 + \|S_h(x,t)\|^2 + \|I_h(x,t)\|^2).
\end{aligned}
$$

For f_2, by virtue of Lemma 2.1 and fundamental inequality, we can obtain

$$
\begin{aligned}
f_2 \leq & \overline{\Lambda}_a^2 + \|S_a(x,t)\|^2 + \overline{\beta}_a^2 K^2\|S_a(x,t)\|^2 + \|I_a^2(x,t)\|^2\\
&+ \overline{\Lambda}_h^2 + \|S_h(x,t)\|^2 + \overline{\beta}_h^2 K^2\|S_h(x,t)\|^2 + \|I_h(x,t)\|^2.
\end{aligned}
$$

Hence, equation (2.8) be equal to

$$
\begin{aligned}
&d(\|S_a(x,t)\|^2 + \|I_a(x,t)\|^2 + \|S_h(x,t)\|^2 + \|I_h(x,t)\|^2)\\
\leq & \{M_1(\|S_a(x,t)\|^2 + \|I_a(x,t)\|^2 + \overline{\Lambda}_a^2 + \|S_h(x,t)\|^2 + \|I_h(x,t)\|^2) + \overline{\Lambda}_a^2 + \overline{\Lambda}_h^2\}dt,
\end{aligned}
\tag{2.9}
$$

where $M_1 = max\{1+\overline{\beta}_a^2K^2-2d_1\lambda_1, 1+\overline{\beta}_h^2K^2-2d_1\lambda_1, 1-2d_1\lambda_1\}$.

Moreover, we can integrate both sides of (2.8) from 0 to $\tau_k \wedge T$, $T>0$

$$\begin{aligned}&(\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2)\\&-(\|S_{a,0}\|^2+\|I_{a,0}\|^2+\|S_{h,0})\|^2+\|I_{h,0}\|^2)\\&\le\int_0^{\tau_k\wedge T}\{M_1(\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\overline{\Lambda}_a^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2)+\overline{\Lambda}_h^2\}dt.\end{aligned} \tag{2.10}$$

Using the Gronwall inequality

$$\begin{aligned}&\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2\\&\le(\|S_{a,0}\|^2+\|I_{a,0}\|^2+\|S_{h,0})\|^2+\|I_{h,0}\|^2+(\overline{\Lambda}_a^2+\overline{\Lambda}_h^2)\cdot(\tau_k\wedge T))e^{M_1\cdot(\tau_k\wedge T)}.\end{aligned} \tag{2.11}$$

As $k\to\infty$, equation (2.11) be equal to

$$\begin{aligned}&\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2\\&\le(\|S_{a,0}\|^2+\|I_{a,0}\|^2+\|S_{h,0})\|^2+\|I_{h,0}\|^2+(\overline{\Lambda}_a^2+\overline{\Lambda}_h^2)T)e^{M_1T}.\end{aligned} \tag{2.12}$$

Define

$$\zeta_k=\inf_{\|\mathcal{N}(x,t)\|>k,0<t<\infty}(\|S_a(x,t)\|^2+\|I_a(x,t)\|^2+\|S_h(x,t)\|^2+\|I_h(x,t)\|^2),\ \text{for any } k>k_0. \tag{2.13}$$

Combine (2.12) and (2.13) to get

$$\zeta_kP(\tau_k\le T)\le(\|S_{a,0}\|^2+\|I_{a,0}\|^2+\|S_{h,0})\|^2+\|I_{h,0}\|^2+(\overline{\Lambda}_a^2+\overline{\Lambda}_h^2)T)e^{M_1T}.$$

When $k\to\infty$, we can get $P(\tau_\infty\le T)=0$, therefore, $\tau_\infty=\infty$. The theorem is proved. □

Remark 2.1. As $t\to\infty$, from Lemma 2.1 and Theorem 2.1, we can obtain that there exists an invariant set,

$$\begin{aligned}\mathbb{D}_1&:=\{(S_{a,0},I_{a,0})\in X_+,\int_{\mathbb{O}}(S_a(x,t)+I_a(x,t))dx<\frac{\bar{\Lambda}_a|\Omega|}{\underline{\mu}_a}\},\\\mathbb{D}_2&:=(S_{h,0},I_{h,0})\in X_+,\int_{\mathbb{O}}(S_h(x,t)+I_h(x,t))dx<\frac{\bar{\Lambda}_h|\Omega|}{\underline{\mu}_h}\}.\end{aligned}$$

3. Global stability and uniform persistence of the system

3.1. Basic reproduction number

To obtain that the basic reproduction number, we consider the following linearize equations around the disease-free equilibrium $E^0=(S_a^0(x),0,S_h^0(x),0)$,

$$\frac{\partial I_a}{\partial t}=d_1\int_\Omega\mathbb{J}(x-y)I_a(y,t)dy-d_1I_a(x,t)+\beta_a(x)S_a^0(x)I_a(x,t)-(\mu_a(x)+\delta_a(x))I_a(x,t). \tag{3.1}$$

Next, we define the linear operators on X,

$$Bv(x):=\beta_a(x)S_a^0(x)v(x),$$

$$Gv(x) := d_1 \int_\Omega \mathbb{J}(x-y)v(x)dy,$$
$$Hv(x) := -d_1 v(x) - (\mu_a(x) + \delta_a(x))v(x).$$

Then, we can obtain that the following abstract form for (3.1)

$$\frac{dI_a}{dt} = BI_a(t) + (G-H)I_a(t).$$

By virtue of [22, Theorem 3.12], $T(t)$ denotes that the uniformly continuous semigroups of $G-H$, we have

$$(G-H)^{-1}v(x) = \int_0^\infty T(t)v(x).$$

The next generation operator $\mathbb{K} = B(G-H)^{-1} = \beta_a(x)S_a^0(x)\int_0^\infty T(t)v(x)$ is defined as

$$R_0 := \beta_a(x)S_a^0(x)/(d_1 \int_\Omega \mathbb{J}(x-y)dy + d_1 + \mu_a(x) + \delta_a(x)).$$

Meanwhile, by virtue of [7], for system (3.1), there exists a principal eigenvalue λ_0 with respect to the following equaiton

$$\lambda\varsigma(x) = d_1 \int_\Omega \mathbb{J}(x-y)\varsigma(y)dy - d_1\varsigma(x) + \beta_a(x)S_a^0(x)\varsigma(x) - (\mu_a(x) + \delta_a(x))\varsigma(x). \tag{3.2}$$

Hence, we can obtain that the following lemma.

Lemma 3.1. *$sign(R_0 - 1) = sign\lambda_0$.*

3.2. Global stability and uniform persistence

Now, we have the following global stability result.

Theorem 3.1. *If $R_0 < 1$, for the disease-free equilibrium $(S_a^0(x), 0, S_h^0(x), 0)$ of system (2.2), we have*

$$\lim_{t\to\infty} S_a(x,t) = S_a^0(x), \lim_{t\to\infty} I_a(x,t) = 0, \lim_{t\to\infty} S_h(x,t) = S_h^0(x), \lim_{t\to\infty} I_h(x,t) = 0.$$

Proof. Let $u_1(x,t) = S_a(x,t) - S_a^0(x)$, we have

$$\frac{\partial u_1(x,t)}{\partial t} = d_1 \int_\Omega \mathbb{J}(x-y)u_1(y,t)dy - d_1u_1(x,t) - \mu_a(x)u_1(x,t) - \beta_a(x)S_a(x,t)I_a(x,t). \tag{3.3}$$

Let $U_1(t) = \int_\Omega u_1^2(x,t)dx$, we can obtain

$$\begin{aligned}
&\frac{dU_1(t)}{dt}\\
=&2\int_\Omega u_1(x,t)\frac{\partial u_1(x,t)}{\partial t}dx\\
=&2\int_\Omega u_1(x,t)\{d_1\int_\Omega \mathbb{J}(x-y)u_1(y,t)dy - d_1u_1(x,t) - \mu_a(x)u_1(x,t) - \beta_a(x)S_a(x,t)I_a(x,t)\}dx\\
=&2\{d_1\int_\Omega\int_\Omega \mathbb{J}(x-y)u_1(y,t)u_1(x,t)dydx - \int_\Omega u_1^2(x,t)dx\} - 2\int_\Omega \mu_a(x)u_1^2(x,t)dx\\
&-2\int_\Omega\{\beta_a(x)S_H(x,t)I_V(x,t)\}u_1(x,t)dx.
\end{aligned} \tag{3.4}$$

On account of $\beta_H(x)$, $b(x)$, $S_H(x,t)$ and $I_V(x,t)$ are bounded, we know that there is a sufficiently small positive number k such that

$$\frac{\beta_H(x)b(x)}{N_H+m}S_H(x,t)I_V(x,t) \geq k. \tag{3.5}$$

By virtue of Lemma 2.2 and equation (3.5), we have

$$\frac{dU_1(t)}{dt} \leq -2(d_1\lambda_1 + \underline{\mu}_a)U_1(t) + 2kU_1^{\frac{1}{2}}(t).$$

By calculating, we have

$$U_1^{\frac{1}{2}}(t) \leq ce^{(-d_1\lambda_1 - \underline{\mu}_a)t} - \frac{k}{d_1\lambda_1 + \underline{\mu}_a}.$$

Moreover

$$\|u_1(t)\|^{\frac{1}{2}} \leq ce^{(-d_1\lambda_1 - \underline{\mu}_a)t} - \frac{k}{d_1\lambda_1 + \underline{\mu}_a}.$$

Owing to k is a sufficiently small positive number, hence, as $t \to \infty$, $u_1(x,t) \to 0$ uniformly on $x \in \Omega$. Furthermore, we obtain that $S_a(x,t) \to S_a^0(x)$.

Next, let $U_2(t) = \int_\Omega \varsigma_0(x)I_a(x,t)dx$, where $\varsigma_0(x)$ denotes the strictly positive eigenfunction with respect to λ_0 for system (3.2), then

$$\begin{aligned}\frac{dU_2(t)}{dt} &= \int_\Omega \varsigma_0(x)\frac{\partial}{\partial t}I_a(x,t)dx \\ &= \int_\Omega \varsigma_0(x)\{d_1\int_\Omega \mathbb{J}(x-y)I_a(y,t)dy - d_1I_a(x,t) \\ &\quad + \beta_a(x)S_a(x,t)I_a(x,t) - (\mu_a(x)+\delta_a(x))I_a(x,t)\}dx.\end{aligned} \tag{3.6}$$

By virtue of (2.5) and (3.2), we have

$$\begin{aligned}\int_\Omega \varsigma_0(x)d_1\int_\Omega \mathbb{J}(x-y)I_a(y,t)dydx &= \int_\Omega I_a(y,t)\int_\Omega \mathbb{J}(y-x)\varsigma_0(x)dxdy \\ &= \int_\Omega I_a(x,t)\int_\Omega \mathbb{J}(x-y)\varsigma_0(y)dydx \\ &= \int_\Omega I_a(x,t)\{\lambda_0\varsigma_0(x) + d_1\varsigma_0(x) - \beta_a(x)S_a^0(x)\varsigma_0(x) \\ &\quad - (\mu_a(x)+\delta_a(x))\varsigma_0(x)\}dx.\end{aligned} \tag{3.7}$$

Furthermore, substituting (3.7) into (3.6), we have

$$\frac{dU_2(t)}{dt} = \int_\Omega \varsigma_0(x)\{\lambda_0I_a(x,t) - \beta_a(x)\{S_a^0(x) - S_a(x,t)\}I_a(x,t)\}dx \leq \int_\Omega \varsigma_0(x)\lambda_0I_a(x,t)dx.$$

As $R_0 < 1$, we know that $\lambda_0 < 0$, hence, $U_2(t) = e^{\lambda_0 t} \to 0$ for $t \to \infty$, then $I_a(x,t) \to 0$. Using a similar approach, we can obtain that $S_h(x,t) \to S_h^0(x)$ and $I_h(x,t) \to 0$. □

Next, we consider the uniform persistence of system (2.2).

Theorem 3.2. *As $R_0 > 1$, then there exists a function $\Gamma(x)$ satisfy that*

$$\lim_{t\to\infty}\inf(S_a(x,t) + I_a(x,t) + S_h(x,t) + I_h(x,t)) \geq \Gamma(x),$$

hence, the disease uniform persistence.

Proof. As $R_0 > 1$, there exists a $\kappa > 0$ such that $\lambda_0(S_a^0 - \kappa) > 0$. Hence, there exists a $t_1 > 0$ satisfy that $S_a(x,t) > S_{a,0} - \kappa$ for $t \geq t_1$ and $x \in \overline{\Omega}$. For the second equation of system (2.2), we have

$$\frac{\partial I_a}{\partial t} \geq d_1 \int_\Omega \mathbb{J}(x-y) I_a(y,t)dy - d_1 I_a(x,t) + \beta_a(x)(S_{a,0} - k)I_a(x,t) - (\mu_a(x) + \delta_a(x))I_a(x,t).$$

Define $\widetilde{I}_a(x,t) = Me^{\widetilde{\lambda}t}\widetilde{\psi}(x)$, $\widetilde{I}_a(x,t)$ satisfy that the following equation

$$\frac{\partial \tilde{I}_a}{\partial t} = d_1 \int_\Omega \mathbb{J}(x-y) \tilde{I}_a(y,t)dy - d_1 \tilde{I}_a(x,t) + \beta_a(x)(S_{a,0} - k)\tilde{I}_a(x,t) - (\mu_a(x) + \delta_a(x))\tilde{I}_a(x,t),$$

where $\widetilde{\psi}(x)$ is the eigenfunction with respect to $\widetilde{\lambda} < 0$. By virtue of the comparison principle, we know $I_a(x,t) \geq \widetilde{I}_a(x,t)$. Furthermore, $I_a(x,t) \geq Me^{\widetilde{\lambda}t}\widetilde{\psi}(x)$ and

$$\lim_{t\to\infty} I_a(x,t) \geq M\widetilde{\psi}(x).$$

For the first and third equations of system (2.2), by virtue of equation (2.7), we can obtain

$$\begin{aligned}
\frac{\partial S_a}{\partial t} &= -d_1\lambda_e S_a(x,t) + \Lambda_a(x) - \mu_a(x)S_a(x,t) - \beta_a(x)S_a(x,t)I_a(x,t),\\
\frac{\partial S_h}{\partial t} &= -d_2\lambda_e S_h(x,t) + \Lambda_h(x) - \mu_h(x)S_h(x,t) - \beta_h(x)S_h(x,t)I_a(x,t).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\frac{\partial S_a}{\partial t} &\geq -d_1\lambda_e S_a(x,t) + \Lambda_a(x) - \mu_a(x)S_a(x,t),\\
\frac{\partial S_h}{\partial t} &\geq -d_2\lambda_e S_h(x,t) + \Lambda_h(x) - \mu_h(x)S_h(x,t).
\end{aligned}$$

By calculating, we have

$$\lim_{t\to\infty} S_a(x,t) \geq \frac{\Lambda_a(x)}{d_1\lambda_e + \mu_a(x)}, \quad \lim_{t\to\infty} S_h(x,t) \geq \frac{\Lambda_h(x)}{d_2\lambda_e + \mu_h(x)}.$$

For the last equation of the system (2.2), we have

$$\frac{\partial I_h}{\partial t} \geq -d_2\lambda_e I_h(x,t) + \beta_h(x)\frac{\Lambda_h(x)}{d_2\lambda_e + \mu_h(x)}M\widetilde{\psi}(x) - (\mu_h(x) + \delta_h(x) + \gamma(x))I_h(x,t),$$

moreover

$$\lim_{t\to\infty} I_h(x,t) \geq \frac{\beta_h(x)\Lambda_h(x)M\widetilde{\psi}(x)}{d_2\lambda_e + \mu_h(x) + \delta_h(x) + \gamma(x)}.$$

Hence

$$\Gamma(x) := \min\{\frac{\Lambda_a(x)}{d_1\lambda_e + \mu_a(x)}, M\widetilde{\psi}(x), \frac{\Lambda_h(x)}{d_2\lambda_e + \mu_h(x)}, \frac{\beta_h(x)\Lambda_h(x)M\widetilde{\psi}(x)}{d_2\lambda_e + \mu_h(x) + \delta_h(x) + \gamma(x)}\}.$$

The theorem is proved. □

4. Numerical simulations

This section presents numerical simulations to support the theoretical findings. The system (2.2) discrete form is as follows:

$$\frac{dS_{a,j}}{dt} = d_1 \sum_{k=1}^{n} \mathbb{J}(x_j - x_k) S_a(t, x_k)\Delta x - d_1 S_{a,j}(t) + \Lambda_a - \mu_a S_{a,j}(t) - \beta_a S_{a,j}(t) I_{a,j}(t),$$
$$\frac{dIa,j}{dt} = d_1 \sum_{k=1}^{n} \mathbb{J}(x_j - x_k) I_a(t, x_k)\Delta x - d_1 I_{a,j}(t + \beta_a S_{a,j}(t) I_{a,j}(t) - (\mu_a + \delta_a) I_{a,j}(t),$$
$$\frac{dS_{h,j}}{dt} = d_2 \sum_{k=1}^{n} \mathbb{J}(x_j - x_k) S_h(t, x_k)\Delta x - d_1 S_{h,j}(t) + \Lambda_h - \mu_h S_{h,j}(t) - \beta_h S_{h,j}(t) I_{a,j}(t),$$
$$\frac{dI_{h,j}}{dt} = d_2 \sum_{k=1}^{n} \mathbb{J}(x_j - x_k) I_h(t, x_k)\Delta x - d_1 I_{h,j}(t) + \beta_h S_{h,j}(t) I_{a,j}(t) - (\mu_h + \delta_h + \gamma) I_{h,j}(t).$$

We set the parameter values and initial conditions as follows:

Table 2. Value of parameter.

Parameter	Value	Parameter	Value
Λ_h	100 [19, 25]	δ_a	0.0005 per day [26]
δ_h	0.001 per day [11, 26]	μ_a	0.01 [19, 25]
μ_h	0.0015	γ	0.1 per day [11, 26]
β_h	0.000078	$d_1 = d_2$	0.1

initial value:

$$(S_{a,0}(x), I_{a,0}(x), S_{h,0}(x), I_{h,0}(x)) = (0.03\sin x + 0.05\cos x, 0.01\cos x, 0.01\sin x + 0.03\cos x, 0).$$

Moreover, the nonlocal kernel function [9] is selected as follows:

$$\mathbb{J}_x = \begin{cases} A\exp(\frac{1}{x^2-1}), & -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $A = 2.6423$, $x \in [-1, 1] \subset R$ and $\int_R \mathbb{J}(x)dx = \int_{-1}^{1} \mathbb{J}(x)dx \approx 1$.

In this section, we choose to change β_H to illustrate the result of the theorem. In Figure 1, Let $\Lambda_a = 350$, $\beta_a = 0.0000058$, then $R_0 = 0.977348693499753 < 1$, illustrates that the density of infected avian and human populations approaches zero as the time approaches infinity, indicating the extinction of the disease. Figure 2 shows that the solution of system (2.2) converges to a steady state, implying the persistence of the disease, where $\Lambda_a = 1500$, $\beta_a = 0.0000088$.

To examine the effect of the diffusion coefficient on the system. Other parameters are shown in Table 2, let $\Lambda_a = 1500$, $\beta_a = 0.0000088$, $d_1 = 0.075$, and the results are presented in Figure 3. In Figure 4, let $d_1 = 0.05$. When the disease is persistent, the disease will be more stable if the spread rate is small, but if the spread rate becomes large, the number of infected persons will decrease sharply, that is, the nonlocal spread of infected will reduce the number of infected persons. In other words, As the diffusion coefficient increases, the number of infected individuals decreases.

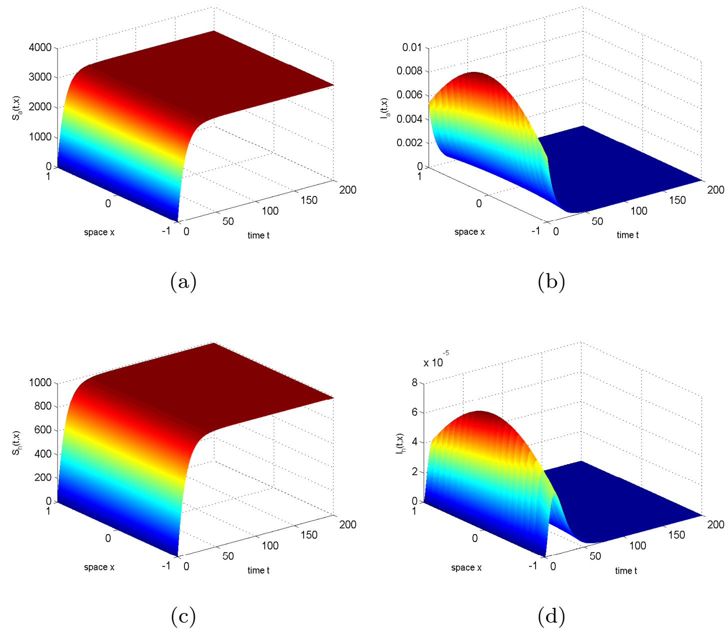

Figure 1. The evolution paths of S_a, I_a, S_h, I_h for system (2.2) with $R_0 = 0.977348693499753 < 1$.

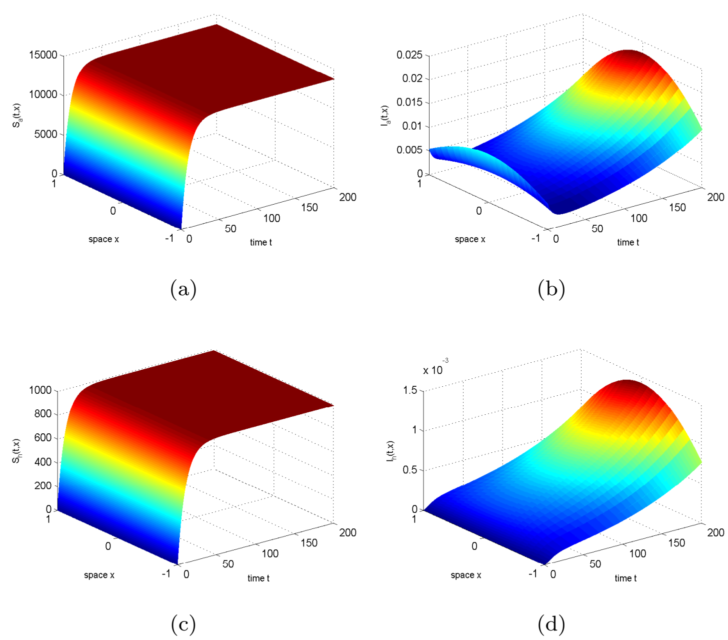

Figure 2. The evolution paths of S_a, I_a, S_h, I_h for system (2.2) with $R_0 = 6.355173770540262 > 1$.

5. Conclusions

From the previous analysis, we note that in the existing studies of avian influenza models, ordinary differential equation models, diffusion equations with Laplace transform or non-local

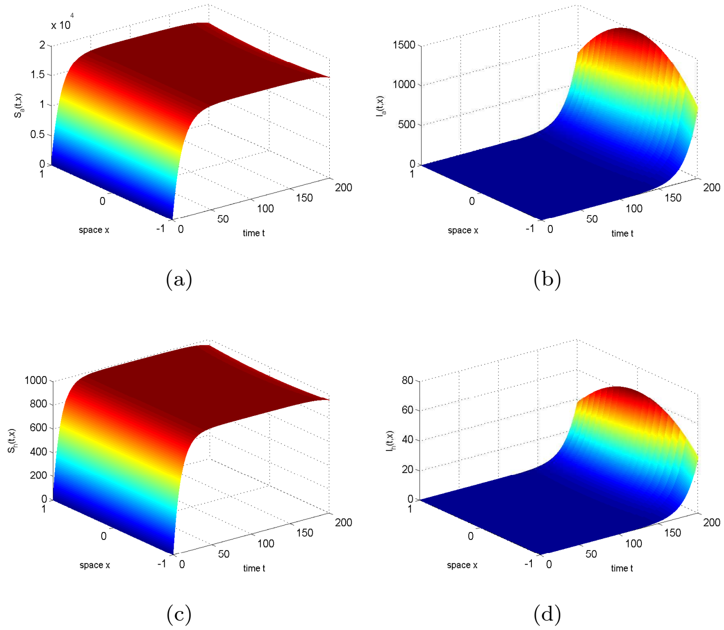

Figure 3. The evolution paths of S_a, I_a, S_h, I_h for system (2.2) with $R_0 = 8.333144545757014 > 1$.

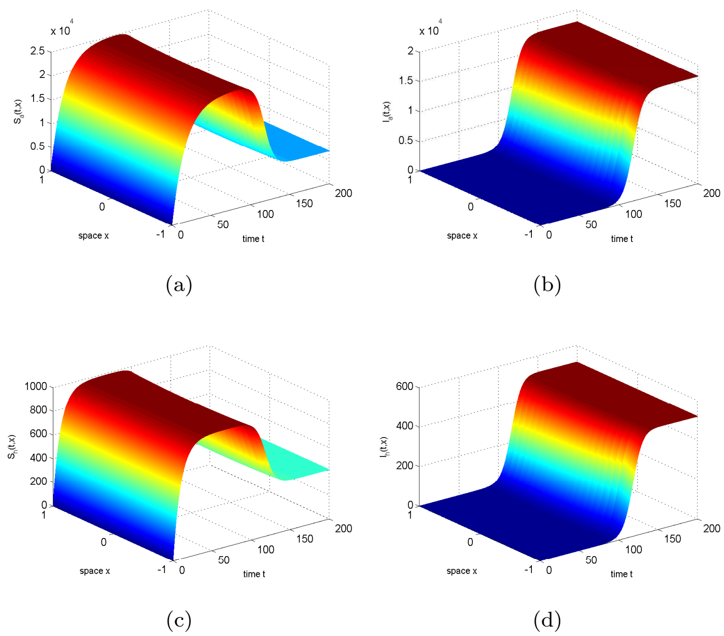

Figure 4. The evolution paths of S_a, I_a, S_h, I_h for system (2.2) with $R_0 = 12.098726446978853 > 1$.

diffusion of transmission rates are considered. However, the above models cannot well model the spatial spread of avian influenza. To better describe the disease spread, the diffusion process is described by integral operators.

In this research, we investigated the long-term behavior of an avian influenza model incorporating nonlocal diffusion. We established the existence, uniqueness, positivity and boundedness of the solution by constructing a Lyapunov function and employing the eigenvalue problem of the nonlocal diffusion term. The basic reproduction number was determined using the next generation matrix method. By utilizing the Lyapunov function and comparison principle, we demonstrated the global stability and uniform persistence of the system. Finally, we conducted numerical simulations to explore the disease's dynamic behavior and the impact of the diffusion coefficient. Our findings reveal that the diffusion rate of infection becomes large, even if the $R_0 > 1$, the infected may also disappear. This means that nonlocal diffusion of infected may suppress the spread of the disease. In addition, in order to just consider the influence of diffusion coefficient on the disease, next, we will try to introduce white noise, Lévy noise, etc. At the same time, the effects of the convolution operator and the Laplacian operator on the disease are compared.

Conflict of interest

The authors declare that they have no conflict of interest.

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