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THE METHOD OF LOWER AND UPPER SOLUTIONS FOR FRACTIONAL DIFFERENTIAL SYSTEM WITH P-LAPLACIAN OPERATORS*

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Abstract This paper focuses on the multi-point boundary value problem for a nonlinear fractional differential system, involving p-Laplacian operator and integral boundary conditions, which arises from many complex processes such as the nonlinear phenomena in non-Newtonian fluids and mathematical modeling. Based on the monotone iterative technique, a new method of lower and upper solutions are proposed. Some new results on the existence of positive solutions for multi-point boundary value problem with integral boundary conditions are established. Finally, an example is presented to illustrate the wide range of potential applications of our main results.

Keywords Fractional differential system, p-Laplacian operators, lower and upper solutions.

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1. Introduction

Differential equations are useful in modern physics, engineering, and in various fields of science. In these days, the theory on existence and uniqueness of boundary value problems of linear and nonlinear fractional equations has attracted the attention of many authors, see [4,7]. There are comprehensive studies in this area. At the same time, it is known that the p-Laplacian operator is also used in analyzing mechanics, physics and dynamic systems, and the related fields of mathematical modeling.

Fractional differential equation with p-Laplacian operator can describe the nonlinear phenomena in non-Newtonian fluids and establishes complex process models, see [2,5,6,8,10,20,24]. Many important results related to the boundary value problems of fractional differential equations with p-Laplacian operator have also been obtained, see [1,3,9,11,13,16,17,19,21,22] and references therein.

Fractional differential system with p-Laplacian operators have also attracted tremendous attention [12,14,15,18,23,25,26], Among them, applying the monotone iterative approach, the

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authors in [14] got the extremal solutions of the following system:

$$\begin{cases} D_{0^+}^{\alpha_1}\Big(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))\Big) = f_1(t,v(t)), \\ D_{0^+}^{\alpha_2}\Big(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))\Big) = f_2(t,u(t)), 0<t<1, \\ u(0) = D_{0^+}^{\beta_1}u(0) = 0, \quad D_{0^+}^{\gamma_1}u(1) = \sum_{j=1}^{m-2} a_{1j}D_{0^+}^{\gamma_1}u(\eta_j) = 0, \\ v(0) = D_{0^+}^{\beta_2}v(0) = 0, \quad D_{0^+}^{\gamma_2}v(1) = \sum_{j=1}^{m-2} a_{2j}D_{0^+}^{\gamma_2}v(\eta_j) = 0, \end{cases}$$

where $0 < \alpha_i, \gamma_i \leqslant 1, 0 < \beta_i \leqslant 2, D_{0^+}^{\alpha_i}, D_{0^+}^{\beta_i}, D_{0^+}^{\gamma_i}$ are the Riemann-Liouville fractional derivatives.

Based on system [14], considering the boundary value conditions of the differential equation,this paper establishes the following multi-point boundary value problem of fractional differential system with integral boundary conditions and p-Laplacian operators.

$$\begin{cases} D_{0^+}^{\alpha_1}\Big(\varphi_{p_1}(D_{0^+}^{\beta_1}u_1(t))\Big) = f_1(t,u_1(t),u_2(t),D_{0^+}^{\beta_1}u_1(t),D_{0^+}^{\beta_2}u_2(t)), \\ D_{0^+}^{\alpha_2}\Big(\varphi_{p_2}(D_{0^+}^{\beta_2}u_2(t))\Big) = f_2(t,u_1(t),u_2(t),D_{0^+}^{\beta_1}u_1(t),D_{0^+}^{\beta_2}u_2(t)), 0<t<1, \\ D_{0^+}^{\beta_1}u_1(0) = u_1(0) = {u_1}'(0) = \cdots = {u_1}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2}u_2(0) = u_2(0) = {u_2}'(0) = \cdots = {u_2}^{(m-2)}(0) = 0, \\ u_1(1) = a_1\int_0^{\eta_1} u_2(s)ds + b_1\int_{\xi_1}^1 u_2(s)ds, \quad D_{0^+}^{\beta_1}u_1(1) = \varepsilon_1 D_{0^+}^{\beta_1}u_1(\gamma_1), \\ u_2(1) = a_2\int_0^{\eta_2} u_1(s)ds + b_2\int_{\xi_2}^1 u_1(s)ds, \quad D_{0^+}^{\beta_2}u_2(1) = \varepsilon_2 D_{0^+}^{\beta_2}u_2(\gamma_2), \end{cases} \tag{1.1}$$

where $1 < \alpha_i \leqslant 2$, $1 \leqslant n-1 < \beta_1 \leqslant n$, $1 \leqslant m-1 < \beta_2 \leqslant m$, $m,n \geqslant 2$, $D_{0^+}^{\alpha_i}, D_{0^+}^{\beta_i}$ are the Riemann-Liouville derivative operators. $a_i, b_i, \varepsilon_i > 0$ are constants, $\eta_i, \xi_i, \gamma_i \in (0,1)$ and satisfes $0 < \eta_i \leqslant \xi_i < 1$, $1 - \beta_1^{-1}(a_2\eta_2^{\beta_1} + b_2(1-\xi_2^{\beta_1})) > 0$, $1 - \beta_2^{-1}(a_1\eta_1^{\beta_2} + b_1(1-\xi_1^{2})) > 0$, φ_{p_i} is the p-Laplacian operator defined by $\varphi_{p_i}(s) = |s|^{p_i-2}s$, $\varphi_{p_i}^{-1} = \varphi_{q_i}$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $p_i > 1$, $f_i : [0,1] \times [0,+\infty)^2 \times (-\infty,0]^2 \to [0,+\infty)$ is continuous function, $i = 1,2$.

The upper and lower solution method provides an effective tool for the existence of solutions in fractional differential systems by constructing appropriate comparison functions, combining the fixed point theorem and monotonic iteration.The purpose of this paper is to establish a method of lower and upper solutions which is used to study the existence of positive solutions of boundary value problem (1.1).

2. Preliminary results

We say a vector function $(u_1(t), u_2(t))$ is a positive solution of boundary value problem (1.1) if it satisfies boundary value problem (1.1) and $u_i(t) \geqslant 0$, for $t \in [0,1], i = 1,2$.

Lemma 2.1. *For any given function $h_i \in C[0,1]$ and real numbers $d_i \in \mathbb{R}, i = 1,2$, the following* *boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}u_1(t))\right)=h_1(t),\\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2}u_2(t))\right)=h_2(t), 0<t<1,\\ D_{0^+}^{\beta_1}u_1(0)=u_1(0)={u_1}'(0)=\cdots={u_1}^{(n-2)}(0)=0,\\ D_{0^+}^{\beta_2}u_2(0)=u_2(0)={u_2}'(0)=\cdots={u_2}^{(m-2)}(0)=0,\\ u_1(1)=a_1\int_0^{\eta_1}u_2(s)ds+b_1\int_{\xi_1}^1 u_2(s)ds, D_{0^+}^{\beta_1}u_1(1)=d_1,\\ u_2(1)=a_2\int_0^{\eta_2}u_1(s)ds+b_2\int_{\xi_2}^1 u_1(s)ds, D_{0^+}^{\beta_2}u_2(1)=d_2, \end{cases} \tag{2.1}$$

has a unique solution, which is given by

$$\begin{aligned} u_1(t)=&-\int_0^1 G_1(t,s)\varphi_{q_1}\left(\varphi_{p_1}(d_1)s^{\alpha_1-1}-\int_0^1 H_1(s,\tau)h_1(\tau)d\tau\right)ds\\ &-\frac{t^{\beta_1-1}}{1-k_1k_2}\left[k_1\int_0^1\left(a_2\int_0^{\eta_2}G_1(\tau,s)d\tau+b_2\int_{\xi_2}^1 G_1(\tau,s)d\tau\right)\varphi_{q_1}\right.\\ &\times\left(\varphi_{p_1}(d_1)s^{\alpha_1-1}-\int_0^1 H_1(s,\tau)h_1(\tau)d\tau\right)ds\\ &+\int_0^1\left(a_1\int_0^{\eta_1}G_2(\tau,s)d\tau+b_1\int_{\xi_1}^1 G_2(\tau,s)d\tau\right)\varphi_{q_2}\\ &\left.\times\left(\varphi_{p_2}(d_2)s^{\alpha_2-1}-\int_0^1 H_2(s,\tau)h_2(\tau)d\tau\right)ds\right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_2(t)=&-\int_0^1 G_2(t,s)\varphi_{q_2}\left(\varphi_{p_2}(d_1)s^{\alpha_2-1}-\int_0^1 H_2(s,\tau)h_2(\tau)d\tau\right)ds\\ &-\frac{t^{\beta_2-1}}{1-k_1k_2}\left[k_2\int_0^1\left(a_1\int_0^{\eta_1}G_2(\tau,s)d\tau+b_1\int_{\xi_1}^1 G_2(\tau,s)d\tau\right)\varphi_{q_2}\right.\\ &\times\left(\varphi_{p_2}(d_2)s^{\alpha_2-1}-\int_0^1 H_2(s,\tau)h_2(\tau)d\tau\right)ds\\ &+\int_0^1\left(a_2\int_0^{\eta_2}G_1(\tau,s)d\tau+b_2\int_{\xi_2}^1 G_1(\tau,s)d\tau\right)\varphi_{q_1}\\ &\left.\times\left(\varphi_{p_1}(d_1)s^{\alpha_1-1}-\int_0^1 H_1(s,\tau)h_1(\tau)d\tau\right)ds\right], \end{aligned} \tag{2.3}$$

and

$$\begin{cases} D_{0^+}^{\beta_1}u_1(t)=\varphi_{q_1}\left(\varphi_{p_1}(d_1)t^{\alpha_1-1}-\int_0^1 H_1(t,s)h_1(s)ds\right),\\ D_{0^+}^{\beta_2}u_2(t)=\varphi_{q_2}\left(\varphi_{p_2}(d_2)t^{\alpha_2-1}-\int_0^1 H_2(t,s)h_2(s)ds\right), \end{cases} \tag{2.4}$$

where

$$G_i(t,s)=\frac{1}{\Gamma(\beta_i)}\begin{cases}(t(1-s))^{\beta_i-1}-(t-s)^{\beta_i-1}, & 0\le s\le t\le 1,\\ (t(1-s))^{\beta_i-1}, & 0\le t\le s\le 1,\end{cases} \tag{2.5}$$

$$H_i(s,\tau)=\frac{1}{\Gamma(\alpha_i)}\begin{cases}(s(1-\tau))^{\alpha_i-1}-(s-\tau)^{\alpha_i-1}, & 0\le\tau\le s\le 1,\\ (s(1-\tau))^{\alpha_i-1}, & 0\le s\le\tau\le 1,\end{cases} \tag{2.6}$$

$$k_1=\beta_2^{-1}(a_1\eta_1^{\beta_2}+b_1(1-\xi_1^{\beta_2})),\ k_2=\beta_1^{-1}(a_2\eta_2^{\beta_1}+b_2(1-\xi_2^{\beta_1})). \tag{2.7}$$

Proof. Let $\varphi_{p_i}(D_{0^+}^{\beta_i}u_i(t))=\bar{u}_i(t)$, we can easily show that boundary value problem (2.1) can be decomposed into the following two coupled boundary value problems:

$$\begin{cases}D_{0^+}^{\alpha_i}\bar{u}_i(t)=h_i(t), & t\in(0,1),\\ \bar{u}_i(0)=0,\bar{u}_i(1)=\varphi_{p_i}(d_i), & i=1,2,\end{cases} \tag{2.8}$$

and

$$\begin{cases}D_{0^+}^{\beta_i}u_i(t)=\varphi_{q_i}(\bar{u}_i(t)), i=1,2,\\ u_1(0)={u_1}'(0)=\cdots={u_1}^{(n-2)}(0)=0, u_1(1)=a_1\displaystyle\int_0^{\eta_1}u_2(s)ds+b_1\int_{\xi_1}^1u_2(s)ds,\\ u_2(0)={u_2}'(0)=\cdots={u_2}^{(m-2)}(0)=0, u_2(1)=a_2\displaystyle\int_0^{\eta_2}u_1(s)ds+b_2\int_{\xi_2}^1u_1(s)ds.\end{cases} \tag{2.9}$$

By the standard way, we can get that boundary value problem (2.8) has a unique solution, which is given by

$$\begin{aligned}\bar{u}_i(t)&=\frac{1}{\Gamma(\alpha_i)}\left(\int_0^t(t-s)^{\alpha_i-1}h_i(s)ds-\int_0^1t^{\alpha_i-1}(1-s)^{\alpha_i-1}h_i(s)ds\right)+\varphi_{p_i}(d_i)t^{\alpha_i-1}\\ &=\varphi_{p_i}(d_i)t^{\alpha_i-1}-\int_0^1H_i(t,s)h_i(s)ds\\ &=\varphi_{p_1}(D_{0^+}^{\beta_i}u_i(t)),\quad i=1,2.\end{aligned} \tag{2.10}$$

Next, we consider the system (2.9).

By the equation and boundary conditions of zero point, we have

$$\begin{cases}u_1(t)=I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(t))+\varsigma t^{\beta_1-1},\\ u_2(t)=I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(t))+\bar{\varsigma}t^{\beta_2-1}.\end{cases} \tag{2.11}$$

Using the boundary conditions of integral,we may obtain

$$\begin{cases}\varsigma-k_1\bar{\varsigma}=a_1\displaystyle\int_0^{\eta_1}I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds+b_1\int_{\xi_1}^1I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds-I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1)),\\ \bar{\varsigma}-k_2\varsigma=a_2\displaystyle\int_0^{\eta_2}I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds+b_2\int_{\xi_2}^1I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds-I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1)),\end{cases}$$

$$\begin{aligned}\varsigma=\frac{1}{1-k_1k_2}\Bigg\{&\left(a_1\int_0^{\eta_1}I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds+b_1\int_{\xi_1}^1I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds-I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1))\right)\\ &+k_1\left(a_2\int_0^{\eta_2}I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds+b_2\int_{\xi_2}^1I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds-I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1))\right)\Bigg\},\end{aligned}$$

$$
\begin{aligned}
\bar{\varsigma} &= \frac{1}{1-k_1k_2}\left\{\left(a_2\int_0^{\eta_2} I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds + b_2\int_{\xi_2}^1 I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds - I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1))\right)\right.\\
&\quad\left.+k_2\left(a_1\int_0^{\eta_1} I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds + b_1\int_{\xi_1}^1 I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds - I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1))\right)\right\},\\
u_1(t) &= I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(t))\\
&\quad+\frac{t^{\beta_1-1}}{1-k_1k_2}\left[\left(a_1\int_0^{\eta_1} I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds + b_1\int_{\xi_1}^1 I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds - I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1))\right)\right.\\
&\quad\left.+k_1\left(a_2\int_0^{\eta_2} I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds + b_2\int_{\xi_2}^1 I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds - I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1))\right)\right]\\
&= -\int_0^1 G_1(t,s)\varphi_{q_1}(\bar{u}_1(s))ds + \frac{t^{\beta_1-1}}{1-k_1k_2}\left[-k_1k_2I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1)) + a_1\int_0^{\eta_1} I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds\right.\\
&\quad+b_1\int_{\xi_1}^1 I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds\\
&\quad\left.+k_1\left(a_2\int_0^{\eta_2} I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds + b_2\int_{\xi_2}^1 I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds - I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1))\right)\right]\\
&= -\int_0^1 G_1(t,s)\varphi_{q_1}(\bar{u}_1(s))ds + \frac{t^{\beta_1-1}}{1-k_1k_2}\left(-k_1I_1 - I_2\right),\\
I_1 &= k_2I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(1)) - a_2\int_0^{\eta_2} I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds - b_2\int_{\xi_2}^1 I_{0^+}^{\beta_1}\varphi_{q_1}(\bar{u}_1(s))ds\\
&= \frac{1}{\Gamma(\beta_1)}[\int_0^1 a_2\int_0^{\eta_2}\tau^{\beta_1-1}(1-s)^{\beta_1-1}\varphi_{q_1}(\bar{u}_1(s))d\tau ds\\
&\quad-a_2\int_0^{\eta_2}\int_0^s (s-\tau)^{\beta_1-1}\varphi_{q_1}(\bar{u}_1(\tau))d\tau ds\\
&\quad+\int_0^1 b_2\int_{\xi_2}^1\tau^{\beta_1-1}(1-s)^{\beta_1-1}\varphi_{q_1}(\bar{u}_1(s))d\tau ds - b_2\int_{\xi_2}^1\int_0^s (s-\tau)^{\beta_1-1}\varphi_{q_1}(\bar{u}_1(\tau))d\tau ds]\\
&= a_2\int_0^{\eta_2}\int_0^1 G_1(\tau,s)\varphi_{q_1}(\bar{u}_1(s))dsd\tau + b_2\int_{\xi_2}^1\int_0^1 G_1(\tau,s)\varphi_{q_1}(\bar{u}_1(s))dsd\tau,\\
I_2 &= k_1I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(1)) - a_1\int_0^{\eta_1} I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds - b_1\int_{\xi_1}^1 I_{0^+}^{\beta_2}\varphi_{q_2}(\bar{u}_2(s))ds\\
&= a_1\int_0^{\eta_1}\int_0^1 G_2(\tau,s)\varphi_{q_2}(\bar{u}_2(s))dsd\tau + b_1\int_{\xi_1}^1\int_0^1 G_2(\tau,s)\varphi_{q_2}(\bar{u}_2(s))dsd\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u_1(t) &= -\int_0^1 G_1(t,s)\varphi_{q_1}(\bar{u}_1(s))ds - \frac{t^{\beta_1-1}}{1-k_1k_2}\left[k_1\left(a_2\int_0^1(\int_0^{\eta_2} G_1(\tau,s)d\tau)\varphi_{q_1}(\bar{u}_1(s))ds\right.\right.\\
&\quad\left.+b_2\int_0^1(\int_{\xi_2}^1 G_1(\tau,s)d\tau)\varphi_{q_1}(\bar{u}_1(s))ds\right) + a_1\int_0^1(\int_0^{\eta_1} G_2(\tau,s)d\tau)\varphi_{q_2}(\bar{u}_2(s))ds\\
&\quad\left.+b_1\int_0^1(\int_{\xi_1}^1 G_2(\tau,s)d\tau)\varphi_{q_2}(\bar{u}_2(s))ds\right],
\end{aligned}
\tag{2.12}
$$

similarly,

$$\begin{aligned}
&u_2(t)\\
=&-\int_0^1 G_2(t,s)\varphi_{q_2}(\bar{u}_2(s))ds\\
&-\frac{t^{\beta_2-1}}{1-k_1k_2}\left[k_2\left(a_1\int_0^1(\int_0^{\eta_1}G_2(\tau,s)d\tau)\varphi_{q_2}(\bar{u}_2(s))ds+b_1\int_0^1(\int_{\xi_1}^1 G_2(\tau,s)d\tau)\varphi_{q_2}(\bar{u}_2(s))ds\right)\right.\\
&\left.+a_2\int_0^1(\int_0^{\eta_2}G_1(\tau,s)d\tau)\varphi_{q_1}(\bar{u}_1(s))ds+b_2\int_0^1(\int_{\xi_2}^1 G_1(\tau,s)d\tau)\varphi_{q_1}(\bar{u}_1(s))ds\right]. \qquad (2.13)
\end{aligned}$$

Therefore, boundary value problem (2.1) has a unique solution which is given by (2.2), (2.3) and we can easily get that $D_{0^+}^{\beta_i}u_i(t), i=1,2$ is given by (2.4). From Lemma 2.1, it is obvious that boundary value problem (1.1) is equivalent to the integral system composed of (2.12) and (2.13).

Where

$$\bar{u}_1(s)=\varphi_{p_1}\left(\varepsilon_1 D_{0^+}^{\beta_1}u_1(\gamma_1)s^{\alpha_1-1}-\int_0^1 H_1(s,\tau)f_1(\tau,u_1(\tau),u_2(\tau),D_{0^+}^{\beta_1}u_1(\tau),D_{0^+}^{\beta_2}u_2(\tau))d\tau\right), \qquad (2.14)$$

$$\bar{u}_2(s)=\varphi_{p_2}\left(\varepsilon_2 D_{0^+}^{\beta_2}u_2(\gamma_2)s^{\alpha_2-1}-\int_0^1 H_2(s,\tau)f_2(\tau,u_1(\tau),u_2(\tau),D_{0^+}^{\beta_1}u_1(\tau),D_{0^+}^{\beta_2}u_2(\tau))d\tau\right). \qquad (2.15)$$

□

From (2.5) and (2.6), we can easily prove that $G_i(t,s)$ and $H_i(t,s)$ satisfy the following Lemma.

Lemma 2.2. *Suppose the functions $G_i(t,s)$ and $H_i(t,s)$ are defined by* (2.5) *and* (2.6)*, then $G_i(t,s)$ and $H_i(t,s)$ are continuous and $G_i(t,s)\geq 0$, $H_i(t,s)\geq 0$, for $(t,s)\in[0,1]\times[0,1], i=1,2$.*

3. The method of lower and upper solutions

In this section, we present the method of lower and upper solutions and existence theorems of positive solutions for boundary problem (1.1) based on the monotone iterative technique.

Definition 3.1. Let $(x_1,x_2)\in AC^n[0,1]\times AC^m[0,1]$, we say that (x_1,x_2) is a lower solution of boundary value problem (1.1) if

$$\begin{cases}
D_{0^+}^{\alpha_i}\left(\varphi_{p_i}(D_{0^+}^{\beta_i}x_i(t))\right)\leq f_i(t,x_1(t),x_2(t),D_{0^+}^{\beta_1}x_1(t),D_{0^+}^{\beta_2}x_2(t)), t\in(0,1), i=1,2,\\
D_{0^+}^{\beta_1}x_1(0)=x_1(0)={x_1}'(0)=\cdots={x_1}^{(n-2)}(0)=0,\\
D_{0^+}^{\beta_2}x_2(0)=x_2(0)={x_2}'(0)=\cdots={x_2}^{(m-2)}(0)=0,\\
x_1(1)\leq a_1\int_0^{\eta_1}x_2(s)ds+b_1\int_{\xi_1}^1 x_2(s)ds, \qquad D_{0^+}^{\beta_1}x_1(1)\geq\varepsilon_1 D_{0^+}^{\beta_1}x_1(r_1),\\
x_2(1)\leq a_2\int_0^{\eta_2}x_1(s)ds+b_2\int_{\xi_2}^1 x_1(s)ds, \qquad D_{0^+}^{\beta_2}x_2(1)\geq\varepsilon_2 D_{0^+}^{\beta_2}x_2(r_1).
\end{cases} \qquad (3.1)$$

Let $(y_1, y_2) \in AC^n[0,1] \times AC^m[0,1]$, we say that (x_1, x_2) is a upper solution of boundary value problem (1.1) if

$$\begin{cases} D_{0^+}^{\alpha_i}\left(\varphi_{p_i}(D_{0^+}^{\beta_i} y_i(t))\right) \geq f_i(t, y_1(t), y_2(t), D_{0^+}^{\beta_1} y_1(t), D_{0^+}^{\beta_2} y_2(t)), t \in (0,1), i = 1,2, \\ D_{0^+}^{\beta_1} y_1(0) = y_1(0) = {y_1}'(0) = \cdots = {y_1}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2} y_2(0) = y_2(0) = {y_2}'(0) = \cdots = {y_2}^{(m-2)}(0) = 0, \\ y_1(1) \geq a_1 \int_0^{\eta_1} y_2(s)ds + b_1 \int_{\xi_1}^1 y_2(s)ds, \qquad D_{0^+}^{\beta_1} y_1(1) \leq \varepsilon_1 D_{0^+}^{\beta_1} y_1(r_1), \\ y_2(1) \geq a_2 \int_0^{\eta_2} y_1(s)ds + b_2 \int_{\xi_2}^1 y_1(s)ds, \qquad D_{0^+}^{\beta_2} y_2(1) \leq \varepsilon_2 D_{0^+}^{\beta_2} y_2(r_1). \end{cases} \tag{3.2}$$

Denote that $E = \{(u_1, u_2) : (u_1, u_2) \in C[0,1] \times C[0,1], D_{0^+}^{\beta_1} u_1, D_{0^+}^{\beta_2} u_2 \in C[0,1], u_1(0) = {u_1}'(0) = \cdots = {u_1}^{(n-2)}(0) = 0, u_2(0) = {u_2}'(0) = \cdots = {u_2}^{(m-2)}(0) = 0\}$ and endowed with the norm $\| (u_1, u_2) \| = \| u_1 \|_\infty + \| u_2 \|_\infty + \| D_{0^+}^{\beta_1} u_1 \|_\infty + \| D_{0^+}^{\beta_2} u_2 \|_\infty$, where $\| u_i \|_\infty = \max_{0 \leq t \leq 1} | u_i(t) |$ and $\| D_{0^+}^{\beta_i} u_i \|_\infty = \max_{0 \leq t \leq 1} | D_{0^+}^{\beta_i} u_i(t) |, i = 1,2$. Then $(E, \| \cdot \|)$ is a Banach space. We denote that $P = \{(u_1, u_2) : (u_1, u_2) \in E, u_i(t) \geq 0, D_{0^+}^{\beta_i} u_i(t) \leq 0, t \in [0,1], i = 1,2\}$. Then P is a normal cone on E. We denote $(x_1, x_2) \leq (y_1, y_2)$ if and only if $(y_1 - x_1, y_2 - x_2) \in P$, for $(x_1, x_2), (y_1, y_2) \in E$.

In this section, we assume the following condition holds:

(H1) $f_i \in C([0,1] \times [0,+\infty) \times [0,+\infty) \times (-\infty,0] \times (-\infty,0], [0,+\infty)), f_i(t, u_1, v_1, w_1, z_1) \leq f_i(t, u_2, v_2, w_2, z_2)$, for $0 \leq u_1 \leq u_2, 0 \leq v_1 \leq v_2, w_1 \geq w_2 \geq 0, z_1 \geq z_2 \geq 0$ and any $t \in [0,1]$.

Theorem 3.1. *Assume that (H1) holds, boundary value problem* (1.1) *has a nonnegative lower solution* $(x_0, y_0) \in P$ *and an upper solution* $(u_0, v_0) \in P$ *such that* $(x_0, y_0) \leq (u_0, y_0)$. *The boundary value problem* (1.1) *has the maximal lower solution* (x^*, y^*) *and the mininal upper solution* (u^*, v^*) *on* $[x_0, u_0] \times [y_0, v_0] \subset P$, *both* (x^*, y^*) *and* (u^*, v^*) *are positive solutons of boundary valne problems* (1.1). *Furthermore,*

$$(0,0) \leq (x_0, y_0) \leq (x^*, y^*) \leq (u^*, v^*) \leq (u_0, v_0),$$
$$(D_{0^+}^{\beta_1} u_0, D_{0^+}^{\beta_2} v_0) \leq (D_{0^+}^{\beta_1} u^*, D_{0^+}^{\beta_2} v^*) \leq (D_{0^+}^{\beta_1} x^*, D_{0^+}^{\beta_2} y^*) \leq (D_{0^+}^{\beta_1} x_0, D_{0^+}^{\beta_2} y_0) \leq (0,0).$$

Proof. The proof is divided into the following three steps.

Step 1. We will obtain the lower solution sequence $\{(x_k, y_k)\}$ and the upper solutiom sequence $\{(u_k, v_k)\}$. According to Lemma 2.1, for the given$(x_0, y_0) \in P$, the following boundary value problem

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1} x_1(t))\right) = f_1(t, x_0(t), y_0(t), D_{0^+}^{\beta_1} x_0(t), D_{0^+}^{\beta_2} y_0(t)), \\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2} y_1(t))\right) = f_2(t, x_0(t), y_0(t), D_{0^+}^{\beta_1} x_0(t), D_{0^+}^{\beta_2} y_0(t)), t \in (0,1), \\ D_{0^+}^{\beta_1} x_1(0) = x_1(0) = {x_1}'(0) = \cdots = {x_1}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2} y_1(0) = y_1(0) = {y_1}'(0) = \cdots = {y_1}^{(m-2)}(0) = 0, \\ x_1(1) = a_1 \int_0^{\eta_1} y_0(s)ds + b_1 \int_{\xi_1}^1 y_0(s)ds, \qquad D_{0^+}^{\beta_1} x_1(1) = \varepsilon_1 D_{0^+}^{\beta_1} x_0(r_1), \\ y_1(1) = a_2 \int_0^{\eta_2} x_0(s)ds + b_2 \int_{\xi_2}^1 x_0(s)ds, \qquad D_{0^+}^{\beta_2} y_1(1) = \varepsilon_2 D_{0^+}^{\beta_2} y_0(r_2), \end{cases} \tag{3.3}$$

has a unique solution (x_1, y_1). Since (x_0, y_0) is a lower solution of boundary value problems (1.1), then

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t))\right) \le f_1(t, x_0(t), y_0(t), D_{0^+}^{\beta_1}x_0(t), D_{0^+}^{\beta_2}y_0(t)), \\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2}y_0(t))\right) \le f_2(t, x_0(t), y_0(t), D_{0^+}^{\beta_1}x_0(t), D_{0^+}^{\beta_2}y_0(t)), \\ D_{0^+}^{\beta_1}x_0(0) = x_0(0) = {x_0}'(0) = \cdots = {x_0}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2}y_0(0) = y_0(0) = {y_0}'(0) = \cdots = {y_0}^{(m-2)}(0) = 0, \\ x_0(1) \le a_1\int_0^{\eta_1} y_0(s)ds + b_1\int_{\xi_1}^1 y_0(s)ds, \qquad D_{0^+}^{\beta_1}x_0(1) \ge \varepsilon_1 D_{0^+}^{\beta_1}x_0(r_1), \\ y_0(1) \le a_2\int_0^{\eta_2} x_0(s)ds + b_2\int_{\xi_2}^1 x_0(s)ds, \qquad D_{0^+}^{\beta_2}y_0(1) \ge \varepsilon_2 D_{0^+}^{\beta_2}y_0(r_2). \end{cases} \tag{3.4}$$

(3.3) minus (3.4), and we can get that

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t))\right) \ge 0, \\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2}y_1(t)) - \varphi_{p_2}(D_{0^+}^{\beta_2}y_0(t))\right) \ge 0, \quad t \in (0,1), \\ D_{0^+}^{\beta_1}x_1(0) - D_{0^+}^{\beta_1}x_0(0) = x_1(0) - x_0(0) = {x_1}'(0) - {x_0}'(0) = \cdots = {x_1}^{(n-2)}(0) - {x_0}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2}y_1(0) - D_{0^+}^{\beta_2}y_0(0) = y_1(0) - y_0(0) = {y_1}'(0) - {y_0}'(0) = \cdots = {y_1}^{(m-2)}(0) - {y_0}^{(m-2)}(0) = 0, \\ x_1(1) - x_0(1) \ge 0, \qquad D_{0^+}^{\beta_1}x_1(1) - D_{0^+}^{\beta_1}x_0(1) \le 0, \\ y_1(1) - y_0(1) \ge 0, \qquad D_{0^+}^{\beta_2}y_1(1) - D_{0^+}^{\beta_2}y_0(1) \le 0. \end{cases}$$

Let

$$\varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t)) := \omega(t), \quad \varphi_{p_2}(D_{0^+}^{\beta_2}y_1(t)) - \varphi_{p_2}(D_{0^+}^{\beta_2}y_0(t)) := \varpi(t).$$

Since $D_{0^+}^{\beta_1}x_1(0) - D_{0^+}^{\beta_1}x_0(0) = 0$, then $\varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t)) = \varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t) - D_{0^+}^{\beta_1}x_0(t)) = 0$ which is $\omega(0) = 0$. And since $D_{0^+}^{\beta_1}x_1(1) - D_{0^+}^{\beta_1}x_0(1) \le 0$, we get that $\omega(1) = \varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t)) \le 0$.

Let

$$h(t) = D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t))\right)$$

and $\varphi_{p_1}(d_1) = \omega(1)$, then we obtain the following boundary value problem

$$\begin{cases} D_{0^+}^{\alpha_1}\omega(t) = h(t) \ge 0, \quad t \in (0,1), \\ \omega(0) = 0, \qquad \omega(1) = \varphi_{p_1}(d_1) \le 0. \end{cases}$$

By (2.10) and Lemma 2.2

$$\omega(t) = \varphi_{p_1}(D_{0^+}^{\beta_1}x_1(t)) - \varphi_{p_1}(D_{0^+}^{\beta_1}x_0(t)) = \varphi_{p_1}(d_1)t^{\alpha_1 - 1} - \int_0^1 H_1(t,s)h(s)ds \le 0, \quad t \in [0,1].$$

From the monotonicity of p-Laplacian operator φ_{p_1}, we have

$$D_{0^+}^{\beta_1}x_1(t) - D_{0^+}^{\beta_1}x_0(t) = D_{0^+}^{\beta_1}(x_1(t) - x_0(t)) := \delta(t) \le 0.$$

Similarly, we can get that

$$\varpi(t) = \varphi_{p_2}(D_{0^+}^{\beta_2} y_1(t)) - \varphi_{p_2}(D_{0^+}^{\beta_2} y_0(t)) \le 0, \quad t \in [0,1],$$

and

$$D_{0^+}^{\beta_2} y_1(t) - D_{0^+}^{\beta_2} y_0(t) = D_{0^+}^{\beta_2}(y_1(t) - y_0(t)) := \overline{\delta}(t) \le 0.$$

Then we obtain the following boundary value problem

$$\begin{cases} D_{0^+}^{\beta_1}(x_1(t) - x_0(t)) := \delta(t) \le 0, \\ D_{0^+}^{\beta_2}(y_1(t) - y_0(t)) := \overline{\delta}(t) \le 0, \quad t \in (0,1), \\ x_1(0) - x_0(0) = {x_1}'(0) - {x_0}'(0) = \cdots = {x_1}^{(n-2)}(0) - {x_0}^{(n-2)}(0) = 0, \quad x_1(1) - x_0(1) := a \ge 0, \\ y_1(0) - y_0(0) = {y_1}'(0) - {y_0}'(0) = \cdots = {y_1}^{(m-2)}(0) - {y_0}^{(m-2)}(0) = 0, \quad y_1(1) - y_0(1) := \overline{a} \ge 0. \end{cases}$$

Thus, we have

$$x_1(t) - x_0(t) = at^{\beta_1 - 1} - \int_0^1 G_1(t,s)\delta(s)ds \ge 0,$$
$$y_1(t) - y_0(t) = \overline{a}t^{\beta_2 - 1} - \int_0^1 G_2(t,s)\overline{\delta}(s)ds \ge 0.$$

So, we can get that $(x_0, y_0) \le (x_1, y_1)$.

From the condition (H_1), we get

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1} x_1(t))\right) = f_1(t, x_0(t), y_0(t), D_{0^+}^{\beta_1} x_0(t), D_{0^+}^{\beta_2} y_0(t)) \\ \qquad\qquad \le f_1(t, x_1(t), y_1(t), D_{0^+}^{\beta_1} x_1(t), D_{0^+}^{\beta_2} y_1(t)), \\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2} y_1(t))\right) = f_2(t, x_0(t), y_0(t), D_{0^+}^{\beta_1} x_0(t), D_{0^+}^{\beta_2} y_0(t)) \\ \qquad\qquad \le f_2(t, x_1(t), y_1(t), D_{0^+}^{\beta_1} x_1(t), D_{0^+}^{\beta_2} y_1(t)), \\ D_{0^+}^{\beta_1} x_1(0) = x_1(0) = {x_1}'(0) = \cdots = {x_1}^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta_2} y_1(0) = y_1(0) = {y_1}'(0) = \cdots = {y_1}^{(m-2)}(0) = 0, \\ x_1(1) = a_1 \int_0^{\eta_1} y_0(s)ds + b_1 \int_{\xi_1}^1 y_0(s)ds \le a_1 \int_0^{\eta_1} y_1(s)ds + b_1 \int_{\xi_1}^1 y_1(s)ds, \\ y_1(1) = a_2 \int_0^{\eta_2} x_0(s)ds + b_2 \int_{\xi_2}^1 x_0(s)ds \le a_2 \int_0^{\eta_2} x_1(s)ds + b_2 \int_{\xi_2}^1 x_1(s)ds, \\ D_{0^+}^{\beta_1} x_1(1) = \varepsilon_1 D_{0^+}^{\beta_1} x_0(r_1) \ge \varepsilon_1 D_{0^+}^{\beta_1} x_1(r_1), \\ D_{0^+}^{\beta_2} y_1(1) = \varepsilon_2 D_{0^+}^{\beta_2} y_0(r_2) \ge \varepsilon_2 D_{0^+}^{\beta_2} y_1(r_1). \end{cases}$$

Then (x_1, y_1) is a lower solution of boundary value problem (1.1). Starting from the initial (x_0, y_0), by the following iterative scheme

$$D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1} x_k(t))\right) = f_1(t, x_{k-1}(t), y_{k-1}(t), D_{0^+}^{\beta_1} x_{k-1}(t), D_{0^+}^{\beta_2} y_{k-1}(t)),$$
$$D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2} y_k(t))\right) = f_2(t, x_{k-1}(t), y_{k-1}(t), D_{0^+}^{\beta_1} x_{k-1}(t), D_{0^+}^{\beta_2} y_{k-1}(t)), t \in (0,1), \tag{3.5}$$

$$D_{0^+}^{\beta_1}x_k(0)=x_k(0)=x_k{}'(0)=\cdots=x_k{}^{(n-2)}(0)=0,$$
$$D_{0^+}^{\beta_2}y_k(0)=y_k(0)=y_k{}'(0)=\cdots=y_k{}^{(m-2)}(0)=0,$$
$$x_k(1)=a_1\int_0^{\eta_1}y_{k-1}(s)ds+b_1\int_{\xi_1}^1 y_{k-1}(s)ds,\quad D_{0^+}^{\beta_1}x_k(1)=\varepsilon_1 D_{0^+}^{\beta_1}x_{k-1}(r_1),$$
$$y_k(1)=a_2\int_0^{\eta_2}x_{k-1}(s)ds+b_2\int_{\xi_2}^1 x_{k-1}(s)ds,\quad D_{0^+}^{\beta_2}y_k(t)=\varepsilon_2 D_{0^+}^{\beta_2}y_{k-1}(r_2),\quad k=1,2,\cdots.$$

We can obtain the sequence $\{(x_k,y_k)\}$, where $(x,y)=(x_k(t),y_k(t))$ are lower solutions of boundary value problem (1.1), and $(x_{k-1},y_{k-1})\le(x_k,y_k)$, so that $\{(x_k,y_k)\}$ is monotonically increasing. Starting from the initial function (u_0,v_0), by the following iterative scheme

$$\begin{cases} D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}u_k(t))\right)=f_1(t,u_{k-1}(t),v_{k-1}(t),D_{0^+}^{\beta_1}u_{k-1}(t),D_{0^+}^{\beta_2}v_{k-1}(t)),\\ D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2}v_k(t))\right)=f_2(t,u_{k-1}(t),u_{k-1}(t),D_{0^+}^{\beta_1}u_{k-1}(t),D_{0^+}^{\beta_2}v_{k-1}(t)),t\in(0,1),\\ D_{0^+}^{\beta_1}u_k(0)=u_k(0)=u_k{}'(0)=\cdots=u_k{}^{(n-2)}(0)=0,\\ D_{0^+}^{\beta_2}v_k(0)=v_k(0)=v_k{}'(0)=\cdots=v_k{}^{(m-2)}(0)=0,\\ u_k(1)=a_1\int_0^{\eta_1}v_{k-1}(s)ds+b_1\int_{\xi_1}^1 v_{k-1}(s)ds,\quad D_{0^+}^{\beta_1}u_k(1)=\varepsilon_1 D_{0^+}^{\beta_1}u_{k-1}(r_1),\\ v_k(1)=a_2\int_0^{\eta_2}u_{k-1}(s)ds+b_2\int_{\xi_2}^1 u_{k-1}(s)ds,\quad D_{0^+}^{\beta_2}v_k(t)=\varepsilon_2 D_{0^+}^{\beta_2}v_{k-1}(r_2),\quad k=1,2,\cdots. \end{cases} \tag{3.6}$$

We can get the sequence $\{(u_k,v_k)\}$, where $(u,v)=(u_k(t),v_k(t))$ are upper solutions of boundary value problem (1.1), and $\{(u_k,v_k)\}$ is monotonically decreasing.

Step 2. We prove that $(x_k,y_k)\le(u_k,v_k)$, if $(x_{k-1},y_{k-1})\le(u_{k-1},v_{k-1})$, $\quad k=1,2,\cdots$. Since $(x_{k-1},y_{k-1})\le(u_{k-1},v_{k-1})$ and $(D_{0^+}^{\beta_1}x_{k-1}(t),D_{0^+}^{\beta_2}y_{k-1}(t))\ge(D_{0^+}^{\beta_1}u_{k-1}(t),D_{0^+}^{\beta_2}v_{k-1}(t))$, and from (H_1), $f_i(t,x_{k-1}(t),y_{k-1}(t),D_{0^+}^{\beta_1}x_{k-1}(t),D_{0^+}^{\beta_2}y_{k-1}(t))\le f_i(t,u_{k-1}(t),v_{k-1}(t),D_{0^+}^{\beta_1}u_{k-1}(t),$ $D_{0^+}^{\beta_2}v_{k-1}(t))$, $i=1,2,\cdots$. By (3.5) and (3.6), we can get

$$\begin{cases} \quad D_{0^+}^{\alpha_1}\left(\varphi_{p_1}(D_{0^+}^{\beta_1}u_k(t))-\varphi_{p_1}(D_{0^+}^{\beta_1}x_k(t))\right)\\ =f_1(t,u_{k-1}(t),v_{k-1}(t),D_{0^+}^{\beta_1}u_{k-1}(t),D_{0^+}^{\beta_2}v_{k-1}(t))\\ \quad -f_1(t,x_{k-1}(t),y_{k-1}(t),D_{0^+}^{\beta_1}x_{k-1}(t),D_{0^+}^{\beta_2}y_{k-1}(t))\ge0,\\ \quad D_{0^+}^{\alpha_2}\left(\varphi_{p_2}(D_{0^+}^{\beta_2}v_k(t))-\varphi_{p_2}(D_{0^+}^{\beta_2}y_k(t))\right)\\ =f_2(t,u_{k-1}(t),v_{k-1}(t),D_{0^+}^{\beta_1}u_{k-1}(t),D_{0^+}^{\beta_2}v_{k-1}(t))\\ \quad -f_2(t,x_{k-1}(t),y_{k-1}(t),D_{0^+}^{\beta_1}x_{k-1}(t),D_{0^+}^{\beta_2}y_{k-1}(t))\ge0,\\ D_{0^+}^{\beta_1}u_k(0)-D_{0^+}^{\beta_1}x_k(0)=u_k(0)-x_k(0)=u_k{}'(0)-x_k{}'(0)=\cdots=u_k{}^{(n-2)}(0)-x_k{}^{(n-2)}(0)=0,\\ D_{0^+}^{\beta_2}v_k(0)-D_{0^+}^{\beta_2}y_k(0)=v_k(0)-y_k(0)=v_k{}'(0)-y_k{}'(0)=\cdots=v_k{}^{(m-2)}(0)-y_k{}^{(m-2)}(0)=0,\\ u_k(1)-x_k(1)\ge0,\qquad D_{0^+}^{\beta_1}u_k(1)-D_{0^+}^{\beta_1}x_k(1)\le0,\\ v_k(1)-y_k(1)\ge0,\qquad D_{0^+}^{\beta_2}v_k(1)-D_{0^+}^{\beta_2}y_k(1)\le0. \end{cases}$$

Similarly, we can get that $(x_k, y_k) \le (u_k, v_k)$, in the same way as the above. Therefore,

$$(x_0, y_0) \le (x_1, y_1) \le \cdots \le (x_k, y_k) \le \cdots \le (u_k, v_k) \le \cdots \le (u_1, v_1) \le (u_0, v_0).$$

Since P is a normal cone on E, the $\{(x_k, y_k)\}$ and $\{(u_k, v_k)\}$ are uniforming bounded. Because $H_i, G_i, \varphi_{p_i}, \varphi_{q_i}$ and f_i are continuous, we can easily get that $\{(x_k, y_k)\}$ and $\{(u_k, v_k)\}$ are equicontinuous. Hence, the $\{(x_k, y_k)\}$ and $\{(u_k, v_k)\}$ are relatively compact. Then there exist (x^*, y^*) and (u^*, v^*) such that

$$\lim_{k\to\infty}(x_k, y_k) = (x^*, y^*), \quad \lim_{k\to\infty}(D_{0^+}^{\beta_1} x_k, D_{0^+}^{\beta_2} y_k) = (D_{0^+}^{\beta_1} x^*, D_{0^+}^{\beta_2} y^*), \tag{3.7}$$

and

$$\lim_{k\to\infty}(u_k, v_k) = (u^*, v^*), \quad \lim_{k\to\infty}(D_{0^+}^{\beta_1} u_k, D_{0^+}^{\beta_2} v_k) = (D_{0^+}^{\beta_1} u^*, D_{0^+}^{\beta_2} v^*), \tag{3.8}$$

which imply that (x^*, y^*) is the maximun lower solution, (u^*, v^*) is the minimal upper solution of boundary value problem (1.1) in $[x_0, u_0] \times [y_0, v_0] \subset P$, and $(x^*, y^*) \le (u^*, v^*)$.

Step 3. We prove that both (x^*, y^*) and (u^*, v^*) are the solutions of boundary value problem (1.1). According to Lemma 2.1 and (3.5), we can get that

$$\begin{aligned}
x_k(t) =& -\int_0^1 G_1(t,s)\varphi_{q_1}(\bar{x}_{k-1}(s))ds - \frac{t^{\beta_1-1}}{1-k_1k_2}\left[k_1\left(a_2\int_0^1\int_0^{\eta_2} G_1(\tau,s)d\tau\varphi_{q_1}(\bar{x}_{k-1}(s))ds\right.\right.\\
&\left.+b_2\int_0^1\int_{\xi_2}^1 G_1(\tau,s)d\tau\varphi_{q_1}(\bar{x}_{k-1}(s))ds\right) + a_1\int_0^1\int_0^{\eta_1}(G_2(\tau,s)d\tau)\varphi_{q_2}(\bar{y}_{k-1}(s))ds\\
&\left.+b_1\int_0^1\int_{\xi_1}^1 (G_2(\tau,s)d\tau)\,\varphi_{q_2}(\bar{y}_{k-1}(s))ds\right],\\
y_k(t) =& -\int_0^1 G_2(t,s)\varphi_{q_2}(\bar{y}_{k-1}(s))ds\\
&-\frac{t^{\beta_2-1}}{1-k_1k_2}\left[k_2\left(a_1\int_0^1\int_0^{\eta_1} G_2(\tau,s)d\tau\varphi_{q_2}(\bar{y}_{k-1}(s))ds\right.\right.\\
&\left.+b_2\int_0^1\int_{\xi_1}^1 G_2(\tau,s)d\tau\varphi_{q_2}(\bar{y}_{k-1}(s))ds\right)\\
&\left.+a_2\int_0^1\int_0^{\eta_2} G_1(\tau,s)d\tau\varphi_{q_1}(\bar{x}_{k-1}(s))ds + b_2\int_0^1\int_{\xi_2}^1 G_1(\tau,s)d\tau\varphi_{q_1}(\bar{x}_{k-1}(s))ds\right],
\end{aligned}$$

where

$$\begin{aligned}
\bar{x}_{k-1}(s) =& \varphi_{p_1}\left(\varepsilon_1 D_{0^+}^{\beta_1} x_{k-1}(\gamma_1)s^{\alpha_1-1}\right.\\
&\left.-\int_0^1 H_1(s,\tau)f_1(\tau, x_{k-1}(\tau), y_{k-1}(\tau), D_{0^+}^{\beta_1} x_{k-1}(\tau), D_{0^+}^{\beta_2} y_{k-1}(\tau))d\tau\right),\\
\bar{y}_{k-1}(s) =& \varphi_{p_2}\left(\varepsilon_2 D_{0^+}^{\beta_2} y_{k-1}(\gamma_2)s^{\alpha_2-1}\right.\\
&\left.-\int_0^1 H_2(s,\tau)f_2(\tau, x_{k-1}(\tau), y_{k-1}(\tau), D_{0^+}^{\beta_1} x_{k-1}(\tau), D_{0^+}^{\beta_2} y_{k-1}(\tau))d\tau\right).
\end{aligned}$$

From (3.7), and by the continuity of φ_{p_i}, f_i, G_i, H_i and Lebesgue dominated convergence theorem, we can show that (u^*, v^*) is a solution of boundary value problem (1.1).

In the same way, we can show that (u^*, v^*) is also a solution of boundary value problem (1.1). Furthermore,

$$(0,0) \le (x_0, y_0) \le (x^*, y^*) \le (u^*, v^*) \le (u_0, v_0),$$
$$(0,0) \ge (D_{0+}^{\beta_1} x_0, D_{0+}^{\beta_2} y_0) \ge (D_{0+}^{\beta_1} x^*, D_{0+}^{\beta_2} y^*) \ge (D_{0+}^{\beta_1} u^*, D_{0+}^{\beta_2} v^*) \ge (D_{0+}^{\beta_1} u_0, D_{0+}^{\beta_2} v_0).$$

The proof is completed. □

4. Illustration

We consider the following multi-point boundary value problem of nonlinear fractional differential system with P-Laplacian operators:

$$\begin{cases} D_{0+}^{\frac{3}{2}}\left(\varphi_{\frac{3}{2}}(D_{0+}^{\frac{3}{2}} u(t))\right) = f_1(t, u(t), v(t), D_{0+}^{\frac{3}{2}} u(t), D_{0+}^{\frac{5}{2}} v(t)), \\ D_{0+}^{\frac{3}{2}}\left(\varphi_{\frac{4}{3}}(D_{0+}^{\frac{5}{2}} v(t))\right) = f_2(t, u(t), v(t), D_{0+}^{\frac{3}{2}} u(t), D_{0+}^{\frac{5}{2}} v(t)), \quad 0 < t < 1, \\ D_{0+}^{\frac{3}{2}} u(0) = u(0) = 0, \\ D_{0+}^{\frac{5}{2}} v(0) = v(0) = v'(0) = 0, \\ u(1) = \frac{1}{2}\int_0^{\eta_1} v(s)ds + \frac{2}{3}\int_{\xi_1}^1 v(s)ds, \quad 0 < \eta_1 < \xi_1 < 1, \\ v(1) = \frac{1}{8}\int_0^{\eta_2} u(s)ds + \frac{3}{16}\int_{\xi_2}^1 u(s)ds, \quad 0 < \eta_2 < \xi_2 < 1, \\ D_{0+}^{\frac{3}{2}} u(1) = 2D_{0+}^{\frac{3}{2}} u(\frac{1}{4}), \\ D_{0+}^{\frac{5}{2}} v(1) = \frac{3}{2} D_{0+}^{\frac{5}{2}} v(\frac{2}{3}). \end{cases} \tag{4.1}$$

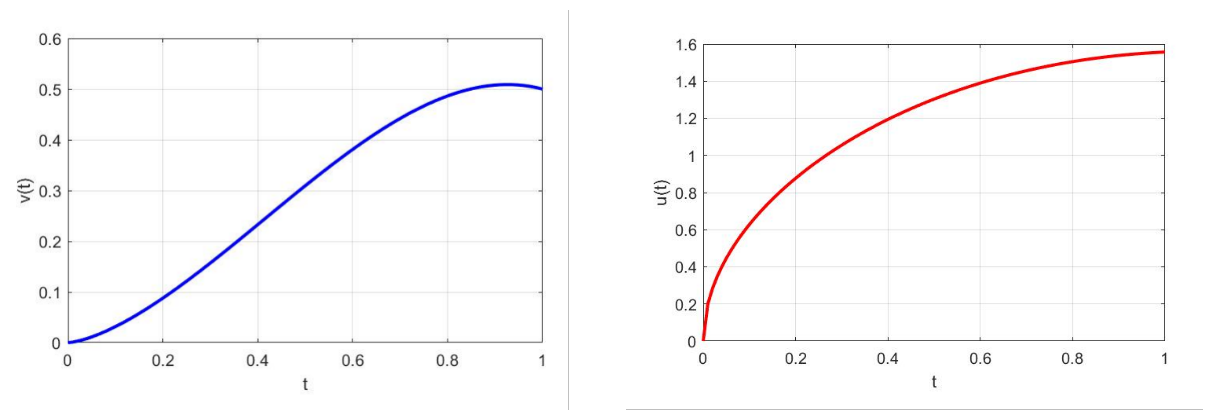

Figure 1. The approximate solution of the system.

Assume that

$$f_1(t, u, v, w, z) = \frac{\Gamma(\frac{1}{4})}{32\Gamma(\frac{3}{4})t^{\frac{3}{2}}}(te^{t(u+v)-3} - \frac{1}{2}w - \frac{16}{105\sqrt{\pi}}z),$$

$$f_2(t,u,v,w,z)=\frac{\Gamma(\frac{1}{3})}{72\Gamma(\frac{5}{6})}(\frac{105\sqrt{\pi}}{4})^{\frac{1}{3}}\frac{1}{t}(t^2e^{t(\frac{u}{2}+v)-2}-w-\frac{16}{105\sqrt{\pi}}z).$$

We can easily check that $(x_0,y_0)=(x_0(t),y_0(t))=(0,0)$ is a lower and $(u_0,v_0)=(u_0(t),v_0(t))$ is an upper solution of boundary value problem (4.1), where $u_0(t)=2t^{\frac{1}{2}}-\frac{\sqrt{\pi}}{4}t^2, v_0(t)=t^{\frac{3}{2}}-\frac{1}{2}t^{\frac{7}{2}}$. All conditions in Theorem 3.1 hold. Then boundary value problem (4.1) has the maximal lower solution (x^*,y^*) and the minimal upper solution (u^*,v^*), both (x^*,y^*) and (u^*,v^*) are solutions of boundary value problem (4.1). □

5. Conclusion

Nonlinear fractional differential system with P-Laplacian operators and integral boundary conditions are an important research area, which has shown high value in both theoretical and applications. With the continuous advancement of computational technology and the emergence of new methods, research in this field will undoubtedly expand to deeper levels and broader fields in the future. At the same, we also expect that these research results can provide more effective tools and methods for solving practical problems.

With the continuous development and improvement of the theory of fractional calculus, the research boundary value problems of nonlinear fractional differential systems will be more in-depth and extensive. This paper only makes a major study on the existence of system solutions, and in subsequent research, stability and persistence of solutions can be further discussed. In addition, we should also continue to study equations with practical value, and the solution of these problems will lay a more theoretical for the research of fractional differential equations, and strive to solve more complex practical problems.

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