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VARIATION INEQUALITIES FOR THE COMMUTATORS OF APPROXIMATE IDENTITIES WITH LIPSCHITZ FUNCTIONS*

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Abstract This paper is devoted to establishing the boundedness of the variation operators for commutators generated by approximate identities with Lipschitz functions in the weighted Lebesgue spaces and the endpoint spaces. As applications, we obtain the corresponding boundedness results for λ-jump operator, the number of up-crossing, heat semigroups, Poisson semigroups and maximal operator.

Keywords Variation operators, approximate identities, commutators, Lipschitz functions, boundedness.

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1. Introduction

Let $(X, \mathcal{F}, \mu)$ be an arbitrary σ finite measure space. For $1 < p < \infty$, let $\{T_r\}_{r>0}$ be a family of bounded operators on $L^p(X, \mathcal{F}, \mu)$ and $\lim_{r\to 0} T_r f$ exists in a certain sense. For the study of the convergence properties and convergence rate of the operator family $\{T_r\}$, we usually consider the square function $\left(\sum_{i=1}^{\infty} |T_{r_i} f - T_{r_{i+1}} f|^2\right)^{1/2}$ or more general ρ-variation operator $\mathcal{V}_\rho(\mathcal{T}_* f)$

$$\mathcal{V}_\rho(\mathcal{T}_* f)(x) := \sup_{r_i \searrow 0} \Big(\sum_{i=1}^{\infty} |T_{r_{i+1}} f(x) - T_{r_i} f(x)|^\rho \Big)^{1/\rho},$$

where the supremum is taken over all sequences $\{r_i\}$ that decreasing to 0.

The research on the variation operator of the operator family originates from the martingale theory and ergodic theory in probability theory. Studying the bounded properties and related inequalities of the variation operator on the function space can not only replace the traditional dense subset convergence method to study the pointwise convergence of the operator family, but also use these inequalities to measure the convergence speed of the operator family and the oscillation when the operator family approaches the limit. Since Lepingle [17] improved the classical Doob's maximal inequality and Bougain [4] proved the similar variational estimates for the Birkhoff ergodic mean of a binary system by using the variational inequality in the study of martingale theory, the improvement and generalization of these results have opened up a new direction for ergodic theory and harmonic analysis, and have attracted considerable research interest in recent years. We refer readers to [1, 2, 5, 6, 8, 10–13, 26, 27].

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On the other hand, the commutator of singular integral operators is an important research object in harmonic analysis. It can not only describe some function spaces but also play an extremely important role and significance in partial differential equations. Therefore, the study of variation operators associated with the commutators of singular integral operators in function spaces, especially in endpoint spaces, has attracted the attention of many scholars. In 2013, Betancor etc [3] established the L^p-boundedness of the variation operator of the commutator of the Riesz transform under the Euclidean background and the Schrödinger background. Later, Liu and Wu [18] studied the boundedness of the variation operator of the commutator family generated by the Calderón-Zygmund singular integral with the standard kernel and the BMO function, and established a boundedness criterion on the weighted L^p space. As an application, the weighted L^p-boundedness of the variation operators for the commutators of the Hilbert transform and the Hermitian Riesz transform has been obtained. Subsequently, significant progress has been made in the study of variational inequalities for the commutators of singular integral operators with rough kernels. Recently, Wen and Hou [24] established the variational inequalities for the commutator families generated by $b \in \mathrm{BMO}(\mathbb{R}^n)$ and approximate identities on $L^p(1 < p < \infty)$ space and endpoint spaces. For the latest research on commutators, we may refer to [22, 25]. Inspired by these results, in this paper, we devote to establish variational inequalities for commutator families generated by approximate identity operators and Lipschitz functions in weighted Lebesgue space, and to derive boundedness estimates on the corresponding endpoint spaces. Since the boundedness of commutators is closely tied to the smoothness of the function, and Lipschitz functions, while possessing certain smoothness, are not necessarily bounded, the study of variational inequalities for commutator families involving Lipschitz functions is of equal importance to those involving BMO functions. In order to express the results of this paper, we will review some essential definitions and representations.

In the context of operator theory, the commutator $[b, T]$ is defined for a locally integrable function b and an operator T (which may be linear or nonlinear) as

$$[b, T]f(x) = T((b(x) - b(\cdot))f)(x).$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space. We investigate the family of approximate identities $\Phi \star f$, which is defined as follows

$$\Phi \star f(x) := \{\phi_t * f(x)\}_{t>0}, \tag{1.1}$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Let $1 \le p \le \infty$ and $0 < \beta < 1$. We say the function $b \in \mathrm{Lip}_\beta^p(\mathbb{R}^n)$, if

$$||b||_{\mathrm{Lip}_\beta^p(\mathbb{R}^n)} = \sup_{B \ni x} \frac{1}{|B|^{\beta/n}} \Big(\frac{1}{|B|} \int_B |b(x) - (b)_B|^p dx \Big)^{1/p} < \infty, \tag{1.2}$$

where B is the ball in $\mathbb{R}^n$ and $(b)_B = |B|^{-1} \int_B b(x)dx$.

When $p = 1$, $\mathrm{Lip}_\beta^p(\mathbb{R}^n)$ is the homogeneous Lipschitz space $\mathrm{Lip}(\beta)$. García-Cuerva [9] proved that as long as $1 \le p \le \infty$, Lip_β^p space are uniform with respect to p, and the norm $\|\cdot\|_{\mathrm{Lip}_\beta^p(\mathbb{R}^n)}$ is equivalent for different p.

Based on the definition of $\mathrm{Lip}(\beta)$, one can easily verify that for $f \in \mathrm{Lip}(\beta)$, $0 < \beta \le 1$,

$$\frac{1}{2}\|f\|_{\mathrm{Lip}(\beta)} \le \sup_{B \ni x} \inf_{C_B} \frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |f(x) - C_B| dx \le \|f\|_{\mathrm{Lip}(\beta)}. \tag{1.3}$$

For $m \in \mathbb{N}$, $\vec{b} = (b_1, b_2, \cdots, b_m) \in \mathrm{Lip}(\vec{\beta})$, $i = 1, \cdots, m$, where $b_i \in \mathrm{Lip}(\beta_i)$, $\vec{\beta} = (\beta_1, \cdots, \beta_m)$ and $0 < \beta = \sum_{i=1}^m \beta_i < n$, the iterated commutator family generated by approximate identities with Lipschitz functions $(\Phi \star f)_{\vec{b}} := \{\phi_{t,\vec{b}} * f\}_{t>0}$, for $f \in \bigcup_{1\le p<\infty} L^p(\mathbb{R}^n)$, which is given by

$$\phi_{t,\vec{b}} * (f)(x) = [b_m, \cdots, [b_2, [b_1, \phi_t]]](f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m [b_j(x) - b_j(y)] \frac{1}{t^n} \phi(\frac{x-y}{t}) f(y) dy. \quad (1.4)$$

In this paper, we will establish the weighted (L^p, L^q)-type estimates of the ρ-variation operators for the iterated commutators $\phi_{t,\vec{b}}$. The main theorems can be formulated as follows.

Theorem 1.1. *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\vec{b} = (b_1, \cdots, b_m)$ and $b_i \in \mathrm{Lip}(\beta_i)$ with $0 < \beta_i \le 1$. Let $\rho > 2$, $\Phi = \{\phi_t\}_{t>0}$ and $\Phi_{\vec{b}} = \{\phi_{t,\vec{b}}\}_{t>0}$ be given by (1.1) and (1.4), respectively. If $0 < \beta = \sum_{i=1}^m \beta_i < n$ and $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$, then for any $1 < p < n/\beta$ with $1/q = 1/p - \beta/n$, and $\omega \in A_{(p,q)}$, $\mathcal{V}_\rho((\Phi \star f)_{\vec{b}})$ is bounded from $L^p(\mathbb{R}^n, \omega(x)^p dx)$ to $L^q(\mathbb{R}^n, \omega(x)^q dx)$.*

When $m = 1$, the 1st-order commutator generated by approximate identities with Lipschitz functions is defined as

$$(\Phi \star f)_b = [b, \phi_t](f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{1}{t^n} \phi(\frac{x-y}{t}) f(y) dy. \quad (1.5)$$

And for $n/\beta \le p \le \infty$, we obtain the following un-weighted results only for the variation operators associated with the 1st-order commutator.

Theorem 1.2. *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, $0 < \beta < 1$. Let $\rho > 2$, $\Phi = \{\phi_t\}_{t>0}$ and $\Phi_b = \{\phi_{t,b}\}_{t>0}$ be given by (1.1) and (1.5), respectively. If $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$, then for any $n/\beta < p < \infty$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{V}_\rho((\Phi \star f)_b)\|_{\mathrm{Lip}(\beta/n - 1/p)} \le C \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p}.$$

Theorem 1.3. *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, $0 < \beta < 1$. Let $\rho > 2$, $\Phi = \{\phi_t\}_{t>0}$ and $\Phi_b = \{\phi_{t,b}\}_{t>0}$ be given by (1.1) and (1.5), respectively. If $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$, then for $p = n/\beta$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{V}_\rho((\Phi \star f)_b)\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^{n/\beta}}.$$

Remark 1.1. We note that the arguments in proving Theorems 1.2-1.3 are not applicable to the cases of high order commutators $\Phi_{\vec{b}}$ $(m > 1)$. It remains uncertain whether the results for $\mathcal{V}_\rho(\Phi_{\vec{b}})$ also hold when $m > 1$, although this is a highly intriguing question.

When $f \in H^1(\mathbb{R}^n)$, combined with atomic decomposition, we can establish the following result.

Theorem 1.4. *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, and $0 < \beta < 1$. Then for $\rho > 2$ and $f \in H^1(\mathbb{R}^n)$, $\mathcal{V}_\rho((\Phi \star f)_b)$ is bounded from $H^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.*

This paper is organized as follows. In Section 2, we review some basic concepts and preliminary results. Section 3 is devoted to proving our main results. Finally, in Section 4, we will give some applications of the main results, such as the λ-jump operator, the number of up-crossing, heat semigroups, Poisson semigroups and maximal operators.

In this paper, the symbol C is employed to denote positive constants, which may take different values in different occurrences. We use $f \lesssim g$ to denote $f \leqslant Cg$. For any ball $B := B(x_0, r) \subset \mathbb{R}^n$, x_0 denotes its center, and r represents its radius. The notation χ_B represents the characteristic function of B. Given $s \in [1, \infty]$, we denote its conjugate index by s', where $1/s + 1/s' = 1$.

2. Preliminaries

2.1. Weights

Let ω be a non-negative locally integrable function on $\mathbb{R}^n$.

(i) We say that $\omega \in A_p$ for $1 < p < \infty$, if

$$[\omega]_{A_p} = \sup_B \left(\frac{1}{|B|}\int_B \omega(y)dy\right)\left(\frac{1}{|B|}\int_B \omega(y)^{1-p'}dy\right)^{p-1} < \infty,$$

where and below, the supremum runs over all balls in $\mathbb{R}^n$, $1/p' + 1/p = 1$.

(ii) A weight ω belongs to the class A_1, if

$$[\omega]_{A_1} = \sup_B \left(\frac{1}{|B|}\int_B \omega(y)dy\right)\|\omega^{-1}\|_{L^\infty(B)} < \infty.$$

(iii) A weight $\omega(x)$ is said to belong to the class $A_{(p,q)}$, $1 < p \le q < \infty$, if

$$[\omega]_{A_{(p,q)}} = \sup_B \left(\frac{1}{|B|}\int_B \omega(y)^q dy\right)^{1/q}\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\right)^{1/p'} < \infty.$$

Note that the A_p classes are nested and increase with p, namely $A_p \subset A_q$, and $A_\infty = \bigcup_{p\ge 1} A_p$.

The following properties of $A_{(p,q)}$ weights are presented in [5] and will be utilized in the following estimates.

Lemma 2.1 (see [7]). *Let $1 < p \le q < \infty$. If $\omega \in A_{(p,q)}$, then there exists $r \in (1, p)$ such that $w^r \in A_{(p/r,q/r)}$.*

2.2. Maximal functions

Here we review some classes of maximal functions, the Hardy-Littlewood maximal function is defined as

$$M(f)(x) := \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|dy,$$

and the sharp maximal function is defined by

$$M^\sharp(f)(x) := \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y) - (f)_B|dy \approx \sup_{B\ni x}\inf_c \frac{1}{|B|}\int_B |f(y) - c|dy. \tag{2.3}$$

The operator M is bounded on $L^p(\omega)$ if and only if $\omega \in A_p$ for $1 < p < \infty$, as shown by Muckenhoupt [20].

On the other hand, the fractional maximal operator M_β is defined by

$$M_\beta(f)(x) := \sup_{B\ni x} \frac{1}{|B|^{1-\frac{\beta}{n}}} \int_B |f(y)|dy,$$

and its variant $M_{\beta,s}$ is defined by

$$M_{\beta,s}(f)(x) := \sup_{B\ni x} \left(\frac{1}{|B|^{1-\frac{\beta s}{n}}} \int_B |f(y)|^s dy \right)^{1/s}, \qquad s > 0.$$

The following lemmas will play a pivotal role in the proof of our main theorems.

Lemma 2.2 (see [14]). *Let $1 < p < \infty$, $\omega \in A_\infty$. Then*

$$\|M(f)\|_{L^p(\omega)} \le \|M^\sharp(f)\|_{L^p(\omega)}, \tag{2.4}$$

for all f such that the left hand is finite.

Lemma 2.3 (see [21]). *Let $0 < \beta < n$, $1 < p < n/\beta$, and $1/q = 1/p - \beta/n$. If $\omega \in A_{(p,q)}$, then*

$$\|M_\beta(f)\|_{L^q(\omega^q)} \le \|f\|_{L^p(\omega^p)}. \tag{2.5}$$

The following lemma follows directly from Lemma 2.1 and Lemma 2.3.

Lemma 2.4 (see [28]). *Let $0 < \beta < n$, $1 < r < p < n/\beta$, and $1/q = 1/p - \beta/n$. If $\omega \in A_{(p,q)}$, then*

$$\|M_{\beta,r}(f)\|_{L^q(\omega^q)} \le \|f\|_{L^p(\omega^p)}. \tag{2.6}$$

2.3. Atomic decomposition

To prove Theorem 1.4, we introduce the atomic definition and the properties of the H^1 norm.

Definition 2.1. Let B be a ball, we say that a function $a(x)$ is an $(1,\infty)$-atom if it satisfies:

(i) supp $a \subset B$;

(ii) $\|a\|_{L^\infty} \le |B|^{-1}$;

(iii) $\int_B a(x)dx = 0$.

Lemma 2.5 (see [23]). *A function $f \in L^1(\mathbb{R}^n)$ belongs to $H^1(\mathbb{R}^n)$ if and only if $f = \sum_i \lambda_i a_i$ in H^1 norm or L^1 norm, where $a_i s$ are $(1,\infty)$-atoms, $\lambda_i \in \mathbb{C}$ with $\sum_i |\lambda_i| < \infty$. Furthermore,*

$$\|f\|_{H^1(\mathbb{R}^n)} \cong \inf\{\sum_i |\lambda_i|\},$$

where the infimum is taken over all the above atomic decomposition of f.

In this paper, we also need the following lemma in [19]. This lemma obtained some unweighted results concerning approximate identities.

Lemma 2.6 (see [19]). *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\rho > 2$. Then for any $1 < p < \infty$, we have*

(i) $\mathcal{V}_\rho(\Phi \star f)$ is bounded on $L^p(\mathbb{R}^n)$;

(ii) $\mathcal{V}_\rho(\Phi \star f)$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$;

(iii) $\mathcal{V}_\rho(\Phi \star f)$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

3. Proof of the main results

3.1. The weighted (L^p, L^q)-type estimates

This section is primarily concerned with proving Theorem 1.1. To begin with, we recall a weighted result on $\mathcal{V}_\rho(\Phi \star f)$, which will be applied in subsequent proofs.

Lemma 3.1 (see [15]). *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\rho > 2$, and $\Phi = \{\phi_t\}_{t>0}$ be given by (1.1). If $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n)$ for some $1 < p_0 < \infty$, then for any $1 < p < \infty$, $\omega \in A_p$,*

$$\|\mathcal{V}_\rho(\Phi \star f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}. \tag{3.1}$$

To prove Theorem 1.1, we first need to establish the following proposition.

Proposition 3.1. *Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\vec{\beta} = (\beta_1, \cdots, \beta_n)$ with $0 < \beta = \sum_{i=1}^m \beta_i < n$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_i \in \mathrm{Lip}(\beta_i)(i = 1, 2, \cdots, m)$, Φ and $\Phi_{\vec{b}}$ being as in Theorem 1.1. If $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$. Then for $\rho > 2$, we have*

$$\begin{aligned} M^\sharp(\mathcal{V}_\rho((\Phi \star f)_{\vec{b}}))(x) \lesssim{}& \|\vec{b}\|_{\mathrm{Lip}(\beta)}\{M_{\beta,s}(\mathcal{V}_\rho(\Phi \star f))(x) + M_{\beta,s}(f)(x)\} \\ &+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{\mathrm{Lip}(\beta_\sigma)} M_{\beta_\sigma,s}(\mathcal{V}_\rho((\Phi \star f)_{\vec{b}_{\sigma'}}))(x) \end{aligned}$$

hold for any $s > 1$.

Proof. Without loss of generality, we only prove the case $m = 2$. That is, we need to prove the following results

$$\begin{aligned} M^\sharp(\mathcal{V}_\rho((\Phi \star f)_{b_1,b_2}))(x) \lesssim{}& \|b_1\|_{\mathrm{Lip}(\beta_1)}\|b_2\|_{\mathrm{Lip}(\beta_2)}\{M_{\beta,s}(\mathcal{V}_\rho(\Phi \star f))(x) + M_{\beta,s}(f)(x)\} \\ &+ \|b_1\|_{\mathrm{Lip}(\beta_1)} M_{\beta_1,s}(\mathcal{V}_\rho((\Phi \star f)_{b_2}))(x) \\ &+ \|b_2\|_{\mathrm{Lip}(\beta_2)} M_{\beta_2,s}(\mathcal{V}_\rho((\Phi \star f)_{b_1}))(x). \end{aligned}$$

For $x \in \mathbb{R}^n$, let $B := B(x_0, r)$, we write $f = f_1 + f_2$, where $f_1 = f\chi_{4B}$. Then for $y \in B$, one can see that

$$\begin{aligned} \mathcal{V}_\rho((\Phi \star f)_{b_1,b_2})(y) \leq{}& |b_1(y) - (b_1)_{4B}|\,|b_2(y) - (b_2)_{4B}|\,\mathcal{V}_\rho(\Phi \star f)(y) \\ &+ |b_1(y) - (b_1)_{4B}|\,\mathcal{V}_\rho((\Phi \star f)_{b_2})(y) \\ &+ |b_2(y) - (b_2)_{4B}|\,\mathcal{V}_\rho((\Phi \star f)_{b_1})(y) \\ &+ \mathcal{V}_\rho(\Phi \star ((b_1(y) - (b_1)_{4B})(b_2(y) - (b_2)_{4B})f_1))(y) \\ &+ \mathcal{V}_\rho(\Phi \star ((b_1(y) - (b_1)_{4B})(b_2(y) - (b_2)_{4B})f_2))(y). \end{aligned}$$

From the definition of $M^\sharp$, we only need to prove that

$$\begin{aligned} &\frac{1}{|B|}\int_B |\mathcal{V}_\rho((\Phi \star f)_{b_1,b_2})(y) - \mathcal{V}_\rho(\Phi \star ((b_1(y) - (b_1)_{4B})(b_2(y) - (b_2)_{4B})f_2))(x_0)|\,dy \\ &\lesssim \|b_1\|_{\mathrm{Lip}(\beta_1)}\|b_2\|_{\mathrm{Lip}(\beta_2)}\{M_{\beta,s}(\mathcal{V}_\rho(\Phi \star f))(x) + M_{\beta,s}(f)(x)\} \\ &\quad + \|b_1\|_{\mathrm{Lip}(\beta_1)} M_{\beta_1,s}(\mathcal{V}_\rho((\Phi \star f)_{b_2}))(x) \\ &\quad + \|b_2\|_{\mathrm{Lip}(\beta_2)} M_{\beta_2,s}(\mathcal{V}_\rho((\Phi \star f)_{b_1}))(x). \end{aligned}$$

For $x \in \mathbb{R}^n$, it is not difficult to see that

$$\begin{aligned}
&\frac{1}{|B|}\int_B |\mathcal{V}_\rho((\Phi\star f)_{b_1,b_2})(y) - \mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(x_0)|\,dy\\
\le&\frac{1}{|B|}\int_B |b_1(y)-(b_1)_{4B}|\,|b_2(y)-(b_2)_{4B}|\,\mathcal{V}_\rho(\Phi\star f)(y)dy\\
&+\frac{1}{|B|}\int_B |b_1(y)-(b_1)_{4B}|\,\mathcal{V}_\rho((\Phi\star f)_{b_2})(y)dy\\
&+\frac{1}{|B|}\int_B |b_2(y)-(b_2)_{4B}|\,\mathcal{V}_\rho((\Phi\star f)_{b_1})(y)dy\\
&+\frac{1}{|B|}\int_B \mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_1))(y)dy\\
&+\frac{1}{|B|}\int_B \big|\mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(y)\\
&\quad-\mathcal{V}\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(x_0)\big|dy\\
=:&\sum_{i=1}^{5} I_i.
\end{aligned}$$

For I_1, using Hölder inequality, we have

$$\begin{aligned}
I_1 &= \frac{1}{|B|}\int_B |b_1(y)-(b_1)_{4B}|\,|b_2(y)-(b_2)_{4B}|\,\mathcal{V}_\rho(\Phi\star f)(y)dy\\
&\le\Big(\frac{1}{|B|}\int_B |b_1(y)-(b_1)_{4B}|^{2s'}\Big)^{\frac{1}{2s'}}\Big(\frac{1}{|B|}\int_B |b_2(y)-(b_2)_{4B}|^{2s'}\Big)^{\frac{1}{2s'}}\\
&\quad\times\Big(\frac{1}{|B|}\int_B |\mathcal{V}_\rho(\Phi\star f)(y)|^s\,dy\Big)^{\frac{1}{s}}\\
&\lesssim \|b_1\|_{\mathrm{Lip}(\beta_1)}\|b_2\|_{\mathrm{Lip}(\beta_2)}M_{\beta,s}(\mathcal{V}_\rho(\Phi\star f))(x).
\end{aligned}$$

As for I_2, we have

$$\begin{aligned}
I_2 &\le \Big(\frac{1}{|B|}\int_B |b_1(y)-(b_1)_{4B}|^{s'}\,dy\Big)^{\frac{1}{s'}}\Big(\frac{1}{|B|}\int_B |\mathcal{V}_\rho((\Phi\star f)_{b_2})(y)|^s\,dy\Big)^{\frac{1}{s}}\\
&\lesssim \|b_1\|_{\mathrm{Lip}(\beta_1)}M_{\beta_1,s}(\mathcal{V}_\rho((\Phi\star f)_{b_2}))(x).
\end{aligned}$$

By symmetry, we can deduce that

$$I_3 \lesssim \|b_2\|_{\mathrm{Lip}(\beta_2)}M_{\beta_2,s}(\mathcal{V}_\rho((\Phi\star f)_{b_1}))(x).$$

In order to estimate I_4, we choose $1<\mu,q<\infty$ satisfying $\mu q = s$. Form Lemma 2.6, we have

$$\begin{aligned}
I_4 &= \frac{1}{|B|}\int_B \mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_1))(y)dy\\
&\lesssim\Big(\frac{1}{|B|}\int_B \Big|\mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_1))(y)\Big|^q dy\Big)^{\frac{1}{q}}\\
&\lesssim\Big(\frac{1}{|B|}\int_{4B} |b_1(y)-(b_1)_{4B}|^q\,|b_2(y)-(b_2)_{4B}|^q\,|f(y)|^q\,dy\Big)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \Big(\frac{1}{|B|}\int_{4B}|b_1(y)-(b_1)_{4B}|^{2q\mu'}\,dy\Big)^{\frac{1}{2q\mu'}}\Big(\frac{1}{|B|}\int_{4B}|b_2(y)-(b_2)_{4B}|^{2q\mu'}\,dy\Big)^{\frac{1}{2q\mu'}}$$
$$\times\Big(\frac{1}{|B|}\int_{4B}|f(y)|^{q\mu}\,dy\Big)^{\frac{1}{q\mu}}$$
$$\lesssim \|b_1\|_{\mathrm{Lip}(\beta_1)}\|b_2\|_{\mathrm{Lip}(\beta_2)}M_{\beta,s}(f)(x).$$

Finally, we consider I_5. Note that

$$\begin{aligned}
&\Big|\mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(y)\\
&\quad-\mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(x_0)\Big|\\
\leq&\Big\|\{\phi_t*((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2)(y)\\
&\quad-\phi_t*((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2)(x_0)\}_{t>0}\Big\|_{\mathcal{V}_\rho}\\
=&\sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n\setminus 4B}\Big\{[\phi_{t_k}(y-z)-\phi_{t_{k+1}}(y-z)]-[\phi_{t_k}(x_0-z)-\phi_{t_{k+1}}(x_0-z)]\Big\}\\
&\quad\times(b_1(z)-(b_1)_{4B})(b_2(z)-(b_2)_{4B})f(z)dz\Big|^\rho\Big)^{1/\rho}\\
\leq&\int_{\mathbb{R}^n\setminus 4B}|f(z)|\,|b_1(z)-(b_1)_{4B}|\,|\,|b_2(z)-(b_2)_{4B}|\\
&\quad\times\big\|\{\phi_t(y-z)-\phi_t(x_0-z)\}_{t>0}\big\|_{\mathcal{V}_\rho}dz.
\end{aligned}$$

By using Schwartz space properties and the mean value theorem, we obtain that

$$\begin{aligned}
&\big\|\{\phi_t(y-z)-\phi_t(x_0-z)\}_{t>0}\big\|_{\mathcal{V}_\rho}\\
\leq&\sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{t_{k+1}}^{t_k}\frac{\partial}{\partial t}(\phi_t(y-z)-\phi_t(x_0-z))dt\Big|\Big)\\
\leq&\int_0^\infty\Big|\frac{\partial}{\partial t}(\phi_t(y-z)-\phi_t(x_0-z))\Big|\,dt\\
\lesssim&|y-x_0|\int_0^\infty\frac{1}{t^{n+2}}\Big(1+\frac{|z-x_0|}{t}\Big)^{-(n+2)}dt\\
=&\frac{|y-x_0|}{|z-x_0|^{n+1}}\int_0^\infty\frac{t^n}{(t+1)^{n+2}}dt\\
\lesssim&\frac{|y-x_0|}{|z-x_0|^{n+1}},
\end{aligned}$$

where $z\in\mathbb{R}^n\setminus 4B$. Thus,

$$\begin{aligned}
&\Big|\mathcal{V}_\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(y)\\
&\quad-\mathcal{V}\rho(\Phi\star((b_1(y)-(b_1)_{4B})(b_2(y)-(b_2)_{4B})f_2))(x_0)\Big|\\
\leq&\int_{\mathbb{R}^n\setminus 4B}|f(z)|\,|b_1(z)-(b_1)_{4B}|\,|b_2(z)-(b_2)_{4B}|\,\frac{|y-x_0|}{|z-x_0|^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=2}^{\infty} \int_{2^{j+1}B\setminus 2^jB} \frac{|b_1(z)-(b_1)_{4B}|\,|b_2(z)-(b_2)_{4B}|\,|f(z)|}{(2^jr)^{n+1}} \cdot r dz \\
&\lesssim \sum_{j=2}^{\infty} \frac{1}{2^j} \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b_1(z)-(b_1)_{4B}|^{s'} |b_2(z)-(b_2)_{4B}|^{s'} dz \Big)^{1/s'} \\
&\quad \times \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(z)|^s dz \Big)^{1/s} \\
&\lesssim \sum_{j=2}^{\infty} \frac{1}{2^j} \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b_1(z)-(b_1)_{2^{j+1}B}+(b_1)_{2^{j+1}B}-(b_1)_{4B}|^{2s'} dz \Big)^{1/2s'} \\
&\quad \times \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b_2(z)-(b_2)_{2^{j+1}B}+(b_2)_{2^{j+1}B}-(b_2)_{4B}|^{2s'} dz \Big)^{1/2s'} \\
&\quad \times M_{\beta,s}(f)(x) \left|2^{j+1}B\right|^{-\frac{\beta}{n}} \\
&\leq \sum_{j=2}^{\infty} \frac{1}{2^j} (\|b_1\|_{Lip(\beta_1)} \left|2^{j+1}B\right|^{\beta_1/n} + \|b_1\|_{\mathrm{Lip}(\beta_1)} (\frac{j+1}{2}) \left|2^{j+1}B\right|^{\beta_1/n}) \\
&\quad \times (\|b_2\|_{\mathrm{Lip}(\beta_2)} \left|2^{j+1}B\right|^{\beta_2/n} + \|b_2\|_{Lip(\beta_2)} (\frac{j+1}{2}) \left|2^{j+1}B\right|^{\beta_2/n}) \\
&\quad \times M_{\beta,s}(f)(x) \left|2^{j+1}B\right|^{-\frac{\beta}{n}} \\
&= \Big(\sum_{j=2}^{\infty} \frac{j+3}{2^{j+1}} \Big) \|b_1\|_{\mathrm{Lip}(\beta_1)} \|b_2\|_{\mathrm{Lip}(\beta_2)} M_{\beta,s}(f)(x) \\
&\lesssim \|b_1\|_{\mathrm{Lip}(\beta_1)} \|b_2\|_{\mathrm{Lip}(\beta_2)} M_{\beta,s}(f)(x).
\end{aligned}
$$

It implies that

$$
I_5 \lesssim |b_1\|_{\mathrm{Lip}(\beta_1)} \|b_2\|_{Lip(\beta_2)} M_{\beta,s}(f)(x).
$$

This completes the proof of Proposition 3.1. □

Next, we prove Theorem 1.1, the proof is standard; see, for example [18, 24].

Proof. We first verify that $\left\|M(\mathcal{V}_\rho((\Phi\star f)_{\vec{b}}))\right\|_{L^q(\omega^q)}$ is finite. By the weighted L^q boundedness of M, it suffices to verify that $\left\|\mathcal{V}_\rho((\Phi\star f)_{\vec{b}})\right\|_{L^q(\omega^q)}$ is finite. For simplicity, we will only check that $\|\mathcal{V}_\rho((\Phi\star f)_b)\|_{L^q(\omega^q)}$ is finite, as the other cases are analogous. Assume that b and ω are both bounded functions, then by Lemma 3.1, we have

$$
\begin{aligned}
\|\mathcal{V}_\rho((\Phi\star f)_b)\|_{L^q(\omega^q)} &\leq \|b\|_\infty \|\omega\|_\infty \|\mathcal{V}_\rho(\Phi\star f)\|_{L^q} + \|\omega\|_\infty \|\mathcal{V}_\rho(\Phi\star(bf))\|_{L^q} \\
&\lesssim \|b\|_\infty \|\omega\|_\infty \|f\|_{L^q} + \|\omega\|_\infty \|bf\|_{L^q} \\
&\lesssim \|b\|_\infty \|\omega\|_\infty \|f\|_{L^q} \\
&< \infty,
\end{aligned}
$$

where $f\in C_c^\infty(\mathbb{R}^n)$.

Then, we perform induction on m. When $m=1$, through the combination of Lemmas 2.2, 2.4, 3.1, and Proposition 3.1, we have

$$
\|\mathcal{V}_\rho((\Phi\star f)_b)\|_{L^q(\omega^q)} \lesssim \|M(\mathcal{V}_\rho((\Phi\star f)_b))\|_{L^q(\omega^q)}
$$

$$
\begin{aligned}
&\lesssim \left\| M^{\sharp}(\mathcal{V}_\rho((\Phi\star f)_b)) \right\|_{L^q(\omega^q)} \\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)} \big(\|M_{\beta,s}(\mathcal{V}_\rho(\Phi\star f))\|_{L^q(\omega^q)} + \|M_{\beta,s}(f)\|_{L^q(\omega^q)} \big) \\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)} \big(\|\mathcal{V}_\rho(\Phi\star f)\|_{L^p(\omega^p)} + \|f\|_{L^p(\omega^p)} \big) \\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p(\omega^p)} .
\end{aligned}
$$

Now we turn to consider the case $m \geqslant 2$. Assume that the theorem holds for $m-1$, and we shall prove it for m. The same reasoning as employed above, together with the induction hypothesis, leads to the conclusion that

$$
\begin{aligned}
\left\|\mathcal{V}_\rho((\Phi\star f)_{\vec{b}})\right\|_{L^q(\omega^q)} &\lesssim \left\|M(\mathcal{V}_\rho((\Phi\star f)_{\vec{b}}))\right\|_{L^q(\omega^q)} \\
&\lesssim \left\|M^{\sharp}(\mathcal{V}_\rho((\Phi\star f)_{\vec{b}}))\right\|_{L^q(\omega^q)} \\
&\lesssim \|\vec{b}\|_{\mathrm{Lip}(\beta)} \big\{ \|M_{\beta,s}(\mathcal{V}_\rho(\Phi\star f))\|_{L^q(\omega^q)} + \|M_{\beta,s}(f)\|_{L^q(\omega^q)} \big\} \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma\in C_j^m} \|\vec{b}_\sigma\|_{\mathrm{Lip}(\beta_\sigma)} \left\| M_{\beta_{\sigma'},s}(\mathcal{V}_\rho((\Phi\star f)_{\vec{b}_{\sigma'}})) \right\|_{L^q(\omega^q)} \\
&\lesssim \|\vec{b}\|_{\mathrm{Lip}(\beta)} \big\{ \|\mathcal{V}_\rho(\Phi\star f)\|_{L^p(\omega^p)} + \|f)\|_{L^p(\omega^p)} \big\} \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma\in C_j^m} \|\vec{b}_\sigma\|_{\mathrm{Lip}(\beta_\sigma)} \left\| \mathcal{V}_\rho((\Phi\star f)_{\vec{b}_{\sigma'}}) \right\|_{L^{p_\sigma}(\omega^{p_\sigma})} \\
&\lesssim \|\vec{b}\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p(\omega^p)} + \sum_{i=1}^{m-1} \sum_{\sigma\in C_j^m} \|\vec{b}_\sigma\|_{\mathrm{Lip}(\beta_\sigma)} \|\vec{b}_{\sigma'}\|_{Lip(\beta_{\sigma'})} \\
&\quad \times \|f\|_{L^p(\omega^p)} \\
&\lesssim \|\vec{b}\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p(\omega^p)} ,
\end{aligned}
$$

where $\beta_{\sigma'} = \beta - \beta_\sigma$, $1/q = 1/p - \beta/n$.

For the general case of Lipschitz functions b, the application of the Lebesgue dominated convergence theorem, combined with reasoning analogous to the derivation of Theorem 1.1 in [24], we can establish Theorem 1.1. The details are omitted. □

3.2. The $(L^p, \dot{\Lambda}_{(\beta/n-1/p)})$ -type estimates

In this section, we will prove Theorems 1.2 and 1.3. The proofs of these theorems are based on the un-weighted results of Theorem 1.1.

Proof. For $x \in \mathbb{R}^n$, let B be any ball containing x, define $f_1(y) = f(y)\chi_{4B}$ and $f_2(y) = f(y) - f_1(y)$. Let

$$C_B = \mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_2))(x_0).$$

According to (1.3), we need only verify that

$$\frac{1}{|B|}\int_B |\mathcal{V}_\rho((\Phi\star f)_b)(y) - C_B|\,dy \lesssim \|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p}|B|^{\beta/n-1/p}.$$

We write

$$\frac{1}{|B|}\int_B |\mathcal{V}_\rho((\Phi\star f)_b)(y) - C_B|\,dy$$

$$
\begin{aligned}
&\leq \frac{1}{|B|}\int_B |b(y)-(b)_{4B}|\,\mathcal{V}_\rho(\Phi\star f)(y)dy\\
&\quad+\frac{1}{|B|}\int_B \mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_1))(y)dy\\
&\quad+\frac{1}{|B|}\int_B \left|\mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_2))(y)-\mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_2))(x_0)\right|dy\\
&=: I+II+III.
\end{aligned}
$$

First, for the term I, by Hölder inequality, we have

$$
\begin{aligned}
I &\leq \Big(\frac{1}{|B|}\int_B |b(y)-(b)_{4B}|^{p'}\,dy\Big)^{1/p'}\Big(\frac{1}{|B|}\int_B \mathcal{V}_\rho(\Phi\star f)(y)^p dy\Big)^{1/p}\\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)}|B|^{\beta/n}\|f\|_{L^p}|B|^{-1/p}\\
&= \|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p}|B|^{\beta/n-1/p}.
\end{aligned}
$$

Next, we choose $1<p_1<p<n/\beta$ satisfy $1/q_1=1/p_1-\beta/n$, and combine with Theorem 1.1, we can get

$$
\begin{aligned}
II &\leq \frac{1}{|B|}\Big(\int_B |\mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_1))(y)|^{q_1}\,dy\Big)^{1/q_1}|B|^{1-1/q_1}\\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)}\frac{1}{|B|}\left(\int_{\mathbb{R}^n}|f_1(y)|^{p_1}dy\right)^{1/p_1}|B|^{1-1/q_1}\\
&= \|b\|_{\mathrm{Lip}(\beta)}\frac{1}{|B|}\left(\int_{4B}|f(y)|^{p_1}dy\right)^{1/p_1}|B|^{1-1/q_1}\\
&\leq \|b\|_{\mathrm{Lip}(\beta)}\frac{1}{|B|}\left(\int_{\mathbb{R}^\ltimes}|f(y)|^{p}dy\right)^{1/p}|B|^{1-1/q_1}|B|^{1/p_1-1/p}\\
&\leq \|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p}|B|^{\beta/n-1/p}.
\end{aligned}
$$

Now, we consider III. Note that

$$
\begin{aligned}
&\Big|\mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_2))(y)-\mathcal{V}\rho(\Phi\star((b-(b)_{4B})f_2))(x_0)\Big|\\
\leq&\Big\|\Big\{\phi_t*((b-(b)_{4B})f_2)(y)-\phi_t*((b-(b)_{4B})f_2)(x_0)\Big\}_{t>0}\Big\|_{\mathcal{V}_\rho}\\
=&\sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n\setminus 4B}\Big\{[\phi_{t_k}(y-z)-\phi_{t_{k+1}}(y-z)]-[\phi_{t_k}(x_0-z)-\phi_{t_{k+1}}(x_0-z)]\Big\}\\
&\times(b(z)-(b)_{4B})f(z)dz\Big|^\rho\Big)^{1/\rho}\\
\leq&\int_{\mathbb{R}^n\setminus 4B}|f(z)|\,|b(z)-(b)_{4B}|\,\big\|\{\phi_t(y-z)-\phi_t(x_0-z)\}_{t>0}\big\|_{\mathcal{V}_\rho}dz.
\end{aligned}
$$

For $z\in\mathbb{R}^n\setminus 4B$, by Theorem 1.1, we have

$$
\begin{aligned}
&\Big|\mathcal{V}_\rho(\Phi\star((b-(b)_{4B})f_2))(y)-\mathcal{V}\rho(\Phi\star((b-(b)_{4B})f_2))(x_0)\Big|\\
\lesssim&\int_{\mathbb{R}^n\setminus 4B}|f(z)|\,|b(z)-(b)_{4B}|\,\frac{|y-x_0|}{|z-x_0|^{n+1}}dz
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=2}^{\infty} r \int_{2^{j+1}B\setminus 2^jB} \frac{|b(z)-(b)_{4B}|\,|f(z)|}{(2^jr)^{n+1}} dz \\
&\lesssim \sum_{j=2}^{\infty} \frac{1}{2^j} \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(z)-(b)_{4B}|^{p'} dz \Big)^{1/p'} \\
&\quad \times \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(z)|^p dz \right)^{1/p} \\
&\lesssim \sum_{j=2}^{\infty} \frac{1}{2^j} \Big(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(z)-(b)_{2^{j+1}B}+(b)_{2^{j+1}B}-(b)_{4B}|^{p'} dz \Big)^{1/p'} \\
&\quad \times \left(\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f(z)|^p dz \right)^{1/p} \\
&\lesssim \sum_{j=2}^{\infty} \frac{j+3}{2^{j+1}} \|b\|_{\mathrm{Lip}(\beta)} |2^{j+1}B|^{\beta/n} \|f\|_{L^p} |2^{j+1}B|^{-1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p} |B|^{\beta/n-1/p}.
\end{aligned}
$$

This completes the proof of Theorem 1.2. □

Theorem 1.3 can be viewed as the endpoint case where $p = n/\beta$ in Theorem 1.2, and its proof is similar to that of Theorem 1.2. Hence, we omit the details.

3.3. Estimates on $H^1(\mathbb{R}^n)$ space

Proof. Let $f \in H^1(\mathbb{R}^n)$, then by atomic decomposition and Lemma 2.5, we only need to prove it for f being a finite sum $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j| \leq 2\|f\|_{H^1(\mathbb{R}^n)}$, where a_j ia an $(1,\infty)$ atom. Indeed, assume that $\mathcal{V}_\rho((\Phi \star f)_b)$ is bounded from $H^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ for such f , then for the general f, one can select a sequence $\{f_k\}_k$ with f_k being a finite sum as above such that f_k converges to f in H^1 norm or almost everywhere when $k \to \infty$. Consequently, by a limit argument, Theorem 1.4 follows from the L^2-boundedness of $\mathcal{V}_\rho((\Phi \star f)_b)$.

In the subsequent discussion, we assume that $f = \sum_j \lambda_j a_j$ is a finite sum satisfying $\sum_j |\lambda_j| \leq 2\|f\|_{H^1(\mathbb{R}^n)}$. Note that

$$
\begin{aligned}
\mathcal{V}_\rho((\Phi \star f)_b)(y) \leq& \sum_j |\lambda_j| \mathcal{V}_\rho((\Phi \star a_j)_{b_1,b_2})(y)\chi_{4B_j}(y) \\
&+ \mathcal{V}_\rho\Big(\Phi \star \Big(\sum_j \lambda_j a_j (b(y)-(b)_{B_j}) \Big)\Big)(y)\chi_{(4B_j)^c}(y) \\
&+ \sum_j |\lambda_j| \left| b(y)-(b)_{B_j} \right| \mathcal{V}_\rho(\Phi \star a_j)(y)\chi_{(4B_j)^c}(y) \\
=:& \sum_{i=1}^{3} A_i.
\end{aligned}
$$

For A_1, using Theorem 1.1 and Hölder inequality, we have

$$
\int_{\mathbb{R}^n} \mathcal{V}_\rho((\Phi \star a_j)_b)(y)\chi_{4B_j}(y)dy
$$

$$
\begin{aligned}
&\lesssim\Big(\int_{4B_j}\mathcal{V}_\rho((\Phi\star a_j)_b)(y)^2dy\Big)^{1/2}|B_j|^{1/2}\\
&\lesssim\|b\|_{\mathrm{Lip}(\beta)}|B_j|^{1/2}\Big(\int_{B_j}|a_j(y)|^2dy\Big)^{1/2}\\
&\lesssim\|b\|_{\mathrm{Lip}(\beta)}|B_j|^{1/2}|B_j|^{-1/2}\\
&=\|b\|_{\mathrm{Lip}(\beta)}.
\end{aligned}
$$

Applying the Chebyshev inequality and noting that $\sum_j|\lambda_j|\leq 2\|f\|_{H^1(\mathbb{R}^n)}$, we obtain

$$
\begin{aligned}
|\{y\in\mathbb{R}^n: A_1>\alpha/3\}|&\leq\frac{3}{\alpha}\sum_j|\lambda_j|\int_{\mathbb{R}^n}\mathcal{V}_\rho((\Phi\star a_j)_b)(y)\chi_{4B_j}(y)dy\\
&\lesssim\frac{1}{\alpha}\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{H^1(\mathbb{R}^n)}.
\end{aligned}
$$

Next, we consider A_2, by Lemma 2.6, and

$$
\Big(\int_{B_j}\big|(b(y)-(b)_{B_j})\big|^2dy\Big)^{1/2}\lesssim|b\|_{\mathrm{Lip}(\beta)},
$$

we deduce that

$$
\begin{aligned}
|\{y\in\mathbb{R}^n: A_2>\alpha/3\}|&\leq\frac{1}{\alpha}\Big\|\sum_j\lambda_j(b(y)-(b)_{B_j})a_j\Big\|_{L^1(\mathbb{R}^n)}\\
&\leq\frac{1}{\alpha}\sum_j|\lambda_j|\Big\|(b(y)-(b)_{B_j})a_j\Big\|_{L^1(\mathbb{R}^n)}\\
&\lesssim\frac{1}{\alpha}\sum_j|\lambda_j|\Big(\int_{B_j}\big|(b(y)-(b)_{B_j})\big|^2dy\Big)^{1/2}\\
&\quad\times\Big(\int_{B_j}|a_j(y)|^2dy\Big)^{1/2}\\
&\lesssim\frac{1}{\alpha}\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{H^1(\mathbb{R}^n)}.
\end{aligned}
$$

Now, we deal with A_3, We apply the mean value theorem and Minkowski's inequality, we conclude that

$$
\begin{aligned}
&\mathcal{V}_\rho(\Phi\star a_j)(y)\\
=&\sup_{t_k\downarrow 0}\Big(\sum_k\Big|\int_{\mathbb{R}^n}\big\{[\phi_{t_k}(y-x)-\phi_{t_{k+1}}(y-x)]\\
&-[\phi_{t_k}(y-y_j)-\phi_{t_{k+1}}(y-y_j)]\big\}a_j(x)dx\Big|^\rho\Big)^{1/\rho}\\
\leq&\int_{B_j}|a_j(x)|\|\{\phi_t(y-x)-\phi_t(y-y_j)\}_{t>0}\|_{\mathcal{V}_\rho}dx\\
\lesssim&\int_{B_j}|a_j(x)|\frac{|x-y_j|}{|y-y_j|^{n+1}}dx.
\end{aligned}
$$

Observe that

$$\int_{B_j} |a_j(x)|dx \leq 1.$$

And we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|b(y) - (b)_{B_j}\right| \mathcal{V}_\rho(\Phi \star a_j)(y)\chi_{(4B_j)^c}(y)dy \\
\leq & \sum_{i=2}^{\infty} \int_{2^{i+1}B_j \setminus 2^i B_j} \left|b(y) - (b)_{B_j}\right| \int_{B_j} |a_j(x)| \frac{|x - y_j|}{|y - y_j|^{n+1}} dxdy \\
\lesssim & \sum_{i=2}^{\infty} \frac{1}{2^i} \Big(\frac{1}{|2^{i+1}B_j|} \int_{2^{i+1}B_j} \left|b(y) - (b)_{B_j}\right| dy \Big) \\
\lesssim & \sum_{i=2}^{\infty} \frac{1}{2^i} \Big(\frac{1}{|2^{i+1}B_j|} \int_{2^{i+1}B_j} \left|b(y) - (b)_{2^{i+1}B_j} + (b)_{2^{i+1}B_j} - (b)_{B_j}\right| dy \Big) \\
\lesssim & \|b\|_{\mathrm{Lip}(\beta)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\{y \in \mathbb{R}^n : A_3 > \alpha/3\}| &\leq \frac{3}{\alpha} \int_{\mathbb{R}^n} \sum_j |\lambda_j| \left|b(y) - (b)_{B_j}\right| \mathcal{V}_\rho(\Phi \star a_j)(y)\chi_{(4B_j)^c}(y)dy \\
&\lesssim \frac{1}{\alpha} \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{H^1(\mathbb{R}^n)}.
\end{aligned}$$

Consequently, we arrive at Theorem 1.4. □

4. Applications

This section presents applications of the main theorem.

4.1. λ-jump operators and the number of up-crossing

We present an application involving λ-jump operators and corresponding the number of up-crossing related to the operators sequence $\{F_\varepsilon\}$, which provides concrete quantitative information about the convergence properties of $\{F_\varepsilon\}$.

Definition 4.1. The λ-jump operator associated with a sequence $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon>0}$ applied to a function f at a point x is denoted by $N_\lambda(\mathcal{F})(x)$ and defined by

$$\begin{aligned}
N_\lambda(\mathcal{F})(x) := \sup\{& n \in \mathbb{N} : \exists s_1 < t_1 \leq s_2 < t_2 < \cdots \leq s_n < t_n \\
& s.t. \quad |F_{s_i}f(x) - F_{t_i}f(x)| > \lambda, i = 1, 2, \cdots, n\}.
\end{aligned} \tag{4.1}$$

Proposition 4.1 (see [16]). *The behavior of the λ-jump operators is governed by the ρ-variation operator. More precisely, we have*

$$\lambda(N_\lambda(\mathcal{F})(x))^{1/\rho} \leq \mathcal{V}_\rho(\mathcal{F}f)(x).$$

Through the application of Theorems 1.1, 1.2, 1.3 and Proposition 4.1, we establish the following results.

Theorem 4.1. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_i \in \mathrm{Lip}(\beta_i)$ $(i = 1, \cdots, m)$ and $0 < \beta = \sum_{i=1}^m \beta_i < n$, $\rho > 2$. Let $\Phi = \{\phi_\varepsilon\}_{\varepsilon>0}$ and $\Phi_{\vec{b}} = \{\phi_{\varepsilon,\vec{b}}\}_{\varepsilon>0}$ be given by (1.1) and (1.4), respectively. If $\mathcal{V}_\rho(\Phi \star f)$ is bounded in $L^s(\mathbb{R}^n, dx)$ for some $1 < s < \infty$, then for $1 < p < n/\beta$ with $1/q = 1/p - \beta/n$ and $\omega \in A_{(p,q)}$, we obtain*

$$\left\| N_\lambda((\Phi \star f)_{\vec{b}})^{1/\rho} \right\|_{L^q(\omega^q)} \leq \frac{C(p,q,\rho)}{\lambda} \|\vec{b}\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p(\omega^p)}.$$

Theorem 4.2. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, $0 < \beta < 1$, and $\rho > 2$. Let $\Phi = \{\phi_\varepsilon\}_{\varepsilon>0}$ and $\Phi_b = \{\phi_{\varepsilon,b}\}_{\varepsilon>0}$ be given by (1.1) and (1.5), respectively. If $\mathcal{V}_\rho(\Phi\star f)$ is bounded in $L^s(\mathbb{R}^n, dx)$ for some $1 < s < \infty$, then we obtain*

$$\left\| N_\lambda((\Phi \star f)_b)^{1/\rho} \right\|_{\mathrm{Lip}(\beta/n-1/p)} \leq \frac{C(p,\rho)}{\lambda} \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p},$$

where $n/\beta < p < \infty$, and

$$\left\| N_\lambda((\Phi \star f)_b)^{1/\rho} \right\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq \frac{C(\rho)}{\lambda} \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^{n/\beta}}.$$

Also, for fixed $0 < \alpha < \gamma$, we consider the number of up-crossing associated with a sequence $\mathcal{F} = \{F_\varepsilon\}_{\varepsilon>0}$ applied to a function f at a point x, which is defined by

$$\begin{aligned} U(\mathcal{F}, f, \alpha, \gamma, x) = \sup \Big\{ & n \in \mathbb{N} : \exists s_1 < t_1 < s_2, t_2 < \cdots < s_n < t_n \\ & s.t. \quad F_{s_i} f(x) < \alpha,\ F_{t_i} f(x) > \gamma, \quad i = 1, 2, \cdots, n \Big\}. \end{aligned} \tag{4.2}$$

It can be easily verified that

$$U(\Phi, f, \alpha, \gamma, x) \leq N_{\gamma-\alpha}(\Phi \star f)(x). \tag{4.3}$$

Combining this with Theorems 4.1 and 4.2, we immediately obtain the following results.

Theorem 4.3. *Under the hypotheses of Theorem 4.1 or Theorem 4.2, it follows that*

$$\|U(\Phi_{\vec{b}}, f, \alpha, \gamma, \cdot)^{1/\rho}\|_{L^q(\omega^q)} \leq \frac{C(p,q,\rho)}{\gamma - \alpha} \|\vec{b}\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p(\omega^q)}$$

or

$$\|U(\Phi_b, f, \alpha, \gamma, \cdot)^{1/\rho}\|_{\mathrm{Lip}(\beta/n-1/p)} \leq \frac{C(p,\rho)}{\gamma - \alpha} \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^p},$$

where $n/\beta < p < \infty$, and

$$\|U(\Phi_b, f, \alpha, \gamma, \cdot)^{1/\rho}\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq \frac{C(\rho)}{\gamma - \alpha} \|b\|_{\mathrm{Lip}(\beta)} \|f\|_{L^{n/\beta}}.$$

4.2. On the heat semigroup and the Poisson semigroup

To the end, we consider the heat semigroup $\mathcal{W} := \{\mathrm{e}^{t\Delta}\}_{t>0}$ and the Poisson semigroup $\mathcal{P} := \{\mathrm{e}^{-t\sqrt{-\Delta}}\}_{t>0}$ associated to $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. It is easy to verify that the heat kernel $W_t(x) := (\pi t)^{-n/2} \mathrm{e}^{-|x|^2/t}$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and satisfies $\int_{\mathbb{R}^n} W_t(x)dx = 1$. Similar to (1.4) and (1.5), we can define its commutator. Then, Theorems 1.1, 1.2, 1.3 hold for the variation operators associated with $\mathcal{W}$ and their commutators. Therefore, we establish the following corresponding results.

Theorem 4.4. *Suppose that $W_t(x) \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} W_t(x)dx = 1$, $b_i \in \mathrm{Lip}(\beta_i)$ $(i = 1, \cdots, m)$ and $0 < \beta = \sum_{i=1}^m \beta_i < n$. If $\mathcal{V}_\rho(\mathcal{W} \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$ and $\rho > 2$, then for any $1 < p < n/\beta$ with $1/q = 1/p - \beta/n$, $\omega \in A_{(p,q)}$, $\mathcal{V}_\rho((\mathcal{W} \star f)_{\vec{b}})$ is bounded from $L^p(\mathbb{R}^n, \omega(x)^p dx)$ to $L^q(\mathbb{R}^n, \omega(x)^q dx)$.*

Theorem 4.5. *Suppose that $W_t(x) \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} W_t(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, $0 < \beta < 1$. If $\mathcal{V}_\rho(\mathcal{W}\star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$ and $\rho > 2$, then for any $n/\beta < p < \infty$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{V}_\rho((\mathcal{W} \star f)_b)\|_{\mathrm{Lip}(\beta/n-1/p)} \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p}.$$

Theorem 4.6. *Suppose that $W_t(x) \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} W_t(x)dx = 1$, $b \in \mathrm{Lip}(\beta)$, $0 < \beta < 1$. If $\mathcal{V}_\rho(\mathcal{W} \star f)$ is bounded in $L^{p_0}(\mathbb{R}^n, dx)$ for some $1 < p_0 < \infty$ and $\rho > 2$, then for $p = n/\beta$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{V}_\rho((\mathcal{W} \star f)_b)\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^{n/\beta}}.$$

Similarly, the variation operators associated with $\mathcal{P}$ and their commutators satisfy the conclusions as well.

4.3. Maximal operator

For any given $f \in L^p(\mathbb{R}^n)$, from the the definition of variation, we can obtain the following pointwise control:

$$A^*f(x) \leq \mathcal{V}_\rho(\mathcal{A}f)(x), \quad \rho \geq 1$$

where the maximal operator A^* is defined as

$$A^*f(x) = \sup_{t>o} |A_t(f)(x)|.$$

This demonstrates that the properties of the variation operator in many cases imply the corresponding properties of the maximal operator. Therefore, we have the following inference.

Corollary 4.1. *Under the same assumptions as those in Theorems 1.1-1.3, we define the maximal operator of approximate identities as $\Phi^* \star f = \sup_{t>o} |\phi_t \star f(x)|$, then*

$$\|\Phi^* \star f\|_{L^q(\omega^q)} \leq C\|f\|_{L^p(\omega^q)}$$

or

$$\|\Phi^* \star f\|_{\mathrm{Lip}(\beta/n-1/p)} \leq C\|f\|_{L^p} \quad n/\beta < p < \infty,$$

and

$$\|\Phi^* \star f\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C\|f\|_{L^{n/\beta}}.$$

Remark 4.1. Let $\mathcal{F} = \{F_\epsilon\}_{\epsilon>0}$ be a family of operators, and define its oscillation operator $\mathcal{O}(\mathcal{F}f)$ as follows:

$$\mathcal{O}(\mathcal{F}f)(x) := \Big(\sum_{i=1}^{\infty} \sup_{t_{i+1}\leq\varepsilon_{i+1}<\varepsilon_i\leq t_i} |F_{\varepsilon_{i+1}}f(x) - F_{\varepsilon_i}f(x)|^2\Big)^{1/2},$$

where $\{t_i\}_{i\in\mathbb{N}}$ is a real decreasing sequence that converges to zero. Based on the proofs of the theorems in this paper, we conclude that the results for oscillation operators associated with commutators of approximate identities also hold correspondingly.

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