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FEEDBACK CONTROL OF CHAOS IN THE MODIFIED KDV-BURGERS-KURAMOTO EQUATION VIA A SINGLE TIME-DELAY

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Abstract In this paper, we investigate the time-delayed feedback control of a novel three-dimensional chaotic system which is found from a class of modified KdV-Burgers-Kuramoto (mKBK) equation. First, the local stability and the occurrence of Hopf bifurcation are studied by introducing a single time-delayed feedback term into the chaotic system. Then, the dynamical properties of bifurcated periodic solutions are investigated by applying the algorithm depending on the normal form theory and center manifold theorem. Finally, numerical simulations are presented to illustrate the effectiveness of the theoretical results.

Keywords Control of chaos, modified KdV-Burgers-Kuramoto equation, Hopf bifurcation, stability.

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1. Introduction

Finding exact solutions for nonlinear partial differential equations (PDEs) have received considerable attention in recent years. Many methods such as Lie symmetry method, tanh-function method, Weis-Tabour-Carneville transform, generalized F-expansion method, and so on have been developed to solve these solutions [3, 5, 7, 14, 19, 25]. Notably, recent advances in analytical and numerical techniques have further enriched the toolbox for tackling nonlinear PDEs. For instance, studies in [9, 12, 20, 23, 30, 31] have demonstrated innovative approaches such as Painlevé analysis, neural networks to resolving high-dimensional nonlinearities in fluid-like systems, while studies in [4, 11, 27] explored the process of pattern formations in dissipative systems using fractional-order and adaptive strategies. These developments underscore the growing synergy between theoretical analysis and computational methods in nonlinear dynamics.

However, Chaos may exist in nonlinear equations and make the dynamics much more complicated. Therefore, it is of great significance to explore complex dynamics such as chaos in the nonlinear equations. There are many works concentrating on these aspects. For instance, Jhangeer etc. [15] studied the bifurcation of nonlinear dynamical systems for chaotic behavior in traveling wave solutions and analyzed the formation of patterns. Wang etc. [28] studied the traveling wave dynamics of the KdV-Burgers-Kuramoto (KBK) equation with Marangoni effect perturbations. Lavrova and Kudryashov [18] explored the nonlinear dynamics of the generalized Kuramoto-Sivashinsky equation with varying degrees of nonlinearity.

In most cases, chaos phenomenon which appears in nonlinear equations is undesirable and shall be controlled. Consequently, chaos control with regard to nonlinear equations has aroused

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great attention by many researchers. For instance, Luo etc. [21] investigated the chaos control problem of microelectromechanical system resonators by using the analog circuits. Adéchinan etc. [2] studied chaos, coexisting attractors and chaos control in a nonlinear dissipative chemical oscillator using the Melnikov method. Abro etc. [1] presented the chaos control and chaotic characteristics of brushless DC motors through fractal-fractional differentiations techniques. Zheng etc. [32] analyzed chaotic motion and control of the driven-damped double Sine-Gordon equation by Melnikov method. Amongst many control methods, the time-delayed feedback control technique developed by Pyragas [24] has proven to be a simple and feasible method for controlling continuous chaotic systems with a wide spectrum of applications [6, 8, 10, 22, 26, 29]. Motivated by this method, in this paper, we attempt to study a single delay feedback control of a class of novel chaotic system recently found in the modified KdV-Burgers-Kuramoto (mKBK) equation. The rest of this paper is structured as follows. In Section 2, a brief introduction of mKBK chaotic system is presented. The local stability and the occurrence of Hopf bifurcation are investigated for mKBK chaotic system with a single delay feedback. In Section 3, the explicit formulae including determining the direction and stability of bifurcation periodic solutions are derived by applying normal form theory and the central manifold theorem. In Section 4, numerical simulations which show good agreement with theoretical results are presented. Finally, Section 5 gives a brief conclusion.

2. Local stability and occurrence of Hopf bifurcation

The standard KBK equation, given by:

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{2.1}$$

where α, β, γ are real constants, is widely recognized for its role in modeling physical processes in unstable systems [16, 17]. It unifies several well-known equations, such as the KdV equation ($\alpha = \gamma = 0$, and $\beta \neq 0$), the Burgers equation ($\beta = \gamma = 0$, and $\alpha \neq 0$), and the Kuramoto equation ($\beta = 0$, $\alpha \neq 0$, and $\gamma \neq 0$). These equations are fundamental in describing wave phenomena, turbulence, and pattern formation in various physical contexts, including fluid dynamics, plasma physics, and chemical reactions.

The mKBK equation, as studied in the paper, introduces a quadratic nonlinear term into the standard KBK equation, resulting in:

$$u_t + (u + u^2)u_x + u_{xx} + bu_{xxx} + u_{xxxx} = 0. \tag{2.2}$$

This modification enhances the nonlinearity of the system, leading to richer dynamical behaviors, including chaos. The mKBK equation is particularly relevant in scenarios where higher-order nonlinear effects cannot be neglected, such as in the study of complex wave interactions, soliton dynamics, and chaotic systems. The inclusion of the term u^2u_x allows for a broader exploration of nonlinear phenomena, making the mKBK equation a valuable tool for understanding intricate dynamical systems in both theoretical and applied contexts.

Performing the traveling wave transformation as follows

$$u(x, t) = \vartheta(\xi) = \vartheta(x - at). \tag{2.3}$$

Substituting Eq. (2.3) into Eq. (2.2) and performing one integration, we get a third-order ordinary differential equation: $c - a\vartheta + \frac{1}{2}\vartheta^2 + \frac{1}{3}\vartheta^3 + \vartheta' + b\vartheta'' + \vartheta''' = 0$, which is equivalent to the following three-dimensional autonomous system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = ax - \frac{1}{2}x^2 - \frac{1}{3}x^3 - y - bz - c, \end{cases} \tag{2.4}$$

where a represents the wave speed and c is an integration constant.

It is found recently that system (2.4) exhibits chaotic dynamical behaviors when $a = 1, b = 1, c = -0.5$, as illustrated in Figure 1. For convenience, in what follows we refer to system (2.4) as mKBK chaotic system. It is easy to calculate that mKBK chaotic system has three equilibrium points $E_1(-2.496, 0, 0), E_2(-0.433254, 0, 0)$ and $E_3(1.40226, 0, 0)$ when $a = 1, b = 1, c = -0.5$. Without loss of generality, we denote $E^*(x^*, 0, 0)$ as the equilibrium point of mKBK chaotic system.

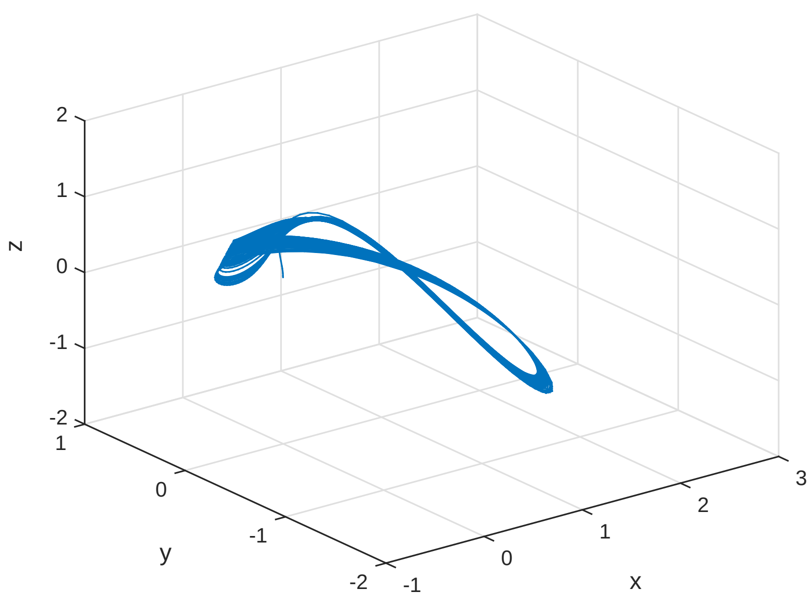

Figure 1. Chaotic attractor of system (2.4) with $a = 1, b = 1, c = -0.5$.

In order to apply delayed feedback control technique to realize chaos control of mKBK chaotic system, we attempt to introduce a single delay feedback term into the system and express it as follows:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = z(t) + k_{22}[y(t) - y(t-\tau)], \\ \dot{z}(t) = ax(t) - \frac{1}{2}x^2(t) - \frac{1}{3}x^3(t) - y(t) - bz(t) - c, \end{cases} \tag{2.5}$$

where $\tau(> 0)$ is the delay and k_{22} is the scaling parameter.

It is obvious that system (2.5) and mKBK chaotic system have the same equilibrium points.

Taking $x(t) = \tilde{x}(t) + x^*, y(t) = \tilde{y}(t), z(t) = \tilde{z}(t)$ and substituting them into system (2.5) yields

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{y}(t), \\ \dot{\tilde{y}}(t) = \tilde{z}(t) + k_{22}[\tilde{y}(t) - \tilde{y}(t-\tau)], \\ \dot{\tilde{z}}(t) = a\tilde{x}(t) - \frac{1}{2}\tilde{x}^2(t) - \frac{1}{3}\tilde{x}^3(t) - \tilde{y}(t) - b\tilde{z}(t) - x^*\tilde{x}(t) - x^{*2}\tilde{x}(t) - x^*\tilde{x}^2(t). \end{cases} \tag{2.6}$$

The corresponding characteristic equation of system (2.6) appears as

$$s^3 - (k_{22} - b)s^2 - (k_{22}b - 1)s + x^{*2} + x^* - a + (k_{22}s^2 + k_{22}bs)\mathrm{e}^{-s\tau} = 0. \tag{2.7}$$

Particularly, when $\tau = 0$, Eq. (2.7) is reduced to $s^3 + bs^2 + s + x^{*2} + x^* - a = 0$. It follows from Routh-Huruitz criterion that all roots of the equation have negative real parts if $b > 0$ and $b + a - x^* - x^{*2} > 0$ hold.

Rewrite Eq. (2.7) as

$$s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 + (\beta_2 s^2 + \beta_1 s)\mathrm{e}^{-s\tau} = 0, \tag{2.8}$$

where $\alpha_0 = x^{*2} + x^* - a, \alpha_1 = 1 - k_{22}b, \alpha_2 = b - k_{22}, \beta_2 = k_{22}, \beta_1 = k_{22}b$.

Let $s = \mathrm{i}\xi(\xi > 0)$ be the pure imaginary root of Eq. (2.8). Substituting it into Eq. (2.8), we have

$$-\xi^3\mathrm{i} - \alpha_2\xi^2 + \alpha_1\xi\mathrm{i} + \alpha_0 + (-\beta_2\xi^2 + \beta_1\xi\mathrm{i})(\cos\xi\tau - \mathrm{i}\sin\xi\tau) = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} \alpha_2\xi^2 - \alpha_0 = \beta_1\xi\sin\xi\tau - \beta_2\xi^2\cos\xi\tau, \\ \xi^3 - \alpha_1\xi \quad = \beta_1\xi\cos\xi\tau + \beta_2\xi^2\sin\xi\tau. \end{cases} \tag{2.9}$$

It follows from Eq. (2.9) that

$$\xi^6 + ({\alpha_2}^2 - {\beta_2}^2 - 2\alpha_1)\xi^4 + ({\alpha_1}^2 - 2\alpha_0\alpha_2 - {\beta_1}^2)\xi^2 + {\alpha_0}^2 = 0. \tag{2.10}$$

Let $\xi = \sqrt{u}$, $p = {\alpha_2}^2 - {\beta_2}^2 - 2\alpha_1$, $q = {\alpha_1}^2 - 2\alpha_0\alpha_2 - {\beta_1}^2$, $r = {\alpha_0}^2$. Then Eq. (2.10) can be rewritten as the following cubic equation

$$u^3 + pu^2 + qu + r = 0. \tag{2.11}$$

Without loss of generality , let $u_j(j = 1, 2, 3)$ be the roots of Eq. (2.11) satisfying $\xi_j = \sqrt{u_j}(j = 1, 2, 3)$. It follows from Eq. (2.9) that

$$\tau_j^{(k)} = \frac{1}{\xi_j}\left[\arccos\left(\frac{\beta_1{\xi_j}^2 - \alpha_1\beta_1 - \alpha_2\beta_2{\xi_j}^2 + \alpha_0\beta_2}{{\beta_2}^2{\xi_j}^2 + {\beta_1}^2}\right) + 2k\pi\right], \tau_0 = \min_{j\in\{1,2,3\}}\left\{\tau_j^{(0)}\right\}, \tag{2.12}$$

where $j = 1, 2, 3$; $k = 0, 1, 2, \cdots$.

Denote by $\rho(u) = u^3 + pu^2 + qu + r, \Delta = p^2 - 3q$, and $u_1^* = \frac{-p+\sqrt{\Delta}}{3}$. Combined with the root distribution of cubic equation, the following results are immediate.

Lemma 2.1. (i) *If $\Delta \le 0$, then the number of roots of Eq. (2.8) with positive real parts when $\tau > 0$ is the same with that of Eq. (2.8) when $\tau = 0$.*
(ii) *If $\Delta > 0, u_1^* > 0$ and $\rho(u_1^*) \le 0$, then the number of roots of Eq. (2.8) with positive real parts for $0 < \tau < \tau_0$ is the same with that of Eq. (2.8) when $\tau = 0$.*

Lemma 2.2. *If* $\rho'(u_j) \neq 0$, *then* $\frac{\mathrm{dRe}s(\tau_j^{(k)})}{\mathrm{d}\tau} \neq 0$ *and* $\mathrm{sign}\left\{\frac{\mathrm{dRe}s(\tau_j^{(k)})}{\mathrm{d}\tau}\right\} = \mathrm{sign}\{\rho'(u_j)\}$ *hold (j=1,2,3).*

Proof. Assume that $s(\tau)$ is the root of Eq. (2.7). Substituting $s(\tau)$ into Eq. (2.7) and taking the derivative of τ yields

$$\left\{3s^2 + 2(b - k_{22})s + 1 - k_{22}b + [2k_{22}s + k_{22}b - \tau(k_{22}s^2 + k_{22}bs)]\mathrm{e}^{-s\tau}\right\}\frac{\mathrm{d}s}{\mathrm{d}\tau} = s(k_{22}s^2 + k_{22}bs)\mathrm{e}^{-s\tau}.$$

Hence we have

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1} = \frac{(3s^2 + 2(b - k_{22})s + 1 - k_{22}b)\mathrm{e}^{s\tau}}{s(k_{22}s^2 + k_{22}bs)} + \frac{2k_{22}s + k_{22}b}{s(k_{22}s^2 + k_{22}bs)} - \frac{\tau}{s}. \tag{2.13}$$

Substituting $s(\tau) = \mathrm{i}\xi_j$ into Eq. (2.13) yields

$$[s(k_{22}s^2 + k_{22}bs)]_{\tau=\tau_j^{(k)}} = -k_{22}b{\xi_j}^2 - \mathrm{i}k_{22}{\xi_j}^3,$$

and

$$\begin{aligned}&[(3s^2 + 2(b - k_{22})s + 1 - k_{22}b)\mathrm{e}^{s\tau}]_{\tau=\tau_j^{(k)}} \\ =\ &[(1 - k_{22}b - 3{\xi_j}^2)\cos\xi_j\tau_j^{(k)} - 2(b - k_{22})\xi_j\sin\xi_j\tau_j^{(k)}] \\ &+\mathrm{i}[2(b - k_{22})\xi_j\cos\xi_j\tau_j^{(k)} + (1 - k_{22}b - 3{\xi_j}^2)\sin\xi_j\tau_j^{(k)}].\end{aligned}$$

Thus we have

$$\begin{aligned}&\left[\frac{\mathrm{dRe}s(\tau)}{\mathrm{d}\tau}\right]\Bigg|^{-1}_{\tau=\tau_j^{(k)}} \\ =\ &\mathrm{Re}\left[\frac{(3s^2 + 2(b - k_{22})s + 1 - k_{22}b)\mathrm{e}^{s\tau}}{s(k_{22}s^2 + k_{22}bs)}\right]\Bigg|_{\tau=\tau_j^{(k)}} + \mathrm{Re}\left[\frac{2k_{22}s + k_{22}b}{s(k_{22}s^2 + k_{22}bs)}\right]\Bigg|_{\tau=\tau_j^{(k)}} \\ =\ &\frac{1}{\Upsilon}\Big\{-k_{22}b{\xi_j}^2[(1 - k_{22}b - 3{\xi_j}^2)\cos\xi_j\tau_j^{(k)} - 2(b - k_{22})\xi_j\sin\xi_j\tau_j^{(k)}] \\ &-k_{22}{\xi_j}^3[2(b - k_{22})\xi_j\cos\xi_j\tau_j^{(k)} + (1 - k_{22}b - 3{\xi_j}^2)\sin\xi_j\tau_j^{(k)}] - k_{22}b^2{\xi_j}^2 - 2{k_{22}}^2{\xi_j}^4\Big\} \\ =\ &\frac{1}{\Upsilon}\Big\{3{\xi_j}^6 + 2(b^2 - 2){\xi_j}^4 + [1 - 2k_{22}b + 2k_{22}({x^*}^2 + x^* - a) - 2b({x^*}^2 + x^* - a)]{\xi_j}^2\Big\} \\ =\ &\frac{1}{\Upsilon}\left(3{\xi_j}^6 + 2p{\xi_j}^4 + q{\xi_j}^2\right) \\ =\ &\frac{1}{\Upsilon}\left\{u_j(3{u_j}^2 + 2pu_j + q)\right\} \\ =\ &\frac{u_j}{\Upsilon}\rho'(u_j),\end{aligned}$$

where $p = {\alpha_2}^2 - {\beta_2}^2 - 2\alpha_1$, $q = {\alpha_1}^2 - 2\alpha_0\alpha_2 - {\beta_1}^2$, $u_j = {\xi_j}^2$, and $\Upsilon = {k_{22}}^2b^2{\xi_j}^4 + {k_{22}}^2{\xi_j}^6$. If $\rho'(u_j) \neq 0$, then $\frac{\mathrm{dRe}s(\tau_j^{(k)})}{\mathrm{d}\tau} \neq 0$ and $\mathrm{sign}\left\{\frac{\mathrm{dRe}s(\tau_j^{(k)})}{\mathrm{d}\tau}\right\} = \mathrm{sign}\{\rho'(u_j)\}$ hold. This completes the proof. □

In summary, we have the following results.

Theorem 2.1. (i) *If $b>0$, $b+a-x^*-x^{*2}>0$, and $\Delta\le 0$, then all roots of Eq. (2.7) have negative real parts for all $\tau\ge 0$ and thus the equilibrium point $E^*(x^*,0,0)$ of system (2.5) is locally asymptotically stable.*
(ii) *If $b>0$, $b+a-x^*-x^{*2}>0$, $\Delta>0$, $u_1^*>0$, and $\rho(u_1^*)\le 0$, then all roots of Eq. (2.7) have negative real parts for $\tau\in[0,\tau_0)$ and thus the equilibrium point $E^*(x^*,0,0)$ of system (2.5) is locally asymptotically stable.*
(iii) *If $b<0$ or $b+a-x^*-x^{*2}<0$, and $\Delta\le 0$, then at least one root of Eq. (2.7) has positive real part for $\tau\ge 0$ and thus the equilibrium point $E^*(x^*,0,0)$ of system (2.5) is unstable.*
(iv) *If $\Delta>0, u_1^*>0, \rho(u_1^*)\le 0$, and $\rho'(u_j)\ne 0$, then system (2.5) undergoes Hopf bifurcation at the equilibrium point $E^*(x^*,0,0)$ for $\tau=\tau_j^{(k)}(j=1,2,3;k=0,1,2,\cdots)$.*

3. Dynamical properties of Hopf bifurcation

In this section, we further study the dynamical properties of Hopf bifurcation by means of the algorithms depending on the center manifold theorem and normal form theory [13].

Suppose that system (2.6) bifurcates at the equilibrium point $(\tilde{x},\tilde{y},\tilde{z})$. Let $x_1=x-\tilde{x}$, $y_1=y-\tilde{y}$, $z_1=z-\tilde{z},\tau=\tau_j+\upsilon$, and $t=\tau t$. Then system (2.6) in $C([-1,0],R^3)$ can be written as

$$\dot{x}_t=N_\upsilon(x_t)+h(\upsilon,x_t), \tag{3.1}$$

where $x_t=(x_{1t},\,x_{2t},x_{3t})^T\in R^3$, and for $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T\in C$,

$$N_\upsilon=(\tau_j+\upsilon)\begin{bmatrix}0 & 1 & 0\\ 0 & k_{22} & 1\\ a-x^*-x^{*2} & -1 & -b\end{bmatrix}\begin{bmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\end{bmatrix}+(\tau_j+\upsilon)\begin{bmatrix}0 & 0 & 0\\ 0 & -k_{22} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1(-1)\\ \varphi_2(-1)\\ \varphi_3(-1)\end{bmatrix},$$

$$h(\upsilon,\varphi)=(\tau_j+\upsilon)\begin{bmatrix}0\\ 0\\ -\frac{1+2x^*}{2}\varphi_1^2(0)-\frac{1}{3}\varphi_1^3(0)\end{bmatrix}.$$

According to Riesz representation theorem, there exists a bounded variation function $\eta(\theta,\upsilon)(\theta\in[-1,0])$ such that

$$N_\upsilon(\varphi)=\int_{-1}^{0}\mathrm{d}\eta(\theta,0)\varphi(\theta). \tag{3.2}$$

In fact, we may take

$$\eta(\theta,\upsilon)=(\tau_j+\upsilon)\begin{bmatrix}0 & 1 & 0\\ 0 & k_{22} & 1\\ a-x^*-x^{*2} & -1 & -b\end{bmatrix}\sigma(\theta)-(\tau_j+\upsilon)\begin{bmatrix}0 & 0 & 0\\ 0 & -k_{22} & 0\\ 0 & 0 & 0\end{bmatrix}\sigma(\theta+1), \tag{3.3}$$

where $\sigma(\cdot)$ is the Dirac delta function.

For $\varphi \in C([-1,0], R^3)$, define

$$L(\upsilon)\varphi = \begin{cases} \dfrac{\mathrm{d}\varphi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1,0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(\upsilon,s)\varphi(s), & \theta = 0 \end{cases}$$

and

$$F(\upsilon)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ h(\upsilon,\varphi), & \theta = 0. \end{cases}$$

Then Eq. (3.1) is equivalent to

$$\dot{x}_t = L(\upsilon)x_t + F(\upsilon)x_t. \tag{3.4}$$

For $\psi \in C'([0,1], (R^3)^*)$, define the adjoint operator L^* of L as

$$L^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in [-1,0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta^T(t,0)\psi(-t), & s = 0 \end{cases}$$

and the bilinear inner product as

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\zeta=0}^{\theta} \bar{\psi}(\zeta-\theta)\mathrm{d}\eta(\theta)\varphi(\zeta)\mathrm{d}\zeta,$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $q(\theta) = (1,\gamma,\delta)^T \mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}$ be the eigenvector of $L(0)$ such that $L(0)q(\theta) = \mathrm{i}\tau_j\xi_j q(\theta)$. Then we have

$$\tau_j \begin{bmatrix} \mathrm{i}\xi_j & -1 & 0 \\ 0 & \mathrm{i}\xi_j - k_{22} + k_{22}\mathrm{e}^{-\mathrm{i}\xi_j\tau_j} & -1 \\ x^{*2} + x^* - a & 1 & \mathrm{i}\xi_j + b \end{bmatrix} q(\theta) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.5}$$

From Eq. (3.5), we can calculate that

$$q(\theta) = (1,\gamma,\delta)^T \mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j} = \left(1, \mathrm{i}\xi_j, \frac{a - x^* - x^{*2} - \mathrm{i}\xi_j}{\mathrm{i}\xi_j + b}\right)^T \mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}. \tag{3.6}$$

Similarly, let $q^*(s) = D(1,\gamma^*,\delta^*)\mathrm{e}^{\mathrm{i}s\xi_j\tau_j}$ be the eigenvector of $L^*(0)$ such that $L^*(0)q^*(\theta) = -\mathrm{i}\tau_j\xi_j q^*(\theta)$. Then we have

$$\begin{aligned} q^*(s) &= D(1,\gamma^*,\delta^*)\mathrm{e}^{\mathrm{i}s\xi_j\tau_j} \\ &= D\left(1, \frac{\mathrm{i}\xi_j(\mathrm{i}\xi_j - b)}{a - x^* - x^{*2}}, \frac{-\mathrm{i}\xi_j}{a - x^* - x^{*2}}\right) e^{\mathrm{i}s\xi_j\tau_j}. \end{aligned}$$

Thus we can obtain that

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \bar{D}(1,\bar{\gamma^*},\bar{\delta^*})(1,\gamma,\delta)^T - \int_{-1}^{0}\int_{\zeta=0}^{\theta}\bar{D}(1,\bar{\gamma^*},\bar{\delta^*})\mathrm{e}^{-\mathrm{i}(\zeta-\theta)\xi_j\tau_j}\mathrm{d}\eta(\theta)(1,\gamma,\delta)^T\mathrm{e}^{-\mathrm{i}\zeta\xi_j\tau_j}\mathrm{d}\zeta \\ &= \bar{D}\left(1+\gamma\bar{\gamma^*}+\delta\bar{\delta^*}-k_{22}\tau_j\delta\bar{\delta^*}\mathrm{e}^{-\mathrm{i}\xi_j\tau_j}\right).\end{aligned}$$

If we choose $D = \frac{1}{1+\bar{\gamma}\gamma^*+\bar{\delta}\delta^*-k_{22}\tau_j\bar{\delta}\delta^*\mathrm{e}^{-\mathrm{i}\xi_j\tau_j}}$, then $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$ hold.

Next we compute the coordinates of center manifold Ξ_0 at $\upsilon = 0$. Assume that x_t is the solution of Eq. (3.4). Define

$$z(t) = \langle q^*, x_t\rangle, W(t,\theta) = x_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta), \tag{3.7}$$

where $z(t)$ and $\bar{z}(t)$ are local coordinates for center manifold Ξ_0 in the direction of q^* and $\bar{q^*}$, respectively.

On the center manifold Ξ_0, we have

$$W(t,\theta) = W(z(t),\bar{z}(t),\theta) = W_{20}(\theta)\frac{z^2(t)}{2} + W_{11}(\theta)z(t)\bar{z}(t) + W_{02}\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3(t)}{6} + \cdots,$$

and

$$\begin{aligned}\dot{z}(t) &= \mathrm{i}\xi_j\tau_j z(t) + \bar{q^*}(0)h(0, W(z(t),\bar{z}(t),\theta) + 2\mathrm{Re}\{z(t)q(0)\}) \\ &= \mathrm{i}\xi_j\tau_j z(t) + \bar{q^*}(0)h_0(z(t),\bar{z}(t)) \\ &= \mathrm{i}\xi_j\tau_j + g(z(t),\bar{z}(t)),\end{aligned}$$

where

$$g(z(t),\bar{z}(t)) = \bar{q^*}(0)h_0(z(t),\bar{z}(t)) = g_{20}\frac{z^2(t)}{2} + g_{11}z(t)\bar{z}(t) + g_{02}\frac{\bar{z}^2(t)}{2} + g_{21}\frac{z^2(t)\bar{z}(t)}{2} + \cdots. \tag{3.8}$$

It follows from $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))^T = W(t,\theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$ and $q(\theta) = (1,\gamma,\delta)^T\mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}$ that

$$\begin{cases} x_{1t}(0) = z(t) + \bar{z}(t) + W_{20}^{(1)}(0)\frac{z^2(t)}{2} + W_{11}^{(1)}(0)z(t)\bar{z}(t) + W_{02}^{(1)}(0)\frac{\bar{z}^2(t)}{2} + o(|(z(t),\bar{z}(t))|^3), \\ x_{2t}(0) = \gamma z(t) + \bar{\gamma}\bar{z}(t) + W_{20}^{(2)}(0)\frac{z^2(t)}{2} + W_{11}^{(2)}(0)z(t)\bar{z}(t) + W_{02}^{(2)}(0)\frac{\bar{z}^2(t)}{2} + o(|(z(t),\bar{z}(t))|^3), \\ x_{3t}(0) = \delta z(t) + \bar{\delta}\bar{z}(t) + W_{20}^{(3)}(0)\frac{z^2(t)}{2} + W_{11}^{(3)}(0)z(t)\bar{z}(t) + W_{02}^{(3)}(0)\frac{\bar{z}^2(t)}{2} + o(|(z(t),\bar{z}(t))|^3). \end{cases}$$

On the other hand, from Eq. (3.8) we have

$$\begin{aligned} g(z(t),\bar{z}(t)) &= \bar{q^*}(0)h_0(z(t),\bar{z}(t)) \\ &= \bar{D}_{\tau_j}(1,\bar{\gamma^*},\bar{\delta^*})\begin{bmatrix} 0 \\ 0 \\ -\frac{1+2x^*}{2}{x_{1t}}^2(0) - \frac{1}{3}{x_{1t}}^3(0)\end{bmatrix} \\ &= -\frac{1+2x^*}{2}\bar{D}_{\tau_j}\bar{\delta^*}[x_{1t}(0)]^2 - \frac{1}{3}\bar{D}_{\tau_j}\bar{\delta^*}[x_{1t}(0)]^3.\end{aligned}$$

Comparing the corresponding coefficients, we have

$$\begin{cases} g_{20} = -(1+2x^*)\bar{D}_{\tau_j}\bar{\delta^*}, \\ g_{11} = -(1+2x^*)\bar{D}_{\tau_j}\bar{\delta^*}, \\ g_{02} = -(1+2x^*)\bar{D}_{\tau_j}\bar{\delta^*}, \\ g_{21} = -\bar{D}_{\tau_j}\bar{\delta^*}\left[(1+2x^*)W_{20}^{(1)}(0) + 2(1+2x^*)W_{11}^{(1)}(0) + 3\right]. \end{cases} \tag{3.9}$$

To determine g_{21}, it is necessary to calculate $W_{20}(\theta)$ and $W_{11}(\theta)$. Since $W(t,\theta) = x_t(\theta) - z(t)q(\theta) - \bar{z}(t)\bar{q}(\theta)$, we have

$$\begin{aligned} \dot{W} &= \dot{x}_t - \dot{z}(t)q - \dot{\bar{z}}(t)\bar{q} \\ &= \begin{cases} LW - 2\mathrm{Re}\left\{\bar{q^*}(\theta)h_0(z(t),\bar{z}(t))q(\theta)\right\}, & \theta \in [-1,0), \\ LW - 2\mathrm{Re}\left\{\bar{q^*}(0)h_0(z(t),\bar{z}(t))q(0)\right\} + h_0, & \theta = 0, \end{cases} \\ &= LW + H(z(t),\bar{z}(t),\theta), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} H(z(t),\bar{z}(t),\theta) &= \begin{cases} 2\mathrm{Re}\left\{\bar{q^*}(\theta)h_0(z(t),\bar{z}(t))q(\theta)\right\}, & \theta \in [-1,0), \\ 2\mathrm{Re}\left\{\bar{q^*}(0)h_0(z(t),\bar{z}(t))q(0)\right\} + h_0, & \theta = 0 \end{cases} \\ &= H_{20}\frac{z^2(t)}{2} + H_{11}z(t)\bar{z}(t) + H_{02}\frac{\bar{z}^2(t)}{2} + \cdots . \end{aligned}$$

Note that

$$W + zq(\theta) + \bar{z}\bar{q}(\theta) = \begin{bmatrix} W^{(1)}(z,\bar{z},\theta) + z\mathrm{e}^{\mathrm{i}\xi_j\tau_j\theta} + \bar{z}\mathrm{e}^{-\mathrm{i}\xi_j\tau_j\theta} \\ W^{(2)}(z,\bar{z},\theta) + z\gamma\mathrm{e}^{\mathrm{i}\xi_j\tau_j\theta} + \bar{z}\bar{\gamma}\mathrm{e}^{-\mathrm{i}\xi_j\tau_j\theta} \\ W^{(3)}(z,\bar{z},\theta) + z\delta\mathrm{e}^{\mathrm{i}\xi_j\tau_j\theta} + \bar{z}\bar{\delta}\mathrm{e}^{-\mathrm{i}\xi_j\tau_j\theta} \end{bmatrix}$$

and

$$h_0 = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \theta \in [-1,0), \\ \begin{bmatrix} 0 \\ 0 \\ h_0^{(3)} \end{bmatrix}, & \theta = 0, \end{cases} \tag{3.11}$$

where

$$h_0^{(3)} = -\frac{1+2x^*}{2}\tau_j[W^{(1)}(z,\bar{z},0) + z + \bar{z}]^2 - \frac{1}{3}\tau_j[W^{(1)}(z,\bar{z},0) + z + \bar{z}]^3. \tag{3.12}$$

On the other hand, on the center manifold Ξ_0, we also have

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{3.13}$$

Expanding the series in Eq. (3.10) and Eq. (3.13), and comparing the corresponding coefficients, we can obtain

$$\begin{cases} (L-2\mathrm{i}\xi_j\tau_j)W_{20}(\theta)=-H_{20}(\theta),\\ LW_{11}(\theta)=-H_{11}(\theta),\\ \cdots . \end{cases} \tag{3.14}$$

Note that for $\theta\in[-1,0)$, we have

$$H(z(t),\bar{z}(t),\theta)=-\bar{q^*}(0)h_0q(\theta)-q^*(0)\bar{h}_0\bar{q}(\theta)=-gq(\theta)-\bar{g}\bar{q}(\theta). \tag{3.15}$$

Expanding the series in the above equation and comparing the corresponding coefficients yields

$$\begin{cases} H_{20}(\theta)=-g_{20}q(\theta)-\bar{g}_{20}\bar{q}(\theta),\\ H_{11}(\theta)=-g_{11}q(\theta)-\bar{g}_{11}\bar{q}(\theta). \end{cases} \tag{3.16}$$

It follows from Eq. (3.14), Eq. (3.16) and the definition of L that

$$\dot{W}_{20}(\theta)=2\mathrm{i}\xi_j\tau_jW_{20}(\theta)+g_{20}q(\theta)+\bar{g_{02}}\bar{q}(\theta). \tag{3.17}$$

Solving the above differential equation yields

$$W_{20}(\theta)=\frac{\mathrm{i}g_{20}}{\xi_j\tau_j}q(0)\mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}+\frac{\mathrm{i}\bar{g}_{02}}{3\xi_j\tau_j}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\theta\xi_j\tau_j}+C_1\mathrm{e}^{2\mathrm{i}\theta\xi_j\tau_j}, \tag{3.18}$$

where $C_1=(C_1^{(1)},C_1^{(2)},C_1^{(3)})\in R^3$ is a constant vector to be determined.

Similarly, we have

$$W_{11}(\theta)=\frac{\mathrm{i}g_{11}}{\xi_j\tau_j}q(0)\mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}+\frac{\mathrm{i}\bar{g}_{11}}{\xi_j\tau_j}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\theta\xi_j\tau_j}+C_2, \tag{3.19}$$

where $C_2=(C_2^{(1)},C_2^{(2)},C_2^{(3)})\in R^3$ is a constant vector to be determined.

When $\theta=0$, we have

$$\begin{aligned} H(z(t),\bar{z}(t),\theta)&=-2\mathrm{Re}\left\{\bar{q^*}(0)h_0(z(t),\bar{z}(t))q(0)\right\}+h_0\\ &=-gq(0)-\bar{g}\bar{q}(0)+h_0, \end{aligned} \tag{3.20}$$

where $h_0=h_{0,z^2}\frac{z^2}{2}+h_{0,z\bar{z}}z\bar{z}+h_{0,\bar{z}^2}\frac{\bar{z}^2}{2}+\cdots$.

It follows from Eq. (3.14) and the definition of L that

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{20}(\theta)=2\mathrm{i}\tau_j\xi_jW_{20}(\theta)-H_{20}(0) \tag{3.21}$$

and

$$\int_{-1}^{0}\mathrm{d}\eta(\theta)W_{11}(\theta)=-H_{11}(0). \tag{3.22}$$

It can be obtained from Eq. (3.20) that

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \begin{bmatrix} 0 \\ 0 \\ h_{0,z^2}^{(3)} \end{bmatrix} \\ &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_j \begin{bmatrix} 0 \\ 0 \\ -(1+2x^*) \end{bmatrix} \end{aligned} \tag{3.23}$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_j \begin{bmatrix} 0 \\ 0 \\ -(1+2x^*) \end{bmatrix}. \tag{3.24}$$

Substituting Eq. (3.18) and Eq. (3.23) into Eq. (3.21) and noting that

$$\left(\mathrm{i}\xi_j\tau_j I - \int_{-1}^{0} \mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}\mathrm{d}\eta(\theta)\right) q(0) = 0, \quad \left(-\mathrm{i}\xi_j\tau_j I - \int_{-1}^{0} \mathrm{e}^{\mathrm{i}\theta\xi_j\tau_j}\mathrm{d}\eta(\theta)\right) \bar{q}(0) = 0,$$

where I is an identity matrix, we get

$$\left(2\mathrm{i}\xi_j\tau_j I - \int_{-1}^{0} \mathrm{e}^{2\mathrm{i}\theta\xi_j\tau_j}\mathrm{d}\eta(\theta)\right) C_1 = \tau_j \begin{bmatrix} 0 \\ 0 \\ -(1+2x^*) \end{bmatrix}.$$

Thus we have

$$\begin{bmatrix} 2\mathrm{i}\xi_j & -1 & 0 \\ 0 & 2\mathrm{i}\xi_j - k_{22} + k_{22}\mathrm{e}^{-2\mathrm{i}\xi_j\tau_j} & -1 \\ x^{*2} + x^* - a & 1 & 2\mathrm{i}\xi_j + b \end{bmatrix} C_1 = \begin{bmatrix} 0 \\ 0 \\ -(1+2x^*) \end{bmatrix}.$$

From the above equation we can calculate that

$$\begin{cases} C_1^{(1)} = \dfrac{-(1+2x^*)}{A}, \\ C_1^{(2)} = \dfrac{-(2\mathrm{i}\xi_j)(1+2x^*)}{A}, \\ C_1^{(3)} = \dfrac{-(2\mathrm{i}\xi_j)(1+2x^*)(2\mathrm{i}\xi_j - k_{22} + k_{22}\mathrm{e}^{-2\mathrm{i}\xi_j\tau_j})}{A}, \end{cases}$$

where

$$A = \begin{vmatrix} 2\mathrm{i}\xi_j & -1 & 0 \\ 0 & 2\mathrm{i}\xi_j - k_{22} + k_{22}\mathrm{e}^{-2\mathrm{i}\xi_j\tau_j} & -1 \\ x^{*2} + x^* - a & 1 & 2\mathrm{i}\xi_j + b \end{vmatrix}.$$

Similarly, substituting Eq. (3.19) and Eq. (3.24) into Eq. (3.22) yields

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ x^{*2}+x^{*}-a & 1 & b \end{bmatrix} C_2 = \begin{bmatrix} 0 \\ 0 \\ -(1+2x^{*}) \end{bmatrix}.$$

Hence, we can calculate that

$$C_2^{(1)} = \frac{-(1+2x^{*})}{B},\ C_2^{(2)} = 0,\ C_2^{(3)} = 0,$$

where

$$B = \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ x^{*2}+x^{*}-a & 1 & b \end{vmatrix}.$$

Through the above analysis, the following parameters can be calculated:

$$\begin{cases} d_1(0) = \dfrac{\mathrm{i}}{2\xi_j\tau_j}\left(g_{11}g_{20} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}\right) + \dfrac{g_{21}}{2}, \\ \mu_2 = -\dfrac{\mathrm{Re}\left\{d_1(0)\right\}}{\mathrm{Re}\left\{s'(\tau_j)\right\}}, \\ \delta_2 = 2\mathrm{Re}\left\{d_1(0)\right\}, \\ \kappa_2 = -\dfrac{\mathrm{Im}\left\{d_1(0)\right\} + \upsilon_2\mathrm{Im}\left\{s'(\tau_j)\right\}}{\tau_j\xi_j}, \end{cases} \tag{3.25}$$

where the sign of μ_2 determines the direction of the Hopf bifurcation: If $\mu_2 < 0(\mu_2 > 0)$, the Hopf bifurcation is subcritical (supercritical); the sign of δ_2 determines the stability of the Hopf bifurcation: If $\delta_2 < 0(\delta_2 > 0)$, the bifurcated periodic solution is stable (unstable); and the sign of κ_2 determines the period of the Hopf bifurcation periodic solution: If $\kappa_2 < 0(\kappa_2 > 0)$, the period decreases (increases).

4. Numerical simulations

In this section, we shall perform some numerical simulations to illustrate the effectiveness of the theoretical results obtained in the previous sections. We consider the mKBK chaotic system with a single delay feedback described as follows:

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = z(t) + k_{22}[y(t) - y(t-\tau)], \\ \dot{z}(t) = x(t) - \dfrac{1}{2}x^2(t) - \dfrac{1}{3}x^3(t) - y(t) - z(t) + \dfrac{1}{2}. \end{cases} \tag{4.1}$$

System (4.1) has three equilibrium points $E_1(-2.496, 0, 0)$, $E_2(-0.433254, 0, 0)$ and $E_3(1.40226, 0, 0)$. Without loss of generality, let $E^*(x^*, 0, 0)$ be the equilibrium point of system

(4.1). Performing the transformation $x(t) = \tilde{x}(t) + x^*, y(t) = \tilde{y}(t), z(t) = \tilde{z}(t)$ and substituting them into system (2.5) yields the following equivalent system

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{y}(t), \\ \dot{\tilde{y}}(t) = \tilde{z}(t) + k_{22}[\tilde{y}(t) - \tilde{y}(t-\tau)], \\ \dot{\tilde{z}}(t) = \tilde{x}(t) - \dfrac{1}{2}\tilde{x}^2(t) - \dfrac{1}{3}\tilde{x}^3(t) - \tilde{y}(t) - \tilde{z}(t) - x^*\tilde{x}(t) - x^{*2}\tilde{x}(t) - x^*\tilde{x}^2(t). \end{cases} \tag{4.2}$$

The corresponding characteristic equation of system (4.2) appears as

$$s^3 - (k_{22} - 1)s^2 - (k_{22} - 1)s + x^{*2} + x^* - 1 + (k_{22}s^2 + k_{22}s)\mathrm{e}^{-s\tau} = 0. \tag{4.3}$$

In what follows we only analyze the case when $x^* = -2.496$ (corresponding to $E_1(-2.496, 0, 0)$) as the analysis for the cases when $x^* = -0.433254$ (corresponding to $E_2(-0.433254, 0, 0)$) and $x^* = 1.40226$ (corresponding to $E_3(1.40226, 0, 0)$) can be done in a similar way. When $x^* = -2.496$, Eq. (4.3) can be expressed as

$$s^3 - (k_{22} - 1)s^2 - (k_{22} - 1)s + 2.626961 + (k_{22}s^2 + k_{22}s)\mathrm{e}^{-s\tau} = 0. \tag{4.4}$$

According to the analysis presented in Section 2, we can obtain that $b = 1 > 0$, $b + a - x^{*2} - x^* = -1.626961 < 0$, $p = {\alpha_2}^2 - {\beta_2}^2 - 2\alpha_1 = b^2 - 2 = -1$, $q = {\alpha_1}^2 - 2\alpha_0\alpha_2 - {\beta_1}^2 = 1 - 2k_{22}b - (x^{*2} + x^* - a)(1 - k_{22}) = -4.253922 + 3.253922k_{22}$, $r = {\alpha_0}^2 = (x^{*2} + x^* - a)^2 = 6.900924$.

Note that $b > 0$, $b + a - x^{*2} - x^* < 0$, from Theorem 2.1 we know that when $\Delta = p^2 - 3q = 13.761766 - 9.761766k_{22} \le 0$, i.e, $k \ge 1.409762$, the equilibrium point of system (4.2) is locally unstable for all $\tau \ge 0$.

Therefore, to achieve control of the chaotic system, we shall consider $k_{22} < 1.409762$. Specially, we may choose $k_{22} = -1$. Then in this case we can calculate that $\Delta = 23.523532$, $\rho(u) = u^3 + pu^2 + qu + r = u^3 - u^2 - 7.507844u + 6.900924$, $u_1 = 0.909161$, $\xi_1 = \sqrt{u_1} = 0.953499$, $u_2 = 2.800868$, $\xi_2 = \sqrt{u_2} = 1.673580$, $\rho'(u_1) = -6.846445$, $\rho'(u_2) = 10.425034$, $u_1^* = \frac{-p+\sqrt{\Delta}}{3} = 1.950035$, $u_2^* = \frac{-p-\sqrt{\Delta}}{3} = -1.283369$, $\rho(u_1^*) = -3.754001$, $\tau_j^{(k)} = \frac{1}{\xi_j}\left\{\arccos\left(\frac{{\xi_j}^2 - 0.626961}{{\xi_j}^2 + 1}\right) + 2k\pi\right\}$, $\tau_1^{(0)} = 1.491809$, $\tau_2^{(0)} = 0.574765$, $\tau_1^{(1)} = 8.801415$, $\tau_2^{(1)} = 4.329103$, $\tau_0 = \min\limits_{j\in\{1,2\}}\left\{\tau_j^{(0)}\right\} = 0.574765$.

It is evident that a single delay feedback term has a little impact on the dynamics of system (4.2) when the value of the delay τ is very small. However, as τ increases, the dynamics of system (4.2) could be affected considerably. According to the analysis presented in Section 2, it is known that Hopf bifurcation occurs when τ reaches the critical value, i.e., $\tau = \tau_j^{(k)}$. Here we take $\tau = \tau_1^{(0)} = 1.491809$ for example, thus we can check that $\Delta > 0$, $u_1^* > 0$, $\rho(u_1^*) < 0$, $\rho'(u_1) \neq 0$. Hence system (4.2) exhibits Hopf bifurcation near the equilibrium point, as shown in Figure 2. In addition, we can determine that $\mu_2 > 0$ and $\delta_2 < 0$ by means of the formulae presented in Section 3, indicating that the bifurcated periodic solution is supercritical and stable. In this circumstance, chaos phenomenon has disappeared. Furthermore, when the delay $\tau = 1.93 \in (\tau_1^{(0)}, \tau_2^{(1)}) = (1.491809, 4.329103)$, all roots of the corresponding characteristic equation have negative real parts and the equilibrium point of system (4.2) is locally stable, as illustrated in Figure 3. Chaos in this sense has been completely controlled.

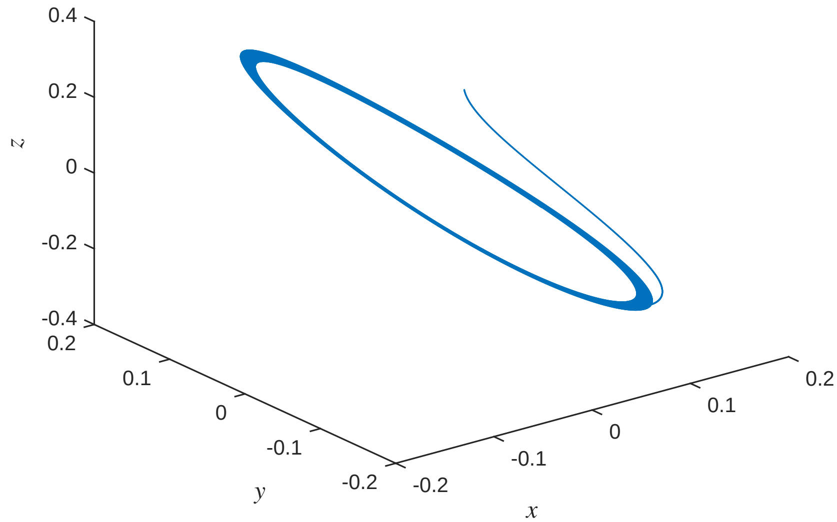

Figure 2. System (4.2) exhibits Hopf bifurcation when $\tau = 1.491809$ and $k_{22} = -1$.

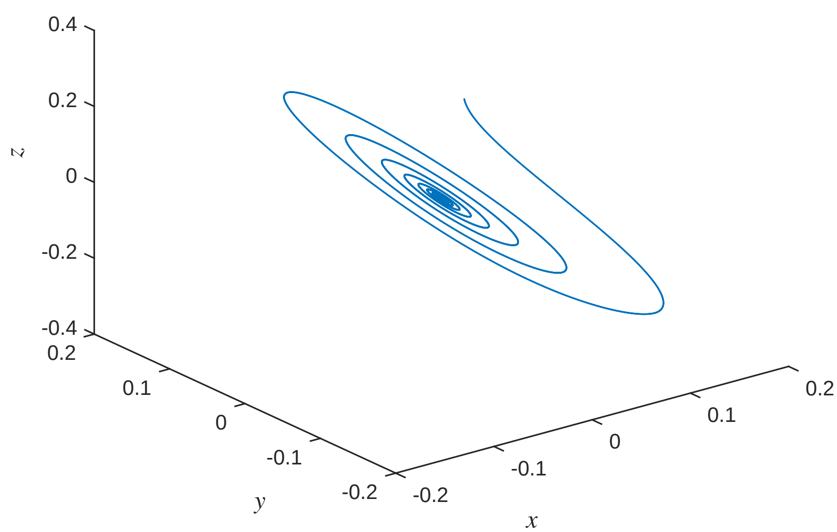

Figure 3. The equilibrium point of system (4.2) is locally stable when $\tau = 1.93$ and $k_{22} = -1$.

5. Conclusion

In this paper, we have studied nonlinear dynamics and chaos control of the novel mKBK chaotic system via a single delay feedback. Some sufficient conditions guaranteeing the local stability and the existence of Hopf bifurcation are obtained by analyzing the corresponding characteristic equation. Explicit formulae which determine the direction of Hopf bifurcation and the stability of bifurcated periodic solutions are derived by means of center manifold theorem and normal form theory. Finally, numerical simulations are carried out to illustrate the effectiveness of the theoretical analysis, which indicate that the proposed delayed feedback control effectively suppresses chaotic behavior when the delay exceeds critical thresholds.

The results may contribute to the growing body of research on chaos control in nonlinear PDEs, particularly in systems with higher-order nonlinearities like the mKBK equation. The success of a single delay feedback highlights its practicality for real-world applications, such as stabilizing fluid flow instabilities, or controlling chaotic oscillations in mechanical systems. Future work in this line could explore the interplay of multiple delays to enhance control robustness, as well as the adaptive delay mechanisms to handle parameter uncertainties.

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