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ON THE PERIODIC ORBITS OF CONTINUOUS THIRD-ORDER DIFFERENTIAL EQUATION WITH PIECEWISE PERTURBATIONS

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Abstract In this paper, we study the sufficient conditions for the existence of periodic solutions of the following differential equation

$$\dddot{x} = -\dot{x} + \varepsilon|\ddot{x}| - \varepsilon\left(\alpha x - \beta\dot{x}\right)^m,$$

where m is a natural number, and α, β and ε are real parameters with $|\varepsilon| > 0$ being small. We apply the averaging method and the Melnikov function method respectively to study the periodic solutions of this type of differential equation. We also provide an example as an application.

Keywords Averaging theory, Melnikov function, periodic solution.

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1. Introduction and statement of the main results

In [13], Llibre et al. used the averaging method to study the periodic orbits analytically of the following differential equation:

$$\dddot{x} = -\dot{x} + \varepsilon|\ddot{x}| - \varepsilon a x^m, \tag{1.1}$$

where m is a natural number and a, ε are real parameters with $|\varepsilon| > 0$ being small.

In the following, we use the averaging method and the Melnikov function method respectively to study the periodic orbits of the following equation:

$$\dddot{x} = -\dot{x} + \varepsilon|\ddot{x}| - \varepsilon\left(\alpha x - \beta\dot{x}\right)^m, \tag{1.2}$$

where m is a natural number, and α, β and ε are real parameters with $|\varepsilon| > 0$ being small. This equation is a generalization of (1.1) and is of the form $\dddot{x} = J\left(\ddot{x}, \dot{x}, x\right)$ which is the so-called jerk equation. It is well known that jerk equation can describe some physical phenomena. Many researchers have shown their interest in the study of periodic solutions of nonlinear jerk equations, such as in [10, 18] and references therein.

To our knowledge, the averaging theory is one of the useful tools for obtaining periodic solutions of differential equations, see for instance, [1, 3, 4, 6, 15, 21]. The Melnikov function method also plays an important role in the study of the number of periodic solutions of differential equations, such as in [5, 8, 9, 12] and references therein. The authors [7] proved that these two methods are equivalent in the study of the number of limit cycles of planar analytic or C^∞

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near-Hamiltonian systems. The authors [11] established the Melnikov function method and the averaging method for finding limit cycles of piecewise smooth near-integrable systems in arbitrary dimension. Further, they showed that these two methods are also equivalent for higher dimensional systems.

The differential equation (1.2) is equivalent to the following differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y + \varepsilon|z| - \varepsilon\left(\alpha x - \beta y\right)^m. \end{cases} \tag{1.3}$$

In this paper, we are mainly interested in discussing how the number of periodic solutions of system (1.3) depends on the parameters α and β, as well as the exponent m. The main results are stated in Theorem 1.1. Later we will prove the following theorem by means of the averaging method and the Melnikov function method, respectively.

Theorem 1.1. *For $|\varepsilon| \neq 0$ sufficiently small, the differential system (1.3) satisfies the following statements:*

If m is even and $\alpha \neq 0$, then by using the first order Melnikov vector function method or the first order averaging theory the system (1.3) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ bifurcating from the periodic solutions of system $(1.3)|_{\varepsilon=0}$ such that

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (-\rho_0^* \cos t, \rho_0^* \sin t, \rho_0^* \cos t) + O(\varepsilon)$$

with

$$\rho_0^* = \left(\frac{2m!!}{\pi\,(m-1)!!\,(\alpha^2+\beta^2)^{m/2}}\right)^{\frac{1}{m-1}}.$$

If m is odd, or m is even and $\alpha = 0$, then the first order Melnikov vector function method or the first order averaging theory cannot detect the periodic solutions of the system (1.3).

The rest of this paper is organized as follows. In Section 2, some preliminaries about the first order averaging theory are presented. In Section 3, we show some preliminary results about the Melnikov vector function method and the averaging theory of perturbed piecewise smooth systems in arbitrary dimension. In Section 4, we prove our main results by using the methods given in Section 2 and Section 3, respectively. In Section 5, an example is provided as an application.

2. The first order averaging theory for differential equations

In this section of this paper we present some preliminaries about the first order averaging theory that we need to find periodic solutions of differential equations.

We consider the following system

$$\dot{x} = \mathcal{F}_0\,(t,x) + \varepsilon \mathcal{F}_1\,(t,x) + \varepsilon^2 \mathcal{R}\,(t,x,\varepsilon)\,, \tag{2.1}$$

where the functions $\mathcal{F}_i : \mathbb{R} \times \mathbf{\Omega} \to \mathbb{R}^n$ for $i = 0, 1$, $\mathcal{R} : \mathbb{R} \times \mathbf{\Omega} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous and $T-$periodic in the first variable, the parameter $|\varepsilon| \neq 0$ is small and $\mathbf{\Omega}$ is an open subset of $\mathbb{R}^n$. Suppose that $\mathcal{F}_0$ is of class C^1, $D\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{R}$ are locally Lipschitz in the second variable.

Now we consider the system

$$\dot{x} = \mathcal{F}_0\left(t, x\right), \tag{2.2}$$

which is called the unperturbed system of (2.1). We suppose that there exists a submanifold of periodic solutions of this system. Let $x(t, z, 0)$ be solution of system (2.2) and V be an open set with $Cl\left(V\right) \subset \mathbf{\Omega}$ such that for each $z \in Cl\left(V\right)$, the solution $x(t, z, 0)$ of the unperturbed system (2.2) is $T-$periodic, where $Cl\left(V\right)$ is defined as the closure of V.

Define $x(t, z, \varepsilon)$ as the solution of system (2.1) that satisfies the initial condition $x(0, z, \varepsilon) = z$. The linearization of system (2.2) about the solution $x(t, z, 0)$ has the form

$$\dot{y} = D_x\mathcal{F}_0\left(t, x\left(t, z, 0\right)\right)y. \tag{2.3}$$

Denote by $M_z(t)$ the fundamental matrix of the linearized system (2.3) such that $M_z(0) = I_n$ is $n \times n$ identity matrix.

The following result provides a way for finding periodic solutions of the differential system (2.1). It can be found in [14].

Theorem 2.1. ([14]) *Consider differential system (2.1). We suppose that there exists an open and bounded set V with $Cl\left(V\right) \subset \mathbf{\Omega}$ such that for each $z \in Cl\left(V\right)$, the solution $x(t, z, 0)$ of system (2.2) is T-periodic. Let $f : Cl\left(V\right) \to \mathbb{R}^n$ be the first order averaged function*

$$f(z) = \frac{1}{T}\int_0^T M_z^{-1}(t)\mathcal{F}_1(t, x(t, z, 0))dt.$$

For each $\mathbf{a} \in V$ satisfying $f(\mathbf{a}) = 0$ there exists a neighborhood U of $\mathbf{a}$ such that $f(z) \neq 0$ for all $z \in \bar{U}\backslash\{\mathbf{a}\}$ and $d_B(f, U, 0) \neq 0$, where $d_B(f, U, 0)$ be the Brouwer degree of f at $\mathbf{a}$. Then for $|\varepsilon| \neq 0$ sufficiently small, the zero $\mathbf{a}$ provide $T-$periodic solution $x(t, \varepsilon)$ of system (2.1) such that $x(0, \varepsilon) \to \mathbf{a}$ as $\varepsilon \to 0$.

In the references [17] and [19] we found the first version of Theorem 2.1 for C^2 differential systems. In [2], Buica et al. set out a more shorter proof of this result.

Remark 2.1. If $f(z)$ is C^1 with $\det(Df(\mathbf{a})) \neq 0$, then $d_B(f, U, 0) \neq 0$, see [16].

3. The first order Melnikov vector function method for piecewise smooth systems

In this section, we show the first order Melnikov vector function method of perturbed piecewise smooth systems in arbitrary dimension.

We first introduce the following definition given in [20].

Definition 3.1. ([20]) Let S be an $(n-1) \times n$ $(n \geq 2)$ matrix. We define $\bar{S}$ to be the $(n-1) \times (n-1)$ matrix satisfying $S = (\beta, \bar{S})$, where $\beta \in \mathbb{R}^n$ is the first column of S.

Next, we consider the following n-dimensional piecewise smooth near-integral system

$$\dot{\mathbf{x}} = \begin{cases} f^+(\mathbf{x}) + \varepsilon g^+(\mathbf{x}), & x_1 \geq 0, \quad (3.1a) \\ f^-(\mathbf{x}) + \varepsilon g^-(\mathbf{x}), & x_1 < 0, \quad (3.1b) \end{cases} \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, $f^\pm$ and $g^\pm$ are C^∞ vector functions defined on an open set $U \subset \mathbf{R}^n$ with $U \bigcap \{x_1 = 0\} \neq \varnothing$, $0 \leq \varepsilon \ll 1$.

We assume that the following basic assumptions hold for the unperturbed system $(3.1)|_{\varepsilon=0}$ as in [11, 20].

(H1) System $(3.1a)|_{\varepsilon=0}$ ($(3.1b)|_{\varepsilon=0}$, resp.) has $n-1$ different C^∞ first integrals $H_i^+(\mathbf{x})$ ($H_i^-(\mathbf{x})$, resp.), $i=1,2,\cdots,n-1$, such that for each $\mathbf{x}\in U^+$ ($\mathbf{x}\in U^-$, resp.), the gradients

$$DH_1^+, DH_2^+, \cdots, DH_{n-1}^+ (DH_1^-, DH_2^-, \cdots, DH_{n-1}^-, resp.)$$

are linearly independent, where $U^+=\{\mathbf{x}\in U | x_1\geq 0\}$ ($U^-=\{\mathbf{x}\in U | x_1<0\}$, resp.).

(H2) Let $\mathbf{H}^\pm(\mathbf{x})=(H_1^\pm(\mathbf{x}), H_2^\pm(\mathbf{x}),\cdots,H_{n-1}^\pm(\mathbf{x}))^T$. There exists an open set $V\subset\mathbb{R}^{n-1}$ such that for each $h\equiv(h_1,h_2,\cdots,h_{n-1})^T\in V$, the curves $L_h^+=\{\mathbf{x}\in U^+|\mathbf{H}^+(\mathbf{x})=h\}$ and $L_h^-=\{\mathbf{x}\in U^-|\mathbf{H}^-(\mathbf{x})=\mathbf{H}^-(\mathbf{A}(\mathbf{h}))\}$ contain no critical point of $(3.1)|_{\varepsilon=0}$ and have two different end points $A(h)$ and $B(h)$ in common satisfying

$$A(h)=(0,a_2(h),\cdots,a_n(h))^T\in U,\ \ B(h)=(0,b_2(h),\cdots,b_n(h))^T\in U.$$

The orbit L_h^+ starts from $A(h)$ and ends at $B(h)$, and L_h^- starts from $B(h)$ and ends at $A(h)$. Thus, $L_h=L_h^+\bigcup L_h^-$ is a closed orbit of (3.1) for $h\in V$.

(H3) The curves $L_h^\pm$, $h\in V$ are not tangent to the switch plane $x_1=0$ at points $A(h)$ and $B(h)$. In other words, for each $h\in V$,

$$J^\pm(x_1,x_2,\cdots,x_n)=\det\frac{\partial(H_1^\pm,H_2^\pm,\cdots,H_{n-1}^\pm)}{\partial(x_2,x_3,\cdots,x_n)}$$

is not equal to zero at each of the points $A(h)$ and $B(h)$.

The authors [11, 20] gave a definition of bifurcation function of system (3.1). From [11], for any integer $k\geq 1$, the bifurcation function $F(h,\varepsilon)$ can be written as

$$F(h,\varepsilon)=\sum_{i=1}^{k}\varepsilon^{i-1}M_i(h)+O(\varepsilon^k) \tag{3.2}$$

for $0<\varepsilon\ll 1$. In the above formula, M_i is called the ith order Melnikov vector function, which plays an important role in studying the number of limit cycles of differential equations. The authors [20] provided an expression of the first order Melnikov vector function $M_1(h)$ and the sufficient conditions for system (3.1) to have periodic orbits as follows.

Lemma 3.1. ([20]) *Let system (3.1) satisfy (H1)-(H3). Then the first order Melnikov vector function $M_1(h)$ has an expression below*

$$M_1(h)=\int_{\widehat{AB}}D\mathbf{H}^+g^+dt+\overline{D\mathbf{H}^+(A)}\left[\overline{D\mathbf{H}^-(A)}\right]^{-1}\int_{\widehat{BA}}D\mathbf{H}^-g^-dt.$$

Further, if $M_1(h_0)=0$ and $\det DM_1(h_0)\neq 0$ for some $h_0\in G$, then for sufficiently small $\varepsilon>0$ there exists a unique periodic orbit near L_{h_0} for system (3.1).

Let

$$\begin{aligned} L_h^+:\ \ \mathbf{x}=q^+(t,h),\ \ 0\leq t\leq T_1(h),\\ L_h^-:\ \ \mathbf{x}=q^-(t,h),\ \ T_1(h)<t\leq T(h)\end{aligned} \tag{3.3}$$

satisfy $q^+(0,h) = A(h)$, $q^+(T_1(h),h) = q^-(T_1(h),h) = B(h)$, and $q^-(T(h),h) = A(h)$, where $T_1(h)$ is defined as the time from $A(h)$ to $B(h)$ along L_h^+, and $T(h)$ is defined as the minimal positive period of the periodic orbit L_h. Define

$$G(\theta,h) = \begin{cases} G^+(\theta,h), & 0 \le \theta \le \theta_1(h), \\ G^-(\theta,h), & \theta_1(h) < \theta \le 2\pi, \end{cases} \tag{3.4}$$

where

$$G^+(\theta,h) = q^+\left(\frac{T(h)}{2\pi}\theta, h\right), \quad G^-(\theta,h) = q^-\left(\frac{T(h)}{2\pi}\theta, h\right).$$

From [11] there exists a variable transformation

$$\mathbf{x} = G(\theta,h), \quad 0 \le \theta \le 2\pi, \quad h \in V,$$

such that system (3.1) carries into a 2π periodic differential equation

$$\frac{dh}{d\theta} = \varepsilon R(\theta,h,\varepsilon), \tag{3.5}$$

where $R(\theta,h,\varepsilon)$ is a piecewise C^∞ smooth function.

For $h_0 \in V$, let $h(\theta,h_0,\varepsilon)$ be the solution of (3.5) satisfying $h(0,h_0,\varepsilon) = h_0$ for $\theta \in [0,2\pi]$. The Poincaré map of (3.5) has the form

$$P(h_0,\varepsilon) = h(2\pi,h_0,\varepsilon) = h_0 + \varepsilon d(h_0,\varepsilon),$$

where $d(h_0,\varepsilon)$ is called a bifurcation function.

From [11], for any integer $k \ge 1$, $d(h_0,\varepsilon)$ can be expressed as

$$d(h_0,\varepsilon) = \sum_{i=1}^{k} \varepsilon^{i-1} f_i(h_0) + O(\varepsilon^k) \tag{3.6}$$

for $0 < \varepsilon \ll 1$, where f_i is called the ith order averaged function. From [11], the first order averaged function $f_1(h)$ has the form

$$\begin{aligned} f_1(h) = &\int_0^{\theta_1(h)} \frac{T(h)}{2\pi} DH^+(G^+)g^+(G^+)d\theta \\ &+ \int_{\theta_1(h)}^{2\pi} \frac{T(h)}{2\pi}\overline{DH^+(A)}\left[\overline{DH^-(A)}\right]^{-1} DH^-(G^-)g^-(G^-)d\theta. \end{aligned} \tag{3.7}$$

Suppose that system (3.1) satisfies assumptions (H1)-(H3). The authors [11] proved that the averaging method and the Melnikov function method are equivalent. That is, if for a given integer $k \ge 1$, $f_k(h) \not\equiv 0$, $f_j(h) \equiv 0$, $j = 1,2,\cdots,k-1$, then the kth order Melnikov function M_k defined in (3.2) and the kth order averaged function f_k defined in (3.6) satisfy $f_k(h) = M_k(h)$. In particular,

$$M_1(h) = f_1(h). \tag{3.8}$$

4. Proofs of Theorem 1.1

In this section, we prove Theorem 1.1 by using the methods given in Section 2 and Section 3, respectively.

The first method. First, we suppose that $m = 2n$ with n a positive integer. By means of a cylindrical change of coordinates $(x, y, z) = (x, \rho\sin\theta, \rho\cos\theta)$ with $\rho > 0$, the differential system (1.3) is written as

$$\begin{cases} \dot{x} = \rho\sin\theta, \\ \dot{\rho} = \varepsilon\cos\theta\left(|\rho\cos\theta| - ((\alpha x - \beta\rho\sin\theta))^{2n}\right), \\ \dot{\theta} = 1 + \dfrac{\varepsilon}{\rho}\sin\theta\left((\alpha x - \beta\rho\sin\theta)^{2n} - |\rho\cos\theta|\right), \end{cases} \tag{4.1}$$

or, it is equivalent to the following system with the new independent variable θ

$$\begin{cases} \dfrac{dx}{d\theta} = x' = \rho\sin\theta + \varepsilon\sin^2\theta\left(|\rho\cos\theta| - (\alpha x - \beta\rho\sin\theta)^{2n}\right) + O\left(\varepsilon^2\right), \\ \dfrac{d\rho}{d\theta} = \rho' = \varepsilon\cos\theta\left(|\rho\cos\theta| - (\alpha x - \beta\rho\sin\theta)^{2n}\right) + O\left(\varepsilon^2\right), \end{cases} \tag{4.2}$$

where the prime denotes the derivative with respect to θ. The unperturbed system of (4.2) is

$$\begin{cases} x' = \rho\sin\theta, \\ \rho' = 0. \end{cases} \tag{4.3}$$

We denote by $\psi(\theta, x_0, \rho_0)$ the solution of the unperturbed system (4.3) satisfying $\psi(0, x_0, \rho_0) = (x_0, \rho_0)$. Then, we have

$$\psi(\theta, x_0, \rho_0) = (x_0 + \rho_0(1 - \cos\theta), \rho_0).$$

Now, note that for all $\rho_0 > 0$, the solution $\psi(\theta, x_0, \rho_0)$ is $2\pi-$periodic. The fundamental matrix associated to the unperturbed system (4.3) is

$$M_{(x_0,\rho_0)}(\theta) = M(\theta) = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}.$$

Note that $M(0) = I_2$, where I_2 is the 2×2 identity matrix. Then, by Theorem 2.1 we need to calculate the zeros of the first order averaged function

$$f(x_0, \rho_0) = \frac{1}{2\pi}\int_0^{2\pi} M(\theta)^{-1}\mathcal{F}_1(\theta, \psi(\theta, x_0, \rho_0))\, d\theta$$

with

$$\mathcal{F}_1(\theta, (x, \rho)) = \begin{pmatrix} \sin^2\theta\left(|\rho\cos\theta| - (\alpha x - \beta\rho\sin\theta)^{2n}\right) \\ \cos\theta\left(|\rho\cos\theta| - (\alpha x - \beta\rho\sin\theta)^{2n}\right) \end{pmatrix}.$$

Thus, we have

$$
\begin{aligned}
f\left(x_0,\rho_0\right) &= \begin{pmatrix} f_1\left(x_0,\rho_0\right) \\ f_2\left(x_0,\rho_0\right) \end{pmatrix} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} (1-\cos\theta)\left(\rho_0|\cos\theta| - (\alpha\rho_0+\alpha x_0-\alpha\rho_0\cos\theta-\beta\rho_0\sin\theta)^{2n}\right) \\ \cos\theta\left(\rho_0|\cos\theta| - (\alpha\rho_0+\alpha x_0-\alpha\rho_0\cos\theta-\beta\rho_0\sin\theta)^{2n}\right) \end{pmatrix} d\theta.
\end{aligned}
\tag{4.4}
$$

Using simple calculations, we find that

$$
\int_0^{2\pi} \rho_0\cos\theta|\cos\theta| d\theta = 0.
$$

For $\rho_0 > 0$ we have that

$$
\begin{aligned}
f_2\left(x_0,\rho_0\right) &= -\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\left(\alpha\rho_0+\alpha x_0-\alpha\rho_0\cos\theta-\beta\rho_0\sin\theta\right)^{2n} d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \rho_0^{2n}\cos\theta\left(\frac{\alpha\rho_0+\alpha x_0}{\rho_0}-\alpha\cos\theta-\beta\sin\theta\right)^{2n} d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \rho_0^{2n}\sum_{k=0}^{2n}(-1)^k C_{2n}^k \alpha^{2n-k} \\
&\quad\times\left(\frac{\rho_0+x_0}{\rho_0}\right)^{2n-k} \cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^k d\theta.
\end{aligned}
$$

If $k=2p$, then $\int_0^{2\pi}\cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2p} d\theta = 0$. So $f_2\left(x_0,\rho_0\right)$ becomes

$$
\begin{aligned}
f_2\left(x_0,\rho_0\right) &= \frac{1}{2\pi}\sum_{p=0}^{n-1} C_{2n}^{2p+1}\rho_0^{2n}\alpha^{2n-2p-1}\left(\frac{\rho_0+x_0}{\rho_0}\right)^{2n-2p-1} \\
&\quad\times\int_0^{2\pi}\cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2p+1} d\theta \\
&= \frac{1}{2\pi}\sum_{p=0}^{n-1} C_{2n}^{2p+1}\rho_0^{2p+1}\alpha^{2n-2p-1}\left(\rho_0+x_0\right)^{2n-2p-1} \\
&\quad\times\int_0^{2\pi}\cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2p+1} d\theta \\
&= \sum_{p=0}^{n-1} C_{2n}^{2p+1}\rho_0^{2p+1}\alpha^{2n-2p-1}\left(\rho_0+x_0\right)^{2n-2p-1} \\
&\quad\times\left(\alpha\frac{(2p+1)!!}{(2p+2)!!}\left(\alpha^2+\beta^2\right)^p\right).
\end{aligned}
$$

We note that the function $f_2\left(x_0,\rho_0\right) = 0$ at $x_0 = -\rho_0$. We put $x_0 = -\rho_0$ in the function $f_1\left(x_0,\rho_0\right)$ we obtain

$$
f_1\left(-\rho_0,\rho_0\right) = \frac{1}{2\pi}\int_0^{2\pi} -\rho_0\cos\theta|\cos\theta| + \rho_0|\cos\theta| d\theta
$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\rho_0^{2n}(1-\cos\theta)\left(\alpha\cos\theta+\beta\sin\theta\right)^{2n}d\theta$$
$$=\frac{1}{\pi}\left(2\rho_0-\pi\rho_0^{2n}\frac{(2n-1)!!}{(2n)!!}\left(\alpha^2+\beta^2\right)^n\right).$$

Solving $f_1\left(-\rho_0,\rho_0\right)=0$ with respect to ρ_0, we get the following solution

$$\rho_0^*=\left(\frac{2\,(2n)!!}{\pi\,(2n-1)!!\,(\alpha^2+\beta^2)^n}\right)^{\frac{1}{2n-1}}.$$

Note that $\alpha^2+\beta^2\neq0$ ensures the existence of the solution $\rho_0^*>0$. We conclude that if $m=2n$ and $\alpha^2+\beta^2\neq0$, then the first order averaged function $f\left(x_0,\rho_0\right)$ has a unique zero $\left(-\rho_0^*,\rho_0^*\right)$ with $\rho_0^*>0$.

To apply the averaging theory of first order, we will show that

$$\det\frac{\partial\left(f_1,f_2\right)}{\partial\left(x_0,\rho_0\right)}\bigg|_{(x_0,\rho_0)=\left(-\rho_0^*,\rho_0^*\right)}\neq0. \tag{4.5}$$

By calculating the partial derivative of $f_2\left(x_0,\rho_0\right)$ with respect to x_0 and ρ_0, we obtain

$$\frac{\partial f_2}{\partial x_0}\left(-\rho_0,\rho_0\right)=\frac{\partial f_2}{\partial\rho_0}\left(-\rho_0,\rho_0\right)$$
$$=\frac{\alpha}{2\pi}C_{2n}^{2n-1}\rho_0^{2n-1}\int_0^{2\pi}\cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2n-1}d\theta$$
$$=\alpha^2\frac{2n\,(2n-1)!!}{(2n)!!}\rho_0^{2n-1}\left(\alpha^2+\beta^2\right)^{n-1}.$$

Analogously by calculating the partial derivative of $f_1\left(x_0,\rho_0\right)$, we obtain

$$\frac{\partial f_1}{\partial x_0}\left(-\rho_0,\rho_0\right)=-\frac{\partial f_2}{\partial x_0}\left(-\rho_0,\rho_0\right),$$
$$\frac{\partial f_1}{\partial\rho_0}\left(-\rho_0,\rho_0\right)=\frac{2}{\pi}-\frac{\partial f_2}{\partial\rho_0}\left(-\rho_0,\rho_0\right)-\frac{1}{2\pi}\int_0^{2\pi}2n\rho_0^{2n-1}\left(\alpha\cos\theta+\beta\sin\theta\right)^{2n}d\theta.$$

Using the above partial derivatives with substituting ρ_0 by ρ_0^*, we find the following matrix

$$Df\left(-\rho_0^*,\rho_0^*\right)=\begin{pmatrix}-\frac{\partial f_2}{\partial\rho_0}\left(-\rho_0^*,\rho_0^*\right) & \begin{array}{c}\frac{2}{\pi}-\frac{\partial f_2}{\partial\rho_0}\left(-\rho_0^*,\rho_0^*\right)\\ -\frac{1}{2\pi}\int_0^{2\pi}2n\left(\rho_0^*\right)^{2n-1}\left(\alpha\cos\theta+\beta\sin\theta\right)^{2n}d\theta\end{array}\\ \frac{\partial f_2}{\partial\rho_0}\left(-\rho_0^*,\rho_0^*\right) & \frac{\partial f_2}{\partial\rho_0}\left(-\rho_0^*,\rho_0^*\right),\end{pmatrix},$$

the determinant of this matrix is

$$-\frac{\partial f_2}{\partial\rho_0}\left(-\rho_0^*,\rho_0^*\right)\left(\frac{2}{\pi}-\frac{1}{2\pi}\int_0^{2\pi}2n\left(\rho_0^*\right)^{2n-1}\left(\alpha\cos\theta+\beta\sin\theta\right)^{2n}d\theta\right)$$
$$=-\alpha^2\frac{2n\,(2n-1)!!}{(2n)!!}\left(\rho_0^*\right)^{2n-1}\left(\alpha^2+\beta^2\right)^{n-1}\left(\frac{2}{\pi}-2n\left(\rho_0^*\right)^{2n-1}\frac{(2n-1)!!}{(2n)!!}\left(\alpha^2+\beta^2\right)^n\right)$$
$$=-\frac{8n\alpha^2}{\pi^2\left(\alpha^2+\beta^2\right)}\left(1-2n\right)$$
$$\neq0$$

when $\alpha \neq 0$ for all natural number n.

Hence, if $\alpha \neq 0$, then (4.5) holds. It follows from Theorem 2.1 and Remark 2.1 that the system (4.2) has the periodic solution of the form

$$(x(\theta,\varepsilon),\rho(\theta,\varepsilon)) = (-\rho_0^* \cos\theta, \rho_0^*) + O(\varepsilon).$$

By means of the previous cylindrical change of coordinates, the system (1.3) has the periodic solution of the form

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (-\rho_0^* \cos\theta, \rho_0^* \sin t, \rho_0^* \cos t) + O(\varepsilon).$$

Now we study the case $m = 2n+1$. Similarly, we have

$$\begin{aligned}
f_2(x_0,\rho_0) &= -\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\left(\alpha\rho_0 + \alpha x_0 - \alpha\rho_0\cos\theta - \beta\rho_0\sin\theta\right)^{2n+1} d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \rho_0^{2n+1}\cos\theta\left(\frac{\alpha\rho_0+\alpha x_0}{\rho_0} - \alpha\cos\theta - \beta\sin\theta\right)^{2n+1} d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \rho_0^{2n+1}\sum_{k=0}^{2n+1}(-1)^k C_{2n+1}^{k}\alpha^{2n+1-k} \\
&\quad \times \left(\frac{\rho_0+x_0}{\rho_0}\right)^{2n+1-k} \cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^k d\theta.
\end{aligned}$$

If $k = 2p$, then $\int_0^{2\pi}\cos\theta\,(\cos\theta+\sin\theta)^{2p}\,d\theta = 0$ as before. So the function $f_2(x_0,\rho_0)$ becomes

$$\begin{aligned}
f_2(x_0,\rho_0) &= \frac{1}{2\pi}\int_0^{2\pi} \rho_0^{2n+1}\sum_{p=0}^{n} C_{2n+1}^{2p+1}\alpha^{2n-2p} \\
&\quad \times \left(\frac{\rho_0+x_0}{\rho_0}\right)^{2n-2p} \cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2p+1} d\theta \\
&= \frac{\rho_0}{2\pi}\sum_{p=0}^{n} C_{2n+1}^{2p+1}\alpha^{2n-2p} \\
&\quad \times \rho_0^{2p}\left(\rho_0+x_0\right)^{2n-2p}\int_0^{2\pi}\cos\theta\left(\alpha\cos\theta+\beta\sin\theta\right)^{2p+1} d\theta \\
&= \rho_0\sum_{p=0}^{n} C_{2n+1}^{2p+1}\rho_0^{2p}\alpha^{2n-2p}\left(\rho_0+x_0\right)^{2n-2p}\left(\alpha\frac{(2p+1)!!}{(2p+2)!!}\left(\alpha^2+\beta^2\right)^p\right).
\end{aligned}$$

Therefore, when $\rho_0 > 0$, for $\alpha \neq 0$ the function $f_2(x_0,\rho_0) \neq 0$ and the first order averaged function $f(x_0,\rho_0)$ do not accept real zeros.

We conclude that if $m = 2n+1$ the first order averaging theory does not detect any periodic solution of the differential system (1.3). This completes the proof of Theorem 1.1.

The second method. Take the change of variables

$$x = w + u, \quad y = u, \quad z = -v.$$

Then the system (1.3) can be transformed into the following 3-dimensional piecewise smooth system

$$\begin{cases} \dot{v} = u + \varepsilon[-v + (\alpha w + \alpha v - \beta u)^m], \\ \dot{u} = -v, & v \geq 0, \\ \dot{w} = \varepsilon[v - (\alpha w + \alpha v - \beta u)^m], \end{cases}$$
$$\begin{cases} \dot{v} = u + \varepsilon[v + (\alpha w + \alpha v - \beta u)^m], \\ \dot{u} = -v, & v < 0, \\ \dot{w} = \varepsilon[-v - (\alpha w + \alpha v - \beta u)^m], \end{cases} \tag{4.6}$$

where $\varepsilon > 0$ is a small parameter. For $\varepsilon = 0$, system (4.6) has 2 different C^∞ first integrals $H_1 = \frac{1}{2}(v^2 + u^2)$ and $H_2(w) = w$. By (3.3) and (3.4), we have

$$G(\theta, h) = (\sqrt{2h_1}\sin\theta, \sqrt{2h_1}\cos\theta, h_2)^T, \quad 0 \leq \theta \leq 2\pi.$$

It follows from (3.7) and (3.8) that the first order Melnikov vector function M_1 of system (4.6) has an expression of the form

$$M_1(h_1, h_2) = f_1(h_1, h_2) = \begin{pmatrix} f_{11}(h_1, h_2) \\ f_{12}(h_1, h_2) \end{pmatrix} = \begin{pmatrix} f_{11}^+(h_1, h_2) + f_{11}^-(h_1, h_2) \\ f_{12}^+(h_1, h_2) + f_{12}^-(h_1, h_2) \end{pmatrix}, \tag{4.7}$$

where

$$\begin{pmatrix} f_{11}^+(h_1, h_2) \\ f_{12}^+(h_1, h_2) \end{pmatrix} = \int_0^\pi DH(G)g^+(G)d\theta,$$
$$\begin{pmatrix} f_{11}^-(h_1, h_2) \\ f_{12}^-(h_1, h_2) \end{pmatrix} = \int_\pi^{2\pi} DH(G)g^-(G)d\theta,$$

with

$$g^+(G) = \begin{pmatrix} -\sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m \\ 0 \\ \sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m \end{pmatrix},$$
$$g^-(G) = \begin{pmatrix} \sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m \\ 0 \\ -\sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m \end{pmatrix}.$$

Then by direct computations we get

$$f_{11}(h_1, h_2) = \sqrt{2h_1}\int_0^\pi \Big(- \sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m\Big)$$

$$\times \sin\theta d\theta + \sqrt{2h_1}\int_{\pi}^{2\pi}\left(\sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta\right.$$
$$\left. -\beta\sqrt{2h_1}\cos\theta)^m\right)\sin\theta d\theta$$
$$= \sqrt{2h_1}\int_0^{2\pi}(\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m \sin\theta d\theta$$
$$= \sqrt{2h_1}\int_0^{2\pi}\sum_{k=0}^{m} C_m^k(\alpha h_2)^{m-k}(2h_1)^{\frac{k}{2}}(\alpha\sin\theta - \beta\cos\theta)^k \sin\theta d\theta. \tag{4.8}$$

Note that $\int_0^{2\pi}(\alpha\sin\theta - \beta\cos\theta)^k \sin\theta d\theta = \left[1-(-1)^k\right]\frac{k!!}{(k+1)!!}\pi\alpha(\alpha^2+\beta^2)^{\frac{k-1}{2}}$. Then we have

$$f_{11}(h_1,h_2) = \alpha\sum_{k=0}^{m} C_m^k(\alpha h_2)^{m-k}(2h_1)^{\frac{k+1}{2}}\frac{\left[1-(-1)^k\right]k!!}{(k+1)!!}\pi(\alpha^2+\beta^2)^{\frac{k-1}{2}}. \tag{4.9}$$

Similarly, by certain calculations, we get

$$f_{12}(h_1,h_2) = \int_0^{\pi}\left(\sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m\right)d\theta$$
$$+\int_{\pi}^{2\pi}\left(-\sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m\right)d\theta$$
$$= 4\sqrt{2h_1} - \int_0^{2\pi}(\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^m d\theta$$
$$= 4\sqrt{2h_1} - \int_0^{2\pi}\sum_{k=0}^{m} C_m^k(\alpha h_2)^{m-k}(2h_1)^{\frac{k}{2}}(\alpha\sin\theta - \beta\cos\theta)^k d\theta. \tag{4.10}$$

Note that $\int_0^{2\pi}(\alpha\sin\theta - \beta\cos\theta)^k d\theta = \left[1-(-1)^{k+1}\right]\frac{(k-1)!!}{k!!}\pi(\alpha^2+\beta^2)^{\frac{k}{2}}$. Then we obtain

$$f_{12}(h_1,h_2) = 4\sqrt{h_1} - \sum_{k=0}^{m} C_m^k(\alpha h_2)^{m-k}(2h_1)^{\frac{k}{2}}\left[1-(-1)^{k+1}\right]\frac{(k-1)!!}{k!!}\pi(\alpha^2+\beta^2)^{\frac{k}{2}}. \tag{4.11}$$

From (4.9) it is not hard to see that $f_{11}(h_1,h_2) = 0$ as $\alpha = 0$. Therefore, in this case, the first order Melnikov vector function method or the first order averaging theory cannot detect the periodic solutions of system (4.6).

As $\alpha \neq 0$, if m is an odd number, then from (4.9) it can be seen that $f_{11}(h_1,h_2)$ and α have the same sign. Hence, there are no h_1 and h_2 such that f_{11} is equal to 0. It means that at this point the first order Melnikov vector function method or the first order averaging theory also fails to detect the periodic solutions of system (4.6).

If m is an even number, then it is easy to check that $f_{11}(h_1,h_2)$ and h_2 have the same sign. Thus, $f_{11}(h_1,h_2) = 0$ only when h_2 is equal to 0. Substituting $h_2 = 0$ into formula (4.11) yields $h_1 = \frac{1}{2}\left(\frac{2m!!}{(m-1)!!\pi(\alpha^2+\beta^2)^{\frac{m}{2}}}\right)^{\frac{2}{m-1}}$.

From the above discussion, it can be seen that as $\alpha \neq 0$ and m is even equations

$$f_{11}(h_1,h_2) = f_{12}(h_1,h_2) = 0$$

have a unique solution

$$h_{10} = \frac{1}{2}\left(\frac{2m!!}{(m-1)!!\pi(\alpha^2+\beta^2)^{\frac{m}{2}}}\right)^{\frac{2}{m-1}}, \quad h_{20} = 0.$$

Further, we calculate

$$\begin{aligned}
&\det \left.\frac{\partial(f_{11}, f_{12})}{\partial(h_1, h_2)}\right|_{(h_{10},h_{20})} \\
&= \det \begin{pmatrix} 0 & \frac{2m(m-1)!!}{m!!}\pi\alpha^2(\alpha^2+\beta^2)^{\frac{m}{2}-1}(2h_{10})^{\frac{m}{2}} \\ 2^{\frac{1}{2}} h_{10}^{-\frac{1}{2}} - \frac{m(m-1)!!}{m!!}\pi(\alpha^2+\beta^2)^{\frac{m}{2}} 2^{\frac{m}{2}} h_{10}^{\frac{m}{2}-1} & 0 \end{pmatrix} \\
&= \frac{8m(2m-1)\alpha^2}{\alpha^2+\beta^2} \neq 0
\end{aligned}$$

when $\alpha \neq 0$. Thus, by Lemma 3.1, for $0 < |\varepsilon| \ll 1$ and m even, system (4.6) has a periodic orbit if $\alpha \neq 0$.

Remark 4.1. From the above proof process, it can be seen that the proof of the second method is simpler and requires less computation.

5. Example

In this section we provide an example as an application of our main results.

Example 5.1. Consider the following perturbed system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -y + \varepsilon|z| - \varepsilon\left(x - \dot{x}\right)^2, \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is a small parameter. Note that system (5.1) is obtained by taking $\alpha = \beta = 1$ and $m = 2$ in (1.3).

The first method. By applying the previous cylindrical change of coordinates, the system (5.1) becomes

$$\begin{cases} \dot{x} = \rho\sin\theta, \\ \dot{\rho} = \varepsilon\cos\theta\left(|\rho\cos\theta| - ((x-\rho\sin\theta))^2\right), \\ \dot{\theta} = 1 + \frac{\varepsilon}{\rho}\sin\theta\left((x-\rho\sin\theta)^2 - |\rho\cos\theta|\right). \end{cases} \tag{5.2}$$

According to the proof process of the first method of Theorem 1.1 and by simple calculations, we have

$$\begin{aligned}
f_1\left(x_0, \rho_0\right) &= \frac{1}{\pi}\left(2\,\rho_0 - 3\pi\rho_0 x_0 - 3\pi{\rho_0}^2 - \pi{x_0}^2\right), \\
f_2\left(x_0, \rho_0\right) &= \rho_0\left(\rho_0 + x_0\right).
\end{aligned}$$

Now, we solve the following nonlinear system

$$\begin{cases} f_1(x_0,\rho_0)=0, \\ f_2(x_0,\rho_0)=0. \end{cases}$$

Since $\rho_0>0$, it is clear that the function $f_2(x_0,\rho_0)$ vanishes at $x_0=-\rho_0$. Substituting value of x_0 in the function $f_1(x_0,\rho_0)$, we get

$$f_1(x_0,\rho_0)=\frac{\rho_0}{\pi}(2-\pi\rho_0).$$

Since $\rho_0>0$, the function $f_1(x_0,\rho_0)$ only vanishes when $\rho_0=\dfrac{2}{\pi}$. So the function $f(x_0,\rho_0)$ has unique zero $(x_0,\rho_0)=\left(-\dfrac{2}{\pi},\dfrac{2}{\pi}\right)$. By computing the Jacobian matrix and the Jacobian of function $f(x_0,r_0)$ at $(x_0,\rho_0)=\left(-\dfrac{2}{\pi},\dfrac{2}{\pi}\right)$, we find respectively

$$Df\left(-\frac{2}{\pi},\frac{2}{\pi}\right)=\begin{pmatrix} -\dfrac{2}{\pi} & -\dfrac{4}{\pi} \\ \dfrac{2}{\pi} & \dfrac{2}{\pi} \end{pmatrix},$$

and

$$\det\left.\frac{\partial(f_1,f_2)}{\partial(x_0,\rho_0)}\right|_{(x_0,\rho_0)=\left(-\frac{2}{\pi},\frac{2}{\pi}\right)}=\frac{4}{\pi^2}\neq 0.$$

Then, for $\varepsilon>0$ sufficiently small, the system (5.2) has the periodic solution of the form

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=\left(-\frac{2}{\pi}\cos t,\frac{2}{\pi}\sin t,\frac{2}{\pi}\cos t\right)+O(\varepsilon).$$

The second method. Take the change of variables

$$x=w+u,\quad y=u,\quad z=-v$$

as before. Then (5.1) can be transformed into the following 3-dimensional piecewise smooth system

$$\begin{cases} \dot v=u+\varepsilon[-v+(\alpha w+\alpha v-\beta u)^2], \\ \dot u=-v, & v\geq 0, \\ \dot w=\varepsilon[v-(\alpha w+\alpha v-\beta u)^2], \end{cases}$$
$$\begin{cases} \dot v=u+\varepsilon[v+(\alpha w+\alpha v-\beta u)^2], \\ \dot u=-v, & v<0, \\ \dot w=\varepsilon[-v-(\alpha w+\alpha v-\beta u)^2], \end{cases} \tag{5.3}$$

where $\varepsilon > 0$ is a small parameter. Obviously, system (5.3)$|_{\varepsilon=0}$ has 2 different C^∞ first integrals $H_1 = \frac{1}{2}(v^2 + u^2)$ and $H_2(w) = w$. From (4.7), (4.8) and (4.10) we obtain that the first order Melnikov vector function M_1 of system (5.3) has an expression of the form

$$M_1(h_1, h_2) = f_1(h_1, h_2) = \begin{pmatrix} f_{11}(h_1, h_2) \\ f_{12}(h_1, h_2) \end{pmatrix},$$

where

$$\begin{aligned} f_{11}(h_1, h_2) = \sqrt{2h_1} \int_0^\pi & \Big(-\sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^2 \Big) \sin\theta d\theta \\ &+ \sqrt{2h_1} \int_\pi^{2\pi} \Big(\sqrt{2h_1}\sin\theta + (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^2 \Big) \sin\theta d\theta \\ &= 4\pi h_1 h_2, \end{aligned}$$

and

$$\begin{aligned} f_{12}(h_1, h_2) = \int_0^\pi & \Big(\sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^2 \Big) d\theta \\ &+ \int_\pi^{2\pi} \Big(-\sqrt{2h_1}\sin\theta - (\alpha h_2 + \alpha\sqrt{2h_1}\sin\theta - \beta\sqrt{2h_1}\cos\theta)^2 \Big) d\theta \\ &= 4\sqrt{2h_1} - 4\pi h_1 - 2\pi h_2^2. \end{aligned}$$

Due to $h_1 > 0$, we can solve equations $f_{11}(h_1, h_2) = f_{12}(h_1, h_2) = 0$ have a unique solution $h_{10} = \frac{2}{\pi^2}$, $h_{20} = 0$. By direct calculations we have

$$\det \frac{\partial(f_{11}, f_{12})}{\partial(h_1, h_2)}\Big|_{(h_{10}, h_{20})} = \det \begin{pmatrix} 0 & \frac{8}{\pi} \\ -3\pi & 0 \end{pmatrix} = 24.$$

Thus, from Lemma 3.1, for $0 < \varepsilon \ll 1$ system (5.3) has a periodic solution.

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