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EXISTENCE RESULTS OF MILD SOLUTIONS FOR IMPULSIVE FRACTIONAL MEASURE DRIVEN DIFFERENTIAL EQUATIONS WITH INFINITE DELAY*

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Abstract The primary focus of this study is the existence and uniqueness of mild solutions for impulsive fractional measure driven differential equations with infinite delay in regular function spaces. First, we rigorously justify the definition of mild solutions for impulsive fractional measure driven differential equations. Then, under conditions of semigroup noncompactness, by utilizing operator semigroup theory, the Kuratowski measure of noncompactness, fixed point theorems, and piecewise estimation techniques, sufficient conditions for the existence of mild solutions are derived. This work extends numerous prior research outcomes, eschewing the need for any priori estimates or noncompactness constraints. Finally, an illustrative example is provided to demonstrate the applicability and efficacy of the theoretical framework.

Keywords Impulsive fractional measure driven differential equations, mild solutions, regular function spaces, kuratowski measure of noncompactness, fixed point theorem.

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1. Introduction

Measure driven differential equations, also called measure differential equations(MDEs), were investigated first in [9,10,35]. These types of equations permit the presence of an infinite number of discontinuities within finite time intervals, effectively capturing the behavior of discontinuous dynamical systems, and as such, they are frequently utilized in fields including mechanics, biomathematics, and economics [4,13,38,41]. Fractional calculus represents a crucial field in the study and application of differentiation and integration at arbitrary orders, serving as a generalization of traditional integer-order calculus. It enjoys widespread applications across various domains, including anomalous diffusion, signal processing and control, fluid mechanics, image processing, and the implementation of fractional order PID controllers [1–3,11,12,36,40].

In recent years, an increasing number of scholars have begun to explore the existence and controllability of mild solutions for fractional measure differential equations, and have achieved a wealth of theoretical results [16,18,25,26,28].

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In [18], Gu and Sun have studied the existence and controllability of mild solutions for fractional measure evolution equations with nonlocal conditions

$$\begin{cases} {}^C D^{\alpha}_{0+}x(t) = Ax(t)dt + (f(t,x(t)) + Bu(t))dg(t), \quad t \in (0,b], \\ x(0) + p(x) = x_0, \end{cases}$$

by using Hausdorff noncompact measure and fixed point theorems, sufficient conditions ensuring the existence and nonlocal controllability of mild solutions were obtained.

In [28], Liu and Liu have investigated the exact controllability for the following fractional measure evolution systems with state-dependent delay and nonlocal conditions

$$\begin{cases} {}^C D^{\alpha}x(t) = Ax(t) + [Bu(t) + f(t, x_{\rho(t,x_t)})]dg(t), \quad t \in [0,a], \\ x(t) + p(x_{t_1}, x_{t_2}, \cdots, x_{t_m})(t) = \varphi(t), \quad t \in (-\infty, 0], \end{cases}$$

without imposing the Lipschitz continuity on the nonlinear term, the exact controllability of the system was achieved by utilizing fractional calculus theory, the Kuratowski measure of noncompactness, and Mönch's fixed point theorem.

In [16], Gou has studied the existence of S-asymptotically ω-periodic mild solutions for the following fractional measure differential equations with nonlocal conditions in Banach spaces

$$\begin{cases} {}^C D^{1+\beta}_t u(t) + \sum_{k=1}^{n} \alpha_k {}^C D^{\gamma_k}_t u(t) = Au(t) + F(t, u(t), u_t)dg(t), \quad t \geq 0, \\ u(t) = Q(\sigma(u), u)(t) + \varphi(t), \quad t \in [-r, 0], \\ u'(0) = Q_0(u) + \psi, \end{cases}$$

by employing the monotone iterative method with upper and lower solutions, the existence of S-asymptotically ω-periodic mild solutions for the equation was obtained. Furthermore, without assuming the generalized monotonicity condition and without requiring the noncompactness measure of the nonlinear term, the existence of upper and lower S-asymptotically ω-periodic mild solutions was established. For more research on measure differential equations, see reference [5–7, 14, 17, 27, 30, 32, 37, 39].

As is well known, impulsive and time delay phenomena are ubiquitous in practical problems across various fields. Therefore, discussing impulsive fractional measure differential equations with infinite delay is of great significance. However, few previous research outcomes on measure differential equations have taken into account the effects of impulsive and time delay factors. To establish the existence of mild solutions for the equations, authors commonly employ compactness conditions of operator semigroups, a priori estimates, and stringent conditions on noncompactness measures, as seen in references [18, 28]

$$M_1 c + \varphi \sup_{t\in[0,b]} \Big(\int_0^t [(t-s)^{\alpha-1}]^q dg(s)\Big)^{\frac{1}{q}} \frac{M_1}{\Gamma(\alpha)} \|h\|_{HLS^p_g} < 1,$$

$$\frac{M_1\gamma}{\Gamma(\alpha)} \sup_{t\in[0,a]} N(t)(\widetilde{M}N(a)+1) < 1.$$

Inspired by the aforementioned studies, this paper investigates the existence of mild solutions for impulsive fractional measure differential equations with infinite delay. Notably, the compactness conditions of operator semigroups and the restrictive a priori estimates concerning the impulsive term are not relied on. Our conclusions extend and improve the results from existing literature [5,6,30,39]. As an application, an example of a noncompact semigroup is provided. Utilizing the measure of noncompactness and Mönch's fixed point theorem, we establish the existence of mild solutions for the following fractional measure differential equations

$$\begin{cases} {}^C D_t^\alpha (x'(t) - g(t, x_t)) = Ax(t) + f(t, x_t, Kx(t))dw(t), & t \in J, t \neq t_i, \\ \Delta x(t_i) = I_i(x_{t_i}), \Delta x'(t_i) = J_i(x_{t_i}), & i = 1, 2, \cdots, n, \\ x_0 = \phi \in \mathcal{B}, x'(0) = x_1 \in X, \end{cases} \tag{1.1}$$

$$\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + h(t, x(t), Kx(t))dw(t), & t \in J, t \neq t_i, \\ \Delta x(t_i) = I_i(x_{t_i}), & i = 1, 2, \cdots, n, \\ x_0 = z_0 \in X, \end{cases} \tag{1.2}$$

where $J = [0, b], b > 0$, ${}^C D_t^\alpha$ is Caputo fractional derivative of order $\alpha \in (0, 1)$. The state variable $x(\cdot)$ takes values in a complex Banach space X. $A : D(A) \subseteq X \to X$ is a sectorial operator. The history $x_t : (-\infty, 0] \to X$ is defined by $x_t(s) = x(t + s)$ for $t \geq 0$ belongs to the phase space $\mathcal{B} \subset G((-\infty, 0], X)$, where $G((-\infty, 0], X)$ denotes the space of regulated functions on $(-\infty, 0]$. $Kx(t) = \int_0^t k(t, s)x(s)ds, k \in C(D, \mathbb{R}^+), D = \{(t, s) : 0 \leq s \leq t \leq b\}$. Here, the fixed times t_i satisfies $0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n < t_{n+1} = b$, $x(t_i^+)$ and $x(t_i^-)$ denote the right and left limits of $x(t)$ at time t_i, and $\triangle x(t_i) = x(t_i^+) - x(t_i^-)$ represents the jump in the state x at time t_i, where I_i determines the size of the jump. Accordingly, J_i and $\triangle x'(t_i)$ have the same meaning. $w : J \to \mathbb{R}$ is a nondecreasing function. $g, f, \phi, I_i, J_i (i = 1, 2, \cdots, n)$ are appropriate functions to be specified later.

2. Preliminaries

In this section, we recall some known facts about regulated functions, explanations, and preliminary results from functional analysis, resolvent operator theory, and fractional calculus, which will be used throughout this article.

Let $G(J, X)$ denote the Banach space consisting of all regulated functions with the norm defined by $\|x\|_\infty = sup_{t\in J}\|x(t)\|$. $A : D(A) \subseteq X \to X$ is a sectorial operator. For details, see [19]. $C(J, X)$ is the Banach space composed of all continuous functions from J into X with the norm $\|\cdot\|_{C(J,X)}$. $L(X)$ is the Banach space composed of all bounded linear operators from X into X with the norm $\|\cdot\|_{L(X)}$.

Let $G_b(J, X) = \{x : J \to X$, $x(t)$ is a regular function at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists, for all $i = 1, 2, \cdots, n\}$. Evidently, $G_b(J, X)$ is a Banach space with norm $\|x\|_{G_b} = sup_{t\in J}\|x(t)\|$. For $x \in G_b(J, X)$ and $i = 1, 2, \cdots, n$, let $\widetilde{x}_i(t) = x(t)$ for $t \in (t_i, t_{i+1}]$ and $\widetilde{x}_i(t_i^+) = x(t_i^+)$, then $\widetilde{x}_i \in G([t_i, t_{i+1}], X)$, and we denote the right limit at zero by $x(0)$. For $V \in G_b$, let $V(t) = \{x(t) : x \in V\}$ and $KV(t) = \{Kx(t) : x \in V\}$. Moreover, for $i = 1, \cdots, n$, we use the concept $\widetilde{V}_i = \{\widetilde{x}_i : x \in V\}$. From Lemma 1.1 in [22], we know that the set $V \subseteq G_b(J, X)$ is relatively compact if and only if each set $\widetilde{V}_i = \{\widetilde{x}_i : x \in V\}$ is relatively compact in $G([t_i, t_{i+1}], X)(i = 0, 1, \cdots, n)$. Let $J_0 = \overline{J}_0 = [0, t_1], \overline{J}_1 = [t_1, t_2], \cdots, \overline{J}_n = [t_n, b]$.

A partition of $[a, b]$ is a finite collection of pairs $\{([t_{i-1}, t_i], e_i), i = 1, 2, \cdots, n\}$, where $[t_{i-1}, t_i]$ are nonoverlapping subintervals of $[a, b]$, $e_i \in [t_{i-1}, t_i], i = 1, 2, \cdots, n$ and $\bigcup_{i=1}^n [t_{i-1}, t_i] = [a, b]$. A gauge δ on $[a, b]$ is a positive function on $[a, b]$. For a given gauge δ we say that a partition is $\delta-$fine if $[t_{i-1}, t_i] \subset (e_i - \delta(e_i), e_i + \delta(e_i)), i = 1, 2, \cdots, n$.

Definition 2.1. [5] A function $x : [a, b] \to X$ is said to be regulated on $[a, b]$, if the limits

$$\lim_{s\to t^-} x(s) = x(t^-), t \in (a, b] \quad \text{and} \quad \lim_{s\to t^+} x(s) = x(t^+), t \in [a, b),$$

exist and are finite. Obviously, $G(J, X)$ is a Banach space endowed with the norm $\|x\|_\infty$.

Definition 2.2. [5] A set $V \subset G([a, b], X)$ is called equiregulated, if there is $\delta > 0, t_0 \in [a, b]$ and for every $\varepsilon > 0$, such that

(i) If $x \in V, t \in [a, b], t - \delta < t < t_0$, then $\|x(t_0^-) - x(t)\| < \varepsilon$,
(ii) If $x \in V, t \in [a, b], t_0 < t < t + \delta$, then $\|x(t) - x(t_0^+)\| < \varepsilon$.

Lemma 2.1. [17, 34] *Let $\{x_n\}_{n=1}^\infty$ be a sequence of functions from $[a, b]$ to X, If x_n converge pointwise to x_0 as $n \to \infty$ and the sequence $\{x_n\}_{n=1}^\infty$ is equiregulated, then x_n converges uniformly to x_0.*

Lemma 2.2. [17, 34] *Let $V \subset G([a, b], X)$. If V is bounded and equiregulated, then the set $\overline{co}(V)$ is also bounded and equiregulated. The set $\overline{co}(V)$ is defined as the closed convex hull of V.*

Next, we will review the definition of Henstock-Lebesgue-Stieljes integral.

Definition 2.3. [17, 34] A functions $\psi : [a, b] \to X$ is said to be Henstock-Lebesgue-Stieltjes integrable w.r.t. $w : [a, b] \to \mathbb{R}$ if there exists a function denoted by

$$(HLS)\int_a^{(\cdot)} : [a, b] \to X,$$

such that, for every $\varepsilon > 0$, there is a gauge δ_ε on $[a, b]$ with

$$\sum_{i=1}^{n} \|\psi(e_i)(w(t_i) - w(t_{i-1})) - ((HLS)\int_0^{t_i} \psi(s)dw(s) - (HLS)\int_0^{t_{i-1}} \psi(s)dw(s))\| < \varepsilon,$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i], e_i), i = 1, 2, \cdots, n\}$ of $[a, b]$.

Denote by $\mathbb{HLS}_w^p([a, b], \mathbb{R})(p > 1)$ the space of all $p-$ordered Henstock-Lebesgue-Stieltjes integral regulated from $[a, b]$ to $\mathbb{R}$ with respect to w, with norm $\|\cdot\|_{\mathbb{HLS}_w^p}$ defined by

$$\|\psi\|_{\mathbb{HLS}_w^p} = ((HLS)\int_a^b \|\psi(s)\|^p dw(s))^{\frac{1}{p}}.$$

Lemma 2.3. [18] *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi \in \mathbb{HLS}_w^p([a, b], \mathbb{R}^+)$ and $w : J \to \mathbb{R}$ be regulated, then the function $H(t) = \int_0^t (t - s)^\varrho \Psi(s)dw(s)$ is regulated and satisfies*

$$H(t) - H(t^-) \le (\int_{t^-}^{t} (t - s)^{q\varrho} dw(s))^{\frac{1}{q}} \Psi(t)(\Delta^- w(t))^{\frac{1}{p}}, \quad t \in (a, b],$$
$$H(t^+) - H(t) \le (\int_{t^+}^{t} (t^+ - s)^{q\varrho} dw(s))^{\frac{1}{q}} \Psi(t)(\Delta^+ w(t))^{\frac{1}{p}}, \quad t \in [a, b),$$

where $\Delta^+ w(t) = w(t^+) - w(t)$ and $\Delta^- w(t) = w(t) - w(t^-)$, with $w(t^+)$ and $w(t^-)$ representing the right and left limits of w at t, respectively.

Definition 2.4. [33] The α order Caputo fractional derivative of the function $f : [0, \infty) \to \mathbb{R}$ is defined as follows

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f'(s)ds = I^{n-\alpha} f^{(n)}(t), \quad n-1 \leq \alpha < n, n \in N,$$

where $\Gamma(\cdot)$ is the Gamma function. Specially, when $0 < \alpha \leq 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds.$$

The Laplace transform of the caputo derivative is given as

$$L\{D_t^\alpha f(t); \lambda\} = \lambda^\alpha F(\lambda) - \sum_{k=0}^{n-1} \lambda^{\alpha-k-1} f^k(0), \quad n-1 \leq \alpha < n.$$

Definition 2.5. The definition of the n dimensional Mittag-Leffler function is as follows

$$E_{(a_1,\ldots,a_n),b}(z_1, \ldots, z_n) = \sum_{k=0}^{\infty} \sum_{l_1+\cdots+l_n=k} \frac{k!}{l_1! \times \cdots \times l_n!} \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)},$$

where $b > 0, a_i > 0, l_i \geq 0, |z_i| < \infty, i = 1, 2, \cdots, n$.

Specifically, the two dimensional Mittag-Leffler function is defined as follows:

$$E_{\zeta,\xi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\zeta + \xi)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\zeta-\xi} e^\mu}{\mu^\zeta - z} d\mu, \zeta, \xi > 0, z \in \mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{\frac{1}{\zeta}}$ counter clockwise. The Laplace transform formula for the Mittag-Leffler function is defined as follows

$$\int_0^\infty e^{-\lambda t} t^{\xi-1} E_{\zeta,\xi}(\vartheta t^\zeta) dt = \frac{\lambda^{\zeta-\xi}}{\lambda^\zeta - \vartheta}, Re\lambda > \vartheta^{\frac{1}{\zeta}}, \vartheta > 0.$$

For more details, refer to [33].

Definition 2.6. Let A be a closed linear operator defined on the domain $D(A)$ in the Banach space X. $\rho(A)$ be the resolvent set of A and $\gamma > 0$. If there exists a strongly continuous function $S_\gamma : \mathbb{R}^+ \to L(X)$ and $\vartheta \geq 0$, such that $\{\lambda^\gamma : Re\lambda > \vartheta\} \subset \rho(A)$ and
(*i*) $\lambda^{\gamma-1}(\lambda^\gamma I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\gamma(t) x dt, Re\lambda > \vartheta, x \in X$,
(*ii*) $(\lambda^\gamma I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_\gamma(t) x dt, Re\lambda > \vartheta, x \in X$,
where $S_\gamma(t)$ is known as the solution operator generated by A, $T_\gamma(t)$ is the $\gamma-$resolvent family generated by A (For details, see [8]).

The Kuratowski measure of noncompactness of a bounded subset V of the Banach space X is defined by

$$\beta(V) = \inf\{\delta > 0 : V \text{can be expressed as the union of a finite number of sets such that the diameter of each set does not exceed } \delta, \ i.e., \ V = \bigcup_{i=1}^n V_i \text{ with } \operatorname{diam}(V_i) \leq \delta, i = 1, 2, \cdots, n.\},$$

where $\operatorname{diam}(V)$ denotes the diameter of a set V.

Lemma 2.4. [6] *Let V, W be bounded sets of X, and $\lambda \in \mathbb{R}$. Then*

(i) $\beta(V) = 0$ if and only if V is relatively compact;
(ii) $V \subseteq W$ implies $\beta(V) \leq \beta(W)$;
(iii) $\beta(\overline{V}) = \beta(V)$;
(iv) $\beta(V \bigcup W) = \max\{\beta(V), \beta(W)\}$;
(v) $\beta(\lambda V) = |\lambda|\beta(V)$, where $\lambda V = \{x = \lambda z : z \in V\}$;
(vi) $\beta(V + W) \leq \beta(V) + \beta(W)$, where $V + W = \{x = y + z : y \in V, z \in W\}$;
(vii) $\beta(coV) = \beta(V)$;
(viii) $|\beta(V) - \beta(W)| \leq 2d_h(V, W)$, where $d_h(V, W)$ denotes the Hausdorff metric of V and W, that is

$$d_h(V, W) = \max\{\sup_{x \in V} d(x, W), \sup_{x \in W} d(x, V)\},$$

and $d(\cdot, \cdot)$ is the distance from an element of X to a subset of X.

Lemma 2.5. [21] *Let $V \subset G([a, b], X)$. If V is bounded and equiregulated, then $\beta(V(t))$ is regulated and satisfies $\beta(V) = \sup\{\beta(V(t)) : t \in [a, b]\}$, where $V(t) = \{x(t) : x \in V\}$.*

Since the Lebesgue-Stieltjes measure is a regular Borel measure, then we refer to Theorem3.1 in [21], the following result can be derived.

Lemma 2.6. [6, 21] *Let $V_0 \subseteq \mathbb{HLS}^p_w(J, X)$ be a countable set. Assume that there exists a positive function $\varsigma \in \mathbb{HLS}^p_w(J, \mathbb{R}^+)$ such that for all $v(t) \in V_0$, $\|v(t)\| \leq \varsigma(t), w - a.e.$ Then we have*

$$\beta(\int_J V_0(t) dw(t)) \leq 2 \int_J \beta(V_0(t)) dw(t).$$

Proof. Let $V_0(t) = \{v(t) : v \in V_0\}$ and $\beta(V_0) = sup\{V_0(t) : t \in J\}$. Using the Heine-Borel theorem and properties of the Kuratowski measure of noncompactness, consider an arbitrary $\epsilon > 0$. There exists a sequence $\{v_i\} \subset V_0$ for $i = 1, 2, \cdots, n$ and $m > 0$ such that we have $V_0 \subset U^n_{i=1} B(v_i(s), \epsilon + m)$, where $B(v_i(s), \epsilon + m)$ denotes a finite number of open balls with radius $r = \epsilon + m$, centered at w_i for $i = 1, 2, \cdots, n$. In fact, $\int_0^t V_0(s) dw(s)$ can be covered by $U^n_{i=1} B(\int_0^t v_i(s) dw(s), \epsilon + m)$, then we have $\beta(\int_0^t V_0(s) dw(s)) \leq \beta(V_0(s)) \leq \beta(V_0)$. there exists a separable closed linear subspace $X_0 \subset X$ such that $\beta(V_0) \leq 2\chi(V_0, X_0) \leq 2 \int \chi(V_0(s), X_0) dw(s) \leq 2 \int \beta(V_0(s)) dw(s)$, where χ is defined by Hausdorff measure of noncompactness. □

Lemma 2.7. [15] *Let $\kappa : J \to [0, \infty)$ be such that $m, \kappa :\in \mathbb{HLS}^p_w(J, \mathbb{R}^+)$. If $m, y : J \to \mathbb{R}$ are such that $m, y \in \mathbb{HLS}^p_w(J, \mathbb{R}^+)$ and*

$$y(t) \leq m(t) + \int_0^t \kappa(s) y(s) dw(s), 0 \leq t \leq b,$$

then we have

$$y(t) \leq m(t) + \int_0^t \frac{m(s)\kappa(s)}{1 + \kappa(s)\Delta^+ w(s)} \frac{\exp(\int_0^t \kappa(\eta) dw(\eta))}{\exp(\int_0^s \kappa(\eta) dw(\eta))} dw(s), 0 \leq s < t \leq b.$$

Remark 2.1. The choice of $\kappa(t)$ and $y(t)$ as nonnegative continuous functions, and $m(t)$ as any continuous function on $0 \leq t \leq b$ reduces Theorem 3.2 to Corollary 1.9.1 of [24], which is a generalized version of the well known integral inequality of Gronwall-Bellman type.

In this work, we adopt the conventional phase space framework for retarded functional differential equations with unbounded delay (see [20]), which will allow us to work with more general phase spaces. Our candidate for the phase space of a measure functional differential equation with infinite delay is a linear space $\mathcal{B} \subset G((-\infty, 0], X)$ equipped with a norm denoted by $\|\cdot\|_{\mathcal{B}}$ [14]. Assume that this normed linear space satisfies $\mathcal{B}$ the following conditions:

(H_1) $\mathcal{B}$ is complete;

(H_2) For all $t_0 \in \mathbb{R}$, all $\sigma > 0$, and all $x : (-\infty, t_0+\sigma] \to \mathbb{R}^n$ which are regulated on $[t_0, t_0+\sigma]$ and $x_{t_0} \in \mathcal{B}$, the following conditions hold: for every $t \in [t_0, t_0 + \sigma]$,

(h_1) $x_t \in \mathcal{B}$;

(h_2) there exists a locally bounded function $\kappa_1 : [0, \infty) \to (0, \infty)$, such that

$$\|x(t)\| \leq \kappa_1(t - t_0)\|x_t\|_{\mathcal{B}};$$

(h_3) there exist locally bounded functions $\kappa_2, \kappa_3 : [0, \infty) \to (0, \infty)$, such that

$$\|x_t\|_{\mathcal{B}} \leq \kappa_2(t - t_0)\|x_{t_0}\|_{\mathcal{B}} + \kappa_3(t - t_0) \sup_{s\in[t_0,t]} \|x(s)\|,$$

where $\kappa_1, \kappa_2, \kappa_3$ are functions independent of x, t_0 and σ;

(H_3) Let $C(t) : \mathcal{B} \to \mathcal{B}$ for $t \geq 0$ be the operator defined by

$$C(t)\varphi(\theta) = \begin{cases} \varphi(t + \theta), & \theta < -t, \\ \varphi(0^-), & -t \leq \theta < 0, \\ \varphi(0), & \theta = 0. \end{cases}$$

Then there exists a continuous function $\kappa_4 : [0, \infty) \to (0, \infty)$, such that $\kappa_4(0) = 0$ and such that $\|C(t)\varphi\|_{\mathcal{B}} \leq (1 + \kappa_4(t))\|\varphi\|_{\mathcal{B}}$ for all $\varphi \in \mathcal{B}$.

Lemma 2.8. [31] *Let Ω be a bounded open subset in the Banach space X and $0 \in \Omega$. Assume that the operator $F : \overline{\Omega} \to X$ is continuous and satisfies the following conditions:*

(1) $x \neq \lambda Fx,\ \forall\ \lambda \in (0, 1), x \in \partial\Omega$,

(2) V is relatively compact if $D \subset \overline{co}(\{0\} \cup F(V))$ for any countable set $V \subset \overline{\Omega}$, then F has a fixed point in $\overline{\Omega}$.

3. Main results

In this chapter, we will discuss the mild solutions of impulsive fractional measure driven differential equations (1.1) and (1.2). Initially, we will employ the Laplace transform method to derive the mild solutions of these equations.

Theorem 3.1. *Let A be a sectorial operator. If $f \in \mathbb{HLS}^1_w(J, \mathbb{R}^+)$ and g satisfy the uniform Hölder condition with the exponent $\xi \in (0, 1]$, then $x : (-\infty, b] \to X$ is a mild solution of the problem (1.1) provided that $x_0 = \phi$, $x(\cdot)|_J \in G_b(J, X)$ and it satisfies the following integral* *equation*

$$x(t)=\begin{cases}S_q(t)\phi(0)+\int_0^t S_q(s)[x_1-g(0,\phi)]ds+\sum_{t_i<t}S_q(t-t_i)I_i(x_{t_i})\\ +\sum_{t_i<t}\int_{t_i}^t S_q(t-s)[J_i(x_{t_i})-g(t_i,x_{t_i}+I_i(x_{t_i}))+g(t_i,x_{t_i})]ds\\ +\int_0^t S_q(t-s)g(s,x_s)ds+\int_0^t T_q(t-s)f(s,x_s,Kx(s))dw(s),t\in J,\end{cases}\tag{3.1}$$

where $S_q(t),T_q(t):\mathbb{R}^+\to L(X)(q=1+\alpha)$ *are defined as follows*

$$S_q(t)=E_{q,1}(At^q)=\frac{1}{2\pi i}\int_{B_r}\frac{e^{\lambda t}\lambda^{q-1}}{\lambda^q-A}d\lambda,$$
$$T_q(t)=t^{q-1}E_{q,q}(At^q)=\frac{1}{2\pi i}\int_{B_r}\frac{e^{\lambda t}}{\lambda^q-A}d\lambda,$$

and B_r *denotes the Bromwich path.*

Proof. If $t\in[0,t_1]$, apply the Riemann-Liouville fractional integrable operator on both sides, we have

$$\begin{aligned}x'(t)&=x_1-g(0,\phi(0))+g(t,x_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Ax(s)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s,Kx(s))dw(s),\\x(t)&=\phi(0)+[x_1-g(0,\phi(0))]t+\int_0^t g(s,x_s)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s(s-\tau)^{\alpha-1}Ax(\tau)d\tau ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s(s-\tau)^{\alpha-1}f(\tau,x_\tau,Kx(\tau))dw(\tau)ds.\end{aligned}$$

If $t\in(t_1,t_2]$, then $x(t_1^+)=x(t_1)+I_1(x_{t_1})$, $x'(t_1^+)=x'(t_1^-)+J_1(x_{t_1})$, we get

$$\begin{aligned}x'(t)&=x_1-g(0,\phi(0))+J_1(x_{t_1})-g(t_1,x_{t_1}+I_1(x_{t_1})\\&\quad+g(t_1,x_{t_1})+g(t,x_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}Ax(s)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s,Kx(s))dw(s),\\x(t)&=\phi(0)+I_1(x_{t_1})+[x_1-g(0,\phi(0))]t\\&\quad+(t-t_1)[J_1(x_{t_1})-g(t_1,x_{t_1}+I_1(x_{t_1}))+g(t_1,x_{t_1})]\\&\quad+\int_0^t g(s,x_s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s(s-\tau)^{\alpha-1}Ax(\tau)d\tau ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t\int_0^s(s-\tau)^{\alpha-1}f(\tau,x_\tau,Kx(\tau))dw(\tau)ds.\end{aligned}$$

Similarly, if $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned} x(t) =& \phi(0) + \sum_{i=1}^{n} I_i(x_{t_i}) + [x_1 - g(0,\phi(0))]t \\ &+ \sum_{i=1}^{n} (t-t_i)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})] \\ &+ \int_0^t g(s,x_s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-\tau)^{\alpha-1} Ax(\tau) d\tau ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-\tau)^{\alpha-1} f(\tau, x_\tau, Kx(\tau)) dw(\tau) ds. \end{aligned}$$

Let $N_i(t) = 1, t > t_i, N_i(t) = 0, t \leq t_i, i = 1, \cdots, n$. The above equation can be represented in the following form

$$\begin{aligned} x(t) =& \phi(0) + \sum_{i=1}^{n} N_i(t) I_i(x_{t_i}) + [x_1 - g(0,\phi(0))]t \\ &+ \sum_{i=1}^{n} N_i(t)(t-t_i)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})] \\ &+ \int_0^t g(s,x_s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-\tau)^{\alpha-1} Ax(\tau) d\tau ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-\tau)^{\alpha-1} f(\tau, x_\tau, Kx(\tau)) dw(\tau) ds. \end{aligned} \tag{3.2}$$

Applying the Laplace transform to Equation (3.2), we have

$$\begin{aligned} L\{x(t)\} =& \frac{\lambda^\alpha \varphi(0)}{\lambda^{\alpha+1} - A} + \frac{\lambda^{\alpha-1}[x_1 - g(0,\varphi)]}{\lambda^{\alpha+1} - A} + \sum_{i=1}^{m} \frac{e^{-\lambda t_i} \lambda^\alpha}{\lambda^{\alpha+1} - A} I_i(x_{t_i}) \\ &+ \sum_{i=1}^{m} \frac{e^{-\lambda t_i} \lambda^{\alpha-1}}{\lambda^{\alpha+1} - A} [J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})] \\ &+ \frac{\lambda^\alpha}{\lambda^{\alpha+1} - A} \int_0^\infty e^{-\lambda t} g(t, x_t) dt + \frac{1}{\lambda^{\alpha+1} - A} \int_0^\infty e^{-\lambda t} f(t, x_t, Kx(t)) dw(t). \end{aligned} \tag{3.3}$$

Apply the inverse Laplace transform to both sides of equation (3.3), we get

$$\begin{aligned} x(t) =& E_{q,1}(At^q)\varphi(0) + \int_0^t E_{q,1}(As^q)[x_1 - g(0,\varphi)]ds \\ &+ \sum_{i=1}^{m} N_i(t) E_{q,1}(A(t-t_i)^q) I_i(x_{t_i}) \\ &+ \sum_{i=1}^{m} N_i(t) \int_{t_i}^t E_{q,1}(A(t-s)^q)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})]ds \\ &+ \int_0^t E_{q,1}(A(t-s)^q) g(s, x_s) ds \\ &+ \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s, x_s, Kx(s)) dw(s), \end{aligned}$$

where $q = 1 + \alpha$. Let $S_q(t) = E_{q,1}(At^q)$ and $T_q(t) = t^{q-1}E_{q,q}(At^q)$ be defined as previously expressed, we have

$$\begin{aligned}x(t) =& S_q(t)\varphi(0) + \int_0^t S_q(s)[x_1 - g(0,\varphi)]ds + \sum_{i=1}^m N_i(t)S_q(t-t_i)I_i(x_{t_i})\\&+ \sum_{i=1}^m N_i(t)\int_{t_i}^t S_q(t-s)[J_i(x_{t_i}) - g(t_i, x_{t_i} + I_i(x_{t_i})) + g(t_i, x_{t_i})]ds\\&+ \int_0^t S_q(t-s)g(s,x_s)ds + \int_0^t T_q(t-s)f(s,x_s,Kx(s))dw(s), t \in J.\end{aligned}$$

Thus, the solution to problem (1.1) is as expressed in the equation (3.1). □

Definition 3.1. A function $x \in G_b(J,X)$ is called to be a solution of the equation (1.2) if $x_0 = z_0$, $x(\cdot)|_J \in G_b(J,X)$ and

$$x(t) = S_q(t)x_0 + \sum_{t_i<t} S_q(t-t_i)I_i(x(t_i)) + \int_0^t T_q(t-s)h(s,x(s),Kx(s))dw(s), t \in J, \tag{3.4}$$

where $S_q(t)$ and $T_q(t)$ have the same meanings as above.

Next, we will discuss the existence of mild solutions for impulsive fractional measure driven differential equations with infinite delay. Initially, let $\|S_\gamma(t)\|_{L(X)} \le M$, $\|T_\gamma(t)\|_{L(X)} \le t^{\gamma-1}M_T$, $t > 0, \gamma \in (0,2)$. For more details, refer to [29]. To facilitate the exposition, we will present some of the assumptions that will be utilized in advance.

(F_1) The following conditions will be satisfied by the function $f : J \times \mathcal{B} \times X \to X$:

(i) The function $f(\cdot,x,y)$ is measurable for all $(x,y) \in \mathcal{B} \times X$, $f(t,\cdot,\cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a function $p_f(t) \in \mathbb{HLS}_w^p(J,\mathbb{R}^+), p > 1$ and a positive constant $d > 0$, such that

$$\|f(t,x,y)\| \le p_f(t)(\|x\|_{\mathcal{B}} + \|y\|) + d,\ t \in J(x,y) \in \mathcal{B} \times X.$$

(iii) For any bounded set $V \subset G(J,X)$, there exist a function $l_f(t) \in \mathbb{HLS}_w^p(J,\mathbb{R}^+)$, $p > 1$, such that

$$\beta(f(t,V_t,KV)) \le l_f(t)(\beta(V_t) + \beta(KV)),\ t \in J,$$

where $V_t = \{x_t : x \in V\} \subseteq \mathcal{B}$.

(F_1') The function $f : J \times \mathcal{B} \times X \to X$ is continuous, and there exists a positive constant $L_f > 0$ such that

$$\|f(t,\phi_1,\phi_2) - f(t,\psi_1,\psi_2)\| \le L_f(\|\phi_1 - \psi_1\|_{\mathcal{B}} + \|\phi_2 - \psi_2\|),\ t \in J.$$

(F_2) The following conditions will be satisfied by the continuous function $g : J \times \mathcal{B} \to X$:

(i) The function $g(\cdot,x)$ is measurable for all $x \in \mathcal{B}$, $g(t,\cdot)$ is continuous for a.e. $t \in J$.

(ii) There exists an integrable function $p_g(t) \in \mathbb{HLS}_w^p(J,\mathbb{R}^+), p > 1$, such that

$$\|g(t,x)\| \le p_g(t)\|x\|_{\mathcal{B}},\ t \in J, x \in X.$$

(iii) For any bounded set $V \subset G(J,X)$, there exists an integrable function $l_g(t) \in \mathbb{HLS}_w^p(J,\mathbb{R}^+), p > 1$, such that

$$\beta(g(t,V_t)) \le l_g(t)\beta(V_t),\ t \in J,$$

where $V_t = \{x_t : x \in V\} \subseteq \mathcal{B}$.

(F_2') The function $g : J \times \mathcal{B} \to X$ is continuous, $g(t, 0) = 0$ and there exists a positive constant $L_g > 0$ such that

$$\|g(t,x) - g(t,y)\| \leq L_g \|x - y\|_{\mathcal{B}},\ t \in J.$$

(F_3) The following conditions will be satisfied by the function $h : J \times \mathcal{B} \times X \to X$:

(i) The function $h(\cdot, x, y)$ is measurable for all $(x, y) \in \mathcal{B} \times X$, $h(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a integrable function $p_h(t) \in \mathbb{HLS}^p_w(J, \mathbb{R}^+), p > 1$ and a positive constant $d > 0$, such that

$$\|h(t,x,y)\| \leq p_h(t)(\|x\|_{\mathcal{B}} + \|y\|) + d,\ t \in J, (x,y) \in \mathcal{B} \times X.$$

(iii) For any bounded set $V \subset G(J, X)$, there exist a function $l_h(t) \in \mathbb{HLS}^p_w(J, \mathbb{R}^+)$, $p > 1$, such that

$$\beta(h(t, V_t, KV)) \leq l_h(t)(\beta(V_t) + \beta(KV)),\ t \in J,$$

where $V_t = \{x_t : x \in V\} \subseteq \mathcal{B}$.

(F_3') The function $h : J \times \mathcal{B} \times X \to X$ is continuous, and there exists a positive constant $L_h > 0$ such that

$$\|h(t, \phi_1, \phi_2) - h(t, \psi_1, \psi_2)\| \leq L_h(\|\phi_1 - \psi_1\|_{\mathcal{B}} + \|\phi_2 - \psi_2\|),\ t \in J.$$

(F_4) $I_i, J_i : \mathcal{B} \to X$ $(i = 1, 2, \cdots, n)$ are continuous functions, there exist constants $c_1, c_2 \geq 0, d_1 > 0$ and $d_2 > 0$ such that

$$\|I_i(x)\| \leq c_1\|x\|_{\mathcal{B}} + d_1, \|J_i(x)\| \leq c_2\|x\|_{\mathcal{B}} + d_2, i = 1, 2, \cdots, n.$$

Theorem 3.2. *Assuming that conditions* (F_1), (F_2) *and* (F_4) *are satisfied, then the equation (1.1) has at least one mild solution.*

Proof. Let $\widehat{\phi}(t) : (-\infty, b] \to X$ be the function defined by

$$\widehat{\phi}(t) = \begin{cases} \phi(t),\ t \in (-\infty, 0], \\ S_q(t)\phi(0) + \int_0^t S_q(s)[x_1 - g(0,\phi)]ds,\ t \in J. \end{cases}$$

Clearly, $sup_{t\in J}\widehat{\phi}(t) \leq M(\|\phi(0)\| + b\|x_1 - g(0, \phi)\|) = M_b$.

Assuming that the space $S(b) = \{y : (-\infty, b] \to X, y_0 = 0, y|_J \in G_b(J, X)\}$ equipped with the norm $\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup_{t\in J}\|y(t)\| = \sup_{t\in J}\|y(t)\|$. The operator $F : S(b) \to S(b)$ is defined as follows:

$$Fy(t) = \begin{cases} \sum_{t_i<t} S_q(t - t_i)I_i(y_{t_i} + \widehat{\phi}_{t_i}) + \sum_{t_i<t} \int_{t_i}^t S_q(t - s)[J_i(y_{t_i} + \widehat{\phi}_{t_i}) \\ -g(t_i, y_{t_i} + \widehat{\phi}_{t_i} + I_i(y_{t_i} + \widehat{\phi}_{t_i})) + g(t_i, y_{t_i} + \widehat{\phi}_{t_i})]ds \\ + \int_0^t S_q(t - s)g(s, y_s + \widehat{\phi}_s)ds \\ + \int_0^t T_q(t - s)f(s, y_s + \widehat{\phi}_s, Ky(s) + K\widehat{\phi}(s))dw(s), t \in J. \end{cases} \tag{3.5}$$

Obviously, the definition of the operator F is meaningful in space $S(b)$. Furthermore, by the Lebesgue Dominated Convergence Theorem, phase space theory, and the given assumptions F_1, F_2, F_4, we can summarize that F is continuous [17, 30]. Let $K_b = \sup_{0\le t\le b} \kappa_3(t)$, $L = \|\widehat{\phi}_t\|_{\mathcal{B}}$, $\|y\|_t = \sup_{0\le s\le t} \|y(s)\|$, then $\|y_t + \widehat{\phi}_t\|_{\mathcal{B}} \le \|y_t\|_{\mathcal{B}} + \|\widehat{\phi}_t\|_{\mathcal{B}} \le K_b\|y\|_t + L$.

Step I. We begin by demonstrating that the set

$$\Omega_0 = \{y \in S(b), y = \lambda F y, \lambda \in (0,1)\},$$

is bounded. In fact, if there exists a $\lambda \in (0,1)$, such that $y = \lambda F y, y \in \Omega_0$.

When $t \in J_0 = [0, t_1]$, under conditions F_1 and F_2, we get

$$\begin{aligned}
\|y(t)\| &\le \lambda \|Fy(t)\| \\
&\le M \int_0^t p_g(s)\|y_s + \widehat{\phi}_s\|_{\mathcal{B}} ds \\
&\quad + M_T t_1^{\alpha} \int_0^t [p_f(s)(\|y_s + \widehat{\phi}_s\|_{\mathcal{B}} + \|Ky(s) + K\widehat{\phi}(s)\|) + d] dw(s) \\
&\le M_T t_1^{\alpha} d[w(t_1) - w(0)] \\
&\quad + [ML + M_T t_1^{\alpha}(L + t_1 k_0 M_b)] \int_0^{t_1} [p_g(s) + p_f(s)] d(w(s) + s) \\
&\quad + [M + M_T t_1^{\alpha}(1 + \kappa_1 k_0 t_1)] K_b \int_0^t [p_g(s) + p_f(s)] \|y\|_s d(w(s) + s),
\end{aligned}$$

where $k_0 = \sup_{(t,s)\in D} k(t,s)$. For $\|y\|_t = \sup_{0\le s\le t} \|y(s)\|$, we have

$$\begin{aligned}
\|y\|_t &\le M_T t_1^{\alpha} d[w(t_1) - w(0)] \\
&\quad + [ML + M_T t_1^{\alpha}(L + t_1 k_0 M_b)] \int_0^{t_1} [p_g(s) + p_f(s)] d(w(s) + s) \\
&\quad + [M + M_T t_1^{\alpha}(1 + \kappa_1 k_0 t_1)] K_b \int_0^t [p_g(s) + p_f(s)] \|y\|_s d(w(s) + s).
\end{aligned} \tag{3.6}$$

Using Lemmas 2.7 and 3.6, there exists a constant $G_0 > 0$, independent of y and $\lambda \in (0,1)$, then $\|y(t)\| \le G_0, t \in J_0$. From condition F_4, we have

$$\begin{aligned}
\|I_1(y_{t_1} + \widehat{\phi}_{t_1})\| &\le c_1(K_b G_0 + L) + d_1 =: \varphi_1, \\
\|J_1(y_{t_1} + \widehat{\phi}_{t_1})\| &\le c_2(K_b G_0 + L) + d_2 =: \psi_1.
\end{aligned}$$

When $t \in J_1$, $\widetilde{y_1} \in G_b(\overline{J}_1, X)$, we have

$$\begin{aligned}
\|\widetilde{y_1}(t)\| &\le M_T t_2^{1+\alpha} d + M[\varphi_1 + b(\psi_1 + 2N)] \\
&\quad + M_T k_0 b^q (M_b + \kappa_1 K_b G_0) \int_0^{t_2} [p_g(s) + p_f(s)] d(w(s) + s) \\
&\quad + (M + M_T t_1^{\alpha})(L + K_b G_0) \int_0^{t_1} [p_g(s) + p_f(s)] d(w(s) + s) \\
&\quad + [M + M_T t_2^{\alpha}(1 + \kappa_1 k_0 b)] K_b \int_{t_1}^t [p_g(s) + p_f(s)] \sup_{t_1\le\tau\le s} \|\widetilde{y_1}(\tau)\| d(w(s) + s),
\end{aligned}$$

where $\|g(t,x_t)\| \le N, (t,x_t) \in J \times V_t$, and $V_t \subset \mathcal{B}$ is bounded.

$$\begin{aligned}\sup_{t_1\le s\le t}\|\widetilde{y_1}(s)\| \le{}& M_T t_2^{1+\alpha} d + M[\varphi_1 + b(\psi_1+2N)]\\&+ M_T k_0 b^q (M_b + \kappa_1 K_b G_0)\int_0^{t_2}[p_g(s)+p_f(s)]d(w(s)+s)\\&+(M+M_T t_1^\alpha)(L+K_b G_0)\int_0^{t_1}[p_g(s)+p_f(s)]d(w(s)+s)\\&+[M+M_T t_2^\alpha(1+\kappa_1 k_0 b)]K_b\int_{t_1}^{t}[p_g(s)+p_f(s)]\sup_{t_1\le\tau\le s}\|\widetilde{y_1}(\tau)\|d(w(s)+s).\end{aligned}$$

Thus, there exists a constant $G_1>0$ independent of $\widetilde{y_1}$ and $\lambda\in(0,1)$, then $\|\widetilde{y_1}(t)\|\le G_1, t\in \overline{J}_1$ and $\|y(t)\|\le G_1, t\in J_1$.

Similarly, we can infer that there exists a constant $G_i>0$, then $\|y(t)\|\le G_i, t\in J_i, i=1,2,\cdots,n$. Let $G=\max\{G_i: 0\le i\le n\}$, we have $\|y(t)\|\le G, t\in J$ and Ω_0 is a bounded set.

Let $R>G$ and $\Omega_R=\{x\in S(b): \|x\|_b<R\}$. Therefore, $0\in\Omega_R$ and Ω_R is a bounded open set, and when $x\in\partial\Omega_R$ and $\lambda\in(0,1)$, we get $x\ne\lambda Fx$.

Step II. $F(\overline{\Omega}_R)$ is equiregulated on J.

When $\theta_0\in J_0=[0,t_1)$, we have

$$\begin{aligned}\|F(y)(\theta)-F(y)(\theta_0^+)\| \le{}& M\int_{\theta_0^+}^{\theta} p_g(s)\|y_s+\widehat{\phi}_s\|_{\mathcal{B}}ds\\&+M_T t_1^\alpha\int_{\theta_0^+}^{\theta}[p_f(s)(\|y_s+\widehat{\phi}_s\|_{\mathcal{B}}+\|Ky(s)+K\widehat{\phi}(s)\|)+d]dw(s)\\\le{}& M_T t_1^\alpha d[w(\theta)-w(\theta_0^+)]+M(K_bR+L)\int_{\theta_0^+}^{\theta}p_g(s)ds\\&+M_T t_1^\alpha(R+L)(k_0t_1+K_b+1)\int_{\theta_0^+}^{\theta}p_f(s)dw(s).\end{aligned}$$

From the given conditions, since w is a regular function, $p_g(s)$ is continuously integrable, and $p_f(s)\in\mathbb{HLS}^p_w(J,\mathbb{R}^+)$, we can conclude that when $\theta\to\theta_0^+$, we have $\|F(y)(\theta)-F(y)(\theta_0^+)\|\to 0$. By employing the same approach, we can similarly demonstrate that $\|F(y)(\theta_0^-)-F(y)(\theta)\|\to 0$ as $\theta\to\theta_0^-$ for every $\theta_0\in J_0=(0,t_1]$.

When $\theta_1\in J_0=(t_1,t_2]$, we have

$$\begin{aligned}\|F(y)(\theta)-F(y)(\theta_0^+)\| \le{}& \|S_q(\theta-t_1)-S_q(\theta_1^+-t_1)\|\varphi_1\\&+M\int_{\theta_1^+}^{\theta}(\psi_1+2N)ds+M\int_{\theta_0^+}^{\theta}p_g(s)\|y_s+\widehat{\phi}_s\|_{\mathcal{B}}ds\\&+M_T t_2^\alpha\int_{\theta_0^+}^{\theta}[p_f(s)(\|y_s+\widehat{\phi}_s\|_{\mathcal{B}}+\|Ky(s)+K\widehat{\phi}(s)\|)+d]dw(s)\\\le{}& \|S_q(\theta-t_1)-S_q(\theta_1^+-t_1)\|\varphi_1+M_T t_2^\alpha d[w(\theta)-w(\theta_0^+)]\\&+M\int_{\theta_1^+}^{\theta}(\psi_1+2N)ds+M(K_bR+L)\int_{\theta_0^+}^{\theta}p_g(s)ds\end{aligned}$$

$$+ M_T t_2^{\alpha}(R+L)(k_0 t_2 + K_b + 1)\int_{\theta_0^+}^{\theta} p_f(s)dw(s).$$

Accordingly, we can deduce that when $\theta \to \theta_1^+$, we have $\|F(y)(\theta) - F(y)(\theta_1^+)\| \to 0$. By employing the same approach, we can similarly demonstrate that $\|F(y)(\theta_1^-) - F(y)(\theta)\| \to 0$ as $\theta \to \theta_1^-$ for every $\theta_1 \in J_1 = (t_1, t_2]$.

Similarly, we can prove that for any $\theta_i \in J_i = (t_i, t_{i+1}], i = 1, 2, \cdots, n$, when $\theta \to \theta_i^+$ then $\|F(y)(\theta) - F(y)(\theta_i^+)\| \to 0$ and $\|F(y)(\theta) - F(y)(\theta_i^-)\| \to 0$ as $\theta \to \theta_0^-$ for every $\theta_i \in J_i = (t_i, t_{i+1}]$. So $F(\overline{\Omega}_R)$ is equiregulated on J.

Step III. Claim that F is continuous on $\overline{\Omega}_R$.

Suppose $\{y^{(n)}\}_{n=1}^{\infty} \subset \overline{\Omega}_R$ such that $y^{(n)} \to x$ as $n \to \infty$. Furthermore, from axiom H_2, we know that

$$\begin{aligned}\|y_t^{(n)} - y_t\|_{\mathcal{B}} &\le \kappa_3(t)\sup\{\|y^{(n)}(s) - y(s)\| : 0 \le s \le t\} + \kappa_2(t)\|y_0^{(n)} - y_0\|_{\mathcal{B}} \\ &\le K_b \sup\{\|y^{(n)}(s) - y(s)\| : 0 \le s \le t\} \\ &\le K_b\|y^{(n)} - y\|_{\infty} \\ &\to 0 \quad (n \to \infty), t \in [0, b].\end{aligned}$$

Thus, by F_1, F_2 and F_4, for $t \in [0, b]$, we see that

$$\begin{aligned}&g(t, y_t^{(n)} + \widehat{\phi}_t) \to g(t, y_t + \widehat{\phi}_t) \quad \text{as } n \to \infty, \\ &f(t, y_t^{(n)} + \widehat{\phi}_t, Ky_t^{(n)} + K\widehat{\phi}_t) \to f(t, y_t + \widehat{\phi}_t, Ky_t + K\widehat{\phi}_t) \quad \text{as } n \to \infty.\end{aligned}$$

Moreover, for $t \in [0, t_1]$, we have

$$\begin{aligned}&\|F(y)^{(n)}(t) - F(y)(t)\| \\ \le& M\int_0^t \|g(s, y_s^{(n)} + \widehat{\phi}_s) - g(s, y_s + \widehat{\phi}_s)\|ds \\ &+ M_T t_1^{\alpha}\int_0^t \|f(s, y_s^{(n)} + \widehat{\phi}_s, Ky_s^{(n)} + K\widehat{\phi}_s) - f(s, y_s + \widehat{\phi}_s, Ky_s + K\widehat{\phi}_s)\|dw(s).\end{aligned}$$

Similarly, for $t \in (t_1, t_2]$, we have

$$\begin{aligned}&\|F(y)^{(n)}(t) - F(y)(t)\| \\ \le& K_b M c_1\|y^{(n)} - y\|_{\infty} + K_b M c_2\int_{t_1}^t \|y^{(n)} - y\|_{\infty}ds \\ &+ 2K_b M\int_{t_1}^t (p_g(s) + c_1)\|y_{t_1}^{(n)} - y_{t_1}\|ds + M\int_0^t \|g(s, y_s^{(n)} + \widehat{\phi}_s) - g(s, y_s + \widehat{\phi}_s)\|ds \\ &+ M_T t_2^{\alpha}\int_0^t \|f(s, y_s^{(n)} + \widehat{\phi}_s, Ky_s^{(n)} + K\widehat{\phi}_s) - f(s, y_s + \widehat{\phi}_s, Ky_s + K\widehat{\phi}_s)\|dw(s).\end{aligned}$$

Similarly, one can demonstrate that for any $t \in (t_i, t_{i+1}], i = 1, 2, \cdots, n$, from the above inequalities and the dominated convergence theorem for the Henstock-Lebesgue-Stieltjes integral, we infer that $\|F(y)^{(n)}(t) - F(y)(t)\| \to 0$ as $n \to \infty$.

Moreover, by Step II, it can shown that $\{F(y^{(n)})\}_{n=1}^{\infty}$ is equiregulated. Therefore, by Lemma 2.1, we get that $\{F(y^{(n)})\}$ converge uniformly to $\{F(y)\}$. Thus, F is a continuous operator.

Step IV. We demonstrate that all conditions of Lemma 2.8 are met.

Let $V \subset \overline{co}(\{0\} \cup F(V)(t))$ and the set $V \subset \overline{\Omega}_R$ be countable, we have

$$V(t) \subset \overline{co}(\{0\} \cup F(V)(t)), t \in [0, b].$$

With conditions F_1, F_2, F_4 satisfied, and given that operators $S_q(t), T_q(t)(t \in J)$ are strongly continuous, it is not difficult to prove that $F\widetilde{V}(t), t \in \overline{J}_i(i = 0, 1, 2, \cdots, n)$ is equicontinuous. In the following proof, we shall not differentiate between $V|_{J_i}$ and $\widetilde{V}_i$ $(i = 1, 2, \cdots, n)$, where $V|_{J_i}$ is a subset of V on $J_i(i = 0, 1, 2, \cdots, n)$, and $\widetilde{V}$ is equicontinuous on $\overline{J}_i, (i = 0, 1, 2, \cdots, n)$.

When $t \in J_0$, based on conditions $F_1(iii)$, $F_2(iii)$, Lemma 2.6, and the properties of non-compactness measures, we get

$$\begin{aligned}
&\beta(V(t)) \leq \beta((FV)(t)) \\
&\qquad \leq 2M \int_0^t l_g(s)\beta(V_s)ds + 2M_T t_1^\alpha \int_0^t l_f(s)[\beta(V_s) + \beta(KV(s))]dw(s), \\
&\sup_{0 \leq s \leq t} \beta(V(s)) \leq 2 \int_0^t [Ml_g(s) + M_T t_1^\alpha (1 + \kappa_1 k_0 t_1) l_f(s)]\beta(V_s) d(w(s) + s) \\
&\qquad \leq 2K_b \int_0^t [Ml_g(s) + M_T t_1^\alpha (1 + \kappa_1 k_0 t_1) l_f(s)] \sup_{0 \leq \tau \leq s} \beta(V(\tau)) d(w(s) + s).
\end{aligned}$$

Therefore, $\beta(V(t)) = 0, t \in J_0$. The set $V \subset G(J_0, X)$ is relatively compact.
Since

$$\begin{aligned}
&\beta(I_1(V_{t_1} + y_{t_1})) = \beta(J_1(V_{t_1} + y_{t_1})) = 0, \\
&\beta(g(t_1, V_{t_1} + y_{t_1})) = \beta(g(t_1, V_{t_1} + y_{t_1} + I_1(V_{t_1} + y_{t_1}))) = 0.
\end{aligned}$$

When $t \in J_1$, we get

$$\begin{aligned}
\sup_{t_1 \leq s \leq t} \beta(V(s)) \leq 2K_b \int_{t_1}^t &[Ml_g(s) + M_T t_1^\alpha (1 + \kappa_1 k_0 t_2) l_f(s)] \\
&\times \sup_{t_1 \leq \tau \leq s} \beta(V(\tau)) d(w(s) + s).
\end{aligned}$$

Thus, $\beta(V(t)) = 0, t \in \overline{J}_1$. The set $V \subset G(\overline{J}_1, X)$ is relatively compact.

Similarly, we can readily demonstrate that every set $V \subset G(\overline{J}_i, X), i = 1, 2, \cdots, n$ is relatively compact. Therefore, the set $V \subset S(b)$ is relatively compact. According to Lemma 2.8, we can deduce that F has a fixed point $y \in \overline{\Omega}_R$, and $y + \widehat{\phi}$ is a mild solution to problem (1.1). □

Theorem 3.3. *Assuming that conditions (F_1'), (F_2') and (F_4) are satisfied, then the equation (1.1) has a unique mild solution.*

Proof. From conditions (F_1') and (F_2'), we get

$$\begin{aligned}
&\|g(t, x)\| \leq L_g \|x\|_{\mathcal{B}}, \beta(g(t, V_t)) \leq L_g \beta(V_t), t \in J, x \in \mathcal{B}, \\
&\|f(t, x, y)\| \leq L_f(\|x\|_{\mathcal{B}} + \|y\|) + \sup_{t \in J} \|f(t, 0, 0)\|, t \in J, x \in \mathcal{B}, y \in X, \\
&\beta(f(t, V_t, KV(t))) \leq L_f[\beta(V_t) + \beta(KV(t))], V \subset G_b(J, X).
\end{aligned}$$

Hence, according to Theorem 3.2, we can deduce that equation (1.1) has at least one mild solution.

Assuming $u \in \overline{\Omega}_R$ and $v \in \overline{\Omega}_R$ are two fixed points of the operator F.

If $t \in J_0 = [0, t_1]$, we get

$$\begin{aligned}\|u(t) - v(t)\| &= \|Fu(t) - Fv(t)\| \\ &\leq ML_g \int_0^t \|u_s - v_s\|_{\mathcal{B}} ds \\ &\quad + M_T t_1^{\alpha} L_f \int_0^t (\|u_s - v_s\|_{\mathcal{B}} + k_0 t_1 \|u(s) - v(s)\|) dw(s) \\ &\leq [ML_g + M_T t_1^{\alpha} L_f (1 + k_0 t_1 \kappa_1)] K_b \int_0^t \|u - v\|_s d(w(s) + s),\end{aligned}$$

which can prove that $u(t) = v(t), t \in J_0$, and $u_{t_1} = v_{t_1}$.

If $t \in \overline{J}_1$, we have

$$\begin{aligned}\|\widetilde{u}(t) - \widetilde{v}(t)\| &= \|F\widetilde{u}(t) - F\widetilde{v}(t)\| \\ &\leq [ML_g + M_T t_2^{\alpha} L_f (1 + k_0 t_2 \kappa_1)] K_b \int_{t_1}^t \sup_{t_1 \leq \tau \leq s} \|\widetilde{u}(\tau) - \widetilde{v}(\tau)\| d(w(s) + s),\end{aligned}$$

which can prove that $\widetilde{u}(t) = \widetilde{v}(t), t \in \overline{J}_1, u(t) = v(t), t \in J_1$, and $u_{t_2} = v_{t_2}$.

Similarly, we can readily demonstrate that $u(t) = v(t), t \in J_i, i = 1, 2, \cdots, n$. Hence $u(t) = v(t), t \in J$. □

Theorem 3.4. *Assuming that function $h : J \times \mathcal{B} \times X \to X$ satisfies F_3 and $I_i : X \to X (i = 1, 2, \cdots, n)$ satisfies F_4, then the problem (1.2) has at one mild solution.*

Proof. The operator $\Phi : G_b(J, X) \to G_b(J, X)$ is defined as follows

$$\Phi x(t) = S_q(t) x_0 + \sum_{t_i < t} S_q(t - t_i) I_i(x(t_i)) + \int_0^t T_q(t - s) h(s, x(s), Kx(s)) dw(s). \tag{3.7}$$

The definition of Φ is valid and continuous. Now we prove that the set

$$\Omega = \{z \in S(b), z = \lambda \Phi z, \lambda \in (0, 1)\},$$

is bounded. In fact, if there exists a $\lambda \in (0, 1)$, such that $z = \lambda \Phi z, z \in \Omega$. Let $p_h^* = \sup_{s \in J} p_h(s)$. If $t \in J_0 = [0, t_1]$, we get

$$\begin{aligned}\|z(t)\| &\leq \|\Phi z(t)\| \\ &\leq M_T \int_0^t (t - s)^{\alpha - 1} p_h(s) (\|z(s)\| + \|Kz(s)\| + d) dw(s) \\ &\leq M_T \alpha^{-1} b^{\alpha} d(w(t_1) - w(0)) + M_T p_h^* \int_0^t [(t - s)^{\alpha - 1} + k_0 \alpha^{-1} b^{\alpha}] \|z(s)\| dw(s).\end{aligned} \tag{3.8}$$

Using Lemmas 2.7 and 3.8, there exists a constant $G_0 > 0$, independent of z and $\lambda \in (0, 1)$, then $\|z(t)\| \leq G_0, t \in J_0$. The proof process below is identical to that of Theorem 3.2, hence we omit it. □

Theorem 3.5. *Assuming that conditions (F_3') and (F_4) are satisfied, then the equation (1.2) has a unique mild solution.*

Proof. The proof process is similar to that of Theorem 3.3. Therefore, we have shown that problem (1.2) has a unique mild solution. □

4. Application

Example 4.1. Consider the following impulsive fractional measure driven differential equations with infinite delay of the form

$$\begin{cases} D_t^\alpha\left[u_t'(t,x)-\int_{-\infty}^t\int_0^\pi b(s-t,\eta,x)u(s,\eta)d\eta ds\right] \\ =\frac{\partial^2}{\partial x^2}u(t,x)+(\int_{-\infty}^t \mu(t,s-t)u(s,x)ds+\int_0^t e^{t-r}u(r,x)dr)dw(t), t\in[0,1]\setminus\{t_i\}, \\ u(t,0)=u(t,\pi)=0, t\in[0,1], \\ u(\theta,x)=\varphi(\theta,x),\theta\in(-\infty,0],x\in[0,\pi], \\ \frac{\partial}{\partial t}u(0,x)=z(x),x\in[0,\pi], \\ \Delta u(t_i,x)=\int_{-\infty}^{t_i} q_i(s-t_i)u(s,x)ds, i=1,2,\cdots,n, \\ \Delta u'(t_i,x)=\int_{-\infty}^{t_i}\overline{q}_i(s-t_i)\frac{u(s,x)}{1+|u(s,x)|}ds, i=1,2,\cdots,n, \end{cases} \tag{4.1}$$

where $J=[0,1]$, D_t^α is Caputo fractional derivative of order $\alpha\in(0,1)$, $0<t_1<t_2<\cdots<t_n<1$ and $z\in X,\varphi\in\mathcal{B}$.

Let the phase space $\mathcal{B}=PC_0\times L^2(\rho,X)$ in [23], with the norm

$$\|\psi\|_{\mathcal{B}}=|\psi(0)|+(\int_{-\infty}^0\rho(s)|\psi(s)|^2ds)^{\frac{1}{2}},$$

where ρ is a positive Lebesgue integrable function. We take $X=L^2[0,\pi]$ and define the operator $A:D(A)\subset X\to X$ by $Ay=y''$ with domain $D(A)=\{y\in X: y',y''\in X, y(0)=y(\pi)=0\}$. It is commonly recognized that A acts as the infinitesimal generator for a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X and $\|C(t)\|_{L(X)}=1$.

Define $g:J\times\mathcal{B}\to X$, $f:J\times\mathcal{B}\times X\to X$ and $I_i,J_i:\mathcal{B}\to X$, respectively, as

$$g(t,\phi)(x)=\int_{-\infty}^0\int_0^\pi b(\theta,\eta,x)\phi(\theta,\eta)d\eta d\theta,$$

$$f(t,\phi,Bu(t))(x)=\int_{-\infty}^0\mu(t,\theta)\phi(\theta,x)d\theta+Bu(t,x),$$

$$I_i(\phi)(x)=\int_{-\infty}^0 q_i(\theta)\phi(\theta,x)d\theta, i=1,2,\cdots,n,$$

$$J_i(\phi)(x)=\int_{-\infty}^0\overline{q}_i(\theta)\frac{\phi(\theta,x)}{1+|\phi(\theta,x)|}d\theta,\ i=1,2,\cdots,n,$$

$$w(t)=\begin{cases} 0, t\le 0, \\ t+\frac{1}{2}\kappa(t), 0\le t\le 1-\frac{1}{2}, \\ \cdots \\ t+(1-\frac{1}{n})\kappa(t-(1-\frac{1}{n-1})), 1-\frac{1}{n-1}<t\le 1-\frac{1}{n}, n>2, n\in\mathbb{N}, \\ \cdots \\ t+1, t=1. \end{cases}$$

Take,

$$\kappa(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Obviously, w is a nondecreasing and left continuous function, $\phi(\theta, x) = \phi(\theta)(x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$, $u(t,x) = u(t)(x)$ and $Bu(t,x) = \int_0^t e^{t-s}u(s,x)ds$, $k(t,s) = e^{t-s}$. Thus, the system (4.1) can be modeled as the abstract form given by (1.1). Assuming the system (4.1) satisfies the following conditions:

(i) $b(s,\eta,x), \frac{\partial b(s,\eta,x)}{\partial x}$ are measurable, $b(s,\eta,0) = b(s,\eta,\pi) = 0$ and

$$L_g := \max\{[\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{\rho(s)}(\frac{\partial^k b(s,\eta,x)}{\partial x^k})^2 d\eta ds dx]^{\frac{1}{2}} : k = 0, 1\} < \infty.$$

(ii) $\mu \in C(\mathbb{R}^2, \mathbb{R})$ and $d(t) = (\int_{-\infty}^0 \mu^2(t,\theta)\rho^{-1}(\theta)d\theta)^{\frac{1}{2}} \in C(J, \mathbb{R}^+)$.

(iii) $q_i \in C(\mathbb{R}, \mathbb{R}^+)$ and $c_i = (\int_{-\infty}^0 q_i^2(\theta)\rho^{-1}(\theta)d\theta)^{\frac{1}{2}} < \infty, i = 1, 2, \cdots, n$.

(iv) $\overline{q}_i \in C(\mathbb{R}, \mathbb{R}^+)$ and $d_i = (\int_{-\infty}^0 \overline{q}_i^2(\theta)\rho^{-1}(\theta)d\theta)^{\frac{1}{2}} < \infty, i = 1, 2, \cdots, n$.

Moreover, g, I are bounded linear operators and $\|J_i(\phi)\| \leq d_i\|\phi\|_{\mathcal{B}}, i = 1, 2, \cdots, n$ for $(t, \phi, Bu(t)), (t, \psi, Bv(t) \in J \times \mathcal{B} \times X \to X$, we have

$$\begin{aligned}
&\|f(t,\phi,Bu(t)) - f(t,\psi,Bv(t))\| \\
&\leq (\int_0^\pi (\int_{-\infty}^0 \mu(t,\theta)[\phi(\theta,x) - \psi(\theta,x)]d\theta)^2 dx)^{\frac{1}{2}} \\
&\quad + (\int_0^\pi (Bu(t,x) - Bv(t,x))^2 dx)^{\frac{1}{2}} \\
&\leq (\int_{-\infty}^0 \frac{\mu^2(t,\theta)}{\rho(\theta)} d\theta \int_{-\infty}^0 \rho(\theta)|\phi(\theta,\cdot) - \psi(\theta,\cdot)|_{L^2}^2 d\theta)^{\frac{1}{2}} + |Bu(t) - Bv(t)|_L \\
&\leq d(t)\|\phi - \psi\|_{\mathcal{B}} + \|Bu - Bv\| \\
&\leq L_f(\|\phi - \psi\|_{\mathcal{B}} + \|Bu - Bv\|),
\end{aligned}$$

where $L_f = max\{\sup_{0 \leq t \leq 1} d(t), 1\}$. All conditions of Theorem 3.3 being satisfied, we can conclude that system (4.1) possesses at least one mild solution. We can take $H = 1$, $M(t) = \gamma^{\frac{1}{2}}(-t)$ and $K(t) = 1 + (\int_{-t}^0 \rho(\theta)d\theta)^{\frac{1}{2}}, t \geq 0$. When we set $\rho(s) = e^{-s}$, $\overline{q}_1(s) = s$, we get $c_1 = (\int_{-\infty}^0 s^2 e^s ds)^{\frac{1}{2}} = \sqrt{2}$. However,

$$[1 + (\int_{-1}^0 \rho(t)dt)^{\frac{1}{2}}][L_g + \max\{(\int_{-\infty}^0 \mu^2(s)\rho^{-1}(s)ds)^{\frac{1}{2}}, 1\} + \sum_{i=1}^n (c_i + d_i)] > 1,$$

with the condition of being less than 1 no longer met, our findings diverge from the previously known results.

Example 4.2. Consider the following impulsive fractional measure driven differential equations

$$\begin{cases} D_t^{\frac{1}{2}} u(t,x) = \dfrac{\partial}{\partial x} u(t,x) + h(t, u(t,x), Bu(t,x))dw(t), t \in [0,1] \setminus \{t_1\}, \\ u(t,0) = u(t,\pi) = 0, t \in [0,1], \\ \Delta u(t_1, x) = \dfrac{u(t_1,x)}{1 + |u(t_1,x)|}, t_1 = \dfrac{1}{2}, x \in [0,\pi], \\ u(0,x) = z_0(x), x \in [0,\pi], \end{cases} \tag{4.2}$$

where $D_t^{\frac{1}{2}}$ is the Caputo fractional derivative operator. Define

$$h(t,u(t),Bu(t))(x)=\sqrt{\frac{2}{\pi}}\frac{e^{-t}}{e^t+e^{-t}}\big[u(t,x)+\int_0^t e^{t-s}u(s,x)ds\big],t\in[0,1]\setminus\{t_1\},$$
$$I_1(u(t_1))(x)=\frac{u(t_1,x)}{1+|u(t_1,x)|},t_1=\frac{1}{2},x\in[0,\pi],u\in X.$$

Thus, the system (4.2) can be modeled as the abstract form given by (1.2) and we have

$$\begin{aligned}&|h(t,u,Bu)-h(t,v,Bv)|_{L^2}\le\sqrt{\frac{2}{\pi}}(|u-v|_{L^2}+|Bu-Bv|_{L^2}),\\&t\in[0,1],u,v\in X,|I_1(u(t_1))|_{L^2}\le|u(t_1)|_{L^2}.\end{aligned}\tag{4.3}$$

All conditions of Theorem 3.4 being satisfied, we can conclude that system (4.2) possesses at least one mild solution. But

$$M_T\alpha^{-1}b^{\alpha}L_h(1+k_0b)]b=\frac{M}{\Gamma(\frac{3}{2})}\sqrt{\pi}=2>1,$$

with the condition of being less than 1 no longer met and $I_1(\cdot)$ does not satisfy the Lipschitz condition.

Remark 4.1. The results of this study demonstrate that for the infinite delay impulsive neutral type second-order measure differential equation, the existence of mild solutions can be ascertained without relying on the compactness conditions of the impulsive term, restrictive a priori estimates, or noncompactness measure estimates.

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