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PERIODIC FLOWS IN SWITCHING DYNAMICAL SYSTEMS THROUGH DISCRETE IMPLICIT MAPPINGS*

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Abstract In this paper, we investigate the periodic flow in a switching dynamical system through an implicit mapping method. By the given accuracy and the transport law, discrete implicit mappings at switching points are obtained and the corresponding interpolation points are achieved. Discrete implicit mappings at non-switching points are obtained by the discretization of differential equations of the switching system and the corresponding interpolation points are also determined. Then the periodic flow expressed by interpolation points in one period is determined. A two-order impulsive system with a pulse at a fixed time is presented as an example. The implicit mapping method may provide a plan for the periodic flows in discontinuous dynamical systems.

Keywords Switching systems, implicit mappings, mapping structures, periodic flows.

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1. Introduction

Switching systems are a class of dynamical systems composed of multiple continuous or discrete subsystems and switch between different subsystems by switching rules or conditions. Many models such as chemical process control models [11], electrical control models [3], network control models [26], traffic control models [28], robot control models [39], biological population models [29] and autonomous driving models [22] can be depicted by switching systems. In the past decades, great attention has been payed to periodic motions in switching systems. Periodic motions in switching systems widely exist in operating modes of switching power converters in electrical engineering [25], intermittent therapy in medicine [27], grazing management in ecology [21], anti-lock braking systems in the automotive industry [8] and have quickly attracted the attention of scholars because of the great significance in economics and science.

Many scholars were interested in the periodic motions of switching systems. Loparo and Aslanis [13, 14] determined the existence of closed trajectories in a class of linear switching systems composed of two subsystems on the phase plane. State-space decomposition theorem and attainability properties of the switching systems were presented. Peleties and Decarlo [23] generalized above results to the switching dynamical systems composed of mth subsystems. Branicky [2, 3] developed the Bendixson Theorem and applied it to judge the existence of closed orbits in a class of two-dimensional continuous switching systems. Savkin and Matveev [24] proved that a class of switching server systems with one server and an arbitrary number of buffers had a set of periodic trajectories that attracted all other trajectories of the systems. Yang and Chen [35]

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determined the existence of closed orbits of a switching system composed of a subsystem with a saddle and a subsystem with a center (focus). Zhang etc. [37] constructed a dynamical circuit model with periodic switching and determined the existence of a periodic flow of the switching system with period-doubling bifurcation and saddle-node bifurcation. Zheng and Fu [38] investigated the dynamics of the periodic motion for switched van der Pol equation with impulsive effect by the theory of mapping dynamics in switching systems. Switching sets and discrete mappings were briefly reviewed. Periodicity of the flow of the switching system was analyzed from the perspective of mapping structures. Llibre, Oliveira and Rodrigues [12] developed an average theory and studied the existence and stability of periodic solutions in the Michelson continuous and discontinuous piecewise linear differential equations by the average theory. Li and Luo [10] obtained the analytical conditions for the existence of periodic solutions of a class of second-order time-delayed discontinuous dynamical systems, and analytically predicted the constraints of two kinds of periodic orbits of the systems. Makarenkov [20] determined the existence of stable limit cycles in a class of planar switching systems composed of two subsystems and deduced the period estimation of the limit cycles. Zhang [36] established a typical switched model alternating between a Duffing oscillator and van der Pol oscillator. The limit cycle of the switched model was located by shooting methods. Wang and Guo [30] gave an algorithm for computing the Lyapunov constants of a class of quadratic switching Liénard systems with three switching lines and obtained a center condition of three limit cycles bifurcating from the focus. Baymout and Benterki [1] determined an exact upper bound for the number of limit cycles of a class of discontinuous piecewise differential systems in Double-struck capital R-3.

Luo [16] developed an implicit mapping method for periodic motions of a class of continous dynamical systems. By the implicit mapping method, mapping structures based on the implicit mappings obtained by the discretization of differential equations of nonlinear dynamical systems were employed to predict analytically the periodic flows of the dynamical systems. Corresponding stability and bifurcations of the periodic motions were determined by eigenvalue analysis. Guo and Luo [6] studied symmetric and asymmetric period-1 motions in a periodically forced, time-delayed and hardening Duffing oscillator by the implicit mappings. Corresponding stability and bifurcation of the period-1 motions of the time-delayed Duffing oscillator were determined by eigenvalue analysis. Luo and Xing [17] investigated bifurcation trees of period-3 motions to chaos in a periodically forced, time-delayed hardening Duffing oscillator by the implicit mapping method. Luo and Xing [18] determined the bifurcation trees for the stable and unstable solutions of period-3 motions to chaos in a class of time-delayed, Duffing oscillators. Xu and Luo [33] investigated amplitude-frequency characteristics of periodic motions in a class of periodically forced van der Pol oscillators using the implicit mapping method. Stability and bifurcation analysis of the periodic motions were completed through the eigenvalue analysis. Nonlinear frequency-amplitude characteristics of the periodic motions were analyzed from the finite Fourier series analysis. Xu, Chen and Luo [32] presented bifurcation trees of period-1 to period-2 motion in a kind of nonlinear rotor systems by the implicit mapping method and the amplitude - frequency characteristics of the periodic motions were determined. Xu and Luo [34] studied bifurcation trees of period-1 motion to chaos in a flexible nonlinear rotor system through the implicit mapping method. Corresponding stability and bifurcation of the periodic motions were discussed by eigenvalue analysis. Stable and unstable periodic motions on the bifurcation tree in the flexible rotor system were achieved. Guo and Luo [4] determined symmetric and asymmetric periodic motions of a nonlinear oscillator with electromagnetic resonant shunt tuned mass damper inerter through the implicit mapping method. Guo and Luo [7] confirmed analytical bifurcation trees of period-3 motions to chaos in a periodically forced nonlinear-spring pendulum through the implicit mapping method. Xing and Luo [31] presented the period-1 motions to twin spiral homoclinic orbits in the Rossler system and predicted the period-1 motions varying with a system parameter semi-analytically through the implicit mapping method. Guo and Luo [5] studied the periodic motions in a 5D Lorenz system by the discrete mappings. A bifurcation tree was given to demonstrate the stable orbits and unstable motions.

So far, the implicit mapping method was mostly used to study the periodic motions of continuous dynamical systems. As far as we know, the periodic motions of discontinuous dynamical systems were rarely investigated by the implicit mapping method except that Luo and Zhu [19] investigated periodic motions in a periodically forced Duffing oscillator through the implicit mapping method.

Hence in this paper, the periodic flow in a switching dynamical system is determined by the implicit mapping method. The remaining contents of this paper consist of three parts. In Section 2, general concepts of switching systems are given and the periodic flow in a switching dynamical system is determined by the implicit mapping method. Through the transport mappings and the given accuracy, discrete implicit mappings are constructed at the switching points and the corresponding interpolation points are achieved. Discrete implicit mappings at the non-switching points are obtained by the discretization of differential equations of the switching system and the corresponding interpolation points are also determined. Then the periodic flow expressed by interpolation points in one period is determined. In Section 3, a two-order impulsive system with a pulse at a fixed time is presented as an example. In Section 4, our main result and further work are presented in conclusion.

2. Preliminary knowledge and the periodic flow of a switching dynamical system

2.1. Preliminary knowledge

In this paper, we need the following definitions [15].

Definition 2.1. A dynamical system composed of many subsystems and corresponding switching rules is called a switching system. In mathematics, it is generally expressed as a $C^{r_i}-$continuous system on the open domain $\Omega_i \subset R^n$

$$\dot{X}^{(i)} = F^{(i)}(X^{(i)}, t, p^{(i)}), X^{(i)} = (x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})^T \in \Omega_i, t \in [t_{k-1}, t_k], \tag{2.1.1}$$

and

$$\dot{X}^{(i)}(t_{k-1}) = X_{k-1}^{(i)}. \tag{2.1.2}$$

The time is t and $X^{(i)} = \dfrac{dX^{(i)}}{dt}$. $p^{(i)} = (p_1^{(i)}, p_2^{(i)}, \cdots, p_m^{(i)})^T \in R^m$ is a parameter vector. $k \in \mathbf{N}^*$, where $\mathbf{N}^*$ is the set of all natural numbers. The dynamical system in Eq. (2.1.1) has a continuous flow as

$$x^{(i)}(t) = \Phi^{(i)}(X_{k-1}^{(i)}, t, p^{(i)}),$$

where $X_{k-1}^{(i)} = \Phi^{(i)}(X_{k-1}^{(i)}, t_{k-1}, p^{(i)})$ for $i = 1, 2, \cdots, m$. We give the following hypotheses of the ith subsystem to study the properties of the switching system with multiple subsystems.

(H1) There are $F^{(i)}(X^{(i)}, t, p^{(i)}) \in C^{r_i}$ and $\Phi^{(i)}(X_k^{(i)}, t, p^{(i)}) \in C^{r_i+1}$ on Ω_i for $t \in [t_{k-1}, t_k]$;

(H2) There are $\|F^{(i)}\| \leq k_1^{(i)}$ and $\|\Phi^{(i)}\| \leq k_2^{(i)}$ on Ω_i for $t \in [t_{k-1}, t_k]$, where $k_1^{(i)}$ and $k_2^{(i)}$ are known constants;

(H3) There is $X^{(i)} = \Phi^{(i)}(t) \notin \partial\Omega_i$ for $t \in [t_{k-1}, t_k]$;

(H4) The switching of two subsystems has time continuity.

To investigate the switching system, a set of dynamical systems in finite time intervals will be introduced first. From such a set of dynamical systems, the dynamical subsystems in a resultant switching system can be selected.

Definition 2.2. From dynamical systems in Eq. (2.1.1), a set of dynamical systems on the open domain Ω_i in the time interval $t_k \in [t_{k-1}, t_k]$ for $i = 1, 2, \cdots m$ is defined as

$$G \equiv \{S_i | i = 1, 2, \cdots, m\}, \tag{2.1.3}$$

where

$$S_i = \left\{ \dot{X}^{(i)} = F^{(i)}(X^{(i)}, t, p^{(i)}) \in R^n \;\middle|\; \begin{array}{l} X^{(i)} \in \Omega_i \subset R^n, p^{(i)} \subset R^m, \\ X^{(i)}(t_{k-1}) = X_{k-1}^{(i)}, t \in [t_{k-1}, t_k], \\ k \in \mathbf{N}^* \end{array} \right\}.$$

From Assumptions (H1)-(H4), the subsystem possesses a finite solution in the finite time interval and such a solution will not reach the corresponding domain boundary. From the set of subsystems, the corresponding set of solutions for such subsystems can be defined as follows.

Definition 2.3. For the ith dynamical subsystems in Eq. (2.1.1) with an initial condition (2.1.2), there is a unique solution $X^{(i)}(t) = \Phi^{(i)}(X_{k-1}^{(i)}, t, p^{(i)})$, where $t \in [t_{k-1},\, t_k]$, $k \in N^*$, $i = 1,\, 2,\, \cdots,\, m$. a set of solutions for the ith subsystem in Eq. (2.1) with an initial condition (2.1.2) on the open domain Ω_i is defined as

$$S = \{\theta^{(i)} |\; i = 1, 2, \cdots, m\},$$

where

$$\theta^{(i)} = \{X^{(i)}(t) |\; X^{(i)}(t) = \Phi^{(i)}(X_{k-1}^{(i)}, t, p^{(i)}), t \in [t_{k-1}, t_k], k \in \mathbf{N}^*\}.$$

Nextly we will investigate the periodic flow of a switching dynamical system of two subsystems by the implicit mapping method.

2.2. The periodic flow of a switching dynamical system

Theorem 2.1. *Consider a switching system*

$$\begin{cases} \dot{X} = F(X, t, p), \quad X \in \Omega \subset R^n,\ p \in R^m,\ t \neq \sigma,\ t \in [t_0, t_0 + T], \\ g(X(t), X(t^+), p) = 0,\ t = \sigma. \end{cases} \tag{2.2.1}$$

The vector function $F(X, t, p)$ is continuous and $\|F(X, t, p)\| \leq L$ on the open domain Ω for $\Omega \subset R^n$, where L is a positive constant. $p = (p_1, p_2, \cdots, p_m)^T \in R^m$ is a parameter vector. The system in Eq. (2.2.1) has a periodic flow $X(t)$ with finite norm $\|X\|$ and a period T. $t = \sigma$ is the only switching time. The periodic flow $X(t)$ is continuous on the left and discontinuous on the right at the point of $t = \sigma$ on the interval $[t_0, t_0 + T]$. $g(X(t), X(t^+), p)$ is the transport law *at the switching time $t = \sigma$. Then there is a set of discrete time $t_k (k = 0, 1, \cdots, N)$ with $N \to \infty$ during one period T, and the corresponding solution $X(t_k)$ and vector field $F(X(t_k), t_k, p)$ are exact. Assuming that a discrete node X_k is on the approximate solution of the periodic flow satisfying $\|X(t_k) - X_k\| \le \varepsilon_k$ and $\|F(X(t_k), t_k, p) - F(X_k, t_k, p)\| \le \delta_k$ for given accuracy $\varepsilon_k > 0$ and $\delta_k > 0$. If the switching time is not on the interval $[t_{k-1}, t_k]$, there is a mapping $P_k : X_{k-1} \to X_k$ $(k = 1, 2, \cdots, N)$, namely*

$$X_k = P_k X_{k-1}, \text{ with } g_k(X_{k-1}, X_k, p) = 0, \ k = 1, 2, \cdots, N, \tag{2.2.2}$$

where $g_k(X_{k-1}, X_k, p)$ is an implicit vector function.

For any small $\varepsilon_m > 0$ and $\delta_m > 0$, there are $\|X(\sigma^+) - X_m\| \le \varepsilon_m$ and $\|F(X(\sigma^+), \sigma, p) - F(X_m, \sigma, p)\| \le \delta_m$. If the switching time is at the point of t_m, namely $t_m = \sigma$, where $m \ne k$ and $m < N$, there is an implicit mapping $P_m : X_{m-1} \to X_m$, i.e.,

$$X_m = P_m X_{m-1}, \ g_m(X_{m-1}, X_m, p) = 0, \tag{2.2.3}$$

where $g_m(X_{m-1}, X_m, p)$ is an implicit vector function, X_{m-1} and X_m are the approximation of $X(t_{m-1})$ and $X(\sigma)$ respectively.

Consider a mapping structure as

$$P = P_N \circ P_{N-1} \circ \cdots \circ P_{m+1} \circ P_m \circ P_{m-1} \circ \cdots \circ P_2 \circ P_1 : X_0 \to X_N.$$

For $X_N = PX_0$, if there is a set of points X_k^* computed by

$$\begin{aligned}
&g_k(X_{k-1}^*, X_k^*, p) = 0, \ k = 1, 2, \cdots, m-1, m+1, \cdots, N, \\
&g_m(X_{m-1}^*, X_m^*, p) = 0, \\
&X_0^* = X_N^*,
\end{aligned}$$

then X_m^* is the approximation of the point $X(\sigma^+)$, and the points X_k^* $(k = 0, 1, 2, \cdots, m-1, m+1, \cdots, N)$ are approximations of points $X(t_k)$ in the periodic solution.

Proof. Because the system in Eq. (2.2.1) has a periodic flow $X(t)$ with finite norm $\|X\|$ and a period T, there is a set of discrete time $t_k (k = 0, 1, \cdots, N)$ with $(N \to \infty)$ on the interval $[t_0, t_0 + T]$, where $t_0 = 0$, $t_N = T$ and $t_k = t_{k-1} + h_k$.

(1) If the switching time $t = \sigma$ is not on the interval $[t_{k-1}, t_k]$, Eq. (2.2.1) is equivalent to the following equation

$$X(t) = X(t_{k-1}) + \int_{t_{k-1}}^{t} F(X, t, p) dt, \tag{2.2.4}$$

where $t \in [t_{k-1}, t_k]$. For the given accuracy $\delta_k > 0$, there is an approximate function $\mathbf{R}_k(t, p) = F(\frac{1}{2}[X(t_{k-1}) + X(t_k)], t_{k-1} + \frac{h}{2}, p)$ on the interval $[t_{k-1}, t_k]$ satisfying the inequality $\|\mathbf{R}_k(t, p) - F(X, t, p)\| \le \delta_k$. For $t \in [t_{k-1}, t_k]$, Eq. (2.2.4) is approximately expressed as

$$\begin{aligned}
X(t) &= X(t_{k-1}) + \int_{t_{k-1}}^{t} [\mathbf{R}_k(t, p) + o(\delta_k)] dt, \\
\bar{X}(t) &= \bar{X}(t_{k-1}) + \int_{t_{k-1}}^{t} \mathbf{R}_k(t, p) dt
\end{aligned}$$

and

$$\bar{X}(t_k) = \bar{X}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{R}_k(t,p)dt. \tag{2.2.5}$$

Let $\bar{X}(t_{k-1}) = X_{k-1}$ and $\bar{X}(t_k) = X_k$. For the given accuracy $\varepsilon_{k-1} > 0$ and $\varepsilon_k > 0$ under $\|X(t_{k-1}) - X_{k-1}\| \le \varepsilon_{k-1}$ and $\|X(t_k) - X_k\| \le \varepsilon_k$, Eq. (2.2.5) gives

$$X_k = X_{k-1} + \bar{g}_k(X_{k-1}, X_k, p), \quad \bar{g}_k(X_{k-1}, X_k, p) = \int_{t_{k-1}}^{t_k} \mathbf{R}_k(t,p)dt. \tag{2.2.6}$$

Thus, a discrete mapping

$$P_k : X_{k-1} \to X_k,\ X_k = P_k X_{k-1} \tag{2.2.7}$$

is obtained by the implicit vector function

$$g_k(X_{k-1}, X_k, p) = X_k - X_{k-1} - \int_{t_{k-1}}^{t_k} \mathbf{R}_k(t,p)dt = 0.$$

From the discrete mapping, two points $X(t_{k-1})$ and $X(t_k)$ for the time interval $[t_{k-1}, t_k]$ can be approximated by X_{k-1} and X_k respectively. Hence $\|F\| \le L$ on the open domain Ω, where L is a positive constant, we have

$$\begin{aligned}
&\|F(X(t_{k-1}), t_{k-1}, p) - F(X_{k-1}, t_{k-1}, p)\| \le L\|X(t_{k-1}) - X_{k-1}\| \le L \cdot \varepsilon_{k-1} = \delta_{k-1},\\
&\|F(X(t_k), t_k, p) - F(X_k, t_k, p)\| \le L\|X(t_k) - X_k\| \le L \cdot \varepsilon_k = \delta_k.
\end{aligned}$$

(2) If the switching time $t = \sigma$ is at the point of t_m, namely $t_m = \sigma$, there is

$$g(X(t_m), X(t_m^+), p) = 0. \tag{2.2.8}$$

For $t \in [t_{m-1}, t_m]$,

$$X(t) = X(t_{m-1}) + \int_{t_{m-1}}^{t} F(X,t,p)dt \tag{2.2.9}$$

and

$$X(t_m) = X(t_{m-1}) + \int_{t_{m-1}}^{t_m} F(X,t,p)dt. \tag{2.2.10}$$

By Eq. (2.2.8) and Eq. (2.2.10), we have

$$g(X(t_{m-1}) + \int_{t_{m-1}}^{t_m} F(X,t,p)dt, X(t_m^+), p) = 0. \tag{2.2.11}$$

For the given accuracy $\delta_m > 0$, there is an approximate function $\mathbf{R}_m(t,p) = F(\frac{1}{2}[X(t_{m-1}) + X(t_m)], t_{m-1} + \frac{h}{2}, p)$ on the interval $[t_{m-1}, t_m]$ satisfying the inequality $\|\mathbf{R}_m(t,p) - F(X,t,p)\| \le \delta_m$. For $t \in [t_{m-1}, t_m]$, Eq. (2.2.11) can be approximated as

$$g(X(t_{m-1}) + \int_{t_{m-1}}^{t_m} [\mathbf{R}_m(t,p) + o(\delta_m)]dt,\ X(t_m^+),\ p) = 0,$$

$$g(\bar{X}(t_{m-1}) + \int_{t_{m-1}}^{t_m} \mathbf{R}_m(t,p)dt,\ \bar{X}(t_m^+),\ p) = 0. \tag{2.2.12}$$

Let $\bar{X}(t_{m-1}) = X_{m-1}$ and $\bar{X}(t_m^+) = X_m$. For the given accuracy $\varepsilon_{m-1} > 0$ and $\varepsilon_m > 0$ under $\|X(t_{m-1}) - X_{m-1}\| \leq \varepsilon_{m-1}$ and $\|X(t_m^+) - X_m\| \leq \varepsilon_m$, Eq. (2.2.12) can be expressed by

$$g_m(X_{m-1} + \int_{t_{m-1}}^{t_m} \mathbf{R}_m(t,p)dt,\ \ X_m, p) = 0, \tag{2.2.13}$$

where the points X_m and X_{m-1} are approximations of points $X(t_m^+)$ and $X(t_{m-1})$ respectively. So a discrete mapping $P_m : X_{m-1} \to X_m$, i.e.,

$$X_m = P_m X_{m-1} \tag{2.2.14}$$

can be obtained by the implicit vector function (2.2.13). The vector function $F(X,t,p)$ satisfies the following inequalities

$$\|F(X(t_{m-1}),\ t_{m-1},\ p) - F(X_{m-1},\ t_{m-1},\ p)\| \leq L\|X(t_{m-1}) - X_{m-1}\| \leq L \cdot \varepsilon_{m-1} = \delta_{m-1},$$
$$\|F(X(t_m^+), t_m,\ p) - F(X_m, t_m,\ p)\| \leq L\|X((t_m^+)) - X_m\| \leq L \cdot \varepsilon_m = \delta_m.$$

For $t \in (t_m, t_{m+1}]$, Eq. (2.2.1) can be transformed into

$$X(t) = X(t_{m+1}) + \int_{t_{m+1}}^{t} F(X,t,p)dt. \tag{2.2.15}$$

When $t \to t_m^+$, Eq. (2.2.15) is equivalent to

$$\lim_{t\to t_m^+} X(t) = X(t_{m+1}) + \lim_{t\to t_m^+} \int_{t_{m+1}}^{t} F(X,t,p)dt. \tag{2.2.16}$$

Since $F(X,t,p)$ is a continuous vector function on the open domain Ω for $\Omega \subset R^n$, we have

$$\lim_{t\to t_m^+} \int_{t_{m+1}}^{t} F(X,t,p)dt = \int_{t_{m+1}}^{t_m} F(X,t,p)dt.$$

Hence Eq. (2.2.16) can be transformed into the following equation

$$X(t_{m+1}) = X(t_m^+) + \int_{t_m}^{t_{m+1}} F(X,t,p)dt. \tag{2.2.17}$$

For a small $\delta_{m+1} > 0$, if there is an approximate function $\mathbf{R}_{m+1}(t,p) = F(\frac{1}{2}[X(t_m) + X(t_{m+1})],$ $t_m + \frac{h}{2}, p)$ satisfying the inequality

$$\|\mathbf{R}_{m+1}(t,p) - F(X,t,p)\| \leq \delta_{m+1},$$

for $t \in (t_m, t_{m+1}]$, Eq. (2.2.17) can be approximately expressed as

$$\bar{X}(t_{m+1}) = \bar{X}(t_m^+) + \int_{t_m}^{t_{m+1}} \mathbf{R}_{m+1}(t,p)dt. \tag{2.2.18}$$

Let $\bar{X}(t_{m+1}) = X_{m+1}$. For any small $\varepsilon_{m+1} > 0$, under $\|X(t_{m+1}) - X_{m+1}\| \leq \varepsilon_{m+1}$, Eq. (2.2.18) gives

$$X_{m+1} = X_m + \int_{t_m}^{t_{m+1}} \mathbf{R}_{m+1}(t,p)dt, \tag{2.2.19}$$

where two points X_m and X_{m+1} are approximations of points $X(t_m^+)$ and $X(t_{m+1})$ respectively. An implicit mapping $P_{m+1} : X_m \to X_{m+1}$, i.e.,

$$X_{m+1} = P_{m+1}X_m, \tag{2.2.20}$$

can be obtained by the implicit vector function

$$g_{m+1}(X_m, X_{m+1}, p) = X_{m+1} - X_m - \int_{t_m}^{t_{m+1}} \mathbf{R}_{m+1}(t,p)dt = 0. \tag{2.2.21}$$

The vector function $F(X,t,p)$ satisfies the following inequalities

$$\|F(X(t_{m+1}), t_{m+1}, p) - F(X_{m+1}, t_{m+1}, p)\| \leq L\|X(t_{m+1}) - X_{m+1}\| \leq L \cdot \varepsilon_{m+1} = \delta_{m+1}.$$

Once the discrete mappings in Eqs. (2.2.7), (2.2.13) and (2.2.20) exist, then the periodic flow of the switching system (2.2.1) can be formed by

$$P = P_N \circ P_{N-1} \circ \cdots \circ P_{m+1} \circ P_m \circ P_{m-1} \circ \cdots \circ P_2 \circ P_1 : X_0 \to X_N,$$

i.e.,

$$\begin{aligned}
&P_1 : X_0 \to X_1 \Rightarrow g_1(X_0, X_1, p) = 0,\\
&P_2 : X_1 \to X_2 \Rightarrow g_2(X_1, X_2, p) = 0,\\
&\vdots\\
&P_{m-1} : X_{m-2} \to X_{m-1} \Rightarrow g_{m-1}(X_{m-2}, X_{m-1}, p) = 0,\\
&P_m : X_{m-1} \to X_m \Rightarrow g_m(X_{m-1}, X_m, p) = 0,\\
&P_{m+1} : X_m \to X_{m+1} \Rightarrow g_{m+1}(X_m, X_{m+1}, p) = 0,\\
&\vdots\\
&P_N : X_{N-1} \to X_N \Rightarrow g_N(X_{N-1}, X_N, p) = 0.
\end{aligned} \tag{2.2.22}$$

With the periodic condition, we have

$$X_0 = X_N. \tag{2.2.23}$$

Solving Eqs. (2.2.22) and (2.2.23) gives X_k^* and X_m^*, i.e.,

$$\begin{aligned}
&g_k(X_{k-1}^*, X_k^*, p) = 0, (k = 1, 2, \cdots, m-1, m+1, \cdots, N),\\
&g_m(X_{m-1}^*, X_m^*, p) = 0,\\
&g_{m+1}(X_m^*, X_{m+1}^*, p) = 0,\\
&X_0^* = X_N^*.
\end{aligned} \tag{2.2.24}$$

The periodic flow of the system can be determined by X_m^* and X_k^*, where X_m^* is approximation of the point $X(t_m^+)$, and the points X_k^* $(k = 0, 1, 2, \cdots, m-1, m+1, \cdots, N)$ are approximations of points $X(t_k)$ of the periodic solution.

(3) If the switching time $t = \sigma$ is on the interval (t_{k-1}, t_k), we can choose the discrete point $t_i = \sigma$, where $i \neq k-1$, k and $i < N$, then the investigation is the same with case (2). The theorem is proved. □

If the system in Eq. (2.2.1) has two fixed switching time points, we can also investigate the periodic flow through the implicit mapping method in Theorem 2.2.1, The discrete implicit mappings at the switching time points can be constructed by the given accuracy and the transport laws. Discrete implicit mappings at non-switching points can be obtained by the discretization of differential equations of the switching system. The corresponding interpolation points are obtained by the implicit mapping structure made up of the discrete implicit mappings. The periodic flow expressed by the interpolation points in one period can be determined.

3. An example

Consider an impulsive system as

$$\begin{cases} \dot{x} = y, \quad t \neq 1,\ t \in [0,6], \\ \dot{y} = 2\cos t - y + 0.01x, \quad t \neq 1, t \in [0,6], \\ x(t^+) = x(t) + e_1 x(t), \quad t = 1, \\ y(t^+) = y(t) + e_2 y(t), \quad t = 1, \end{cases} \tag{3.1}$$

where the impulse parameters $e_1 = 0.2$ and $e_2 = 0.1802$. Let $X(t) = (x(t), y(t))^T$. The vector function $F(X, t, p) = (y,\ 2\cos t - y + 0.01x)^T$ is continuous on $\Omega = [-1, 2.5] \times [-1.5, 1.5]$ for $\Omega \subset R^n$. The analytical solution of Eq. (3.1) satisfying the conditions $x(1) = 1.251$, $y(1) = 1.271$ on the interval $[0, 1]$ can be expressed by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.1082e^{0.0099t} + 0.3315e^{-1.0099t} + 0.9415\cos t - 191.2623\sin t \\ 0.01097e^{0.0099t} - 0.3348e^{-1.0099t} + 1.0094\cos t - 191.2526\sin t \end{bmatrix}. \tag{3.2}$$

The analytical solution of Eq. (3.1) satisfying the conditions $x(1^+) = 1.5012$ and $y(1^+) = 1.5$ on the interval $(1, 6]$ can be expressed by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.9415\cos t - 191.2623\sin t + 1.2674 \\ 1.0094\cos t - 191.2526\sin t + 161.8936 \end{bmatrix}. \tag{3.3}$$

The analytical solution of Eq. (3.1) on the interval $[0, 1]$ and $(1, 6]$ can be sketched in Figure 1. By Figure 1, we can determine the system in Eq. (3.1) has a periodic flow $X(t) = (x(t), y(t))^T$ with finite norm $\|x\|$ and a period $T = 6$. $t = 1$ is the only impulsive time. The periodic flow $X(t)$ is continuous on the left and discontinuous on the right at the point of $t = 1$ on the interval $[0, 6]$. Then we will study the periodic solution in the system by the implicit mapping method.

There is a set of discrete time $t_0 < t_1 < \cdots < t_k < \cdots < t_N$ on the interval $[0, 6]$, where $t_0 = 0$, $t_N = 6$ and $t_k = t_{k-1} + h$.

(1) If the impulsive time is not on the interval $[t_{k-1}, t_k]$, Eq. (3.1) is integrated by t as follows

$$\begin{cases} x(t_k) = x(t_{k-1}) + \displaystyle\int_{t_{k-1}}^{t_k} y(t)dt, \ t \in [t_{k-1}, t_k], \\ y(t_k) = y(t_{k-1}) + \displaystyle\int_{t_{k-1}}^{t_k} [2\cos t - y(t) + 0.01x(t)]dt. \end{cases} \tag{3.4}$$

Let

$$\mathbf{P}_k^{(1)}(t, p) = 0.5h[y(t_{k-1}) + y(t_k)]$$

and

$$\mathbf{P}_k^{(2)} = 2h\cos(t_{k-1} + 0.5h) - 0.5h[y(t_{k-1}) + y(t_k)] + 0.005h[x(t_{k-1}) + x(t_k)],$$

where the vector functions $\mathbf{P}_k^{(1)}(t, p)$ and $\mathbf{P}_k^{(2)}(t, p)$ are approximations of functions $y(t)$ and $2\cos t - y + 0.01x$ on the interval $t \in [t_{k-1}, t_k]$ respectively. Then Eq. (3.4) can be changed into

$$\begin{cases} g_k^{(1)}(t, p) = x_k - x_{k-1} - \dfrac{y_{k-1} + y_k}{2}h = 0, \qquad t \in [t_{k-1}, t_k], \\ g_k^{(2)}(t, p) = y_k - y_{k-1} - 2h\cos(t_{k-1} + \dfrac{h}{2}) + \dfrac{h}{2}(y_{k-1} + y_k) - 0.005h(x_{k-1} + x_k) = 0, \end{cases} \tag{3.5}$$

where $X_{k-1} = (x_{k-1}, y_{k-1})^T$ and $X_k = (x_k, y_k)^T$. The points X_{k-1} and X_k are approximations of points $X(t_{k-1})$ and $X(t_k)$ respectively. $g_k(t, p) = (g_k^{(1)}(t, p), g_k^{(2)}(t, p))^T$ is a vector function connecting X_{k-1} with X_k. An implicit mapping

$$P_k : X_{k-1} \to X_k, \ X_k = A_k X_{k-1} + D_k, \ k = 1, \ 2, \ \cdots, \ N \tag{3.6}$$

can be determined by the implicit vector functions (3.5), where

$$D_k = \frac{1}{1 + 0.5h - 0.0025h^2} \begin{bmatrix} h^2\cos(t_{k-1} + 0.5h) \\ 2h\cos(t_{k-1} + 0.5h) \end{bmatrix},$$

and

$$A_k = \frac{1}{1 + 0.5h - 0.0025h^2} \begin{bmatrix} 1 + 0.5h + 0.0025h^2 & h \\ 0.01h & 1 - 0.5h + 0.0025h^2 \end{bmatrix}.$$

(2) If the impulsive time is at the point of t_m, namely $t_m = 1$, by the switching conditions and Eq. (3.1), we have

$$\begin{cases} x(t_m^+) = 1.2x(t_{m-1}) + \displaystyle\int_{t_{m-1}}^{t_m} 1.2y(t)dt, \ t \in [t_{m-1}, t_m], \\ y(t_m^+) = 1.1802y(t_{m-1}) + \displaystyle\int_{t_{m-1}}^{t_m} 1.1802[2\cos t - y(t) + 0.01x(t)]dt. \end{cases} \tag{3.7}$$

Let $\mathbf{P}_m^{(1)}(t, p) = 0.5[y(t_{m-1}) + y(t_m)]$ and $\mathbf{P}_m^{(2)}(t, p) = 2\cos(t_{m-1} + 0.5h) - 0.5[y(t_{m-1}) + y(1)] + 0.005h[x(t_{m-1}) + x(t_m)]$. The vector functions $\mathbf{P}_m^{(1)}(t, p)$ and $\mathbf{P}_m^{(2)}(t, p)$ are the approximations of functions $y(t)$ and $2\cos t - y + 0.01x$ on the interval $t \in [t_{m-1}, t_m]$ respectively. Then Eq. (3.10) can be changed into

$$\begin{cases} g_m^{(1)}(t,p) = x_m - 1.2x_{m-1} - 0.6h[y_{m-1} + y(t_m)] = 0, \\ g_m^{(2)}(t,p) = y_m - 1.1802y_{m-1} - 2.3604h\cos(t_{m-1} + 0.5h) \\ \qquad\qquad + 0.5901h[y_{m-1} + y(t_m)] - 0.005901h[x_{m-1} + x(t_m)] \\ \qquad = 0, \end{cases} \tag{3.8}$$

where $X_{m-1} = (x_{m-1}, y_{m-1})^T$ and $X_m = (x_m, y_m)^T$. The points X_{m-1} and X_m are approximations of points $X(t_{m-1})$ and $X(t_m^+)$. $g_m(t,p) = (g_m^{(1)}(t,p), g_m^{(2)}(t,p))^T$ is an implicit vector function connecting X_{m-1} with X_m. An implicit mapping

$$P_m : X_{m-1} \to X_m,\ X_m = A_m X_{m-1} + D_m, \tag{3.9}$$

can be determined by (3.8), where

$$A_m = \begin{bmatrix} 1.2 & 0.6h \\ 0.005901h & 1.1802(1 - 0.5h) \end{bmatrix}$$

and

$$D_m = \begin{bmatrix} 0.6hy_m \\ 1.1802h[2\cos(t_{m-1} + 0.5h) - 0.5y(1) + 0.005x(1)] \end{bmatrix}.$$

For $t \in (t_m, t_{m+1}]$, Eq. (3.1) is integrated by t as follows

$$\begin{cases} x(t) = x(t_{m+1}) + \displaystyle\int_{t_{m+1}}^{t} y(t)dt, \qquad t \in (t_m, t_{m+1}], \\ y(t) = y(t_{m+1}) + \displaystyle\int_{t_{m+1}}^{t} [2\cos t - y(t) + 0.01x(t)]dt. \end{cases} \tag{3.10}$$

Let $t \to t_m^+$, then Eq. (3.10) can be expressed as

$$\begin{cases} x(t_m^+) = x(t_{m+1}) + \displaystyle\int_{t_{m+1}}^{t_m} y(t)dt,\ t \in (t_m, t_{m+1}], \\ y(t_m^+) = y(t_{m+1}) + \displaystyle\int_{t_{m+1}}^{t_m} [2\cos t - y(t) + 0.01x(t)]dt. \end{cases} \tag{3.11}$$

Let

$$\mathbf{P}_{m+1}^{(1)}(t,p) = 0.5[y(t_{m+1}) + y(t_m)]$$

and

$$\mathbf{P}_{m+1}^{(2)}(t,p) = 2\cos(t_m + 0.5h) - 0.5[y(t_{m+1}) + y(t_m)] + 0.005h[x(t_{m+1}) + x(t_m)],$$

where the vector functions $\mathbf{P}^{(1)}_{m+1}(t,p)$ and $\mathbf{P}^{(2)}_{m+1}(t,p)$ are approximations of functions $y(t)$ and $2\cos t - y(t) + 0.01x(t)$ respectively on the interval $t \in (t_m, t_{m+1}]$. So Eq. (3.11) can be expressed as

$$\begin{cases} g^{(1)}_{m+1} = x_{m+1} - x_m - 0.5h[y(t_m) + y_{m+1}] = 0,\ t \in (t_m, t_{m+1}], \\ g^{(2)}_{m+1} = y_{m+1} - y_m - 2h\cos(t_m + 0.5h) + 0.5h[y(t_m) + y_{m+1}] \\ \qquad\qquad -0.005h[x(t_m) + x_{m+1}] \\ \qquad = 0, \end{cases} \tag{3.12}$$

where the points X_m and X_{m+1} are approximations of points $X(t_m^+)$ and $X(t_{m+1})$, $g_{m+1}(X_m, X_{m+1}, p) = (g^{(1)}_{m+1}, g^{(2)}_{m+1})^T$ and $g_{m+1}(X_m, X_{m+1}, p)$ is a vector function connecting X_m with X_{m+1}. An implicit mapping

$$P_{m+1} : X_m \to X_{m+1},\ X_{m+1} = A_{m+1}X_m + D_{m+1} \tag{3.13}$$

can be determined by Eq. (3.12), where

$$A_{m+1} = \frac{1}{1 + 0.5h - 0.0025h^2}\begin{bmatrix} 1 + 0.5h & 0.5h \\ 0.005h & 1 \end{bmatrix}$$

and

$$D_{m+1} = \frac{1}{1 + 0.5h - 0.0025h^2}\begin{bmatrix} [0.0025x(1) + \cos(1 + 0.5h)]h^2 + 0.5hy(1) \\ 0.0025h^2y(1) + 2h\cos(1 + 0.5h) - 0.5hy(1) + 0.005hx(1) \end{bmatrix}.$$

(3) If the impulsive time is on the interval (t_{k-1}, t_k), we can choose the discrete point $t_i = 1$, where $i \neq k-1$, k and $i < N$. Then the study is same with the second situation.

By Eqs. (3.6), (3.9) and (3.13), the periodic flow of the impulsive system can be formed by

$$P = P_N \circ P_{N-1} \circ \cdots \circ P_{m+1} \circ P_m \circ P_{m-1} \circ \cdots \circ P_2 \circ P_1 : X_0 \to X_N, \tag{3.14}$$

namely

$$X_N = AX_0 + D, \tag{3.15}$$

where

$$\begin{aligned} A &= A_N \cdot A_{N-1} \cdots A_{m+1} \cdot A_m \cdot A_{m-1} \cdots A_2 \cdot A_1, \\ D &= A_N \cdot A_{N-1} \cdots A_{m+1} \cdot A_m \cdot A_{m-1} \cdots A_2 \cdot D_1 \\ &\quad + A_N \cdot A_{N-1} \cdots A_{m+1} \cdot A_m \cdot D_{m-1} \cdots A_3 \cdot D_2 \\ &\quad + \cdots + A_N \cdot A_{N-1} \cdots A_{m+1} \cdot D_m + A_N \cdot A_{N-1} \\ &\quad \cdots A_m D_{m+1} + \cdots + A_N D_{N-1} + D_N. \end{aligned}$$

Let $h = 0.1$, we have

$$A = \begin{bmatrix} 1.2597 & 1.2366 \\ 0.0124 & 0.0153 \end{bmatrix}, D = \begin{bmatrix} -0.8861 \\ 0.6735 \end{bmatrix}.$$

By Eq. (3.15) and the periodic condition

$$X_0 = X_N, \tag{3.16}$$

we have $x_0 = 0.1465$, $y_0 = 0.6858$. Then we can obtain the following interpolation points on the interval $[0, 6]$

$X_0(0.1465, 0.6858)$, $X_1(0.2213, 0.8109)$, $X_2(0.3080, 0.9223)$, $X_3(0.4051, 1.0193)$,
$X_4(0.5111, 1.1016)$, $X_5(0.6246, 1.1687)$, $X_6(0.7441, 1.2205)$, $X_7(0.8680, 1.2566)$,
$X_8(0.9946, 1.2772)$, $X_9(1.1226, 1.2823)$, $X_{10}(1.5003, 1.5014)$, $X_{11}(1.6371, 1.4655)$,
$X_{12}(1.7807, 1.4054)$, $X_{13}(1.9177, 1.3334)$, $X_{14}(2.048, 1.2500)$, $X_{15}(2.1671, 1.1559)$,
$X_{16}(2.2775, 1.0519)$, $X_{17}(2.3770, 0.9389)$, $X_{18}(2.4649, 0.8178)$, $X_{19}(2.5403, 0.6898)$,
$X_{20}(2.6025, 0.5560)$, $X_{21}(2.6512, 0.4178)$, $X_{22}(2.6859, 0.2763)$, $X_{23}(2.7064, 0.1329)$,
$X_{24}(2.7125, -0.0111)$, $X_{25}(2.7042, -0.1541)$, $X_{26}(2.6818, -0.2950)$,
$X_{27}(2.454, -0.4323)$, $X_{28}(2.5956, -0.5647)$, $X_{29}(2.5328, -0.6909)$,
$X_{30}(2.4577, -0.8097)$, $X_{31}(2.3713, -0.9200)$, $X_{32}(2.2742, -1.0206)$,
$X_{33}(2.1677, -1.1107)$, $X_{34}(2.0527, -1.1892)$, $X_{35}(1.9304, -1.2556)$,
$X_{36}(1.8022, -1.3090)$, $X_{37}(1.6693, -1.3491)$, $X_{38}(1.5331, -1.3754)$,
$X_{39}(1.3949, -1.3876)$, $X_{40}(1.2563, -1.3858)$, $X_{41}(1.1185, -1.3698)$,
$X_{42}(0.9830, -1.3399)$, $X_{43}(0.8512, -1.2964)$, $X_{44}(0.7244, -1.2397)$,
$X_{45}(0.6039, -1.1704)$, $X_{46}(0.4909, -1.0892)$, $X_{47}(0.3866, -0.9969)$,
$X_{48}(0.2920, -0.8945)$, $X_{49}(0.2081, -0.7829)$, $X_{50}(0.1358, -0.6634)$,
$X_{51}(0.0758, -0.5370)$, $X_{52}(0.0287, -0.4051)$, $X_{53}(-0.0050, -0.2690)$,
$X_{54}(-0.0250, -0.1300)$, $X_{55}(-0.0309, 0.0105)$, $X_{56}(-0.0209, 0.1510)$,
$X_{57}(-0.0008, 0.2901)$, $X_{58}(0.0350, 0.4266)$, $X_{59}(0.0843, 0.5589)$,
$X_{60}(0.1465, 0.6858)$.

The approximate periodic solution of Eq. (3.1) connecting with such discrete points on the interval $[0, 6]$ can be sketched in Figure 2. The analytical solution of Eq. (3.1) on the interval $[0, 1]$ and $(1, 6]$ can be sketched in Figure 1.

The approximate periodic solution of Eq. (3.1) on the interval $[0, 6]$ expressed by interpolation points and the analytical solution of Eq. (3.1) on the interval $[0, 1]$ and $(1, 6]$ can be sketched by Figure 3.

By Figure 3, we can determine the approximate periodic solution expressed by interpolation points and the analytical solution of Eq. (3.1) overlap highly on the interval $[0, 6]$. The approximate periodic solution expressed by interpolation points is enough accurate and effective for the investigation of the periodic flow of an impulsive differential system. The implicit mapping method can be further applied to general nonlinear switching systems.

In recent years, the existence of periodic motion of Van der Pol oscillators with pulses has been studied in [19]. The method presented in this paper can be used to give the approximate solution of periodic flow satisfying the given accuracy, which is meaningful in the physical world. We will take this work forward in the future. The implicit mapping method in this paper can also be applied to the study of periodic flow in the practical problem model of switching systems with time switching.

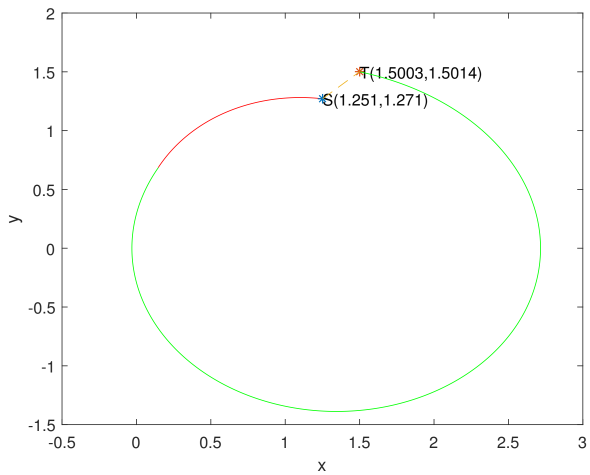

Figure 1. The red curve crossing the point $S(1.251, 1.271)$ depicts the analytical solution of Eq. (3.1) on the interval $[0, 1]$ satisfying the conditions $x(1) = 1.251$ and $y(1) = 1.271$; The green curve crossing the point $T(1.5012, 1.5)$ depicts the analytical solution of Eq. (3.1) satisfying the conditions $x(1^+) = 1.5012$ and $y(1^+) = 1.5$ on the interval $(1, 6]$.

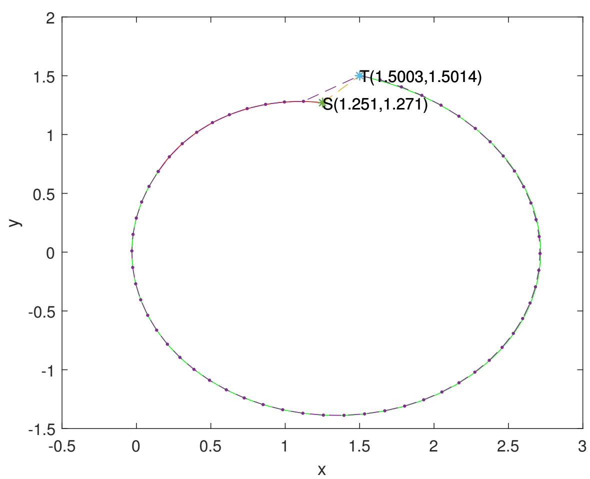

Figure 2. Blue dots connecting with short lines depict the approximate periodic solution of Eq. (3.1) on the interval $[0, 6]$.

4. Conclusion

In this paper, the periodic flow of a switching dynamical system is investigated through the implicit mapping method. Using the transport laws and the given accuracy, we constructed discrete implicit mappings at the switching points and the corresponding interpolation points are obtained. Discrete implicit mappings at the non-switching time are obtained by the discretization of differential equations of the switching system and the corresponding interpolation points are also given. The mapping structure based on discrete implicit mappings is employed for the periodic flow in the switching dynamical system. Then the periodic flow expressed by interpolation points in one period is determined. A two-order impulsive system with a pulse

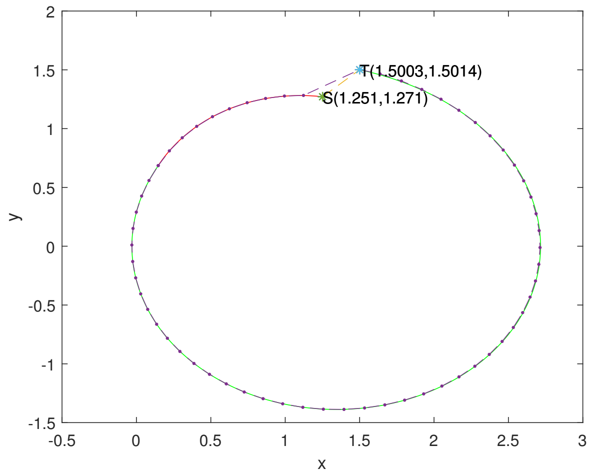

Figure 3. Overlap of the approximate periodic solution of Eq. (3.1) and the analytical solution of Eq. (3.1).

at a fixed time is presented as an example. The method presented in this paper is suitable for switching systems with finite fixed switching time. On this basis, switching systems with arbitrary switching time with a known probability distribution can be further considered. In this case, it is necessary to find the random law of the switching time according to the characteristics of the known probability distribution, and construct the corresponding implicit mapping on the random switching time. We will continue to work on this in the future.

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