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STABILIZATION OF A PARABOLIC PDE SANDWICHED BY TWO NONLINEAR ODES

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Abstract This paper is devoted to the stabilization of a class of nonlinear parabolic ODE-PDE-ODE systems constituting by a parabolic equation sandwiched by two nonlinear ODEs. Different from the related literature where both the two ODE subsystems are linear time invariant (LTI) or only the ODE subsystem proximal to the control input is nonlinear but that distal to the input is LTI, serious nonlinearities are contained in the system under investigation since both the two ODE subsystems (no matter proximal or distal to the input) are all nonlinear which lead to the incapability of the control schemes on this topic. To solve the control problem, a novel control framework is established by smartly combining infinite-dimensional backstepping method with the the finite-dimensional one. Specifically, three steps of backstepping transformations are subsequently introduced for the system, which include two finite-dimensional ones respectively for the distal and proximal ODE subsystems and an infinite-dimensional one for the PDE subsystem. Then, a new target system is obtained under the backstepping transformations while a state-feedback controller is explicitly designed. Finally, by recursive analysis from the target system, desirable stability of the resulting closed-loop system is obtained, i.e., all the states of the resulting closed-loop system are bounded and converge to zero ultimately. A simulation example is provided to validate the effectiveness of the proposed theoretical results.

Keywords ODE-PDE-ODE systems, nonlinearity, backstepping, stabilization, distributed parameter systems.

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1. Introduction

In practice, many dynamic systems are insufficient to be described by pure partial differential equations (PDEs) or ordinary differential equations (ODEs) but must be by a composite one that involves both PDEs and ODEs (see e.g., [2, 5, 7–10, 13–16, 18, 19, 21, 23, 26–30]). Such as the overhead crane with flexible string and payload [21], the tubular chemical reactor with actuator or sensor dynamics [9], etc. For such systems, the resolution of certain control problems requires a skilful incorporation of the schemes from both these two types of systems. However, strong coupling of these two subsystem with distinctive characteristics usually brings huge obstacles in control design and analysis. Thus, many interesting control problems in this field remain unsolved and deserve investigation.

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Over the past two decades, stabilization of the composite PDE and ODE systems in different setups has received extensive attention (see e.g., [2,5,7,9,10,13–16,18,19,23,26,28–30]). As the previous works on this topic, works [2,9,10,13,18,26,29,30] consider one setup that cascaded PDE-ODE systems, where a LTI ODE system is actuated by different classes of PDE actuators, such as the first-order hyperbolic equation, diffusion equation, reaction-diffusion equation and wave equation. Latterly, works [5,7,14,23] consider the counteraction from the PDE actuator to the controlled ODE which leads to heavy interactions of the subsystems and hence results in a more general setup, i.e., coupled PDE-ODE systems. Note that an infinite-dimensional backstepping method has been used to the explicit design of the feedback controller for the systems with above setups. Another representative setup in the field is cascaded ODE-PDE system where a PDE is actuated by an ODE actuator. Specifically, in [19], Lyapunov direct method is used for a Burger's equation with first-order dynamic boundary condition (which implies a first-order actuator). In [15] and [28], infinite-dimensional backstepping method is used for a reaction-diffusion equation with first- and high-order linear actuator dynamics, respectively. In [16,17], two steps of backstepping which involves one infinite- and then one finite-dimensional backstepping transformations is used for reaction-diffusion equation with high-order nonlinear actuator dynamics.

Recently, stabilization has drawn much investigation for sandwiched ODE-PDE-ODE systems which are more general than the PDE-ODE or ODE-PDE ones mentioned above. For such systems with different setups, multiple control methods are proposed for the stabilization controller design. Specifically, one step infinite-dimensional backstepping method is adopted in [1, 20] for two classes of simple hyperbolic ODE-PDE-ODE systems, where both the two ODE subsystems are LTI while that proximal to control input must be one-dimensional. Once the ODE subsystems proximal to control input are multi-dimensional but lower-triangular, two steps of backstepping transformations involving an infinite- and a finite-dimensional ones are subsequently used in [12, 24, 25] for ODE-PDE-ODE systems with PDE subsystems being described by a 2×2 hyperbolic equations and diffusion equation, respectively. In [3, 4], multiple steps of infinite-dimensional backstepping transformations joint with decoupling transformations are used for hyperbolic and parabolic ODE-PDE-ODE systems with more general form where both the two ODEs are LTI multi-dimensional systems without being lower-triangular. It is necessary to point out that, the aforementioned results are severely restricted by the strong assumptions that all the ODE subsystems must be LTI. However, nonlinearities usually (or even ineluctably) exist in practical dynamic systems which render the incapability of the traditional control methods. Then, a natural and nontrivial control problem arises, that is, *for a sandwiched ODE-PDE-ODE system with serious nonlinearities, how does one design a stabilizing controller?*

Towards the problem just raised, only one work (i.e., [6]) is found to make some progress but limited by the system nonlinearities. Specifically, in [6], multi-step infinite-dimensional backstepping method is used for hyperbolic and parabolic ODE-PDE-ODE systems with nonlinear actuator dynamics (which imply a nonlinear ODE subsystem proximal to the control input) but the distal ODE subsystem must be LTI. Remark the fact that all the existing results for ODE-PDE-ODE systems (as well as the PDE-ODE ones as special cases) require the distal ODE subsystem to be LTI, which has been a huge and essential obstacle of the application of the existing schemes. Once the distal ODE subsystem is nonlinear, the existing control schemes would be ineffective. Thus, a powerful control scheme requires to be developed in this field.

This paper is devoted to make remarkable progress on the problem raised above by considering the stabilization of a class of nonlinear parabolic ODE-PDE-ODE systems. Notably, both the two ODE subsystems are nonlinear while heavily coupled with the parabolic equation in-domain and at one boundary, respectively. Then, existing methods in the related literature are ineffective. For this, a novel control framework is established by a skilful combination of the infinite- and finite-dimensional backstepping methods. Specifically, a series of state transformations are firstly introduced for the distal ODE and PDE subsystems, respectively. By finite-dimensional backstepping method, a new PDE-ODE subsystem (i.e., (ξ, v) given by (3.11) below) is obtained by the smart choice of the virtual controls in recursive step. Then, for the new PDE subsystem (i.e., v subsystem), an infinite-dimensional backstepping transformation and its inversion are adopted to change system (ξ, v, Z) into a new one (i.e., (ξ, w, Z)). Finally, for the newly obtained system, finite-dimensional backstepping method is used again under another series of state transformations, which recursively derives the controller while leads to the final target system (i.e., system (ξ, w, ζ) given by (3.16) joint with (4.1) below) . With the help of the further dynamics of the target system, a recursive analysis step is given to show the stability of the final target system which implies that of the original system.

The remainder of the paper is organized as follows. Section 2 formulates the control problem. Section 3 gives the procedure of control design while Section 4 gives the performance analysis of the resulting closed-loop system. Section 5 provides a numerical example to validate the effectiveness of the theoretical results. Section 6 concludes the paper with some remarks. This paper ends with an appendix which collects some useful inequalities and the detailed proof of some propositions as well as some important claims.

Notation. Throughout the paper, the following notations are used. Let $\mathbf{R}$ and $\mathbf{R}^n$ denote the set of all the real numbers and the n-dimensional real space, respectively. For a vector $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}}$, let $\xi_{[i]} = (\xi_1, \cdots, \xi_i)^{\mathrm{T}}$, ξ^{T} denote its transpose and $|\xi|$ denote its Euclidean norm. For a function $w(x):\ [0,\, 1] \to \mathbf{R}$, let $\|w\| = \sqrt{\int_0^1 w^2(x)\mathrm{d}x}$. For a real-valued time-varying function $f(t)$, $f \in \mathcal{L}_p$ means $\left(\int_0^\infty |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} < \infty$ with $p \geq 1$ and particularly $f \in \mathcal{L}_\infty$ means $\sup_{t\geq 0} |f(t)| < \infty$. Let ℓ_p denote the function which belongs to $\mathcal{L}_p$.

2. Problem formulation

In this paper, we consider stabilization of the following systems which are constituted of a parabolic equation sandwiched by two nonlinear ODEs:

$$\begin{cases} \dot{X}_i(t) = X_{i+1}(t) + f_i(X_{[i]}), i = 1, \cdots, n-1, \\ \dot{X}_n(t) = u(0,t) + f_n(X), \\ \partial_t u(x,t) = \partial_x^2 u(x,t) + \lambda u(x,t) + \varepsilon(x) X, \\ \partial_x u(0) = 0, \quad u(1) = Z_1, \\ \dot{Z}_j(t) = Z_{j+1}(t) + g_j\left(Z_{[j]}\right), j = 1, \cdots, m-1, \\ \dot{Z}_m(t) = U(t) + g_m(Z), \end{cases} \tag{2.1}$$

where $(x,t) \in [0,1] \times [0,+\infty)$, $X = X_{[n]} \in \mathbf{R}^n$, $Z = Z_{[m]} \in \mathbf{R}^m$, $u \in \mathbf{R}$ are system states, U is control input of the entire system, $\lambda > 0$ is a positive constant, $\varepsilon(x)$ is a matrix functions with appropriate dimension and $\varepsilon^{(2i-1)}(0) = 0$, $i = 1, 2, \cdots$, f_i and g_j are nonlinear functions which satisfy Assumption 2.1 below. Remark that the above system is of practical significance since it can be degenerated to a class of lower-triangular nonlinear systems or a reaction-diffusion equation which describe most of the dynamics in chemistry and mechanism, while the whole system can describe the dynamics of a chemical reactor with both finite-dimensional nonlinear actuator dynamics [3,6].

Assumption 2.1. *For $i = 1, \cdots, n$ and $j = 1, \cdots, m$, the following inequalities hold*

$$\left|f_i(X_{[i]})\right| \le \sigma_f \sum_{k=1}^{i} |X_k|, \quad \left|g_j(Z_{[j]})\right| \le \sigma_g \sum_{k=1}^{i} |Z_k|, \tag{2.2}$$

with σ_f, σ_g being some positive constants.

The control objective of the paper is to design a state-feedback controller such that all the closed-loop system states are bounded while the original system states, i.e., u, X and Z converge to zero.

The following twofold aspects show the distinctive characteristics of system (2.1) while analyze the ineffectiveness of the existing schemes:

(i) *More serious nonlinearities are involved than the sandwiched ones in* [1,3,4,6,12,20,24,25]. Since both the two ODE subsystems (distal or proximal to the control input) in system (2.1) are nonlinear, the controller cannot be designed by merely using (one- or multiple-step) infinite-dimensional backstepping transformation as in [1,3,4,6,20]. Moreover, the compensation of nonlinearities is required which makes the control design and performance analysis more difficult than those of [12,24,25].

(ii) *More complicated configuration is contained than the PDE-ODE and ODE-PDE systems in* [2,5,7,9,10,13–16,18,19,23,26,28–30]. The sandwiched configuration of system (2.1) involves an additional ODE subsystem (distal or proximal to the control input) than the PDE-ODE and ODE-PDE ones in the literature. This, together with the nonlinearities therein, brings essential technique obstacles of the traditional control schemes in the literature.

3. Control design

This section presents the procedure of control design. Seeing from equation (2.1) that the investigated system possesses lower-triangular structure since $u(0,t)$ and Z_1 can be respectively viewed as the inputs of the distal ODE subsystem and the PDE subsystem while both the X and Z subsystems are lower-triangular. Motivated by such structure, the controller is explicitly designed by a combination of the finite- and infinite-dimensional backstepping methods. Specifically, the control procedure is divided into three parts. First, a finite-dimensional backstepping design is given for the distal ODE subsystem (i.e., X-subsystem). Then, an infinite-dimensional backstepping design is subsequently given for the distal ODE-PDE subsystem (i.e., (X,u)-subsystem). Finally, the control procedure ends with a finite-dimensional backstepping for the whole system (i.e., (X,u,Z)-system).

3.1. Stabilization of X-subsystem

In this subsection, a finite-dimensional backstepping step is provided for the X-subsystem. For this, the following pivotal state transformation is first introduced for system (2.1):

$$\xi_1 = X_1, \quad \xi_i = X_i - \tau_{i-1}, \quad v(x,t) = u(x,t) - N(x)\xi, \quad i = 2, \cdots, n, \tag{3.1}$$

where $\xi = (\xi_1, \cdots, \xi_n)^{\mathrm{T}}$, v are the new system states, τ_i's are the virtual control inputs to be determined later, $N(x)$ is given as follows:

$$N(x) = [h_n \quad 0] \exp\left(\begin{bmatrix} 0 & \lambda I - G \\ I & 0 \end{bmatrix} x\right) \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{3.2}$$

with $h_n = (h_{n,1}, h_{n,2}, \cdots, h_{n,n})$ and

$$G = \begin{bmatrix} h_{1,1} & 1 & & \\ h_{2,1} - q_{1,1} & h_{2,2} - q_{1,2} & 1 & \\ \vdots & \vdots & \ddots & 1 \\ h_{n,1} - q_{n-1,1} & h_{n,2} - q_{n-1,2} & \cdots & h_{n,n} - q_{n-1,n} \end{bmatrix},$$

$h_{i,j}$ and $q_{i,j}$ being some constants to be given in the following derivations. Remark that $N(x)$ given above satisfies the following equations which will be used later

$$\begin{cases} N''(x) + N(x)(\lambda I - G) = 0, \\ N'(0) = 0, \\ N(0) = h_n. \end{cases} \tag{3.3}$$

Then, a recursive step is given as follows.

Step 1. We first choose Lyapunov function $V_1 = \frac{1}{2}\xi_1^2$. Then, computing $\dot{V}_1$ along the solutions of system (2.1) by noting $\xi_1 f_1 \leq \sigma_f \xi_1^2$ with σ_1 being certain positive constant, we have

$$\dot{V}_1 = \xi_1 (X_2 + f_1) = \xi_1 (\xi_2 + \tau_1 + f_1) \leq \xi_1 (\xi_2 + \tau_1 + \sigma_f \xi_1).$$

By choosing

$$\tau_1 = -(c_1 + \sigma_f)\xi_1 \triangleq h_{1,1}\xi_1,$$

we directly have

$$\begin{cases} \dot{\xi}_1 = \xi_2 + \tau_1 + f_1 = \xi_2 + h_{1,1}\xi_1 + f_1, \\ \dot{V}_1 \leq -c_1\xi_1^2 + \xi_1\xi_2. \end{cases} \tag{3.4}$$

Step 2. $V_2 = V_1 + \frac{1}{2}\xi_2^2$. As preparation, $\dot{\tau}_1$ is first given as follows:

$$\dot{\tau}_1 = h_{1,1}^2\xi_1 + h_{1,1}\xi_2 + h_{1,1}f_1 \text{ We then choose}$$

$$\triangleq q_{1,1}\xi_1 + q_{1,2}\xi_2 + q'_{1,1} f_1.$$

Then, $\dot{V}_2$ is given along the solutions of system (2.1) and using (3.4), i.e.,

$$\begin{aligned}\dot{V}_2 &\leq -c_1\xi_1^2 + \xi_1\xi_2 + \xi_2\left(X_3 + f_2 - \dot{\tau}_1\right)\\ &= -c_1\xi_1^2 + \xi_1\xi_2 + \xi_2\left(\xi_3 + \tau_2 + f_2 - q_{1,1}\xi_1 - q_{1,2}\xi_2 - q'_{1,1}f_1\right). \end{aligned} \tag{3.5}$$

By Assumption 2.1 and using Young's inequality, the following estimations are obtained

$$\begin{cases} -q'_{1,1}\xi_2 f_1 \leq |q'_{1,1}|\sigma_f\,|\xi_2\xi_1| \leq \dfrac{1}{2}\xi_1^2 + \dfrac{1}{2}|q'_{1,1}|^2\sigma_f^2\xi_2^2,\\ \xi_2 f_2 \leq \sigma_f\xi_2\left(|\xi_1| + |\xi_2 + h_{1,1}\xi_1|\right)\\ \qquad \leq \dfrac{1}{2}\xi_1^2 + {\sigma_f}^2\xi_2^2 + \sigma_f\cdot\xi_2^2 + \sigma_f^2 h_{1,1}^2\xi_2^2, \end{cases}$$

which gives

$$\begin{aligned}\xi_2\left(-q'_{1,1}f_1 + f_2\right) &\leq \xi_1^2 + \left({\sigma_f}^2 + \sigma_f + \sigma_f^2 h_{1,1}^2 + \frac{1}{2}|q'_{1,1}|^2\sigma_f^2\right)\xi_2^2\\ &\triangleq \xi_1^2 + \sigma_2\xi_2^2.\end{aligned}$$

Substituting above inequality into (3.5) and then choosing

$$\tau_2 = -c_2\xi_2 - \xi_1 + q_{1,1}\xi_1 + q_{1,2}\xi_2 - \sigma_2\xi_2 \triangleq h_{2,1}\xi_1 + h_{2,2}\xi_2,$$

we have

$$\begin{cases}\dot{\xi}_2 = \xi_3 + \tau_2 - \dot{\tau}_1 + f_2 = \xi_3 + \displaystyle\sum_{i=1}^{2}\left(h_{2,i} - q_{1,i}\right)\xi_i - q'_{1,1}f_1 + f_2,\\ \dot{V}_2 \leq -\left(c_1 - 1\right)\xi_1^2 - c_2{\xi_2}^2 + \xi_2\xi_3.\end{cases} \tag{3.6}$$

Step k $(3,\cdots,n-1)$**.** Suppose that the first $k-1$ steps have been completed. Then, we obtain the following virtual control

$$\tau_{k-1} = \sum_{i=1}^{k-1} h_{k-1,i}\xi_i, \tag{3.7}$$

which gives that

$$\begin{cases}\dot{\xi}_{k-1} = \xi_k + \displaystyle\sum_{i=1}^{k-1}\left(h_{k-1,i} - q_{k-2,i}\right)\xi_i - \sum_{i=1}^{k-2} q'_{k-2,i}f_i + f_{k-1},\\ \dot{V}_{k-1} \leq -\displaystyle\sum_{i=1}^{k-1}\left(c_i - (k-i-1)\right)\xi_i^2 + \xi_{k-1}\xi_k,\end{cases}$$

and hence

$$\dot{\tau}_{k-1} = \sum_{i=1}^{k-1} h_{k-1,i}\left(\xi_{i+1} + \sum_{j=1}^{i}\left(h_{i,j} - q_{i-1,j}\right)\xi_j - \sum_{j=1}^{i-1} q'_{i-1,j}f_j + f_i\right)$$

$$\triangleq \sum_{i=1}^{k} q_{k-1,i}\xi_i + \sum_{i=1}^{k-1} q'_{k-1,i} f_i.$$

Choose $V_k = V_{k-1} + \frac{1}{2}\xi_k^2$. Then, computing $\dot{V}_k$ along the solutions of (2.1) leads to

$$\begin{aligned}\dot{V}_k &= \dot{V}_{k-1} + \xi_k \left(\xi_{k+1} + \tau_k + f_k - \dot{\tau}_{k-1}\right)\\ &\le -\sum_{i=1}^{k-1} \left(c_i - (k-1-i)\right)\xi_i^2 + \xi_{k-1}\xi_k\\ &\quad + \xi_k\left(\xi_{k+1} + \tau_k + f_k - \sum_{i=1}^{k} q_{k-1,i}\xi_i - \sum_{i=1}^{k-1} q'_{k-1,i} f_i\right).\end{aligned}$$

By Assumption 2.1 and using Young's inequality, the following estimation is obtained

$$\xi_k\left(f_k - \sum_{i=1}^{k-1} q'_{k-1,i} f_i\right) \le \sum_{i=1}^{k-1} \xi_i^2 + \sigma_k \xi_k^2,$$

which leads to that

$$\dot{V}_k \le -\sum_{i=1}^{k-1} \left(c_i - (k-i)\right)\xi_i^2 + \xi_k\left(\xi_{k-1} + \xi_{k+1} + \tau_k - \sum_{i=1}^{k} q_{k-1,i}\xi_i + \sigma_k\xi_k\right). \tag{3.8}$$

Then, by choosing the following virtual control

$$\tau_k = -c_k\xi_k - \xi_{k-1} + \sum_{i=1}^{k} q_{k-1,i}\xi_i - \sigma_k\xi_k \triangleq \sum_{i=1}^{k} h_{k,i}\xi_i,$$

we obtain that

$$\begin{cases}\dot{\xi}_k = \xi_{k+1} + \sum_{i=1}^{k} \left(h_{k,i} - q_{k-1,i}\right)\xi_i - \sum_{i=1}^{k-1} q'_{k-1,i} f_i + f_k,\\ \dot{V}_k \le -\sum_{i=1}^{k} \left(c_i - (k-i)\right)\xi_i^2 + \xi_k\xi_{k+1}.\end{cases} \tag{3.9}$$

Step n. Choose $V_n = V_{n-1} + \frac{1}{2}\xi_n^2$. Letting $x = 0$ in the last equality of (3.1) and noting $N(0) = h_n$ (given by (3.2)) leads to that $u(0) = v(0) + N(0)\xi$. Then, by the similar derivation as those of (3.8), we obtain that

$$\begin{aligned}\dot{V}_n &\le -\sum_{i=1}^{n-1} \left(c_i - (n-i)\right)\xi_i^2 - c_n\xi_n^2\\ &\quad + \xi_n\left(\xi_{n-1} + v(0,t) + N(0)\xi - \sum_{i=1}^{n} q_{n-1,i}\xi_i + \sigma_n\xi_n + c_n\xi_n\right).\end{aligned}$$

By letting

$$N(0)\xi = -\xi_{n-1} + \sum_{i=1}^{n} q_{n-1,i}\xi_i - \left(c_n + \sigma_n\right)\xi_n$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-2} q_{n-1,i}\xi_i + (q_{n-1,n-1} - 1)\,\xi_{n-1} + (q_{n-1,n} - c_n - \sigma_n)\,\xi_n \\
&\triangleq \sum_{i=1}^{n} h_{n,i}\xi_i \\
&= h_n\xi,
\end{aligned}
$$

with $N(0) = h_n$, we obtain that

$$
\begin{cases}
\dot{\xi}_n = v(0) + \displaystyle\sum_{i=1}^{n} (h_{n,i} - q_{n-1,i})\,\xi_i - \sum_{i=1}^{n-1} q'_{n-1,i} f_i + f_n, \\
\dot{V}_n \le -\displaystyle\sum_{i=1}^{n} (c_i - (n-i))\,\xi_i^2 + \xi_n v(0).
\end{cases}
\tag{3.10}
$$

3.2. Stabilization of (X, u)-subsystem

In this subsection, an infinite-dimensional backstepping step is used for the subsequent control design for the stabilization of (X, u)-subsystem. First, as preparation, by transformation (3.1), (X, u)-subsystem is changed into a new one, i.e., (ξ, v)-subsystem whose dynamics are given in the following proposition with detailed proof being postponed in Part A of Appendix at the end of the paper.

Proposition 3.1. *By transformation* (3.1) *with the choice of* τ_i *given above, the following new system is obtained:*

$$
\begin{cases}
\dot{\xi} = G\xi + \Psi(H\xi) + Bv(0), \\
\partial_t v = \partial_x^2 v + \lambda v - N(x)\Psi(H\xi) + \varepsilon(x)H\xi - N(x)Bv(0), \\
\partial_x v(0) = 0, \\
v(1) = Z_1 - N(1)\xi,
\end{cases}
\tag{3.11}
$$

where G *has been given before,* $B = (0, \cdots, 0, 1)^{\mathrm{T}} \in \mathbf{R}^n$ *and*

$$
\Psi = \begin{bmatrix} f_1 \\ f_2 - q'_{1,1} f_1 \\ \vdots \\ f_n - \displaystyle\sum_{i=1}^{n-1} q'_{n,i} f_1 \end{bmatrix},
$$

$$
H = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
h_{1,1} & 1 & \cdots & \cdots & 0 \\
h_{2,1} & h_{2,2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & 1
\end{bmatrix}.
$$

Then, we give the following infinite-dimensional backstepping transformation and its inversion:

$$\begin{cases} w(x) = v(x) - \int_0^x \kappa(x,y)v(y)\mathrm{d}y, \\ v(x) = w(x) + \int_0^x \iota(x,y)w(y)\mathrm{d}y, \end{cases} \tag{3.12}$$

where $\kappa(x,y) = \sum_{i=0}^{+\infty} K_i(\xi,\eta)$, $\iota(x,y) = \sum_{i=0}^{+\infty} L_i(\xi,\eta)$ with $\xi = x+y$, $\eta = x-y$ and

$$\begin{cases} K_0 = L_0 = -\frac{\lambda}{4}(\xi+\eta) + \int_0^\eta N(\xi)B\mathrm{d}\xi, \\ K_{i+1} = \frac{\lambda}{2}\int_0^\eta \int_0^\tau K_i(\tau,s)\mathrm{d}s\mathrm{d}\tau + \frac{\lambda}{4}\int_\eta^\xi \int_0^\tau K_i(\tau,s)\mathrm{d}s\mathrm{d}\tau \\ \qquad - \frac{1}{2}\int_0^\eta \int_s^{2\eta-s} N(\frac{\tau-s}{2})K_i(\tau,s)\mathrm{d}\tau\mathrm{d}s, \\ L_{i+1} = -\frac{\lambda}{2}\int_0^\eta \int_0^\tau L_i(\tau,s)\mathrm{d}s\mathrm{d}\tau - \frac{\lambda}{4}\int_\eta^\xi \int_0^\tau L_i(\tau,s)\mathrm{d}s\mathrm{d}\tau. \end{cases} \tag{3.13}$$

Remark that the detailed expressions of κ and ι have been given in [22] (by letting $\varepsilon = 1$, $b = q = 0$, $f(x) \equiv 0$, $\lambda(x) \equiv \lambda$, $g(x) = -N(x)B$ therein). Moreover, the following properties hold for κ and ι which will be used later: ① They are smooth with respect to x, y on $[0,1]$. ② Their partial derivatives with respective to x, y in various orders are uniformly bounded for $0 \le y \le x \le 1$, i.e., there exists an positive constant $\mathcal{C}$ such that

$$\sup_{0\le y\le x\le 1} |\partial_x^i \partial_y^j \kappa(x,y)| \le \mathcal{C}, \quad \sup_{0\le y\le x\le 1} |\partial_x^i \partial_y^j \iota(x,y)| \le \mathcal{C}, \quad \forall i,j = 0,1,\cdots. \tag{3.14}$$

③ κ satisfies the following equations

$$\begin{cases} \partial_x^2\kappa(x,y) - \partial_y^2\kappa(x,y) = \lambda\kappa(x,y), \\ \partial_y\kappa(x,0) = -N(x)B + \int_0^x \kappa(x,y)N(y)B\mathrm{d}y, \\ \kappa(x,x) = -\frac{\lambda x}{2}. \end{cases} \tag{3.15}$$

By infinite-dimensional backstepping transformation (3.12), system (3.11) is changed into a new one which is given in the following proposition. Its proof has been postponed to part B of Appendix for readability.

Proposition 3.2. *System* (3.11) *is changed into the following new one by* (3.12)*:*

$$\begin{cases} \dot{\xi} = G\xi + \Psi(H\xi) + Bw(0,t), \\ \partial_t w = \partial_x^2 w + \Phi(x,\xi), \\ \partial_x w(0) = 0, \\ w(1) = Z_1 - \chi, \end{cases} \tag{3.16}$$

with

$$\begin{cases} \Phi = \left(\int_0^x \kappa(x,y)N(y)\mathrm{d}y - N(x) \right) \Psi(H\xi) \\ \qquad - \left(\int_0^x \kappa(x,y)\varepsilon(y)\mathrm{d}y - \varepsilon(x) \right) H\xi, \\ \chi = N(1)\xi + \int_0^1 \kappa(1,y)v(y)\mathrm{d}y. \end{cases} \tag{3.17}$$

Remark that Φ given by (3.17) satisfies the following property which will be used later:

$$\begin{cases} \|\Phi\| \le \mathcal{C}|\xi|, \\ \|\partial_x^i \Phi(x,\xi)\| \le \mathcal{C}|\xi|, \\ |\partial_x^i \Phi(1,\xi)| \le \mathcal{C}|\xi|, \\ \partial_x^{2i-1}\Phi(0,\xi) = 0, \end{cases} \tag{3.18}$$

with $i = 1, 2, \cdots$ and $\mathcal{C}$ being certain positive constant. In fact, the first three inequalities of (3.18) can be directly obtained by (3.14) and the boundedness of $N(x)$, $\varepsilon(x)$. The last equality is derived by noting $N^{(2i-1)}(0) = 0$ and $\varepsilon^{(2i-1)}(0) = 0$.

3.3. Stabilization of entire system (X, u, Z)

In this subsection, a finite-dimensional backstepping stepping step is given for the stabilization of the whole system and hence to complete the control design procedure. For this, define the following transformation

$$\zeta_1 = w(1) = Z_1 - \chi, \quad \zeta_i = Z_i - \alpha_{i-1}, \quad i = 2, \cdots, m, \tag{3.19}$$

with α_i's being certain virtual controls to be determined later.

Step $n+1$**.** We first choose Lyapunov function $V_{n+1} = V_n + \frac{1}{2\mathcal{C}^2}\|w\|^2 + \frac{1}{2\mathcal{C}^2}\zeta_1^2$. Then, computing $\dot{V}_{n+1}$ along the solutions of (3.16) gives that

$$\begin{aligned} \dot{V}_{n+1} &= \dot{V}_n + \frac{1}{\mathcal{C}^2}\int_0^1 w\left(\partial_x^2 w + \Phi(x,\xi)\right)\mathrm{d}x + \frac{1}{\mathcal{C}^2}\zeta_1\dot{\zeta}_1 \\ &\le -\sum_{i=1}^n \left(c_i - (n-i)\right)\xi_i^2 + \xi_n w(0) + \frac{1}{\mathcal{C}^2} w(1)\partial_x w(1) - \frac{1}{\mathcal{C}^2}\|\partial_x w\|^2 \\ &\quad + \frac{1}{\mathcal{C}^2}\int_0^1 w(x)\Phi(x,\xi)\mathrm{d}x + \frac{1}{\mathcal{C}^2}\zeta_1\dot{\zeta}_1. \end{aligned}$$

By Poincare's inequality and then Agmon's inequality (See Lemmas G.2 and G.1 in Part G of Appendix at the end of the paper), we have $w^2(0) \le 3w^2(1) + 5\|\partial_x w\|^2$. Then, by the first equality of (3.18) while using Young's inequality, we obtain that

$$\begin{aligned} &\xi_n w(0) \le \frac{1}{\mu}\xi_n^2 + \frac{3\mu}{4}w^2(1) + \frac{5\mu}{4}\|\partial_x w\|^2, \\ &\frac{1}{\mathcal{C}^2}\int_0^1 w(x)\Phi(x,\xi)\mathrm{d}x \le \frac{1}{4\mathcal{C}^2}w^2(1) + \frac{1}{2\mathcal{C}^2}\|\partial_x w\|^2 + 2|\xi|^2, \end{aligned}$$

with $\mu > 0$ being certain to-be-determined constant. Thus, there holds that

$$\begin{aligned}\dot{V}_{n+1} \leq & -\sum_{i=1}^{n-1}\left(c_i-(n-i)-2\right)\xi_i^2-\left(c_n-\frac{1}{\mu}-2\right)\xi_n^2-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2 \\ & +\frac{1}{\mathcal{C}^2}\zeta_1\left(\frac{3\mu\mathcal{C}^2}{4}\zeta_1+\frac{1}{4}\zeta_1+\partial_x w(1)+g_1-\dot{\chi}+\zeta_2+\alpha_1\right).\end{aligned}$$

By choosing

$$\begin{aligned}\alpha_1 &= -c_{n+1}\zeta_1-\frac{3\mu\mathcal{C}^2}{4}\zeta_1-\frac{1}{4}\zeta_1-g_1+\dot{\chi}-\partial_x w(1) \\ &\triangleq -\partial_x w(1)+\rho_1\left(Z_1,\chi,\dot{\chi}\right),\end{aligned} \tag{3.20}$$

we obtain that

$$\begin{aligned}\dot{V}_{n+1} \leq & -\sum_{i=1}^{n-1}\left(c_i-(n-i)-2\right)\xi_i^2-\left(c_n-\frac{1}{\mu}-2\right)\xi_n^2 \\ & -\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2-\frac{c_{n+1}}{\mathcal{C}^2}\zeta_1^2+\frac{1}{\mathcal{C}^2}\zeta_1\zeta_2.\end{aligned}$$

Step $n+2$**.** First, we choose $V_{n+2}=V_{n+1}+\frac{1}{2\mathcal{C}^2}\zeta_2^2$. By (3.16), we obtain that

$$\begin{aligned}\dot{\alpha}_1 &= -\partial_t\partial_x w(1)+\dot{\rho}_1 \\ &= -\partial_x^3 w(1)-\partial_x\Phi(1,\xi)+\frac{\partial\rho_1}{\partial Z_1}\left(Z_2+g_1\right)+\sum_{i=0}^{1}\frac{\partial\rho_1}{\partial\chi^{(i)}}\chi^{(i+1)},\end{aligned}$$

which helps to give $\dot{V}_{n+2}$, i.e.,

$$\begin{aligned}\dot{V}_{n+2} =& \dot{V}_{n+1}+\frac{1}{\mathcal{C}^2}\zeta_2\dot{\zeta}_2 \\ \leq & -\sum_{i=1}^{n-1}\left(c_i-(n-i)-2\right)\xi_i^2-\left(c_n-\frac{1}{\mu}-2\right)\xi_n^2-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2 \\ & -\frac{c_{n+1}}{\mathcal{C}^2}\zeta_1^2+\frac{1}{\mathcal{C}^2}\zeta_2\left(\zeta_1+\zeta_3+\alpha_2+g_2+\partial_x^3 w(1)-\frac{\partial\rho_1}{\partial Z_1}(Z_2+g_1)\right. \\ & \left.+\partial_x\Phi(1,\xi)-\sum_{i=0}^{1}\frac{\partial\rho_1}{\partial\chi^{(i)}}\chi^{(i+1)}\right).\end{aligned}$$

By (3.18) and using Young's inequality, we have

$$\frac{1}{\mathcal{C}^2}\zeta_2\partial_x\Phi(1,\xi) \leq \frac{1}{\mathcal{C}^2}\zeta_2^2+|\xi|^2,$$

which gives that

$$\dot{V}_{n+2} \leq -\sum_{i=1}^{n-1}\left(c_i-(n-i)-3\right)\xi_i^2-\left(c_n-\frac{1}{\mu}-3\right)\xi_n^2-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2$$

$$-\frac{c_{n+1}}{\mathcal{C}^2}\zeta_1^2 + \frac{1}{\mathcal{C}^2}\zeta_2\Bigg(\zeta_1 + \zeta_2 + \zeta_3 + \alpha_2 + g_2 + \partial_x^3 w(1) - \frac{\partial\rho_1}{\partial Z_2}(Z_2 + g_1) - \sum_{i=0}^{1}\frac{\partial\rho_1}{\partial\chi^{(i)}}\chi^{(i+1)}\Bigg).$$

Then, the choice of the following virtual controller

$$\begin{aligned}\alpha_2 &= -c_{n+2}\zeta_2 - \zeta_1 - \zeta_2 - g_2 - \partial_x^3 w(1) + \frac{\partial\rho_1}{\partial Z_1}(Z_2 + g_1) + \sum_{i=0}^{1}\frac{\partial\rho_1}{\partial\chi^{(i)}}\chi^{(i+1)}\\ &= -\partial_x^3 w(1) - c_{n+2}\Big(Z_2 + \partial_x w(1) - \rho_1(Z_1, \chi, \dot{\chi})\Big) - \zeta_1 - \zeta_2\\ &\quad - g_2 + \frac{\partial\rho_1}{\partial Z_1}(Z_2 + g_1) + \sum_{i=0}^{1}\frac{\partial\rho_1}{\partial\chi^{(i)}}\chi^{(i+1)}\\ &\triangleq -\partial_x^3 w(1) + p_{2,1}\partial_x w(1) + \rho_2\Big(Z_{[2]}, \dot{\chi}_{[2]}\Big),\end{aligned} \tag{3.21}$$

gives that

$$\begin{aligned}\dot{V}_{n+2} \le &-\sum_{i=1}^{n-1}(c_i - (n-i) - 3)\,\xi_i^2 - \left(c_n - \frac{1}{\mu} - 3\right)\xi_n^2 - \left(\frac{1}{2\mathcal{C}^2} - \frac{5\mu}{4}\right)\|\partial_x w\|^2\\ &- \frac{c_{n+1}}{\mathcal{C}^2}\zeta_1^2 - \frac{1}{\mathcal{C}^2}\sum_{i=1}^{2}c_{n+i}\zeta_i^2 + \frac{1}{\mathcal{C}^2}\zeta_2\zeta_3.\end{aligned}$$

Step $n+k$ $(k = 3, \cdots, m-1)$. Suppose that the first $n+k-1$ steps have been completed. Then, we have

$$\begin{aligned}\alpha_{k-1} &= -c_{n+k-1}\zeta_{k-1} - \zeta_{k-2} - g_{k-1} - \zeta_{k-1} - \partial_x^{2k-3}w(1) + \sum_{i=1}^{k-3}p_{k-2,i}\partial_x^{2i+1}w(1)\\ &\quad + \sum_{i=1}^{k-2}\frac{\partial\rho_{k-2}}{\partial Z_i}(Z_{i+1} + g_i) + \sum_{i=0}^{k-2}\frac{\partial\rho_{k-2}}{\partial\chi^{(i)}}\chi^{(i+1)}\\ &\triangleq -\partial_x^{2k-3}w(1) + \sum_{i=1}^{k-2}p_{k-1,i}\partial_x^{2i-1}w(1) + \rho_{k-1}\Big(Z_{[k-1]}, \dot{\chi}_{[k-1]}\Big),\\ \dot{V}_{n+k-1} &\le -\sum_{i=1}^{n-1}\Big(c_i - (n-i) - k\Big)\xi_i^2 - \left(c_n - \frac{1}{\mu} - k\right)\xi_n^2\\ &\quad - \left(\frac{1}{2\mathcal{C}^2} - \frac{5\mu}{4}\right)\|\partial_x w\|^2 - \frac{1}{\mathcal{C}^2}\sum_{i=1}^{k-1}c_{n+i}\zeta_i^2 + \frac{1}{\mathcal{C}^2}\zeta_{k-1}\zeta_k.\end{aligned}$$

Then, along the solutions of (3.16), we obtain

$$\begin{aligned}\dot{\alpha}_{k-1} &= -\partial_t\partial_x^{2k-3}w(1) + \sum_{i=1}^{k-2}p_{k-1,i}\partial_t\partial_x^{2i-1}w(1) + \dot{\rho}_{k-1}\\ &= -\partial_x^{2k-1}w(1) - \partial_x^{2k-3}\Phi(1,\xi)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-2} p_{k-1,i}\left(\partial_x^{2i+1}w(1)+\partial_x^{2i-1}\Phi(1,\xi)\right)\\
&+\sum_{i=1}^{k-1}\frac{\partial\rho_{k-1}}{\partial Z_i}(Z_{i+1}+g_i)+\sum_{i=0}^{k-1}\frac{\partial\rho_{k-1}}{\partial\chi^{(i)}}\chi^{(i+1)}.
\end{aligned}
\tag{3.22}
$$

Choose $V_{n+k}=V_{n+k-1}+\frac{\delta}{2}\zeta_k^2$, the following inequalities hold by using (3.22)

$$
\begin{aligned}
\dot{V}_{n+k}\leq&-\sum_{i=1}^{n-1}\Big(c_i-(n-i)-k\Big)\xi_i^2-\left(c_n-\frac{1}{\mu}-k\right)\xi_n^2-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2\\
&-\frac{1}{\mathcal{C}^2}\sum_{i=1}^{k-1}c_{n+i}\zeta_i^2+\frac{1}{\mathcal{C}^2}\zeta_k\Bigg(\zeta_{k-1}+\zeta_{k+1}+\alpha_k+g_k+\partial_x^{2k-1}w(1)\\
&+\partial_x^{2k-3}\Phi(1,\xi)-\sum_{i=1}^{k-2}p_{k-1,i}\left(\partial_x^{2i+1}w(1)+\partial_x^{2i-1}\Phi(1,\xi)\right)\\
&-\sum_{i=1}^{k-1}\frac{\partial\rho_{k-1}}{\partial Z_i}(Z_{i+1}+g_i)-\sum_{i=0}^{k-1}\frac{\partial\rho_{k-1}}{\partial\chi^{(i)}}\chi^{(i+1)}\Bigg).
\end{aligned}
$$

By (3.18) and using Young's inequality, we obtain

$$
\frac{1}{\mathcal{C}^2}\zeta_k\left(\partial_x^{2k-3}\Phi(1,\xi)-\sum_{i=1}^{k-2}p_{k-1,i}\partial_x^{2i-1}\Phi(1,\xi)\right)\leq\ \frac{1}{\mathcal{C}^2}\zeta_k^2+|\xi|^2,
$$

which, together with the choice of α_k given as follows

$$
\begin{aligned}
\alpha_k=&-c_{n+k}\zeta_k-\zeta_{k-1}-g_k-\zeta_k-\partial_x^{2k-1}w(1)+\sum_{i=1}^{k-2}p_{k-1,i}\partial_x^{2i+1}w(1)\\
&+\sum_{i=1}^{k-1}\frac{\rho_{k-1}}{\partial Z_i}(Z_{i+1}+g_i)+\sum_{i=0}^{k-1}\frac{\partial\rho_{k-1}}{\partial\chi^{(i)}}\chi^{(i+1)}\\
&\triangleq-\partial_x^{2k-1}w(1)+\sum_{i=1}^{k-1}p_{k,i}\partial_x^{2i-1}w(1)+\rho_k\Big(Z_{[k]},\dot{\chi}_{[k]}\Big),
\end{aligned}
\tag{3.23}
$$

leads to that

$$
\begin{aligned}
\dot{V}_{n+k}\leq&-\sum_{i=1}^{n-1}\Big(c_i-(n-i)-k-1\Big)\xi_i^2-\left(c_n-\frac{1}{\mu}-k-1\right)\xi_n^2-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2\\
&-\frac{1}{\mathcal{C}^2}\sum_{i=1}^{k}c_{n+i}\zeta_i^2+\frac{1}{\mathcal{C}^2}\zeta_k\zeta_{k+1}.
\end{aligned}
$$

Step $n+m$. Choose $V_{n+m}=V_{n+m-1}+\frac{\delta}{2}\zeta_m^2$. By the similar derivation as previous step, we obtain that

$$
U=-c_{n+m}\zeta_m-\zeta_{m-1}-g_m-\zeta_m-\partial_x^{2m-1}w(1)+\sum_{i=1}^{m-2}p_{m-1,i}\partial_x^{2i+1}w(1)
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m-1}\frac{\rho_{m-1}}{\partial Z_i}(Z_{i+1}+g_i)+\sum_{i=0}^{m-1}\frac{\partial\rho_{m-1}}{\partial\chi^{(i)}}\chi^{(i+1)}\\
&\triangleq -\partial_x^{2m-1}w(1)+\sum_{i=1}^{m-1}p_{m,i}\partial_x^{2i-1}w(1)+\rho_m\Big(Z_{[m]},\dot{\chi}_{[m]}\Big),
\end{aligned}
\tag{3.24}
$$

which brings that

$$
\begin{aligned}
\dot{V}_{n+m}\le &-\sum_{i=1}^{n-1}\Big(c_i-(n-i)-m-1\Big)\xi_i^2-\left(c_n-\frac{1}{\mu}-m-1\right)\xi_n^2\\
&-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|\partial_x w\|^2-\frac{1}{\mathcal{C}^2}\sum_{i=1}^{m}c_{n+i}\zeta_i^2.
\end{aligned}
$$

Noting that $-\|\partial_x w\|^2\le \frac{1}{2}w^2(1)-\frac{1}{4}\|w\|^2$ (see Poincaré's inequality), we arrive at

$$
\begin{aligned}
\dot{V}_{n+m}\le &-\sum_{i=1}^{n-1}\Big(c_i-(n-i)-m-1\Big)\xi_i^2-\left(c_n-\frac{1}{\mu}-m-1\right)\xi_n^2\\
&-\frac{1}{4}\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\|w\|^2-\left(\frac{c_{n+1}}{\mathcal{C}^2}-\left(\frac{1}{2\mathcal{C}^2}-\frac{5\mu}{4}\right)\frac{1}{2}\right)\zeta_1^2-\frac{1}{\mathcal{C}^2}\sum_{i=2}^{m}c_{n+i}\zeta_i^2.
\end{aligned}
$$

By choosing

$$
\begin{cases}
c_i>n-i+m+1, \quad i=1,\cdots,n-1,\\
0<\mu<\dfrac{2}{5\mathcal{C}^2},\\
c_n>\dfrac{1}{\mu}+m+1,\\
c_{n+1}>\dfrac{1}{4}\left(1-\dfrac{5\mu\mathcal{C}^2}{2}\right),\\
c_{n+i}>0, \quad i=2,\cdots,m,
\end{cases}
$$

we obtain that

$$
\dot{V}_{n+m}\le -\sigma V_{n+m}, \tag{3.25}
$$

with σ being certain positive constant.

4. Stability analysis

This section analyzes the stability of the resulting closed-loop system. As preparation, we first give two propositions which respectively present two pivotal new systems. Specifically, Proposition 4.1 gives the dynamics of ζ_i's which can be directly obtained by transformation (3.19) with the choices of virtual controls and actual control given above whose proof is omitted. Proposition 4.2 gives the further dynamics of w-subsystem whose proof is somewhat long and hence postponed to Part C of Appendix at the end of the paper.

Proposition 4.1. *By the choice of virtual controls* (3.20), (3.21), (3.23) *and actual control* (3.24), *the dynamics of* ζ *is given as follows:*

$$\begin{cases}\dot{\zeta}_1 = a_{1,1}\zeta_1 + \zeta_2 - \partial_x w(1),\\ \dot{\zeta}_2 = a_{2,1}\zeta_1 + a_{2,2}\zeta_2 + \zeta_3 + \partial_x \Phi(1,\xi),\\ \dot{\zeta}_k = a_{k,1}\zeta_{k-1} + a_{k,2}\zeta_k + \zeta_{k+1} + \partial_x^{2k-3}\Phi(1,\xi)\\ \qquad - \displaystyle\sum_{i=1}^{k-2} p_{k-1,i}\partial_x^{2i-1}\Phi(1,\xi), \quad k = 3,\cdots,n,\end{cases} \tag{4.1}$$

with $a_{i,j}$ *being certain constants and* $\zeta_{m+1} = 0$.

Proposition 4.2. *With controller* (3.24) *in loop,* w *given in (3.16) satisfies the following equations:*

$$\begin{cases}\partial_x^k \partial_t w = \partial_x^{k+2} w + \partial_x^k \Phi(x,\xi),\\ \partial_t^k w = \partial_x^{2k} w + \Gamma_k(x,\xi,w(0),\partial_t w(0),\cdots,\partial_t^{k-2}w(0)),\\ \partial_x^{2k-1} w(0) = 0, \quad k = 1,2,\cdots,\\ \partial_t w(1) = -\partial_x w(1) + \Upsilon_1\left(\zeta_{[2]}\right),\\ \partial_t^2 w(1) = -\partial_x^3 w(1) + \Upsilon_2\left(\zeta_{[3]},\partial_x w(1)\right) + \Lambda_2(\xi),\\ \partial_t^k w(1) = -\partial_x^{2k-1} w(1) + \Upsilon_k\Big(\zeta_{[k+1]},\partial_x w(1),\partial_x^3 w(1),\cdots,\partial_x^{2k-3}w(1)\Big)\\ \qquad + \Lambda_k\Big(\xi,w(0),\partial_t w(0),\cdots,\partial_t^{k-3}w(0)\Big), \quad k = 3,\cdots,m,\end{cases} \tag{4.2}$$

where Υ_k *is certain linear function with respect to the arguments therein,* Γ_k *and* Λ_k *are given as follows:*

$$\begin{cases}\Gamma_1 = \Phi(x,\xi),\\ \Gamma_2 = \partial_x^2\Phi(x,\xi) + \partial_\xi\Phi(x,\xi)\left(G\xi + \Psi(H\xi) + Bw(0)\right),\\ \Gamma_k = \partial_x^{2k}\Phi(x,\xi) + \dfrac{\partial\Gamma_{k-1}}{\partial_\xi}\dot{\xi} + \displaystyle\sum_{i=0}^{k-3}\frac{\partial\Gamma_{k-1}}{\partial(\partial_t^i w(0))}\partial_t^{i+1}w(0),\\ \Lambda_2 = b_{2,1}\partial_x\Phi(1,\xi),\\ \Lambda_3 = \displaystyle\sum_{i=1}^{2} b_{2,i}\partial_x^{2i-1}\Phi(1,\xi) + \partial_\xi\Lambda_2(\xi)\dot{\xi},\\ \Lambda_k = \displaystyle\sum_{i=1}^{k-1} b_{k,i}\partial_x^{2i-1}\Phi(1,\xi) + \partial_\xi\Lambda_2(\xi)\dot{\xi} + \sum_{i=0}^{k-4}\frac{\partial\Lambda_{k-1}}{\partial(\partial_t^i w(0))}\partial_t^{i+1}w(0),\end{cases} \tag{4.3}$$

with $b_{i,j}$ *being certain constant.*

In the following, we turn to give the stability results of the resulting closed-loop system. First, a proposition (i.e., Proposition 4.3) is given to present the stability of states ζ, ξ and w. Then, another proposition (i.e., Proposition 4.4) is given to show the integrability and boundedness of some system signals. Finally, the stability of the original system states are given in a theorem (i.e., Theorem 4.1).

Proposition 4.3. *The designed controller* (3.24) *guarantees that* X, ξ, ζ *and* $\sup_{x\in[0,1]} w(x,t)$ *are bounded while*

$$\begin{cases} \lim_{t\to+\infty}\left(|\xi|+|\zeta|+\sup_{x\in[0,1]}|w(x,t)|\right)=0, \\ \partial_x w(1), \partial_t w(1),\ \|\partial_x^2 w\|\in\mathcal{L}_2, w(0), \|\partial_x w\|\in\mathcal{L}_2\cap\mathcal{L}_\infty. \end{cases}$$

Proof. The boundness and convergence of ξ, ζ and $\|w\|$ can be directly obtained by (3.25) which gives that $V_{n+m}(t)\le V_{n+m}(0)e^{-\sigma t}$.

By the first equality of (3.16) while using Young's inequality and noting Υ_1, $\|\Phi\|\in\mathcal{L}_2$, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x w\|^2 &= \partial_x w(1)\partial_t w(1) - \int_0^1 \partial_x^2 w\left(\partial_x^2 w + \Phi(x,\xi)\right)\mathrm{d}x \\ &= \partial_x w(1)\left(-\partial_x w(1)+\Upsilon_1\right) - \|\partial_x^2 w\|^2 - \int_0^1 \partial_x^2 w(x)\Phi(x,\xi)\mathrm{d}x \\ &\le -\frac{1}{2}\left(\partial_x w(1)\right)^2 - \frac{1}{2}\|\partial_x^2 w\|^2 + \ell_1. \end{aligned}$$

Integrating both sides of above inequality leads to $\|\partial_x w\|\in\mathcal{L}_\infty$, $\partial_x w(1)$, $\|\partial_x^2 w\|\in\mathcal{L}_2$, hence $\partial_t w(1)\in\mathcal{L}_2$ (seen from the fourth equality of (4.2)). Then, Agmon's inequality brings that $\lim_{t\to+\infty}\sup_{x\in[0,1]}|w(x,t)|=0$ and $w(x,t)$ is bounded on $[0,\,1]\times[0,\,+\infty)$. □

Proposition 4.4. *The designed controller* (3.24) *guarantees that* (*for* $i=1,2,\cdots,m$)

$$\begin{cases} \alpha_i, \dot{Z}_i, Z_{i+1}, \|\partial_x^{2i} w\|, \|\partial_x^{2i+1} w\|, \partial_x^{2i-1} w(1), \partial_t^i w(1), \partial_t^i w(0), \chi^{(i)} \in\mathcal{L}_\infty\cap\mathcal{L}_2, \\ \|\partial_x^{2i+2} w\|, \partial_x^{2i+1} w(1), \alpha_{i+1}, Z_{i+2}, \partial_t^{i+1} w(1), \chi^{(i+1)}\in\mathcal{L}_2, \end{cases} \tag{4.4}$$

with $Z_{m+1}=U$.

Proof. Such proposition is proven by recursive steps.

Step 1. We choose $E_1=\frac{1}{2}\left(\|\partial_x^2 w\|^2+\|\partial_x^3 w\|^2\right)$. Computing $\dot{E}_1$ along the solutions of system (4.2) by using integration by parts while noting $\partial_t\partial_x w(0)=0$, $\partial_x^3 w(0)=0$, $\|\partial_x^i\Phi\|\in\mathcal{L}_2$ and using Young's inequality, we obtain that

$$\begin{aligned} \dot{E}_1 &= \int_0^1 \partial_x^2 w\partial_t\partial_x^2 w\mathrm{d}x + \int_0^1 \partial_x^3 w\partial_t\partial_x^3 w\mathrm{d}x \\ &= \partial_x^2 w(1)\partial_t\partial_x w(1) - \int_0^1 \partial_x^3 w\left(\partial_x^3 w+\partial_x\Phi(x,\xi)\right)\mathrm{d}x \\ &\quad +\partial_x^3 w(1)\partial_t\partial_x^2 w(1) - \int_0^1 \partial_x^4 w\left(\partial_x^4 w+\partial_x^2\Phi(x,\xi)\right)\mathrm{d}x \\ &\le \partial_x^2 w(1)\partial_t\partial_x w(1)+\partial_x^3 w(1)\partial_t\partial_x^2 w(1) - \frac{1}{2}\|\partial_x^3 w\|^2 - \frac{1}{2}\|\partial_x^4 w\|^2 + \ell_1. \end{aligned} \tag{4.5}$$

By using the following estimation whose proof is given in Part D of Appendix,

$$\textbf{Claim ①:}\ \begin{cases} \partial_x^2 w(1)\partial_x\partial_t w(1) \le \frac{1}{4}\left(\partial_x^3 w(1)\right)^2+\ell_1, \\ \partial_x^3 w(1)\partial_x^2\partial_t w(1) \le -\frac{1}{2}\left(\partial_x^3 w(1)\right)^2+\ell_1, \end{cases}$$

we arrive at

$$\dot{E}_1 = -\frac{1}{4}\left(\partial_x^3 w(1)\right)^2 - \frac{1}{2}\|\partial_x^3 w\|^2 - \frac{1}{2}\|\partial_x^4 w\|^2 + \ell_1,$$

which gives that E_1, $\|\partial_x^2 w\|$, $\|\partial_x^3 w\| \in \mathcal{L}_\infty$ and $\partial_x^3 w(1)$, $\|\partial_x^3 w\|$, $\|\partial_x^4 w\| \in \mathcal{L}_2$ by integrating over $[0,t]$ and $[0,+\infty)$, respectively. Then, (4.2) with (3.17) gives that $\|\partial_t w\|$, $\|\partial_t v\|$, $\dot{\chi} \in \mathcal{L}_\infty$, and moreover, Agmon's inequality brings that $\partial_x w(1) \in \mathcal{L}_\infty$. Thus, there subsequently hold that $\partial_t w(1)$, $\partial_t w(0)$, α_1, Z_2, $\dot{Z}_1 \in \mathcal{L}_\infty \cap \mathcal{L}_2$, $\|\partial_t^2 w\|$, $\ddot{\chi}$, $\partial_t^2 w(1)$, α_2, Z_3, $\partial_t^2 w(1) \in \mathcal{L}_2$ (seen from (4.2) and (3.20), (3.21), respectively).

Step 2. We choose $E_2 = \frac{1}{2}\left(\|\partial_x^4 w\|^2 + \|\partial_x^5 w\|^2\right)$. Computing $\dot{E}_2$ along the solutions of system (4.2) by using integration by parts while noting $\partial_t\partial_x^3 w(0) = 0$, $\partial_x^5 w(0) = 0$, $\|\partial_x^i \Phi\| \in \mathcal{L}_2$ and using Young's inequality, we obtain that

$$\begin{aligned}
\dot{E}_2 &= \int_0^1 \partial_x^4 w \partial_t \partial_x^4 w \mathrm{d}x + \int_0^1 \partial_x^5 w \partial_t \partial_x^5 w \mathrm{d}x \\
&= \partial_x^4 w(1)\partial_t\partial_x^3 w(1) - \int_0^1 \partial_x^5 w\left(\partial_x^5 w + \partial_x^3 \Phi(x,\xi)\right)\mathrm{d}x \\
&\quad + \partial_x^5 w(1)\partial_t\partial_x^4 w(1) - \int_0^1 \partial_x^6 w\left(\partial_x^6 w + \partial_x^4 \Phi(x,\xi)\right)\mathrm{d}x \\
&\leq \partial_x^4 w(1)\partial_t\partial_x^3 w(1) + \partial_x^5 w(1)\partial_t\partial_x^4 w(1) - \frac{1}{2}\|\partial_x^5 w\| - \frac{1}{2}\|\partial_x^6 w\| + \ell_1. \qquad (4.6)
\end{aligned}$$

Then, by using the following claim whose proof is given in Part E of Appendix

$$\textbf{Claim ②:} \begin{cases} \partial_x^4 w(1)\partial_x^3\partial_t w(1) \leq \frac{1}{4}\left(\partial_x^5 w(1)\right)^2 + \ell_1, \\ \partial_x^5 w(1)\partial_x^4\partial_t w(1) \leq -\frac{1}{2}\left(\partial_x^5 w(1)\right)^2 + \ell_1, \end{cases}$$

we arrive at

$$\dot{E}_2 = -\frac{1}{4}\left(\partial_x^5 w(1)\right)^2 - \frac{1}{2}\|\partial_x^5 w\|^2 - \frac{1}{2}\|\partial_x^6 w\|^2 + \ell_1,$$

integrating which over $[0,t]$ and $[0,+\infty)$ respectively gives that E_2, $\|\partial_x^4 w\|$, $\|\partial_x^5 w\| \in \mathcal{L}_\infty$ and $\partial_x^5 w(1)$, $\|\partial_x^5 w\|$, $\|\partial_x^6 w\| \in \mathcal{L}_2$. Then, (4.2) with (3.17) gives that $\|\partial_t^2 w\|$, $\|\partial_t^2 v\|$, $\ddot{\chi} \in \mathcal{L}_\infty$, and moreover, Agmon's inequality brings that $\partial_x^3 w(1) \in \mathcal{L}_\infty$. Thus, there subsequently hold that $\partial_t^2 w(1)$, $\partial_t^2 w(0)$, α_2, Z_3, $\dot{Z}_2 \in \mathcal{L}_\infty \cap \mathcal{L}_2$, $\|\partial_t^3 w\|$, $\chi^{(3)}$, α_3, Z_4, $\partial_t^3 w(1) \in \mathcal{L}_2$ (seen from (4.2) and (3.21), (3.23), respectively).

Step k ($k = 3, \cdots, m$). Suppose that the first $k-1$ steps have been completed. Then, (4.4) holds for $i = 1, \cdots, k-1$. Choose $E_k = \frac{1}{2}\left(\|\partial_x^{2k} w\|^2 + \|\partial_x^{2k+1} w\|^2\right)$. Computing $\dot{E}_k$ along the solutions of system (4.2) by using integration by parts while noting $\partial_t\partial_x^{2k-1} w(0) = 0$, $\partial_x^{2k+1} w(0) = 0$, $\|\partial_x^i \Phi\| \in \mathcal{L}_2$ and using Young's inequality, we obtain that

$$\begin{aligned}
\dot{E}_k &= \int_0^1 \partial_x^{2k} w \partial_t \partial_x^{2k} w \mathrm{d}x + \int_0^1 \partial_x^{2k+1} w \partial_t \partial_x^{2k+1} w \mathrm{d}x \\
&= \partial_x^{2k} w(1)\partial_t\partial_x^{2k-1} w(1) - \int_0^1 \partial_x^{2k+1} w\left(\partial_x^{2k+1} w + \partial_x^{2k-1} \Phi(x,\xi)\right)\mathrm{d}x
\end{aligned}$$

$$+\partial_x^{2k+1}w(1)\partial_t\partial_x^{2k}w(1)-\int_0^1\partial_x^{2k+2}w\left(\partial_x^{2k+2}w+\partial_x^{2k}\Phi(x,\xi)\right)\mathrm{d}x$$
$$\leq\partial_x^{2k}w(1)\partial_t\partial_x^{2k-1}w(1)+\partial_x^{2k+1}w(1)\partial_t\partial_x^{2k}w(1)-\frac{1}{2}\|\partial_x^{2k+1}w\|-\frac{1}{2}\|\partial_x^{2k+2}w\|+\ell_1. \quad (4.7)$$

Then, by using the following claim whose proof is given at Part F of Appendix

$$\textbf{Claim ③:}\begin{cases}\partial_x^{2m+2}w(1)\partial_x^{2m+1}\partial_t w(1)\leq\frac{1}{2}\|\partial_x^{2m+4}w\|^2+\ell_1,\\ \partial_x^{2m+3}w(1)\partial_x^{2m+2}\partial_t w(1)\leq-\frac{1}{2}\big(\partial_x^{2m+3}w(1)\big)^2+\ell_1,\end{cases}$$

we obtain that

$$\dot{E}_k=-\frac{1}{4}\left(\partial_x^{2k+1}w(1)\right)^2-\frac{1}{2}\|\partial_x^{2k+1}w\|^2-\frac{1}{2}\|\partial_x^{2k+2}w\|^2+\ell_1,$$

integrating which over $[0,t]$ and $[0,+\infty)$ respectively gives that E_k, $\|\partial_x^{2k}w\|$, $\|\partial_x^{2k+1}w\|\in\mathcal{L}_\infty$ and $\partial_x^{2k+1}w(1)$, $\|\partial_x^{2k+1}w\|$, $\|\partial_x^{2k+2}w\|\in\mathcal{L}_2$. Then, (4.2) with (3.17) gives that $\|\partial_t^k w\|$, $\chi^{(k)}\in\mathcal{L}_\infty$, and moreover, Agmon's inequality brings that $\partial_x^{2k-1}w(1)\in\mathcal{L}_\infty$. Thus, there subsequently hold that $\partial_t^k w(1)$, $\partial_t^k w(0)$, α_k, Z_{k+1}, $\dot{Z}_k\in\mathcal{L}_\infty\cap\mathcal{L}_2$, $\|\partial_t^{k+1}w\|$, $\chi^{(k+1)}$, α_{k+1}, Z_{k+1}, $\partial_t^{k+1}w(1)\in\mathcal{L}_2$ (seen from (4.2) and (3.23), respectively). This completes the proof. □

It is a position to give the main results of the resulting closed-loop system which are summarized in the following theorem.

Theorem 4.1. *Consider system* (2.1) *under Assumption* 2.1*. The designed controller given by* (3.24) *guarantees that all the system signals of the resulting closed-loop system are bounded while original system states X, Z and $u(x,t)$ converge to zero, i.e.,*

$$\lim_{t\to+\infty}|X|=0,\ \lim_{t\to+\infty}|Z|=0,\ \lim_{t\to+\infty}\sup_{x\in[0,1]}|u(x,t)|=0.$$

Proof. Then, convergence of X is direct since $X=H\xi$ and $\lim_{t\to+\infty}|\xi|=0$ (given by Proposition 4.3). Moreover, by (4.4), we know that Z_i, $\dot{Z}_i\in\mathcal{L}_\infty$ and $Z_i\in\mathcal{L}_2$, $i=1,\cdots,m+1$. Then, by the well-known Barbalat's Lemma, we obtain that $\lim_{t\to+\infty}|Z|=0$.Noting that $\lim_{t\to+\infty}\sup_{x\in[0,1]}|w(x,t)|=0$, transformation (3.12) gives that $\lim_{t\to+\infty}\sup_{x\in[0,1]}|v(x,t)|=0$. Then, the third equality of (3.1) gives that $\lim_{t\to+\infty}\sup_{x\in[0,1]}|u(x,t)|=0$. □

5. Simulation results

In this section, we validate the effectiveness of the proposed theoretical results for system (2.1) with $n=m=2$, $f_1=\sin(X_1)$, $f_2=2X_1\cos(X_2)+3\sin^2(X_2)$, $g_1=\cos(Z_1)$, $g_2=2Z_1\sin(Z_2)$ and $\lambda=5$, $\varepsilon(x)=\big(\cos(x),\cos(x^2)\big)$. Then, we can find $\sigma_f=3$ and $\sigma_g=2$ for Assumption 2.1. In this section, we assume that the initial condition is $X_1(0)=3.8$, $X_2(0)=-1.5$, $Z_1(0)=-8.8$, $Z_2(0)=10.5$ and $u(x,0)=2\cos(x)+x^3$.

By the control design procedure given above, we obtain the controller as follows:

$$U=-c_4\zeta_2-\zeta_1-\zeta_2-g_2-\partial_x^3w(1)+\ddot{\chi}$$
$$-\left(c_3-\frac{3}{4\delta}-\frac{1}{4}\right)(Z_2+g_1-\dot{\chi})-\frac{\partial g_1}{\partial Z_1}(Z_2+g_1),$$

where w has been given by the first line of (3.12) with $\kappa(x,y)$ being given by infinite series (3.13) which is truncated at $i = 15$ for simulation, $\zeta_1 = Z_1 - \chi$, $\zeta_2 = Z_2 - \alpha_1$ and

$$\begin{aligned}
\alpha_1 &= -\partial_x w(1) + \dot{\chi} - \left(c_3 - \frac{3}{4\delta} - \frac{1}{4}\right)(Z_1 - \chi) - g_1,\\
N(x) &= [h_2 \quad 0] \exp\left(\begin{bmatrix} 0 & \lambda I - G \\ I & 0 \end{bmatrix} x\right) \begin{bmatrix} I \\ 0 \end{bmatrix},\\
h_2 &= \left[h_{1,1}^2 - 1, h_{1,1} - \sigma_2 - c_2\right],\\
G &= \begin{bmatrix} h_{1,1} & 1 \\ -1 & -\sigma_2 - c_2 \end{bmatrix},
\end{aligned}$$

with $h_{1,1} = -c_1 - \sigma_f$, $\sigma_2 = \sigma_f^2 + \sigma_f + \frac{3}{2}h_{1,1}^2\sigma_f^2$. Thus, by using the explicit forward Euler method with 20-step discretization in Matlab, we implement the above controller with $\delta = 5$, $c_1 = 50$, $c_2 = 120$, $c_3 = 15$, $c_4 = 2$. Consequently, four simulation figures are obtained (see Figures 1-4 for detail) which indicate that the both the original system states and control input are bounded and converge to zero ultimately.

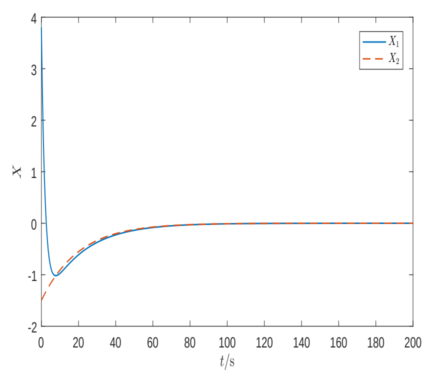

Figure 1. Trajectory of system state X.

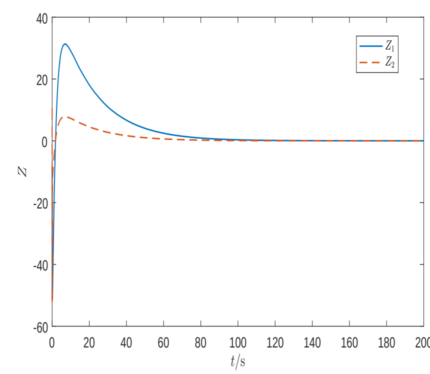

Figure 2. Trajectory of system state Z.

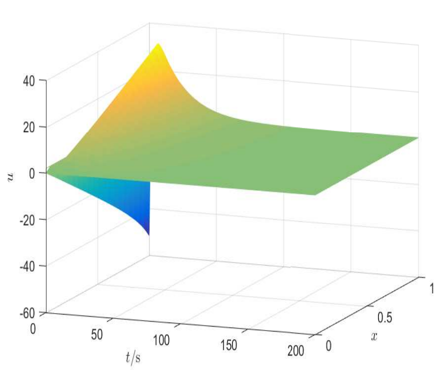

Figure 3. Trajectory of system state u.

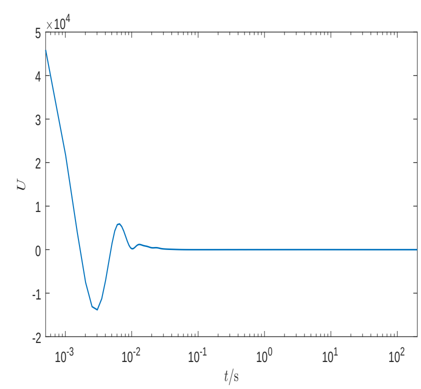

Figure 4. Trajectory of control input U.

6. Concluding remarks

In this paper, stabilization of a class of nonlinear ODE-PDE-ODE systems has been solved. Owing to the presence of serious nonlinearities involving in the two ODE subsystems which are respectively located at both distal and proximal to the control input, the traditional control schemes are incapable. Then, a novel control framework is established in this paper by a smart combination of the infinite- and finite-dimensional backstepping method. By the proposed scheme, a state-feedback controller is explicitly designed which guarantees the desirable stability of the resulting closed-loop system. The future interesting research directions are twofold. First, stabilization of other nonlinear ODE-PDE-ODE systems with PDE subsystem being replaced by other ones, such as wave equation or the first-order hyperbolic equation. Second, stabilization via output feedback. For this, certain observer design schemes for the ODE and PDE systems should be designed to reconstruct the system states, and hence challenge the control design.

Appendix

A. Proof of Proposition 3.1

The last equality of (3.11) can be directly obtained by letting $x = 1$ in the third equality of (3.1) while noting $u(1) = Z_1$ (shown in (2.1)). Moreover, letting $x = 0$ in $\partial_x v(x,t) = \partial_x u(x,t) - N'(x)\xi$ gives the third equality of (3.11) by noting $\partial_x u(0) = 0$ and $N'(0) = 0$. By transformation (3.1) with the choice of τ_i, we directly obtain that $X = H\xi$. Then, the first lines of (3.4), (3.6), (3.9), (3.10) directly give the first equation of (3.11).

By computing the second-order partial derivative with respect to x of v from the third equality of (3.1) while using (3.3), we directly bring that

$$\begin{aligned}\partial_x^2 v &= \partial_x^2 u - N''(x)\xi,\\ &= \partial_x^2 u + N(x)(\lambda I - G)\xi.\end{aligned} \tag{A.1}$$

Moreover, computing the partial derivative with respect to t of v from the third equality of (3.1) while using (2.1) and the first equality of (3.11), we obtain

$$\begin{aligned}\partial_t v &= \partial_x^2 u + \lambda u + \varepsilon(x)X - N(x)\dot{\xi}\\ &= \partial_x^2 u + \lambda u + \varepsilon(x)X - N(x)\big(G\xi + \Psi(H\xi) + Bv(0)\big)\\ &= \partial_x^2 u + \lambda v + \varepsilon(x)X + N(x)(\lambda I - G)\xi - N(x)\big(\Psi(H\xi) + Bv(0)\big).\end{aligned}$$

Subtracting both sides of above equalities from those of (A.1) while using (3.3) leads to the second equality of (3.11).

B. Proof of Proposition 3.2

First, letting $x = 0$ in the first equality of (3.12) directly gives that $w(0) = v(0)$. Then, the first equation of (3.16) is obtained from that of (3.11). Moreover, letting $x = 1$ in the first equality of (3.12) directly gives the fourth equality of (3.16). Second, computing $\partial_x w$ and $\partial_x^2 w$ from the first equality of (3.12) gives that

$$\begin{cases} \partial_x w = \partial_x v - \kappa(x,x)v(x) - \displaystyle\int_0^x \partial_x \kappa(x,y)v(y)\mathrm{d}y, \\ \partial_x^2 w = \partial_x^2 v - \dfrac{\mathrm{d}}{\mathrm{d}x}\kappa(x,x)v(x) - \kappa(x,x)\partial_x v(x) \\ \qquad\quad -\partial_x \kappa(x,x)v(x) - \displaystyle\int_0^x \partial_x^2 \kappa(x,y)v(y)\mathrm{d}y. \end{cases} \tag{B.1}$$

Letting $x = 0$ in the first equality of (B.1) while noting $\kappa(0,0) = 0$, $\partial_x v(0) = 0$ leads to $\partial_x w(0) = 0$, which is the third equality of (3.16).

Moreover, computing the time derivative of both sides of the first equality of (3.12) while using integration by parts, we obtain that

$$\begin{aligned} \partial_t w &= \partial_t v - \int_0^x \kappa(x,y)\partial_t v(y)\mathrm{d}y \\ &= \partial_x^2 v + \lambda v - N(x)\Psi(H\xi) + \varepsilon(x)H\xi - N(x)Bv(0) \\ &\quad - \int_0^x \kappa(x,y)\Big(\partial_x^2 v + \lambda v - N(y)\Psi(H\xi) + \varepsilon(y)H\xi - N(y)Bv(0)\Big)\mathrm{d}y \\ &= \partial_x^2 v + \lambda v - N(x)\Psi(H\xi) + \varepsilon(x)H\xi - N(x)Bv(0) - \kappa(x,x)\partial_x v(x) \\ &\quad + \kappa(x,0)\partial_x v(0) + \partial_y \kappa(x,x)v(x) - \partial_y \kappa(x,0)v(0) - \int_0^x \partial_y^2 \kappa(x,y)v(y)\mathrm{d}y \\ &\quad - \lambda \int_0^x \kappa(x,y)v(y)\mathrm{d}y - \int_0^x \kappa(x,y)\Big(- N(y)\Psi(H\xi) + \varepsilon(y)H\xi\Big)\mathrm{d}y \\ &\quad + \int_0^x \kappa(x,y)N(y)\mathrm{d}yBv(0). \end{aligned}$$

Subtracting both sides of above equality from those of the second equality of (B.1) while using (3.15) gives the second equality of (3.16).

C. Proof of Proposition 4.2

By computing the k-th partial derivatives with respect to x of both sides of the second equality of (3.16) directly gives the first line of (4.2). The second line is obtained by recursive step. First, by computing the partial derivative with respect to t of both sides of the second equality of (3.16) while using the first line of (4.2), we obtain

$$\begin{aligned} \partial_t^2 w &= \partial_t \partial_x^2 w + \partial_t \Phi(x,\xi) \\ &= \partial_x^4 w + \partial_x^2 \Phi(x,\xi) + \partial_\xi \Phi(x,\xi)\dot{\xi} \\ &= \partial_x^4 w + \partial_x^2 \Phi(x,\xi) + \partial_\xi \Phi(x,\xi)\left(G\xi + \Psi(H\xi) + Bw(0)\right) \\ &\triangleq \partial_x^4 w + \Gamma_2(x,\xi,w(0)), \end{aligned}$$

which shows that the second equality of (4.2) holds for $k = 2$. Suppose that the second equality of (4.2) holds for $k = l\,(l = 2, \cdots, m-1)$. Then, we have

$$\partial_t^l w = \partial_x^{2l} w + \Gamma_l(x,\xi,w(0),\partial_t w(0),\cdots,\partial_t^{l-2} w(0)).$$

Computing the partial derivative with respect to t of both sides of above equality leads to that

$$\begin{aligned}\partial_t^{l+1}w &= \partial_t\partial_x^{2l}w + \partial_t\Gamma_l \\ &= \partial_x^{2l+2}w + \partial_x^{2l}\Phi(x,\xi) + \partial_\xi\Gamma_l\dot{\xi} + \sum_{i=0}^{l-2}\frac{\partial\Gamma_l}{\partial(\partial_t^i w(0))}\partial_t^{i+1}w(0) \\ &= \partial_x^{2l+2}w + \Gamma_{l+1}(x,\xi,w(0),\partial_t w(0),\cdots,\partial_t^{l-1}w(0)).\end{aligned}$$

Thus, the second line of (4.2) is obtained.

The third line of (4.2) is derived by recursive step. Suppose that $\partial_x^{2l-1}w(0)$ holds for $l=1,\cdots$. Then, by letting $x=0$ in the first line of (4.2) with $k=2l-1$ while noting $\partial_x^{2l-1}\Phi(0,\xi)=0$ (given by (3.18)), we obtain that $\partial_x^{2l+1}w(0)=\partial_t\partial_x^{2l-1}w(0)-\partial_x^{2l-1}\Phi(0,\xi)=0$, and hence brings the third line of (4.2).

The fourth equality of (4.2) is directly obtained from the first one of (4.1) by noting $w(1)=\zeta_1$. Computing the partial derivative of both sides of the fourth equality of (4.2) with respect to t brings that

$$\begin{aligned}\partial_t^2 w(1) &= -\partial_t\partial_x w(1) + \Upsilon_1\left(\dot{\zeta}_{[2]}\right) \\ &= -\partial_x^3 w(1) + \Upsilon_2\left(\zeta_{[3]},\partial_x w(1)\right) + b_{2,1}\partial_x\Phi(1,\xi) \\ &\triangleq -\partial_x^3 w(1) + \Upsilon_2\left(\zeta_{[3]},\partial_x w(1)\right) + \Lambda_2(\xi),\end{aligned}$$

which is the fifth equality of (4.2). The last equality of (4.2) is derived by recursive step. First, computing the partial derivative with respect to t of both sides of the fifth equality of (4.2) while using its first line with $k=1,3$ and (4.1) gives that

$$\begin{aligned}\partial_t^3 w(1) &= -\partial_t\partial_x^3 w(1) + \Upsilon_2\left(\dot{\zeta}_{[3]},\partial_t\partial_x w(1)\right) + \dot{\Lambda}_2(\xi) \\ &= -\partial_x^5 w(1) + \Upsilon_3\left(\zeta_{[4]},\partial_x w(1),\partial_x^3 w(1)\right) + b_{3,1}\partial_x\Phi(1,\xi) \\ &\quad + b_{3,2}\partial_x^3\Phi(1,\xi) + \partial_\xi\Lambda_2(\xi)\dot{\xi} \\ &\triangleq -\partial_x^5 w(1) + \Upsilon_3\left(\zeta_{[4]},\partial_x w(1),\partial_x^3 w(1)\right) + \Lambda_3(\xi,w(0)).\end{aligned}$$

Suppose that the last equality of (4.2) holds for $k=l\,(l=4,\cdots,m-1)$. Then, computing its time derivative gives that

$$\begin{aligned}\partial_t^{l+1}w(1) &= -\partial_x^{2l+1}w(1) + \Upsilon_{l+1}\left(\zeta_{[l+2]},\partial_x w(1),\cdots,\partial_x^{2l-1}w(1)\right) \\ &\quad + \sum_{i=1}^{l} b_{l+1,i}\partial_x^{2i-1}\Phi(1,\xi) + \partial_\xi\Lambda_l(\xi)\dot{\xi} + \sum_{i=0}^{l-3}\frac{\partial\Lambda_l}{\partial(\partial_t^i w(0))}\partial_t^{i+1}w(0) \\ &\triangleq -\partial_x^{2l+1}w(1) + \Upsilon_{l+1}\left(\zeta_{[l+2]},\partial_x w(1),\cdots,\partial_x^{2l-1}w(1)\right) \\ &\quad + \Lambda_{l+1}(\xi,w(0),\partial_t w(0),\cdots,\partial_t^{l-2}w(0)).\end{aligned}$$

Thus, the last equality of (4.2) holds.

D. Proof of Claim ①

From Proposition 4.3, the first equality of (3.12) implies that $\|\partial_x v\|\in\mathcal{L}_\infty$, $\|\partial_x^2 v\|\in\mathcal{L}_2$. Thus, there holds that $v(0)\in\mathcal{L}_\infty\cap\mathcal{L}_2$, $\dot{\xi}$, $\|v\|\in\mathcal{L}_\infty\cap\mathcal{L}_2$, and hence $\dot{\chi}\in\mathcal{L}_2$, χ, $Z_1\in\mathcal{L}_\infty\cap\mathcal{L}_2$.

Noting that $\partial_x^2 w(1) = \partial_t w(1) - \Phi(1,\xi) \in \mathcal{L}_2$, $\partial_t \partial_x w(1) = \partial_x^3 w(1) + \partial_x \Phi(1,\xi) = \partial_x^3 w(1) + \ell_2$ (seen from (4.2) and Proposition 4.3), we obtain the following inequality by Young's inequality

$$\begin{aligned}\partial_x^2 w(1)\partial_t\partial_x w(1) &= \ell_2\left(\partial_x^3 w(1) + \ell_2\right)\\ &\le \frac{1}{4}\left(\partial_x^3 w(1)\right)^2 + \ell_1,\end{aligned}$$

which gives the first inequality of Claim ①.

Since $\zeta_{[3]}$, $\partial_x w(1) \in \mathcal{L}_2$, we have $\Upsilon_2 \in \mathcal{L}_2$. Moreover, (4.3), together with the integrability of ξ and $w(0)$, gives that Γ_2, $\Lambda_2 \in \mathcal{L}_2$. Then, (4.2) with (4.3) brings that

$$\begin{aligned}\partial_t\partial_x^2 w(1) &= \partial_x^4 w(1) + \partial_x^2\Phi(1,\xi)\\ &= \partial_t^2 w(1) - \Gamma_2(1,\xi,w(0)) + \partial_x^2\Phi(1,\xi)\\ &= -\partial_x^3 w(1) + \Upsilon_2\left(\zeta_{[3]}, \partial_x w(1)\right) + \Lambda_2(\xi) - \Gamma_2(1,\xi,w(0)) + \partial_x^2\Phi(1,\xi)\\ &= -\partial_x^3 w(1) + \ell_2,\end{aligned}$$

by which and then using Young's inequality, we bring that

$$\partial_x^3 w(1)\partial_t\partial_x^2 w(1) = \partial_x^3 w(1)\left(-\partial_x^3 w(1) + \ell_2\right) \le -\frac{1}{2}\left(\partial_x^3 w(1)\right)^2 + \ell_1,$$

which gives the second inequality of Claim ①.

E. Proof of Claim ②

With the proven fact that (4.4) when $i = 1$ in hands, (4.3) brings that

$$\begin{aligned}&\partial_x^4 w(1) = \partial_t^2 w(1) - \Gamma_2(1,\xi,w(0)) \in \mathcal{L}_2,\\ &\partial_t\partial_x^3 w(1) = \partial_x^5 w(1) + \partial_x^3\Phi(1,\xi) = \partial_x^5 w(1) + \ell_2.\end{aligned}$$

Then, by Young's inequality, there holds that

$$\begin{aligned}\partial_x^4 w(1)\partial_t\partial_x^3 w(1) &= \ell_2\left(\partial_x^5 w(1) + \ell_2\right)\\ &\le \frac{1}{4}\left(\partial_x^5 w(1)\right)^2 + \ell_1,\end{aligned}$$

which gives the first inequality of Claim ②.

Moreover, (4.4) with $i = 1$ gives that ξ, $w(0)$, $\partial_t w(0)$, $\partial_x^4\Phi(1,\xi) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\partial_x^3 w(1) \in \mathcal{L}_2$. Then, (4.3) gives that Υ_3, Γ_3, $\Lambda_3 \in \mathcal{L}_2$, and hence (4.2) with (4.3) brings that

$$\begin{aligned}\partial_t\partial_x^4 w(1) &= \partial_x^6 w(1) + \partial_x^4\Phi(1,\xi)\\ &= \partial_t^3 w(1) - \Gamma_3(1,\xi,w(0),\partial_t w(0)) + \partial_x^4\Phi(1,\xi)\\ &= -\partial_x^5 w(1) + \Upsilon_3\left(\zeta_{[4]}, \partial_x w(1), \partial_x^3 w(1)\right) + \Lambda_3(\xi, w(0))\\ &\quad -\Gamma_3(1,\xi,w(0),\partial_t w(0)) + \partial_x^4\Phi(1,\xi)\\ &= -\partial_x^5 w(1) + \ell_2,\end{aligned}$$

by which and then using Young's inequality, we obtain that

$$\partial_x^5 w(1)\partial_t\partial_x^4 w(1) = \partial_x^5 w(1)\left(-\partial_x^5 w(1) + \ell_2\right) \le -\frac{1}{2}\left(\partial_x^5 w(1)\right)^2 + \ell_1,$$

which gives the second inequality of Claim ②.

F. Proof of Claim ③

With the proven fact that (4.4) when $i = k-1$ in hands, (4.3) brings that

$$\partial_x^{2k} w(1) = \partial_t^k w(1) - \Gamma_k(1, \xi, w(0), \cdots, \partial_t^{k-2} w(0)) \in \mathcal{L}_2,$$
$$\partial_t \partial_x^{2k-1} w(1) = \partial_x^{2k+1} w(1) + \partial_x^{2k-1} \Phi(1, \xi) = \partial_x^{2k+1} w(1) + \ell_2.$$

Then, Young's inequality leads to that

$$\begin{aligned} \partial_x^{2k} w(1) \partial_t \partial_x^{2k-1} w(1) &= \ell_2 \left(\partial_x^{2k+1} w(1) + \ell_2 \right) \\ &\leq \frac{1}{4} \left(\partial_x^{2k+1} w(1) \right)^2 + \ell_1. \end{aligned} \tag{G.1}$$

Moreover, (4.4) with $i = k-1$ also gives that $\partial_x^{2k} \Phi(1, \xi)$, ξ, $w(0)$, $\partial_x^i w(0)$, $\partial_x^{2i-1} w(1) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $i = 1, \cdots, k-1$, and $\partial_x^{2i-1} w(1) \in \mathcal{L}_2$. Then, (4.3) gives that Υ_{k+1}, Γ_{k+1}, $\Lambda_{k+1} \in \mathcal{L}_2$, and hence (4.2) with (4.3) brings that

$$\begin{aligned} \partial_t \partial_x^{2k} w(1) &= \partial_x^{2k+2} w(1) + \partial_x^{2k} \Phi(1, \xi) \\ &= \partial_t^{k+1} w(1) - \Gamma_{k+1}\big(1, \xi, w(0), \partial_t w(0), \cdots, \partial_t^{k-1} w(0)\big) + \partial_x^{2k} \Phi(1, \xi) \\ &= -\partial_x^{2k+1} w(1) + \Upsilon_{k+1}\big(\zeta_{[k+2]}, \partial_x w(1), \cdots, \partial_x^{2k-1} w(1)\big) \\ &\quad + \Lambda_{k+1}\big(\xi, w(0), \partial_t w(0) \cdots, \partial_t^{k-2} w(0)\big) \\ &\quad - \Gamma_{k+1}\big(1, \xi, w(0), \partial_t w(0), \cdots, \partial_t^{k-1} w(0)\big) + \partial_x^{2k} \Phi(1, \xi) \\ &= -\partial_x^{2k+1} w(1) + \ell_2, \end{aligned}$$

by which and then using Young's inequality, we obtain that

$$\begin{aligned} \partial_x^{2k+1} w(1) \partial_t \partial_x^{2k} w(1) &= \partial_x^{2k+1} w(1) \left(-\partial_x^{2k+1} w(1) + \ell_2 \right) \\ &\leq -\frac{1}{2} \left(\partial_x^{2k+1} w(1) \right)^2 + \ell_1. \end{aligned} \tag{G.2}$$

G. Useful inequalities and criterions

The following four lemmas give some useful inequalities.

Lemma G.1. ([11]) (*Agmon's inequality*) *For any continuously differentiable function* w *defined on* $[0, 1]$*, there hold*

$$\begin{cases} \max_{[0,1]} w(x)^2 \leq w(0)^2 + 2\|w\|\|w_x\|, \\ \max_{[0,1]} w(x)^2 \leq w(1)^2 + 2\|w\|\|w_x\|. \end{cases}$$

Lemma G.2. ([11]) (*Poincaré's inequality*) *For any continuously differentiable function* w *defined on* $[0, 1]$*, there hold*

$$\begin{cases} \|w\|^2 \leq 2w(0)^2 + 4\|w_x\|^2, \\ \|w\|^2 \leq 2w(1)^2 + 4\|w_x\|^2. \end{cases}$$

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