Journal of Applied Analysis and Computation
Volume 16, Number 1, February 2026, 296–311
Website:http://www.jaac-online.com
DOI:10.11948/20250094

THE PERSISTENCE OF AN AGE-STRUCTURED SYPHILIS MODEL

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Abstract Syphilis, a highly infectious bacterial infection, poses a significant health threat globally due to its high morbidity and mortality rates. Predominantly transmitted through sexual contact, the age distribution among hosts plays a pivotal role in the disease transmission dynamics. In this paper, we first formulate an age-structured epidemic model with four infection stages (primary, secondary, latent and tertiary) and then derive the explicit expression of the basic reproduction number by using the next generation equation. According to the definition of the persistence and applying advanced mathematical techniques, including multiple integral reordering, variable transformations, Laplace transforms, and the method of contradiction, we not only prove the weak persistence but also the strong persistence of the disease.

Keywords Syphilis, age-structured model, the basic reproduction number, persistence.

MSC(2010) 92D30, 35B40, 58Z05.

1. Introduction and model

Syphilis, a sexually transmitted disease caused by the bacterium *Treponema pallidum*, follows a complex pathogenesis. Upon infection, which occurs through mucous membranes or skin breaches, the pathogen establishes a primary chancre at the entry point. This initial lesion serves as a site for bacterial multiplication before *Treponema pallidum* spreads systemically through the bloodstream, avoiding detection by the immune system due to its minimal antigenic variation and adept immune evasion tactics. In the secondary phase, the infection proliferates, manifesting in diverse clinical signs such as skin eruptions, mucosal lesions, and systemic symptoms. Left untreated, syphilis may advance to latent and tertiary stages, impacting various organs and inflicting significant harm.

The consideration of age structure is crucial in modeling biological processes, reflecting physiological variations across different life stages. Building on the foundational work of McKendrick [15], age-structured models have become a staple in addressing biological and epidemiological challenges, as detailed in the seminal works of f Webb [19], Iannelli [11], Capasso [3], Inabacite [13], Wang et al. [18], Zhang et al. [2], Li and Yang [29]. These models, which have been instrumental in understanding disease dynamics, include the renowned epidemic model introduced by Kermack and McKendrick [14]. Given that syphilis primarily spreads through sexual contact, the age distribution within the host population is a pivotal factor in its transmission. The primary goals of this study are to develop an age-structured model for syphilis that encompasses four stages of infection—primary, secondary, latent, and tertiary—and to investigate the model's dynamical properties. It's important to note that the persistence of syphilis in

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age-structured models cannot be directly assessed using the persistence theories found in traditional literature, such as that by Zhao [28]. Establishing persistence in these models requires a foundational approach to defining persistence and the application of advanced mathematical techniques, including multiple integral reordering, variable transformations, Laplace transforms, and the method of contradiction.

Drawing on previous models of syphilis transmission formulated through ordinary differential equations (ODEs) [8, 12, 25, 30], we incorporate the following fundamental compartments: susceptible individuals (S), exposed individuals (E), individuals in the primary stage of infection (I_1), those in the secondary stage (I_2), individuals in the early latent phase (L_1), those in the late latent phase (L_2), and individuals in the final (tertiary) stage of infection (I_3). For a detailed introduction to the process of syphilis, please refer to literatures [20, 22]. To delve into the epidemiological dynamics of syphilis within a stratified age demographic, we define $S(a,t)$, $E(a,t)$, $I_1(a,t)$, $I_2(a,t)$, $L_1(a,t)$, $L_2(a,t)$, and $I_3(a,t)$ as the age-specific distributions of individuals across these stages at time t and age a. Consequently, we introduce the following age-structured syphilis model that captures these dynamics

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) S(a,t) = -\varrho(a,t)S(a,t) - \mu(a)S(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) E(a,t) = \varrho(a,t)S(a,t) - (\delta(a)+\mu(a))E(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) I_1(a,t) = \delta(a)E(a,t) + L_1(a,t) - (\sigma(a)+\mu(a))I_1(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) I_2(a,t) = \sigma(a)I_1(a,t) + L_1(a,t) - (\eta(a)+\mu(a)+\xi(a))I_2(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) L_1(a,t) = \eta(a)I_2(a,t) - (\zeta(a)+\mu(a))L_1(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) L_2(a,t) = \zeta(a)L_1(a,t) - (\theta(a)+\mu(a))L_2(a,t), \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) I_3(a,t) = \xi(a)I_2(a,t) + \theta(a)L_2(a,t) - \mu(a)I_3(a,t), \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &S(0,t)=\lambda, E(0,t)=0, I_1(0,t)=0, I_2(0,t)=0, \\ &L_1(0,t)=0, L_2(0,t)=0, I_3(0,t)=0 \end{aligned} \tag{1.2}$$

and initial conditions

$$\begin{aligned} &S(a,0)=S_0(a)>0, E(a,0)=E_0(a)>0, \\ &I_1(a,0)=I_{10}(a)>0, I_2(a,0)=I_{20}(a)>0, \\ &L_1(a,0)=L_{10}(a)>0, L_2(a,0)=L_{20}(a)>0, I_3(a,0)=I_{30}(a)>0. \end{aligned} \tag{1.3}$$

We denote the infection rates for primary stage syphilis by $\alpha(a)$, for secondary stage by $\beta(a)$, and for early latent stage by $\gamma(a)$, corresponding to the infected individuals $I_1(a,t)$, $I_2(a,t)$, and $L_1(a,t)$, respectively. Additionally, we let $\mathbf{B}(a)$ signify the effective contact rate for individuals of age a. Consequently, the transition rate of susceptible individuals into the exposed category

(those infected but not yet infectious) is given by $\varrho(a,t)S(a,t)$, where

$$\varrho(a,t)=\int_0^{a_{\max}}\frac{\tilde{\alpha}(a,s)I_1(s,t)+\tilde{\beta}(a,s)I_2(s,t)+\tilde{\gamma}(a,s)L_1(s,t)}{N(s,t)}ds, \tag{1.4}$$

in which

$$\begin{aligned}&N(a,t)=S(a,t)+E(a,t)+I_1(a,t)+I_2(a,t)+L_1(a,t)+L_2(a,t)+I_3(a,t),\\&\tilde{\alpha}(a,s)=\mathbf{B}(a)\alpha(s),\tilde{\beta}(a,s)=\mathbf{B}(a)\beta(s),\tilde{\gamma}(a,s)=\mathbf{B}(a)\gamma(s),\end{aligned}$$

and $a_{\max}>0$ represents the maximum age of the population. The exposed individuals then progress into the primary stage $I_1(a,t)$ with rate $\delta(a)$, the secondary stage $I_2(a,t)$ with rate $\sigma(a)$, the early latent stage $L_1(a,t)$ with rate $\eta(a)$, the late latent stage $L_2(a,t)$ with rate $\zeta(a)$, and the tertiary stage $I_3(a,t)$ with rate $\theta(a)$, respectively. $\mu(a)$ is the age dependent mortality rate for all individuals and $d(a)$ is syphilis-related mortality rate. A portion of secondary stage $\xi(a)I_2(a,t)$ will develop into the tertiary stage directly. In this section, we study the persistence for system (1.1) under initial and boundary conditions (1.2)-(1.3). For the sake of simplicity, set

$$\begin{aligned}&s(a,t)=\frac{S(a,t)}{N(a,t)},e(a,t)=\frac{E(a,t)}{N(a,t)},\\&i_j(a,t)=\frac{I_j(a,t)}{N(a,t)},l_j(a,t)=\frac{L_j(a,t)}{N(a,t)},i_3(a,t)=\frac{I_3(a,t)}{N(a,t)},j=1,2,\end{aligned} \tag{1.5}$$

then, we can rewrite system (1.1) as follows:

$$\begin{cases}\left(\dfrac{\partial s}{\partial a}+\dfrac{\partial s}{\partial t}\right)=-\tilde{\varrho}(a,t)s(a,t)+d(a)i_3(a,t)s(a,t),\\\left(\dfrac{\partial e}{\partial a}+\dfrac{\partial e}{\partial t}\right)=\tilde{\varrho}(a,t)s(a,t)-\delta(a)e(a,t),\\\left(\dfrac{\partial i_1}{\partial a}+\dfrac{\partial i_1}{\partial t}\right)=\delta(a)e(a,t)+(1-\epsilon)\omega(a)l_1(a,t)-\sigma(a)i_1(a,t),\\\left(\dfrac{\partial i_2}{\partial a}+\dfrac{\partial i_2}{\partial t}\right)=\sigma(a)i_1(a,t)+\epsilon(a)\omega(a)l_1(a,t)-(\eta(a)+\xi(a))i_2(a,t),\\\left(\dfrac{\partial l_1}{\partial a}+\dfrac{\partial l_1}{\partial t}\right)=\eta(a)i_2(a,t)-\zeta(a)l_1(a,t),\\\left(\dfrac{\partial l_2}{\partial a}+\dfrac{\partial l_2}{\partial t}\right)=\zeta(a)l_1(a,t)-\theta(a)l_2(a,t),\\\left(\dfrac{\partial i_3}{\partial a}+\dfrac{\partial i_3}{\partial t}\right)=\xi(a)i_2(a,t)+\theta(a)l_2(a,t),\end{cases} \tag{1.6}$$

where $\tilde{\varrho}(a,t)=\int_0^{a_{\max}}(\tilde{\alpha}(a,s)i_1(s,t)+\tilde{\beta}(a,s)i_2(s,t)+\tilde{\gamma}(a,s)l_1(s,t))ds$. Corresponding initial values and boundary conditions for system (1.6) are

$$\begin{aligned}&s(0,t)=1,e(0,t)=i_j(0,t)=l_j(0,t)=i_3(0,t)=0,s(a,0)=s_0(a),\\&e(a,0)=e_0(a),i_j(a,0)=i_{j0}(a),l_j(a,0)=l_{j0}(a),i_3(a,0)=i_{30}(a),j=1,2.\end{aligned} \tag{1.7}$$

Assumption 1.1. *We suppose that*

1. *The population is homogeneously mixed and population activities are free from outside interference;*
2. *The natural death rate $\mu(\cdot)$ is locally integrable and $\int_0^{a_{\max}} \mu(a)\,da = +\infty$; $\tilde{\alpha}(\cdot,\cdot), \tilde{\beta}(\cdot,\cdot), \tilde{\gamma}(\cdot,\cdot) \in L^\infty_+((0,a_{\max})\times(0,a_{\max}))$; $\delta(\cdot)$, $\mathbf{B}(\cdot)$, $\sigma(\cdot)$, $\xi(\cdot)$, $\zeta(\cdot)$, $\eta(\cdot)$, $\omega(\cdot)$, $\epsilon(\cdot) \in L^\infty_+(0,a_{\max})$; all these functions are extended by zero outside of $[0,a_{\max}]$.*

According to the definition in [31], we can define the basic reproduction number as

$$\mathcal{R}_0 = \mathbb{F}(0) = \int_0^{a_{\max}} (\alpha(a)\Pi_{i_1}(0,a) + \beta(a)\Pi_{i_2}(0,a) + \gamma(a)\Pi_{l_1}(0,a))da, \tag{1.8}$$

where

$$\begin{aligned}
\Pi_{i_1}(a) &= \int_0^a \delta(s) \int_0^s \mathbf{B}(l) \exp\left\{-\int_l^s \delta(\tau)d\tau\right\} \exp\left\{-\int_s^a (\lambda+\sigma(\tau))d\tau\right\} dlds,\\
\Pi_{i_2}(a) &= \int_0^a \sigma(s)\Pi_{i_1}(\lambda,s) \exp\left\{-\int_s^a (\eta(\tau)+\xi(\tau))d\tau\right\} ds,\\
\Pi_{l_1}(a) &= \int_0^a \eta(s)\Pi_{i_2}(\lambda,s) \exp\left\{-\int_s^a \zeta(\tau)d\tau\right\} ds.
\end{aligned}$$

In our pervious works, we not only studied the well-posedness and the stability of system (1.6) but also presented the optimal control problem and conducted some numerical simulations for system (1.6), the readers refer to [21,23] for details. However, we have not addressed an extremely important issue in the field of infectious disease modeling research: the persistence of the disease. We will focus on analyzing the issue of system (1.6) in this paper.

2. Persistence of the disease

In this section, we prove the persistence of the disease. Note that the concept of persistence can be further classified as weak persistence, strong persistence, uniform weak persistence and uniform persistence [6,7]. Since age-structured models are first-order hyperbolic partial differential equations, we cannot use the comparison principle and then apply the strong repeller theory (Chapter 1 in [28]) to show the uniform persistence of our model. Differing from the method that mentioned in [9] (the persistence theory of general infinite dimensional systems), we will prove the persistence conclusion of the age-structured syphilis model by using the definition of persistence [28]. The classic technique in [17] only provides weak persistence conclusion for age-structured SIR models and does not provide strong persistence conclusion. Kunyia et al. [16] prove the uniform strong ρ-persistence of an age-structured SIRS epidemic model in an innovative way. In this section, for the complex age-structured syphilis model we not only obtain weak persistence conclusion but also strong persistence conclusion.

We first set

$$\begin{aligned}
\Phi_e(a) &= \exp\left\{-\int_0^a \delta(s)ds\right\}, \Phi_1(a) = \exp\left\{-\int_0^a \sigma(s)ds\right\},\\
\Phi_2(a) &= \exp\left\{-\int_0^a \eta(s)ds\right\}, \Phi_3(a) = \exp\left\{-\int_0^a \xi(s)ds\right\},\\
\Phi_l(a) &= \exp\left\{-\int_0^a \zeta(s)ds\right\},
\end{aligned}$$

and chose $a_{\max}=+\infty$. Then, we introduce

$$\begin{aligned}
&s(a,t)=s(a,t), s(a,0)=s_0(a), x_3(a,t)=\frac{i_2(a,t)}{\Phi_2(a)\Phi_3(a)}, x_3(a,0)=\frac{i_{20}(a)}{\Phi_2(a)\Phi_3(a)},\\
&x_1(a,t)=\frac{e(a,t)}{\Phi_e(a)}, x_1(a,0)=\frac{e_0(a)}{\Phi_e(a)}, x_2(a,t)=\frac{i_1(a,t)}{\Phi_1(a)}, x_2(a,0)=\frac{i_{10}(a)}{\Phi_1(a)},\\
&x_4(a,t)=\frac{l_1(a,t)}{\Phi_l(a)}, x_4(a,0)=\frac{l_{10}(a)}{\Phi_l(a)}.
\end{aligned}\tag{2.1}$$

From system (1.6) with $\omega(a)=d(a)=0$, these new variables in (2.1) take the form

$$\begin{aligned}
&\left(\frac{\partial s}{\partial a}+\frac{\partial s}{\partial t}\right)=-\tilde{\varrho}(a,t)s(a,t), \left(\frac{\partial x_1}{\partial a}+\frac{\partial x_1}{\partial t}\right)=\frac{\tilde{\varrho}(a,t)}{\Phi_e(a)}s(a,t),\\
&\left(\frac{\partial x_2}{\partial a}+\frac{\partial x_2}{\partial t}\right)=-\frac{\Phi_e'(a)}{\Phi_1(a)}x_1(a,t), \left(\frac{\partial x_3}{\partial a}+\frac{\partial x_3}{\partial t}\right)=-\frac{\Phi_1'(a)}{\Phi_2(a)\Phi_3(a)}x_2(a,t),\\
&\left(\frac{\partial x_4}{\partial a}+\frac{\partial x_4}{\partial t}\right)=-\frac{\Phi_2'(a)}{\Phi_l(a)}x_3(a,t), s(0,t)=1, x_m(0,t)=0, m=1,2,3,4,
\end{aligned}$$

where $\Phi'(a)=d\Phi(a)/da$, $\tilde{\varrho}(a,t)=\mathbf{B}(a)\Psi(t)$ with

$$\Psi(t)=\int_0^{+\infty}\Big(\alpha(a)\Phi_1(a)x_2(a,t)+\beta(a)\Phi_2(a)\Phi_3(a)x_3(a,t)+\gamma(a)\Phi_l(a)x_4(a,t)\Big)da.$$

We call $\Psi(t)$ the infective force at time t [17]. Denote that $x_m^1(a,t)=x_m(a,t)$ for $t\geq a$ and $x_m^0(a,t)=x_m(a,t)$ for $t<a, m=1,2,3,4$. Thus, we have

$$\begin{aligned}
\Psi(t)=&\int_0^t\alpha(a)\Phi_1(a)x_2^1(a,t)da+\int_0^t\beta(a)\Phi_2(a)\Phi_3(a)x_3^1(a,t)da\\
&+\int_0^t\gamma(a)\Phi_l(a)x_4^1(a,t)da+\int_t^{+\infty}\alpha(a)\Phi_1(a)x_2^0(a,t)da\\
&+\int_t^{+\infty}\beta(a)\Phi_2(a)\Phi_3(a)x_3^0(a,t)da+\int_t^{+\infty}\gamma(a)\Phi_l(a)x_4^0(a,t)da.
\end{aligned}\tag{2.2}$$

Applying the characteristics line method, we obtain that

$$s(a,t)=\begin{cases}\exp\left\{-\int_0^a\tilde{\varrho}(a-s,t-s)ds\right\}, t\geq a\geq 0,\\ s_0(a-t)\exp\left\{-\int_0^t\tilde{\varrho}(a-s,t-s)ds\right\}, a>t\geq 0,\end{cases}$$

$$x_1(a,t)=\int_0^{a\divideontimes t}\frac{s(a-s,t-s)\tilde{\varrho}(a-s,t-s)}{\Phi_e(a-s)}ds+x_{10}(a-t),$$

where $a\divideontimes t=\min\{a,t\}$, $x_{m0}(a-t)=0$ for $a<t, m=1,2,3,4$, and

$$\begin{aligned}
x_2(a,t)&=-\int_{[a,t]_+}a\frac{x_1(\tau,t+\tau-a)}{\Phi_1(\tau)}d(\Phi_e(\tau))+x_{20}(a-t),\\
x_3(a,t)&=-\int_{[a,t]_+}a\frac{x_2(\tau,t+\tau-a)}{\Phi_2(\tau)\Phi_3(\tau)}d(\Phi_1(\tau))+x_{30}(a-t),\\
x_4(a,t)&=-\int_{[a,t]_+}a\frac{x_3(\tau,t+\tau-a)}{\Phi_l(\tau)}d(\Phi_2(\tau))+x_{40}(a-t),
\end{aligned}$$

where $[a,t]_+$ is the positive part of $[a-t]$ which is $(a-t)$ for $a \geq t$ and 0 for $a < t$. Thus, after some calculations, we have

$$
\begin{aligned}
x_2^1(a,t) &= -\int_0^a \int_0^\tau \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(\tau))}{\Phi_1(\tau)},\\
x_2^0(a,t) &= -\int_{a-t}^a \left(\int_{a-t}^\tau \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc + x_{10}(\tau-t) \right) \frac{d(\Phi_e(\tau))}{\Phi_1(\tau)} + x_{20}(a-t),\\
x_3^1(a,t) &= \int_0^a \int_0^\tau \int_0^r \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(r))}{\Phi_1(r)} \frac{d(\Phi_1(\tau))}{\Phi_1(\tau)},\\
x_3^0(a,t) &= -\int_{a-t}^a \left(\int_{a-t}^\tau \left(\int_{a-t}^r \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc + x_{10}(r-t) \right) \frac{d(\Phi_e(r))}{\Phi_1(r)} \right.\\
&\quad \left. + x_{20}(\tau-t) \right) \frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)} + x_{30}(a-t),
\end{aligned}
$$

and

$$
\begin{aligned}
x_4^1(a,t) &= -\int_0^a \int_0^\tau \int_0^r \int_0^l \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(l))}{\Phi_1(l)} \frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)} \frac{d(\Phi_2(\tau))}{\Phi_l(\tau)},\\
x_4^0(a,t) &= -\int_{a-t}^a \left(\int_{a-t}^\tau \left(\int_{a-t}^r \left(\int_{a-t}^l \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc + x_{10}(l-t) \right) \frac{d(\Phi_e(l))}{\Phi_1(l)} \right.\right.\\
&\quad \left.\left. + x_{20}(r-t) \right) \frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)} + x_{30}(\tau-t) \right) \frac{d(\Phi_2(\tau))}{\Phi_l(\tau)} + x_{40}(a-t),
\end{aligned}
$$

where $d(\Phi(a))$ represents the derivative of $\Phi(a)$ with respect to a.

Since $\tilde{\varrho}(a,t) = \mathbf{B}(a)\Psi(t)$, we can obtain the following expression of $\Psi(t)$ from (2.2)

$$
\begin{aligned}
&\Psi(t)\\
&= -\int_0^t \alpha(a)\Phi_1(a) \int_0^a \int_0^\tau \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(\tau))}{\Phi_1(\tau)} da\\
&\quad + \int_0^t \beta(a)\Phi_2(a)\Phi_3(a) \int_0^a \int_0^\tau \int_0^r \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(r))}{\Phi_1(r)} \frac{d(\Phi_1(\tau))}{\Phi_1(\tau)} da\\
&\quad - \int_0^t \gamma(a)\Phi_l(a) \int_0^a \int_0^\tau \int_0^r \int_0^l \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \frac{d(\Phi_e(l))}{\Phi_1(l)} \frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)} \frac{d(\Phi_2(\tau))}{\Phi_l(\tau)} da\\
&\quad + \int_t^{+\infty} \alpha(a)\Phi_1(a) \left[-\int_{a-t}^a \left(\int_{a-t}^\tau \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc + x_{10}(\tau-t) \right) \frac{d(\Phi_e(\tau))}{\Phi_1(\tau)} \right.\\
&\quad \left. + x_{20}(a-t) \right] da\\
&\quad + \int_t^{+\infty} \beta(a)\Phi_2(a)\Phi_3(a) \left[\int_{a-t}^a \left(\int_{a-t}^\tau \left(\int_{a-t}^r \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc \right.\right.\right.\\
&\quad \left.\left.\left. + x_{10}(r-t) \right) \frac{d(\Phi_e(r))}{\Phi_1(r)} + x_{20}(\tau-t) \right) \frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)} + x_{30}(a-t) \right] da\\
&\quad + \int_t^{+\infty} \gamma(a)\Phi_l(a) \left[-\int_{a-t}^a \left(\int_{a-t}^\tau \left(\int_{a-t}^r \left(\int_{a-t}^l \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)} dc + x_{10}(l-t) \right) \right.\right.\right.\\
&\quad \left.\left.\left. \times \frac{d(\Phi_e(l))}{\Phi_1(l)} + x_{20}(r-t) \right) \frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)} + x_{30}(\tau-t) \right) \frac{d(\Phi_2(\tau))}{\Phi_l(\tau)} + x_{40}(a-t) \right] da.
\end{aligned}
\tag{2.3}
$$

For the first integration part in (2.3), we change the order of the integration and obtain that

$$\begin{aligned}&-\int_0^t \alpha(a)\Psi_1(a)\int_0^a\int_0^\tau \frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}da\\ &=-\int_0^t \alpha(a)\Psi_1(a)\int_0^a s(t-a+b)\Psi(t-a+b)\frac{\mathbf{B}(b)}{\Phi_e(b)}db\int_b^a\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}da,\end{aligned}\tag{2.4}$$

then we change the variables $a=a$ and $a-b=r$ and change the order of the integration in (2.4), yields

$$\begin{aligned}&-\int_0^t \alpha(a)\Phi_1(a)\int_0^a s(t-a+b)\Psi(t-a+b)\frac{\mathbf{B}(b)}{\Phi_e(b)}db\int_b^a\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}da\\ &=\int_0^t\Psi(t-r)\int_r^t\alpha(a)\Phi_1(a)s(a-r,t-r)\frac{\mathbf{B}(a-r)}{\Phi_e(a-r)}\int_{a-r}^a\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}dadr\\ &=\int_0^t\Psi(t-r)\int_r^t s(a-r,t-r)K_1(a,a-r)dadr,\end{aligned}$$

where $K_1(a,a-r)=\alpha(a)\frac{\mathbf{B}(a-r)}{\Phi_e(a-r)}\int_{a-r}^a\frac{\Phi_1(a)}{\Phi_1(\tau)}d(\Phi_e(\tau))$. Changing the variable $a-t=w$ in the fourth integration part in (2.3), one has

$$\begin{aligned}&\int_t^{+\infty}\alpha(a)\Phi_1(a)\Big[-\int_{a-t}^a\Big(\int_{a-t}^\tau\frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc\\ &+x_{10}(\tau-t)\Big)\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}+x_{20}(a-t)\Big]da\\ &=-\int_0^{+\infty}\alpha(t+w)\Phi_1(t+w)\int_w^{t+w}\int_w^\tau\frac{s(c,c-w)\Psi(c-w)\mathbf{B}(c)}{\Phi_e(c)}dc\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}dw\\ &-\int_0^{+\infty}\alpha(t+w)\Phi_1(t+w)x_{10}(w)\int_w^{t+w}\frac{d(\Phi_e(\tau))}{\Phi_1(\tau)}dw\\ &+\int_0^{+\infty}\alpha(t+w)\Phi_1(t+w)x_{20}(w)dw\\ &:=\mathbf{A}_1(t).\end{aligned}\tag{2.5}$$

From the expression of $\Phi_e(a)$, it follows that $\psi_1(t)>0$. Similarly, changing the variables and the order of the integration several times, we can obtain

$$\begin{aligned}&\int_0^t\beta(a)\Phi_2(a)\Phi_3(a)\int_0^a\int_0^\tau\int_0^r\frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)}da\\ &=\int_0^t\Psi(t-r)\int_r^t s(a-r,t-r)K_2(a,a-r)dadr,\\ &-\int_0^t\gamma(a)\Phi_l(a)\int_0^a\int_0^\tau\int_0^r\int_0^l\frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc\frac{d(\Phi_e(l))}{\Phi_1(l)}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)}\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}da\\ &=\int_0^t\Psi(t-r)\int_r^t s(a-r,t-r)K_3(a,a-r)dadr,\end{aligned}$$

where

$$\begin{aligned}
K_2(a,a-r)&=\beta(a)\Phi_2(a)\Phi_3(a)\frac{\mathbf{B}(a-r)}{\Phi_e(a-r)}\int_{a-r}^{a}\int_{a-r}^{\tau}\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau))},\\
K_3(a,a-r)&=\gamma(a)\Phi_l(a)\frac{\mathbf{B}(a-r)}{\Phi_e(a-r)}\int_{a-r}^{a}\int_{a-r}^{\tau}\int_{a-r}^{l}\frac{d(\Phi_e(l))}{\Phi_1(l)}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r))}\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}.
\end{aligned}\tag{2.6}$$

Furthermore, we have

$$\begin{aligned}
&\int_t^{+\infty}\beta(a)\Phi_2(a)\Phi_3(a)\left[\int_{a-t}^{a}\left(\int_{a-t}^{\tau}\left(\int_{a-t}^{r}\frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc\right.\right.\right.\\
&\left.\left.\left.+x_{10}(r-t))\frac{d(\Phi_e(r))}{\Phi_1(r)}+x_{20}(\tau-t)\right)\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)}+x_{30}(a-t)\right]da\\
&=\int_0^{+\infty}\beta(t+w)\Phi_2(t+w)\Phi_3(t+w)\int_w^{t+w}\int_w^{\tau}\int_w^{r}\frac{s(c,c-w)\Psi(c-w)\mathbf{B}(c)}{\Phi_e(c)}dc\\
&\quad\times\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(d\tau))}{\Phi_2(\tau)\Phi_3(\tau)}dw\\
&\quad+\int_0^{+\infty}\beta(t+w)\Phi_2(t+w)\Phi_3(t+w)\int_w^{t+w}x_{10}(w)\int_w^{\tau}\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)}dw\\
&\quad-\int_0^{+\infty}\beta(t+w)\Phi_2(t+w)\Phi_3(t+w)x_{20}(w)\int_w^{t+w}\frac{d\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau)}dw\\
&\quad+\int_0^{+\infty}\beta(t+w)\Phi_2(t+w)\Phi_3(t+w)x_{30}(w)dw\\
&:=\mathbf{A}_2(t)\\
&>0,
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\int_t^{+\infty}\gamma(a)\Phi_l(a)\left[-\int_{a-t}^{a}\left(\int_{a-t}^{\tau}\left(\int_{a-t}^{r}\left(\int_{a-t}^{l}\frac{s(c,t-a+c)\tilde{\varrho}(c,t-a+c)}{\Phi_e(c)}dc+x_{10}(l-t)\right)\right.\right.\right.\\
&\left.\left.\left.\times\frac{d(\Phi_e(l))}{\Phi_1(l)}+x_{20}(r-t)\right)\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)}+x_{30}(\tau-t)\right)\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}+x_{40}(a-t)\right]da\\
&=-\int_0^{+\infty}\gamma(t+w)\Phi_l(t+w)\int_w^{t+w}\int_w^{\tau}\int_w^{r}\int_w^{l}\frac{s(c,c-w)\Psi(c-w)\mathbf{B}(c)}{\Phi_e(c)}dc\frac{d(\Phi_e(l))}{\Phi_1(l)}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)}\\
&\quad\times\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}dw-\int_0^{+\infty}\gamma(t+w)\Phi_l(t+w)\int_w^{t+w}x_{10}(w)\int_w^{\tau}\int_w^{r}\frac{d(\Phi_e(l))}{\Phi_1(l)}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)}\\
&\quad\times\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}dw+\int_0^{+\infty}\gamma(t+w)\Phi_l(t+w)\int_w^{t+w}x_{20}(w)\int_w^{\tau}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r)}\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}dw\\
&\quad-\int_0^{+\infty}\gamma(t+w)\Phi_l(t+w)\int_w^{t+w}x_{30}(w)\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}dw\\
&\quad+\int_0^{+\infty}\gamma(t+w)\Phi_l(t+w)x_{40}(w)dw\\
&:=\mathbf{A}_3(t)\\
&>0.
\end{aligned}\tag{2.8}$$

Finally, based on (2.5)-(2.8), we can rewrite the expression of $\Psi(t)$ in (2.2) as follows:

$$\Psi(t) = \int_0^t \Psi(t-r) \int_r^t s(a-r,t-r)(K_1(a,a,-r) + K_2(a,a-r) + K_3(a,a-r))dadr + \sum_{j=1}^{3} \mathbf{A}_j(t). \tag{2.9}$$

Next we study the weak persistence of the disease and have the following result.

Theorem 2.1 (**Uniform weak persistence**). *Under Assumption 1.1, if $\mathcal{R}_0$ in (1.8) with $a_{\max} = +\infty$ is larger than 1, then the disease is uniformly weakly persistent in the sense that the infective force $\Psi(t)$ is not 0 a.e., satisfies $\Psi^\infty = \limsup_{t\to\infty} \Psi(t) > \phi$ with $\phi > 0$ not depending on the initial values.*

Proof. By (2.9), we have $\Psi(t) \geq \int_0^t \Psi(t-r) \int_r^t s(a-r,t-r)(K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr$. Setting $\Psi_\omega = \Psi(t+\omega)$, one has $\Psi_\omega(t) \geq \int_0^t \Psi_\omega(t-r) \int_r^t s(a-r,t-r+\omega)(K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr$. Suppose that $\Psi(t) \leq \phi$ for $t \geq \omega$ and note that

$$s(a,t) = \exp\left\{-\int_0^a \Psi(t-r)\mathbf{B}(a-r)dr\right\}, t \geq a,$$
$$s(a,t) = s_0(a-t)\exp\left\{-\int_0^t \Psi(t-r)\mathbf{B}(a-r)dr\right\}, t < a,$$

then we obtain $s(a-r,t-r) \geq e^{-\phi(a-r)}$ for $a \leq t$ and $\Psi_\omega(t) \geq \int_0^t \Psi_\omega(t-r) \int_r^t e^{-\phi(a-r)}(K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr$, which leads to

$$\begin{aligned}\Psi_{\omega+W}(t) =& \Psi_\omega(t+W) \\ \geq& \int_0^t \Psi_{\omega+W}(t-r) \int_r^{r+W} e^{-\phi(a-r)}(K_1(a,a-r) + K_2(a,a-r) \\ &+ K_3(a,a-r))dadr.\end{aligned} \tag{2.10}$$

Taking Laplace transforms $\widehat{\cdot}$ on the both sides of (2.10) yields $\widehat{\Psi}_{\omega+W}(\nu) \geq \widehat{\Phi}_{\omega+W}(\nu)k(\phi,\nu,W)$, where $k(\phi,\nu,W) = \int_0^{+\infty} e^{-\nu r} \int_r^{W+r} e^{-\phi(a-r)}(K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr$. In fact, by the boundedness of Φ, it follows that the Laplace transform of $\Psi_{\omega+W}$ is defined on $[0,+\infty)$. Moreover, it is obvious that $k(0,0,+\infty) = \mathcal{R}_0 > 1$ and $k(\phi,\nu,W) > 1$ when ϕ and ν are taken sufficiency small and W sufficiency large. Hence, we can verify that $\widehat{\Psi}_{s+W}(\nu) = 0$ i.e., Ψ_{s+W} is 0 a.e. on $[0,+\infty)$. Thus, we can say Ψ is eventually 0 for a.e. $t \geq r$. To completes the proof, it suffices to derive the condition such that every infective force Ψ that is eventually 0 a.e. on $[0,+\infty)$. Since $s(a-r,t-r) \geq e^{-\bar{\Psi}(a-r)}$ for $t \geq a$, where $\bar{\Psi}$ is the supermum of $\Psi(t)$, then we have

$$\Psi(t) \geq \int_0^t \Psi(t-r) \int_r^t e^{-\bar{\Psi}(a-r)}(K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr.$$

Suppose that Ψ is eventually 0, then there admits some $W > 0$ such that $\Psi(t) = 0$ for a.e. $t > W$. We take $W > 0$ such that $\int_W^{+\infty} \Psi(t)dt = 0$, then $\Psi(t) \geq \int_0^t \Psi(t-r)e^{-\bar{\Psi}W} \int_r^t (K_1(a,a-r) + K_2(a,a-r) + K_3(a,a-r))dadr, 0 < t \leq W$, which leads to $\int_W^\infty \int_0^t \Psi(t-r) \int_r^t (K_1(a,a-$

$r)+K_2(a,a-r)+K_3(a,a-r))dadrdt=0$. Further, changing the order of integration several times and recalling that $\Psi(t)=0, t\in(W,+\infty)$, then we obtain

$$\int_0^W \Psi(r)\int_{W-\tau}^{\infty}\int_0^{\tau}(K_1(a,a+s)+K_2(a,a+s)+K_3(a,a+s))dadsdr=0.$$

Choosing some W such that Ψ does not vanish a.e. on $(0,W-\phi_1)$ for $\phi_1\in(0,W)$. Thus, we have

$$\int_{\phi_1}^{\infty}\int_0^{W-\phi_1}(K_1(a,a+r)+K_2(a,a+r)+K_3(a,a+r))dadr=0. \tag{2.11}$$

Taking $\phi_1\to 0$ and changing the order of the integration in (2.11), one has

$$\begin{aligned}&\int_{\phi_1}^{\infty}\int_0^{W-\phi_1}(K_1(a,a+r)+K_2(a,a+r)+K_3(a,a+r))dadr\\ =&\int_0^W\int_b^{+\infty}(K_1(a,b)+K_2(a,b)+K_3(a,b))dadb\\ =&0.\end{aligned}$$

Recalling the expression of $K_i(a,b)$ $(i=1,2,3)$ in (2.6), we have

$$\begin{aligned}&\int_0^W\int_b^{+\infty}\alpha(a)\frac{\mathbf{B}b)}{\Phi_e(b)}\int_b^a\frac{\Phi_1(a)}{\Phi_1(\tau)}d(\Phi_e(\tau))dadb\\ &+\int_0^W\int_b^{+\infty}\beta(a)\Phi_2(a)\Phi_3(a)\frac{\mathbf{B}(b)}{\Phi_e(b)}\int_b^a\int_b^{\tau}\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau))}dadb\\ &+\int_0^W\int_b^{+\infty}\gamma(a)\Phi_l(a)\frac{\mathbf{B}(b)}{\Phi_e(b)}\int_b^a\int_b^{\tau}\int_b^{l}\frac{d(\Phi_e(l))}{\Phi_1(l)}\frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r))}\frac{d(\Phi_2(\tau))}{\Phi_l(\tau)}dadb=0.\end{aligned}$$

Setting $\Phi_z(a+s)/\Phi_z(a)=\tilde{\Phi}_z(s), z=e,1,2,3$, and changing the variables and the order of the integration which mentioned in Theorem 4.2 [17], then we obtain

$$\begin{aligned}&\int_0^W\int_b^{+\infty}\alpha(a)\frac{\mathbf{B}b)}{\Phi_e(b)}\int_b^a\frac{\Phi_1(a)}{\Phi_1(\tau)}d(\Phi_e(\tau))dadb=\int_0^W\alpha(a)\int_a^{+\infty}\mathbf{B}(b)\Gamma_1(b-a)dbda,\\ &\Gamma_1(b)=\int_0^b\tilde{\Phi}_1(b-s)d(\tilde{\Phi}_e(s)),\ \lim_{a\to 0}\int_0^{+\infty}\mathbf{B}(a+b)\Gamma_1(b)db=\int_0^{+\infty}\mathbf{B}(b)\Gamma_1(b)db,\\ &\int_0^W\int_b^{+\infty}\beta(a)\Phi_2(a)\Phi_3(a)\frac{\mathbf{B}(b)}{\Phi_e(b)}\int_b^a\int_b^{\tau}\frac{d(\Phi_e(r))}{\Phi_1(r)}\frac{d(\Phi_1(\tau))}{\Phi_2(\tau)\Phi_3(\tau))}dadb\\ =&\int_0^W\beta(a)\int_a^{+\infty}\mathbf{B}(b)\Gamma_2(b-a)dbda\\ =&\int_0^W\beta(a)\int_0^{+\infty}\mathbf{B}(b+a)\Gamma_2(b)dbda,\ \text{where}\\ &\Gamma_2(b)=\int_0^b\int_0^{\tau}\tilde{\Phi}_1(b-s)d(\tilde{\Phi}_e(s))\tilde{\Phi}_2(b-\tau)\tilde{\Phi}_3(b-\tau)d(\Phi_1(\tau)),\\ &\lim_{a\to 0}\int_0^{+\infty}\mathbf{B}(a+b)\Gamma_2(b)db=\int_0^{+\infty}\mathbf{B}(b)\Gamma_2(b)db,\end{aligned}$$

and

$$\begin{aligned}
&\int_0^W \int_b^{+\infty} \gamma(a)\Phi_l(a)\frac{\mathbf{B}(b)}{\Phi_e(b)} \int_b^a \int_b^\tau \int_b^l \frac{d(\Phi_e(l))}{\Phi_1(l)} \frac{d(\Phi_1(r))}{\Phi_2(r)\Phi_3(r))} \frac{d(\Phi_2(\tau))}{\Phi_l(\tau)} dadb \\
&= \int_0^W \gamma(a) \int_a^{+\infty} \mathbf{B}(b)\Gamma_3(b-a)dbda \\
&= \int_0^W \gamma(a) \int_0^{+\infty} \mathbf{B}(b+a)\Gamma_3(b)dbda, \text{ where} \\
&\Gamma_3(b) = \int_0^b \int_0^\tau \int_0^l \tilde{\Phi}_1(b-s)d(\tilde{\Phi}_e(s))\tilde{\Phi}_2(b-l)\tilde{\Phi}_3(b-l)d(\Phi_1(l))\tilde{\Phi}_l(b-\tau)d(\Phi_2(\tau)), \\
&\lim_{a\to 0}\int_0^{+\infty} \mathbf{B}(b+a)\Gamma_3(b)db = \int_0^{+\infty} \mathbf{B}(b)\Gamma_3(b)db.
\end{aligned}$$

Thus, we have derived the condition such that $\Phi(t)$ is eventually 0 a.e. on $[0,+\infty)$, namely

$$\begin{aligned}
&\int_0^W \alpha(a) \int_0^{+\infty} \mathbf{B}(a+b)\Gamma_1(b)dbda + \int_0^W \beta(a) \int_0^{+\infty} \mathbf{B}(a+b)\Gamma_2(b)dbda \\
&+ \int_0^W \gamma(a) \int_0^{+\infty} \mathbf{B}(a+b)\Gamma_3(b)dbda = 0.
\end{aligned} \tag{2.12}$$

According to the condition of Assumption 1.1, (2.12) is obviously impossible to hold. Hence, the result of Theorem 2.1 holds. This completes the proof. □

To prove the strong persistence of the disease, in addition to Assumption 1.1, we also need to make the following assumption.

Assumption 2.1. *We assume that the initial valus $s_0(a), e_0(a), i_{k0}(a), l_{10}(a) \in L_+^\infty(0, a_{\max})$ are extended by zero outside of $[0, a_{\max}]$, $k = 1, 2$.*

Theorem 2.2 (**Strong persistence**). *Under Assumptions 1.1 and 2.1, if $\mathcal{R}_0 > 1$ and the infective force in (2.9) $\Psi(t) \not\equiv 0$, then the disease is strongly persistent; i.e., $\Psi_\infty = \liminf_{t\to\infty} \Psi(t) > 0$.*

Proof. We proof the theorem by contradiction. Suppose that $\Psi(t) \not\equiv 0$ and $\Psi_\infty = \liminf_{t\to\infty} \Psi(t) = 0$. Since $\Psi(t)$ in (2.9) is uniformly continuous on $t \in [0,+\infty)$, we can verify that $\Psi(t)$ is not zero a.e. (almost everywhere) when $\mathcal{R}_0 > 1$. From Theorem 2.1, we know that $\Psi^\infty > \phi$. Hence, there exists a subsequence satisfying

$$\Psi(s_j) = \phi, \lim_{j\to+\infty} \Psi(t_j + s_j) \to 0, \Psi(t + s_j) \le \phi, \forall\ t \in [0, t_j], \tag{2.13}$$

where $s_j, t_j \in (0,+\infty), j \in \mathbb{N}$ and $s_j \to +\infty (j \to +\infty)$. For $t \in (-\infty,+\infty)$, we define $\Psi_j(t)$ as follows:

$$\Psi_j(t) = \begin{cases} \Psi(t + s_j), t \in [-s_j, +\infty), \\ \Psi(0), t \in (-\infty, -s_j). \end{cases} \tag{2.14}$$

From the expression of $\Psi(t)$ in (2.9), we get

$$\begin{aligned}
\Psi_j(t) = &\int_0^{t+s_j} \Psi(t-r) \int_0^{t+s_j-r} \exp\left\{-\int_0^b \Psi(t-r-s)\mathbf{B}(s)ds\right\} \big(K_1(b+r,b) \\
&+ K_2(b+r,b) + K_3(b+r,b)\big)dbdr + \sum_{m=1}^3 \mathbf{A}_m(t+s_j) \text{ for } t \in [-s_j, +\infty).
\end{aligned} \tag{2.15}$$

According to condition (3) in Assumption 1.1, Assumption 2.1 and the expression of $\mathbf{A}_m(t)(m = 1, 2, 3)$ in (2.7)-(2.8), we know that

$$\mathbf{A}_m(t) \to 0 \ (t \to +\infty). \tag{2.16}$$

Since $\Psi(t)$ is bounded, nonnegative and uniformly continuous on $t \in [0, +\infty)$, it follows that $\Psi_j(t)$ is nonnegative, uniformly bounded and equicontinuous with respect to j on $t \in (-\infty, +\infty)$ [4]. Hence, based on Arzela-Ascoli Theorem, we know that there exists a subsequence of $\{\Psi_j(t)\}$ (still denote as $\{\Psi_j(t)\}$) which is convergence, namely, there admits a nonnegative bounded and continuous function $\Psi(t)$ such that

$$\check{\Psi}(t) = \lim_{j\to+\infty} \Psi_j(t), \forall t \in (-\infty, +\infty). \tag{2.17}$$

Note that $s_j \to +\infty (j \to +\infty)$ and (2.16), we set $j \to +\infty$ in (2.15) and obtain the following result by using Dominated Convergence Theorem [1]

$$\begin{aligned}\Psi(t) = &\int_0^{+\infty} \check{\Psi}(t-r) \int_0^{+\infty} \exp\left\{-\int_0^b \check{\Psi}(t-r-s)\mathbf{B}(s)ds\right\} \big(K_1(b+r, b) \\ &+ K_2(b+r, b) + K_3(b+r, b)\big)dbdr \text{ for } t \in (-\infty, +\infty).\end{aligned} \tag{2.18}$$

Next, we discuss the property of $\{t_j\}$. If $t_j \to +\infty (j \to +\infty)$ is not true, then $\{t_j\}$ has a subsequence (still denoted as $\{t_j\}$) which satisfies $t_j \to t_0 (j \to +\infty)$, further from (2.13). Thus, since $\Psi(t)$ is uniformly continuous on $t \in [0, +\infty)$, it follows that for any fixed $\varpi > 0$, there exists a j_0 such that $\Psi(t + s_j) \le \phi + \varpi, \forall\ t \in [0, t_0]$. Further, we have $\limsup_{j\to+\infty} \Psi(t + s_j) \le \phi, \forall\ t \in [0, t_0]$. Based on (2.17) and the definition of $\Psi(t)$ in (2.9) we have

$$\check{\Psi}(t) \le \phi, \forall\ t \in [0, t_0]. \tag{2.19}$$

Combined (2.13), (2.14) with (2.17), we further obtain that

$$\check{\Psi}(0) = \lim_{j\to+\infty} \Psi(s_j) = \phi > 0,\ \check{\Psi}(t_0) = \lim_{j\to+\infty} \Psi(t_0 + s_j) = 0, t_j \to t_0. \tag{2.20}$$

Thus, we can immediately obtain that $\Psi_j(0) = \Psi(s_j) > \phi/2$ for large enough j. Since $\{\Psi_j(t)\}$ is equicontinuous on $t \in (-\infty, +\infty)$, it follows that there admits a appropriately small $T_0 > 0$ such that

$$\Psi_j(t) > \phi/4 := \phi_1, t \in [0, T_0]. \tag{2.21}$$

Note that functions $\mathbf{B}(a), \alpha(a), \beta(a)$ and $\gamma(a)$ have a positive lower bound for $a \in [0, a_{\max}]$, then we can define the domain of $\exp\left\{-\int_0^b \bar{\Psi}\mathbf{B}(s)ds\right\}(K_1(b+r, b) + K_2(b+r, b) + K_3(b+r, b))$ ($\bar{\Psi}$ represents the supermum of $\Psi(t)$) as $D := \{(b, r)|b > 0, r > 0, 0 < b + r < a_{\max}\}$ and denote the positive lower bound of D as P. Without loss of generality, we assume $T_0 < a_{\max}$, from (2.15), we have

$$\begin{aligned}\Psi_j(t) \ge \int_{t-T_0}^t \phi_1 \bigg(\int_0^{t-r} &\big(K_1(b+r, b) + K_2(b+r, b) \\ &+ K_3(b+r, b)\big) \exp\{-\int_0^b \mathbf{B}(s)\bar{\Psi}ds\}db \bigg) dr.\end{aligned}$$

Because of the inclusion of the integration region at the right end of the above equation in D, it is easy to obtain that

$$\Psi_j(t) \geq \frac{T_0^2}{2} P\phi_1, \forall\ t \in [T_0, a_{\max}]. \tag{2.22}$$

Based on (2.21) and (2.22), we have

$$\Psi_j(t) \geq \phi_2,\ \forall\ t \in [0, a_{\max}], \phi_2 = \min\{\phi_1, \frac{T_0^2}{2} P\phi_1\}, \tag{2.23}$$

which leads to $\check{\Psi}(t) = \lim_{j\to+\infty} \Psi_j(t) > 0, t \in [0, a_{\max}]$. For any fixed $t \in [a_{\max}, \frac{3}{2}a_{\max}]$, from (2.15) and (2.23), we get

$$\begin{aligned}\Psi_j(t) \geq \int_{t-a_{\max}}^{t} \phi_2\Big(\int_0^{t-r} \big(&K_1(b+r,b) + K_2(b+r,b)\\ &+ K_3(b+r,b)\big)\exp\{-\int_0^b \mathbf{B}(s)\bar{\Psi}ds\}db\Big)dr,\end{aligned}$$

it further leads to $\Psi_j(t) \geq \frac{a_{\max}^2}{8} P\phi_2, t \in [a_{\max}, \frac{3}{2}a_{\max}]$. Similarly, we can obtain $\Psi_j(t) \geq \phi_3, t \in [0, \frac{3}{2}a_{\max}], \phi_3 = \min\{\phi_2, \frac{a_{\max}^2}{8} P\phi_2\}$, which implies that $\check{\Psi}(t) = \lim_{j\to+\infty} \Psi_j(t) > 0, t \in [0, \frac{3}{2}a_{\max}]$. By analogy, using the same method, the range that satisfies $\check{\Psi}(t) > 0$ can be increased by $a_{\max}/2$ each time. Thus, we have

$$\check{\Psi}(t) = \lim_{j\to+\infty} \Psi_j(t) > 0, t \in [0, +\infty), \tag{2.24}$$

which leads a contradictions with (2.20). Hence, we can assume $t_j \to +\infty (j \to +\infty)$. Consequently, for any fixed $t^* > 0$, there admits a j_0 such that $t_j > t_0$ when $j > j_0$. Based on $\Psi(t + s_j) \leq \phi, t \in [0, t^*]$ in (2.13), it is easy to obtain that

$$\limsup_{j\to+\infty} \Psi(t + s_j) \leq \phi, t \in [0, +\infty). \tag{2.25}$$

Combined (2.13), (2.14) with (2.17), we immediately have $\check{\Psi}(0) = \phi > 0$. Using (2.14) and (2.25) again, one has

$$\check{\Psi}(t) \leq \phi, t \in [0, +\infty). \tag{2.26}$$

Following the same proof process as starting from (2.20) to obtain (2.24), the following inequality can be obtained from $\check{\Psi}(0) = \phi > 0$

$$\check{\Psi} = \lim_{j\to+\infty} > 0, t \in [0, +\infty). \tag{2.27}$$

Thus, according to (2.18), (2.27) and using the contradiction method again, we can obtain that $\limsup_{t\to+\infty} \check{\Psi}(t) > \phi$, which leads to a contradictions with (2.26). Hence, the assumption about $\Psi_\infty = \liminf_{t\to\infty} \Psi(t) = 0$ is not true. In other words, $\Psi_\infty = \liminf_{t\to\infty} \Psi(t) > 0$ when $\mathcal{R}_0 > 1$, i.e., the disease is strongly persistent. This completes the proof. □

3. Conclusion

In this paper, we considered four infection stages of syphilis and constructed an age-structured compartmental model. Subsequently, by following the approach outlined in [24, 26, 27, 31], we derived the explicit expression for the basic reproduction number $\mathcal{R}_0$.

Compared to previous studies on classical age-structured epidemic models [5, 10, 31], we further discussed uniform weak persistence and strong persistence of the disease following [4, 17]. The difficulty lies in the fact that age-structured models are described by first-order hyperbolic partial differential equations, which cannot be studied directly by employing the persistence theory mentioned in the classical references such as [28]. Specifically, constructing comparison systems under the assumption of dissipation and subsequently deriving persistence conditions via the comparison principle for ordinary differential equations (ODEs) or parabolic equations is widely applicable to biological and epidemic models described by ODEs and reaction-diffusion systems. However this method is not feasible for age-structured epidemic models. Therefore, proving persistence for age-structured epidemic models necessitates starting from the fundamental definitions of persistence and employing techniques such as multiple integral reordering, change of variables, Laplace transform methods, and combined with contradiction method, as illustrated in the proof of Theorems 2.1 and 2.2. The approach in establishing persistence demonstrated in this paper can be extended to other age-structured models in biology and medicine. This is also one of the focuses of our future work.

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability statement. No data associated in the manuscript.

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Received March 2025; Accepted August 2025; Available online August 2025.