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SEASONALITY OF RICKER MUTUALISM MODEL EMPLOYING RATE INDICATOR WITH ALMOST PERIODIC FUNCTION

Seda Igret Araz[1,†]

Abstract In this paper, we implement a rate indicator function featuring an almost periodic component to capture seasonal trends within the Ricker model, which represents a mutualistic framework. For the updated model incorporating this function, we derived parameters that guarantee the uniqueness of the solution and confirmed that the initial conditions do not affect the solution. We anticipate that this study will provide valuable insights into the critical role of the rate indicator function and the influence of almost periodicity in analyzing seasonal trends.

Keywords Almost periodicity, ecological model, seasonality, rate indicator.

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1. Introduction and preliminaries

The concept of rate of change is fundamental in both mathematics and its real-world applications, providing a precise means to describe how one quantity changes in relation to another. Essentially, the rate of change measures the speed or pace at which one variable changes with respect to another, offering valuable insights into dynamic systems across diverse fields, including physics, economics, biology, and engineering. Mathematically, the rate of change is defined as the ratio of the variation observed in one variable relative to that of another. This formulation enables the analysis of how systems evolve over time or in response to other influencing factors. It is particularly useful in modeling relationships that are not constant, as it allows for the examination of both linear and non-linear dynamics. For example, in medicine, healthcare providers assess the variations in a patient's vital signs, including heart rate and blood pressure, to uncover any health concerns. In environmental science, the trend of carbon dioxide levels in the atmosphere over time can reflect changes in the climate. In economics, the rate of change of price with respect to quantity is known as price elasticity. Understanding the rate of change is crucial for analyzing and modeling dynamic systems in the natural world and beyond. Whether we're studying the motion of objects, tracking the growth of populations, or investigating economic trends, the ability to measure and interpret the rate at which things change allows us to make informed predictions and decisions. By using mathematical tools such as the derivative, we can capture the instantaneous rate of change, which gives us deeper insights into how systems evolve at every point in time. The rate of change is not just a mathematical concept—it's a lens through which we can understand the world around us. The function that defines the rate of change, can be called as the rate indicator function, is crucial for accurately representing the real-world dynamics of the model in which it is incorporated. However, differential equations

[†]The corresponding author.
[1]Department of Mathematics Education, Siirt University, 56100 Siirt, Turkey
Email: sedaaraz@siirt.edu.tr(S. Igret Araz)

serve as mathematical representations of rate of change and appear as part of real-world problems. Nonetheless, specific modifications can be incorporated into these comprehensive models based on potential of real-world problems. For instance, periodic or almost periodic functions [6] can be used as parameters of models to capture seasonality behavior in models [1]. A model with almost periodic behavior can still exhibit seasonality, but it would do so with slight irregularities in the timing or magnitude of its cycles. Seasonality in this context becomes more flexible, acknowledging that while patterns are still recurring and predictable in nature, their exact timing or intensity may vary by small amounts. To handle this, time series models that can accommodate variations in periods (e.g., Fourier transforms, stochastic seasonal models, or flexible decomposition methods) are typically employed. In [1], an almost periodic Ross–Macdonald model with age structure for the vector population in a patchy environment is examined. Numerical simulations reveal that the biting rate significantly impacts disease transmission, and human migration may occasionally reduce the transmission risk. A condition is also numerically determined to assess whether a control strategy regarding migration is necessary. Furthermore, the results indicate that extending the vector's maturation period is advantageous for disease control, and the threshold length of the maturation period required for disease outbreak can be computed. Finally, a comparison between the almost periodic and periodic models suggests that the periodic model may either overestimate or underestimate the disease transmission risk. In [11], a periodic model that incorporates vaccinations is introduced, and the existence of a disease-free periodic solution is demonstrated. The basic reproduction number for the model is derived, and it is shown that, depending on its value, either the disease-free periodic solution will be globally stable, or the disease will persist in the population. The numerical simulations for both scenarios are provided, supporting the theoretical results. Monthly measles data from Pakistan, covering the period from January 2019 to December 2021, are fitted to estimate unknown parameters and determine the basic reproduction number. Numerical evaluations will then be conducted to assess the impact of the seasonal contact rate, increased vaccination coverage, vaccine efficacy, and other key parameters on the transmission dynamics of measles. In [10], given that monthly measles case data exhibit seasonal fluctuations, a susceptible-exposed-infectious-recovered (SEIR) model with a periodic transmission rate, based on the model by Earn et al., is proposed to study the seasonal dynamics of measles epidemics and the impact of vaccination. The basic reproduction number is calculated, the model's dynamical behavior is analyzed, and the model is used to simulate the monthly measles case data reported in China. Sensitivity analysis of the basic reproduction number with respect to various model parameters shows that measles can be controlled and eventually eradicated by increasing the immunization rate, improving vaccine management, and raising public awareness of measles. In [17], the persistence of a class of seasonally forced epidemiological models is examined. An abstract theorem on persistence by Fonda is applied, and five different application examples are provided. In [14], a general almost periodic model is proposed to describe a mutualistic interaction between two seasonal species, where climate-mediated shifts alter their population dynamics. In the modeling process, almost periodic functions [20] are used as parameters to capture the loss of synchronicity in the population dynamics of the mutualistic species [19]. Additionally, it is considered that the benefit each species gains from interacting with its partner is modeled by a family of increasing bounded functions, which reflect the fact that the maximum benefit for each species is attained when the partner species reaches high abundance. It is proven that a unique, almost periodic, globally stable solution exists for the model when certain conditions on the model parameters are satisfied. Numerical simulations of the model's solutions reveal significant differences between results obtained within the periodic [4] and almost periodic frameworks [2]. Therefore, in this study, this rate indicator function is specifically analyzed as an almost periodic function, with the goal of exploring the seasonality of mutualistic interactions since. To achieve this, the rate function is integrated into the Ricker's mutualism model [12,13,16,18], and the conditions for a single almost periodic solution are presented.

Now, we shall present some definitions that will be used in this study.

Definition 1.1. Let the function $\gamma \in C(\mathbb{R})$ be almost periodic. We assume that there exists a number $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a number T. For each $\varepsilon > 0$ and $t \in \mathbb{R}$, if the following inequality

$$|\gamma(t+T) - \gamma(t)| < \varepsilon \tag{1.1}$$

holds, then the function γ is called an almost periodic function. The above collection of all almost periodic functions will be denoted by $AP(\mathbb{R})$ which is a Banach space endowed with the usual sup-norm. An almost periodic function has a mean value which is formulated by [6]

$$M[f] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt.$$

$M : AP(\mathbb{R}) \to \mathbb{R}$ is a bounded linear functional with the property that $f \geq 0$ implies $M[f] \geq 0$.

Example 1.1. A well-known example of almost periodic functions is $\cos(t) + \cos\left(\sqrt{2}t\right)$. The graphical representation of this function is simulated in Figure 1.

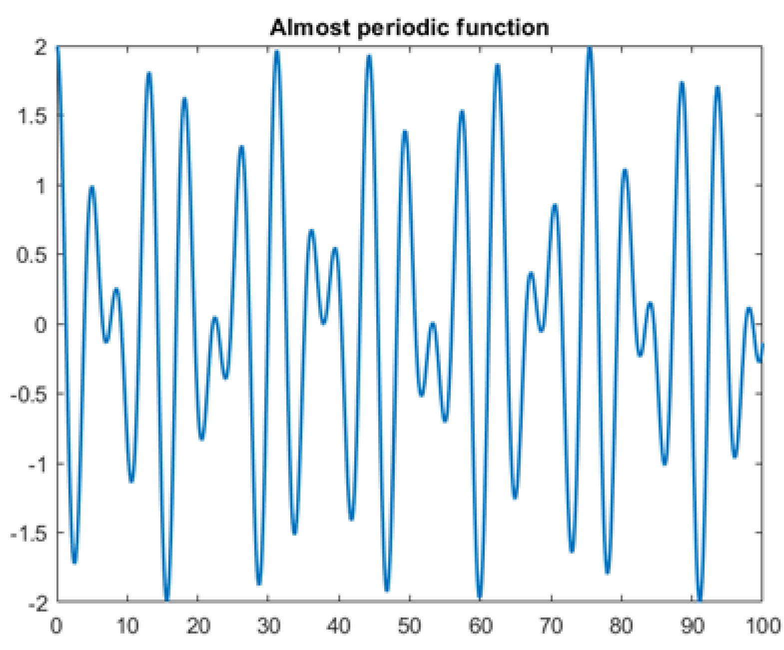

Figure 1. Graphical representation of the almost periodic function presented in Example 1.1.

Definition 1.2. [9] Let $f, g : \mathbb{R} \times D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a couple of differentiable and almost periodic function on t. Assuming that $f(t,u,v)$ and $g(t,u,v)$ are both uniformly almost periodic with respect to $(u,v) \in C$ for every compact $C \subseteq D$. We consider the following system:

$$\begin{aligned} u'(t) &= f(t,u,v), \\ v'(t) &= g(t,u,v). \end{aligned} \tag{1.2}$$

If for all $t \in \mathbb{R}, u \in (x(t), X(t)), v \in (y(t), Y(t))$

$$\begin{aligned} f_v(t,u,v) &\geq 0, \\ g_u(t,u,v) &\geq 0, \end{aligned} \tag{1.3}$$

above system is called as cooperative. If

$$\begin{aligned} f_v(t,u,v) &\leq 0, \\ g_u(t,u,v) &\leq 0, \end{aligned} \tag{1.4}$$

above system is called as competitive. We will say that $(x(t), y(t))$ are a sub-solution pair if

$$\begin{aligned} u'(t) &\leq f(t,u(t),v(t)), \\ v'(t) &\leq g(t,u(t),v(t)), \end{aligned} \tag{1.5}$$

for every $t \in \mathbb{R}$. A super-solution $(X(t), Y(t))$ is defined similarly with the reversing inequalities. We will say that a sub-solution $(x(t), y(t))$ and a super-solution $(X(t), Y(t))$ are ordered if $x(t) \leq X(t)$ and $y(t) \leq Y(t)$ for all $t \in \mathbb{R}$.

Theorem 1.1. *Consider an ordered pair of a sub-solution pair $(x(t), y(t))$ and a super-solution pair $(X(t), Y(t))$ of the above system such that $x(t) < X(t)$, and $y(t) < Y(t)$. Suppose that there is no equilibrium point (u_0, v_0) such that $x(t) \leq u_0 \leq X(t)$ and $y(t) \leq v_0 \leq Y(t)$. If the system is cooperative type, then it has an almost periodic solution satisfying $x(t) \leq u(t) \leq X(t)$ and $y(t) \leq v(t) \leq Y(t)$ for all $t \in \mathbb{R}$. Furthermore, if $(\underline{u}(t), \underline{v}(t)), (\overline{u}(t), \overline{v}(t))$ denote the minimal and maximal almost periodic solutions having initial data satisfying $x(0) \leq u(0) \leq X(0)$ and $y(0) \leq v(0) \leq Y(0)$. Then, any solution of system, converges to the product of strips $(\underline{u}(t), \overline{u}(t)) \times (\underline{v}(t), \overline{v}(t))$.*

2. Model formulation of mutualism interactions

In ecological modeling, understanding how two species interact is central to predicting population dynamics. While competition and predation have been well-studied, mutualism—a biological interaction where both species benefit—presents unique modeling challenges [21]. Classic population models like the Lotka-Volterra equations often oversimplify mutualistic interactions, sometimes predicting unrealistic, unbounded growth. The Ricker mutualism model arises from the need to: incorporate density-dependent regulation, which prevents unrealistic population explosions; account for mutualistic benefits in a nonlinear, saturating, or diminishing way; and allow for more realistic discrete-time dynamics, especially suitable for species with seasonal reproduction [12, 13, 16, 18]. Such a model is governed by the following system of differential equations

$$P_1'(t) = \rho_1 \left(1 - \frac{P_1}{K_1}\right) P_1 + \kappa_1 P_2 P_1, P_1(0) = P_1^0 \geq 0, \tag{2.1}$$

$$P_2'(t) = \rho_2 \left(1 - \frac{P_2}{K_2}\right) P_2 + \kappa_2 P_1 P_2, P_2(0) = P_2^0 \geq 0,$$

where ρ_1, $\rho_2 > 0$ are respectively the intrinsic growth rates of species P_1 and P_2, K_1, $K_2 > 0$ represent respectively the carrying capacity of P_1 and P_2, $\kappa_1 > 0$ represents mutualistic effect of species P_2 on species P_1 and $\kappa_2 > 0$ represents mutualistic effect of species P_1 on species P_2. We will now demonstrate that the assumptions required for the next section are satisfied for the model under consideration and depict the numerical simulation for original model.

Positivity of solutions. We shall check the positivity of the solutions of the model, provided that the initial data are positive.

$$P_1'(t) = \rho_1 \left(1 - \frac{P_1}{K_1}\right) P_1 + \kappa_1 P_2 P_1 \Rightarrow P_1'(t) \geq \rho_1 P_1 - \rho_1 \frac{P_1^2}{K_1} \tag{2.2}$$
$$\Rightarrow P_1(t) \geq \frac{K_1 P_1^0 \exp(\rho_1 t)}{P_1^0 (\exp(\rho_1 t) - 1) + K_1},$$
$$P_2'(t) = \rho_2 \left(1 - \frac{P_2}{K_2}\right) P_2 + \kappa_2 P_1 P_2 \Rightarrow P_2'(t) \geq \rho_2 P_2 - \rho_2 \frac{P_2^2}{K_2} \tag{2.3}$$
$$\Rightarrow P_2(t) \geq \frac{K_2 P_2^0 \exp(\rho_2 t)}{P_2^0 (\exp(\rho_2 t) - 1) + K_2}.$$

We can conclude that the solutions of the system are positive when $P_1^0(\exp(\rho_1 t) - 1) + K_1 > 0$ and $P_2^0(\exp(\rho_2 t) - 1) + K_2 > 0$.

The cooperative system. To examine whether the system is cooperative, we will use definition presented earlier. The above model is cooperative since

$$\frac{\partial}{\partial P_2} \gamma_1(t, P_1, P_2) = \frac{\partial}{\partial P_2} \left(\rho_1 \left(1 - \frac{P_1}{K_1}\right) P_1 + \kappa_1(t) P_2 P_1\right) = \kappa_1 P_1 \geq 0, \tag{2.4}$$
$$\frac{\partial}{\partial P_1} \gamma_2(t, P_1, P_2) = \frac{\partial}{\partial P_1} \left(\rho_2 \left(1 - \frac{P_2}{K_2}\right) P_2 + \kappa_2(t) P_1 P_2\right) = \kappa_2 P_2 \geq 0.$$

Midpoint method for solution of the model. We will simplify above system as follows

$$\gamma_1(t, P_1(t), P_2(t)) = P_1 \left(\rho_1 \left(1 - \frac{P_1}{K_1}\right) + \kappa_1 P_2\right), \tag{2.5}$$
$$\gamma_2(t, P_1(t), P_2(t)) = P_2 \left(\rho_2 \left(1 - \frac{P_2}{K_2}\right) + \kappa_2 P_1\right).$$

To solve this model, we employ the midpoint method [8] by applying the associated integral on above equation and evaluation at the points at $t = t_{n+1}$ and $t = t_n$. Therefore, we have the following

$$P_1(t_{n+1}) = P_1(t_n) + \int_{t_n}^{t_{n+1}} \gamma_1(\tau, P_1(\tau), P_2(\tau)) d\tau, \tag{2.6}$$
$$P_2(t_{n+1}) = P_2(t_n) + \int_{t_n}^{t_{n+1}} \gamma_2(\tau, P_1(\tau), P_2(\tau)) d\tau.$$

Employing the midpoint method [3] yields

$$P_1^{n+1} = P_1^n + h \left[\gamma_1 \left(t_n + \frac{h}{2}, P_1\left(t_n + \frac{h}{2}\right), P_2\left(t_n + \frac{h}{2}\right)\right)\right], \tag{2.7}$$

$$P_2^{n+1} = P_2^n + h\left[\gamma_2\left(t_n + \frac{h}{2}, P_1\left(t_n + \frac{h}{2}\right), P_2\left(t_n + \frac{h}{2}\right)\right)\right].$$

For the above model, the following parameters and initial data are considered

$$P_1(0) = 2; P_1(0) = 1; \rho_1 = 0.1; \rho_2 = 0.9; K_1 = 20; K_2 = 20; \kappa_1 = 0.009; \kappa_2 = 0.008272. \quad (2.8)$$

In Figure 2, the numerical simulation of Ricker's mutualism model is shown.

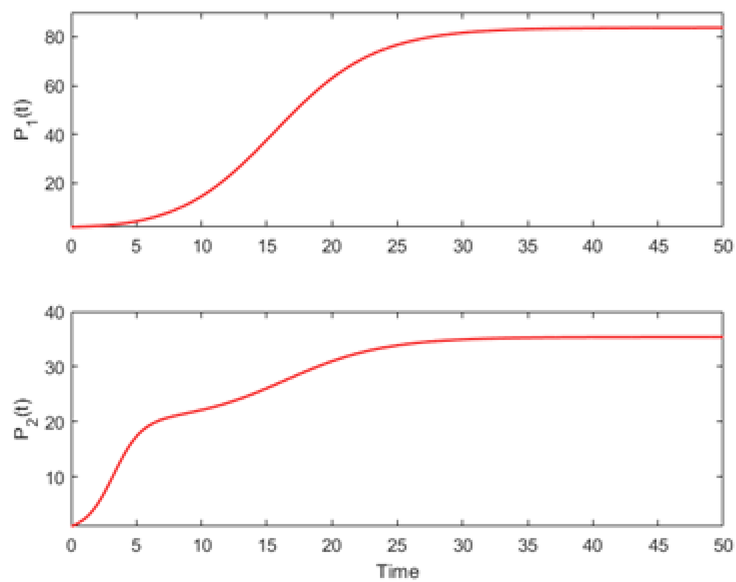

Figure 2. Simulation showing the dynamics of the Ricker mutualism model in the absence of the almost periodic function.

3. Ricker's mutualism model with rate indicator function

In this section, we will incorporate the rate indicator functions [5, 7] into Ricker's mutualism model [2, 10–12] to control the process as needed. The mutualism model under investigation is given by the following system:

$$\begin{aligned} P_1'(t) &= \rho_1\left(1 - \frac{P_1}{K_1}\right)P_1 + \kappa_1 P_2 P_1, \quad P_1(0) = P_1^0 \geq 0, \\ P_2'(t) &= \rho_2\left(1 - \frac{P_2}{K_2}\right)P_2 + \kappa_2 P_1 P_2, \quad P_2(0) = P_2^0 \geq 0. \end{aligned} \quad (3.1)$$

To perform our aim, we will replace the constants κ_1 and κ_2 with $\kappa_1(t)$ and $\kappa_2(t)$. Here, $\kappa_1(t)$ and $\kappa_2(t)$ may be the functions of time that are obtained using experimental data after the times $t_1, t_2, ..., t_n$, which can be considered here as day, week, month depending on data. In this case, we can take such function as follows:

$$\kappa_1(t) = \kappa_1^i + \frac{\kappa_1^{i+1} - \kappa_1^i}{t_{i+1} - t_i}(t - t_i), \quad (3.2)$$

where the term $\frac{\kappa_1^{i+1}-\kappa_1^i}{t_{i+1}-t_i}$ is called as rate. Also, κ_1^{i+1} and κ_1^i are the observed data at times t_{i+1} and t_i, respectively. Similarly, we have

$$\kappa_2(t) = \kappa_2^i + \frac{\kappa_2^{i+1} - \kappa_2^i}{t_{i+1} - t_i}(t - t_i), \tag{3.3}$$

where κ_2^{i+1} and κ_2^i are the observed data at times t_{i+1} and t_i, respectively. Moreover, the rate indicator function can be chosen as any regression model, piecewise interpolation polynomial, periodic, almost periodic, quasi periodic under suitable conditions. Using the idea of rate indicator functions [5,7], our model is modified as follows:

$$\begin{aligned} P_1'(t) &= \rho_1\left(1 - \frac{P_1}{K_1}\right)P_1 + \kappa_1(t)P_2P_1, \quad P_1(0) = P_1^0 \geq 0, \\ P_2'(t) &= \rho_2\left(1 - \frac{P_2}{K_2}\right)P_2 + \kappa_2(t)P_1P_2, \quad P_2(0) = P_2^0 \geq 0. \end{aligned} \tag{3.4}$$

Specifically, it is worth noting that in this study, we will choose the rate indicator function as an almost periodic function.

4. Existence and uniqueness of solution of mutualism model incorporating the rate indicator function with an almost periodic function

In this section, we examine the existence and uniqueness of solutions to the mutualism model incorporating rate indicator functions characterized by almost periodic functions. Furthermore, we perform numerical simulations of the modified model under varying initial conditions. To achieve our aim, we recall our modified model:

$$\begin{aligned} P_1'(t) &= \rho_1\left(1 - \frac{P_1}{K_1}\right)P_1 + \kappa_1(t)P_2P_1, \quad P_1(0) = P_1^0 \geq 0, \\ P_2'(t) &= \rho_2\left(1 - \frac{P_2}{K_2}\right)P_2 + \kappa_2(t)P_1P_2, \quad P_2(0) = P_2^0 \geq 0. \end{aligned} \tag{4.1}$$

For simplicity, we design above system as follows:

$$\begin{aligned} P_1'(t) &= P_1\left(\rho_1 + \gamma_1(t, P_2) - \rho_1\frac{P_1}{K_1}\right), \quad P_1(0) = P_1^0 \geq 0, \\ P_2'(t) &= P_2\left(\rho_2 + \gamma_2(t, P_1) - \rho_2\frac{P_2}{K_2}\right), \quad P_2(0) = P_2^0 \geq 0 \end{aligned} \tag{4.2}$$

where

$$\gamma_1(t, P_2) = \kappa_1(t)P_2, \gamma_2(t, P_1) = \kappa_2(t)P_1.$$

We suppose that the right side of above system is continuous t and C^1 with respect to P_i, $i = 1, 2$. We should also put the following conditions on the functions γ_i, $j = 1, 2$ to be satisfied.

C1) We assume that

$$\gamma_i(t, 0) = 0, \gamma_i(t, P_j) \geq 0, \text{ and } \frac{\partial\gamma_i(t, P_j)}{\partial P_j} \geq 0, \forall t \in \mathbb{R}, P_j \geq 0. \tag{4.3}$$

C2) (Uniqueness of solutions) For some $k_i > 0$,

$$|\gamma_i(t, P_2) - \gamma_i(t, P_1)| \leq k_i |P_2 - P_1|. \tag{4.4}$$

C3) (Boundedness) There is $K > 0$ such that

$$0 \leq |\gamma_i(t, P_j)| \leq K, \text{ for all } P_j \geq 0, \forall t \in \mathbb{R}. \tag{4.5}$$

Given an almost periodic function $u : \mathbb{R} \to \mathbb{R}$, we denote

$$u_* = \inf_{t\in\mathbb{R}} u(t), u^* = \sup_{t\in\mathbb{R}} u(t). \tag{4.6}$$

Theorem 4.1. ([14,15]) *Suppose that $\gamma_i(t, P_j) \geq 0$ and $\kappa_i \geq 0$ are continuous almost periodic functions and that at least one of them is not a constant function. Assume that with $\kappa_i > 0$ and that considered system does not admit equilibrium points with positive coordinates. Suppose that the conditions (C1)-(C3) are satisfied. Then the following statements are valid:*

i) *There exists at least one almost periodic solution $(P_1(t), P_2(t))$ of above system whose components are positive, $P_i(t) > 0, i = 1, 2$.*

ii) *If $k_1 k_2 < \rho_{1*}\rho_{2*}$, there exists a unique almost periodic solution $(P_1^*(t), P_2^*(t))$ in $\mathbb{R}^2_{>0}$. Any solution $(P_1(t), P_2(t))$ of above system with positive initial conditions $P_i(0) > 0, i = 1, 2$, converges asymptotically to $(P_1^*(t), P_2^*(t))$ when $t \to \infty$. Thus*

$$\lim_{t\to\infty} \|(P_1^*(t), P_2^*(t)) - (P_1(t), P_2(t))\| = 0. \tag{4.7}$$

Proof. We shall first prove i). By (C1), we know that the system cooperative. Then, we can consider sub- and super-solution pairs. We suggest for a super-solution pair

$$(X(t), Y(t)) = (P, P), \quad P > 0.$$

We assume that ρ_1 is bounded on a closed interval and $\gamma_1(t, P)$ bounded for all $t \in \mathbb{R}$, $N \geq 0$ since $\gamma_1(t, P)$ is a bounded function by (C3). Then, for P big enough, we can have

$$\begin{aligned} X'(t) = 0 \Rightarrow X'(t) \geq P\left(\rho_1 + \gamma_1(t, P) - \rho_1 \frac{P}{K_1}\right), \quad P_1(0) = P_1^0 \geq 0, \\ Y'(t) = 0 \Rightarrow Y'(t) \geq P\left(\rho_2 + \gamma_2(t, P) - \rho_2 \frac{P}{K_2}\right), \quad P_2(0) = P_2^0 \geq 0. \end{aligned} \tag{4.8}$$

Then, we get a super-solution pair. Similarly, we propose for a sub-solution pair

$$(x(t), y(t)) = (Q, Q), \quad Q > 0.$$

We claim that these constant functions satisfy the inequalities in (C2) and (C3) when $Q > 0$ small enough and by applying A1). Indeed, since the right hand side in the following inequalities is positive for $Q > 0$ small enough,

$$\begin{aligned} x'(t) = 0 \Rightarrow x'(t) \leq Q\left(\rho_1 + \gamma_1(t, Q) - \rho_1 \frac{Q}{K_1}\right), \\ y'(t) = 0 \Rightarrow y'(t) \leq Q\left(\rho_2 + \gamma_2(t, Q) - \rho_2 \frac{Q}{K_2}\right). \end{aligned} \tag{4.9}$$

Then, we have a sub-solution pair. Therefore, by Theorem 1.1 there exists at least one almost periodic solution for considered system [14, 15].

For uniqueness, we need to show that $\overline{N}_1(t) = \underline{N}_1(t)$ and $\overline{N}_2(t) = \underline{N}_2(t)$ for a maximal pair $\left(\overline{N}_1, \overline{N}_2\right)$ and minimal pair $\left(\underline{N}_1, \underline{N}_2\right)$ of almost periodic solutions. To achieve our aim, we consider the following claim:

$$\overline{N}(t) \geq \underline{N}(t) \geq 0, M\left[\overline{N}\right] = M\left[\underline{N}\right]$$

for almost periodic functions $\overline{N}$ and $\underline{N}$. Then, $\overline{N}(t) = \underline{N}(t)$ for every $t \in \mathbb{R}$. The proof of above is presented in [14].

For proof, we consider $M\left[\left(\ln \overline{N}_i\right)'\right] = 0$, then

$$\begin{aligned} M[\rho_i] &= -M\left[\gamma_i\left(t, \overline{N}_j\right)\right] + M\left[\rho_i \overline{N}_i\right], \\ M[\rho_i] &= -M\left[\gamma_i\left(t, \underline{N}_j\right)\right] + M\left[\rho_i \underline{N}_i\right], \end{aligned}$$

such that $i \neq j$. Substracting these two equalities and employing (C2), we have

$$\begin{aligned} M\left[\rho_i\left(\overline{N}_i - \underline{N}_i\right)\right] &= M\left[\gamma_i\left(t, \overline{N}_j\right) - \gamma_i\left(t, \underline{N}_j\right)\right] \\ &\leq k_i M\left[\overline{N}_j - \underline{N}_j\right]. \end{aligned}$$

By (C2), we have

$$0 \leq q_{1*} M\left[\overline{N}_1 - \underline{N}_1\right] \leq k_1 M\left[\overline{N}_2 - \underline{N}_2\right] \leq \frac{k_1 k_2}{q_{2*}} M\left[\overline{N}_1 - \underline{N}_1\right].$$

If $M\left[\overline{N}_1 - \underline{N}_1\right] > 0$, then a contradiction arises in ii). Therefore $M\left[\overline{N}_1\right] = M\left[\underline{N}_1\right]$, which implies $\overline{N}_1 = \underline{N}_1$ by the claim above. From the above inequality we have $\overline{N}_2 = \underline{N}_2$. According to Theorem 4.1, the interaction between species that benefit from each other during different seasons causes population densities of these mutualistic species to experience sustained oscillations that are almost periodic. To depict this scenario, we perform the numerical simulations for Ricker's mutualism model with rate indicator with an almost periodic function in Figures 3 and 4. □

Proposition 4.1. ([14, 15]) *Suppose that $q_i \geq 0$ and $\kappa_i \geq 0$ are continuous almost periodic functions and that at least one of them is not a constant function. Assume that with $\kappa_i > 0$ and that considered system does not admit equilibrium points with positive coordinates. Suppose that the conditions (C1)-(C3) are satisfied. Then, the following statements is valid:*

i) *There exists at least one almost periodic solution $(P_1(t), P_2(t))$ of above system whose components are positive, $P_i(t) > 0, i = 1, 2$.*

ii) *If $\kappa_1^* \kappa_2^* < \rho_{1*}\rho_{2*}$, there exists a unique almost periodic solution $(P_1^*(t), P_2^*(t))$ in $\mathbb{R}^2_{>0}$, which attracts any other positive solution of the considered system when $t \to \infty$.*

Proof. We showed that (C1) and (C2) are satisfied. Applying that $\kappa_i \geq 0$ then (C3) is valid such that $k_1 := \kappa_1, k_2 := \kappa_2$, so the conditions of Theorem 1.1 are satisfied. Therefore the result follows from Theorem 4.1 [14].

We will perform the numerical simulations employing different initial conditions such as $(2.6, 1.6)$, $(2.3, 1.3)$, $(2, 1)$, $(1.7, 0.7)$ and $(1.4, 0.4)$ and the parameters

$$\rho_1 = 0.1; \rho_2 = 0.9. \tag{4.10}$$

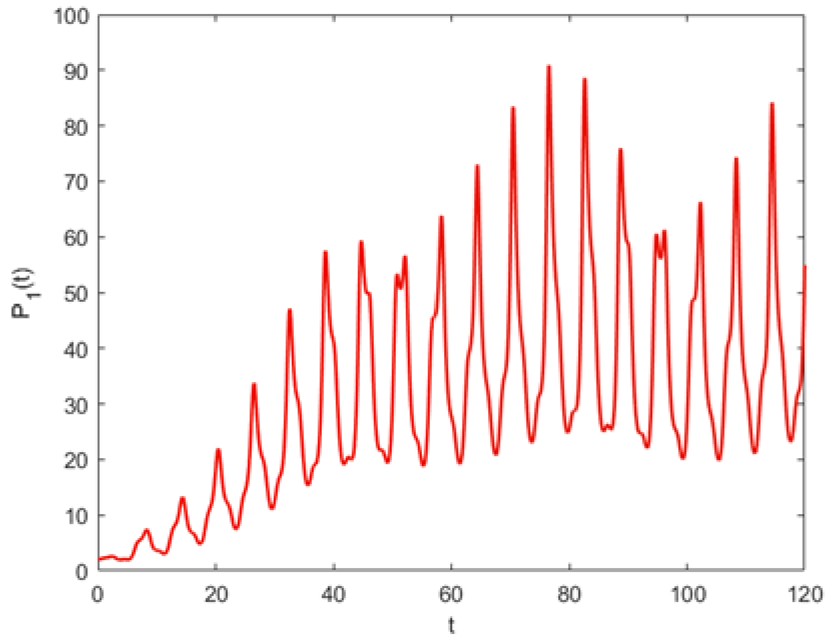

Figure 3. Graphical representation of the dynamics of the function $P_1(t)$ in Ricker mutualism model, highlighting the influence of the rate indicator function.

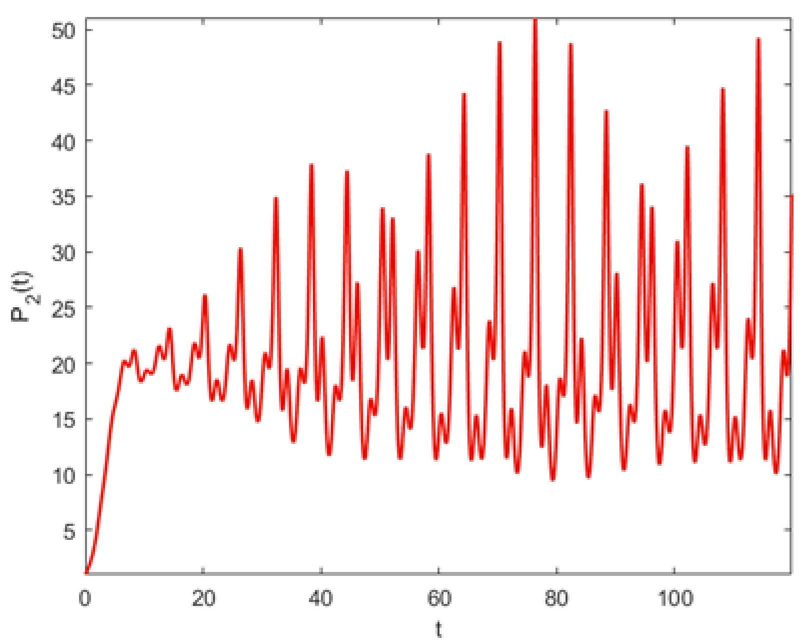

Figure 4. Graphical representation of the dynamics of the function $P_2(t)$ in Ricker mutualism model, highlighting the influence of the rate indicator function.

During simulations, our almost periodic functions were chosen as $\kappa_1(t) = 0.09(2\cos(\pi t) + 3\cos(t))$ and $\kappa_2(t) = 0.08272(3\cos(\pi t) + 2\cos(t))$. With the values of the parameters $\kappa_1^*\kappa_2^* =$

0.00061 and $\rho_{1*}\rho_{2*} = 0.09$, the conditions $\kappa_1^*\kappa_2^* < \rho_{1*}\rho_{2*}$ given in Proposition 4.1 are satisfied. Therefore, all solutions of the considered model converge to a single global attractor almost periodic orbit in the set $\mathbb{R}^2_{>0}$.

In Figures 5 and 6, we present the simulations employing different initial conditions such as $(50,44)$, $(47,38)$, $(32,32)$, $(24,17)$, $(12,7)$, $(5,2)$, $(2,1)$.

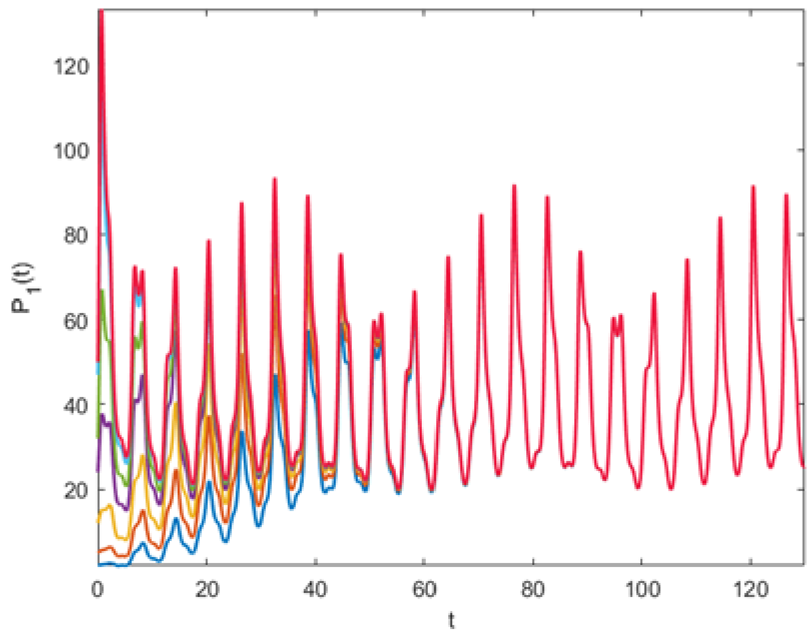

Figure 5. Simulation of the function $P_1(t)$ in the Ricker mutualism model with the rate indicator function, under different initial conditions.

From Figures 5-6, it can be seen that, under certain parameter conditions of the model, the solutions tend to converge to a unique global attractor, as demonstrated by the findings of Theorem 4.1. □

5. Qualitative analysis of Ricker's mutualism model with rate indicator function

Qualitative analysis focuses on understanding the overall behavior of a system, independent of numerical solutions. Particularly for complex or difficult-to-solve systems, this approach is a valuable tool for understanding the system's fundamental characteristics and long-term behavior.

Lyapunov exponents. The Ricker's mutualism model with rate indicator function is integrated using the midpoint method which is well-suited for solving ordinary differential equations (ODEs) and provides a good balance between accuracy and computational efficiency. Lyapunov exponent is calculated by comparing the trajectories of the original system and a perturbed version. For each time step, the logarithm of the absolute differences between the populations of the original and perturbed systems is computed. The average growth rate of these differences over time gives the Lyapunov exponent. This is done in a loop over the time steps. The calculation of the Lyapunov exponent involves numerical differentiation (using the differences between populations

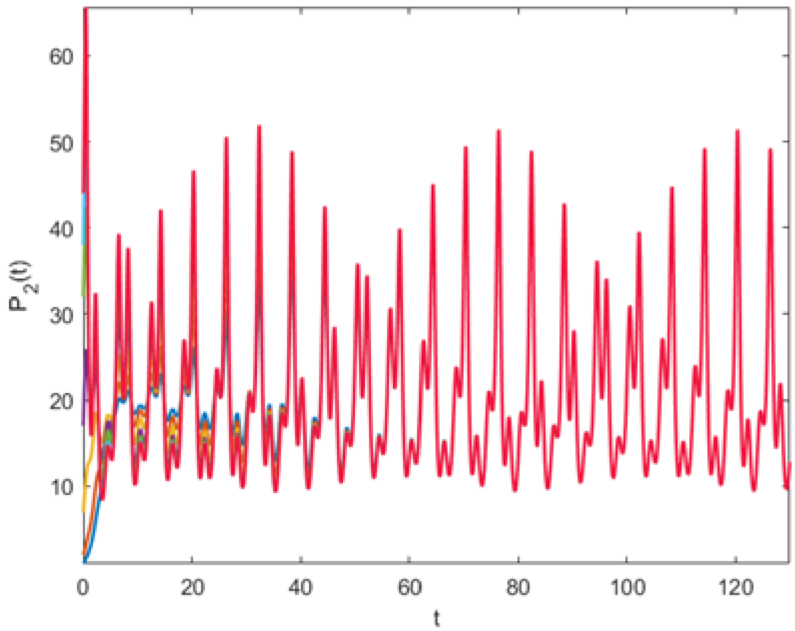

Figure 6. Simulation of the function $P_2\,(t)$ in the Ricker mutualism model with the rate indicator function, under different initial conditions.

at consecutive time steps) to estimate the exponent. In Figure 5, we present the state variables of the associated model and the Lyapunov exponents of Ricker's mutualism model with rate indicator function. The Lyapunov exponents for the considered model are depicted in Figure 7.

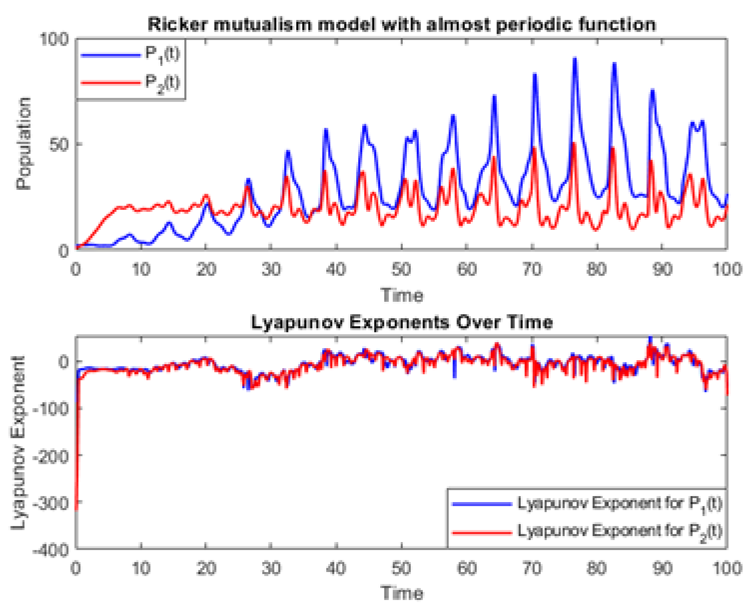

Figure 7. Lyapunov exponents of Ricker's mutualism model with rate indicator function.

Since both Lyapunov exponents are negative in a dynamical system, it indicates that the system is stable, and nearby trajectories converge towards each other over time. The system has a stable fixed point or periodic orbit where trajectories tend to settle. Perturbations from this point will decay, meaning any small deviations will diminish over time. The system behaves like an attractor. As time progresses, trajectories that start near each other will come together, indicating a strong tendency for the system to return to a certain state. In such a stable environment, the system's future states can be predicted with greater accuracy since small changes do not lead to drastically different outcomes. An examination of Figure 7 reveals that while one of the Lyapunov exponents remains negative, the other gradually approaches zero, especially in the later iterations. Although the existence of two negative exponents would typically imply convergence to a fixed point, the near-zero value of one exponent instead indicates the emergence of periodic behavior. Based on this analysis, it can be concluded that the associated model exhibits periodic dynamics and does not have chaotic behavior. To verify our results, we depict the numerical simulations for state variables of Ricker's mutualism model with rate indicator function in Figure 8. The numerical simulation for Ricker's mutualism model having

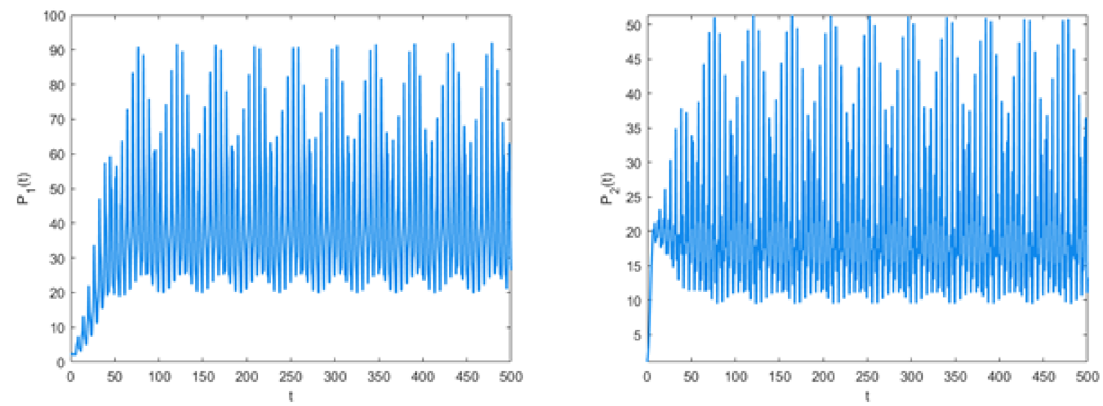

Figure 8. Graphical representation of each variable of the Ricker mutualism model with the rate indicator function over an extended time interval.

an almost periodic global attractor is shown in Figure 9.

Figure 9 presents a phase portrait plotting $P_2(t)$ against $P_1(t)$, revealing the geometric structure of the system's trajectories in state space. The resulting figure exhibits a spiraling, looped structure typical of strange attractors or complex dynamical systems. The trajectories never settle into a fixed point or closed orbit, reinforcing the earlier interpretation that the system does not reach a steady state or simple periodic behavior. Instead, it evolves continuously within a bounded region, which is characteristic of quasiperiodic or chaotic systems influenced by deterministic but non-repeating inputs, such as an almost periodic function.

Bifurcation diagram. In a mutualistic system, such as two species benefiting from each other's presence, a bifurcation might occur if the population sizes of both species are highly sensitive to certain environmental factors. For instance, if the growth rates of the species are close to critical thresholds, a small change in environmental factors (e.g., resource availability) could trigger a bifurcation, leading to a shift from stable coexistence to oscillations or even extinction. A cooperative system can experience a bifurcation, but it depends on the dynamics of the system and how the cooperative interactions are modeled. If the system includes nonlinearity, sensitive parameters, or feedback loops, bifurcations may occur as those parameters change. Therefore,

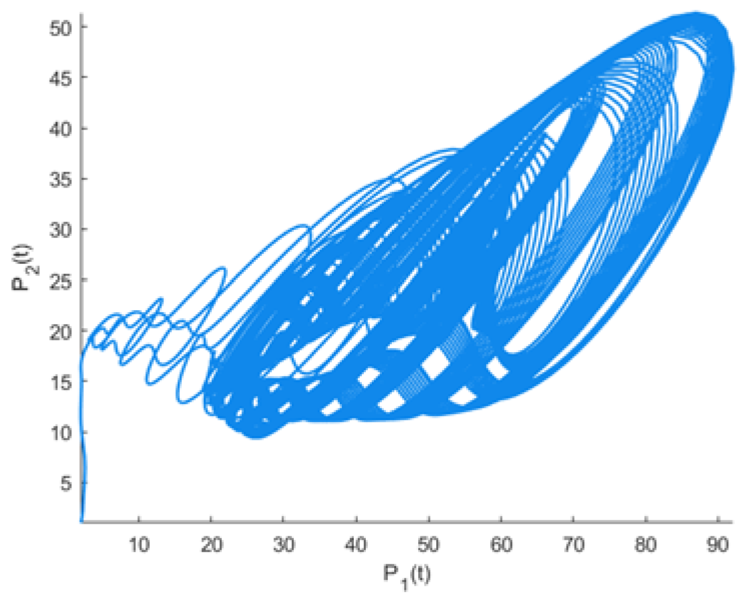

Figure 9. Phase plot of $P_2(t)$ versus $P_1(t)$, depicting the geometric behavior of the system trajectories within the state space.

while cooperation in a system does not automatically imply bifurcation, the presence of certain conditions—especially nonlinearity—can make bifurcations possible.

The bifurcation diagram for Ricker's mutualism model with rate indicator function is performed for varying the parameter K_1 and is simulated in Figure 10. Figure 10 shows a bifurcation diagram that investigates how the populations respond to changes in the system parameter K_1, likely representing a growth rate or interaction strength for one of the species. The horizontal axis represents a range of K_1 values, while the vertical axis shows the corresponding long-term values of $P_1(t)$ and $P_2(t)$. The plot uses blue and red dots for each species, respectively. For lower K_1 values, the population values are widely scattered, indicating a high degree of variability and suggesting chaotic behavior. As K_1 increases, the spread of points narrows, and the system appears to stabilize into more regular, possibly periodic behavior. This transition is a hallmark of bifurcation phenomena, where small changes in parameters lead to qualitative changes in system behavior.

6. Conclusion

The rate of change is an essential concept that underpins much of what we understand about the world around us. It allows us to describe how things evolve over time, make predictions about future behavior, optimize systems, and analyze complex phenomena. The rate of change provides a way to quantify how one quantity changes in response to changes in another. In this study, we employed a rate indicator function, incorporating an almost periodic function, to depict seasonality of the Ricker's mutualism model, which represents a mutualistic system. For the modified version of the model, to which this function was added, we derived conditions that ensure the uniqueness of the solution and demonstrated that the solution was not affected by the initial conditions. We strongly believe that this research will significantly contribute to

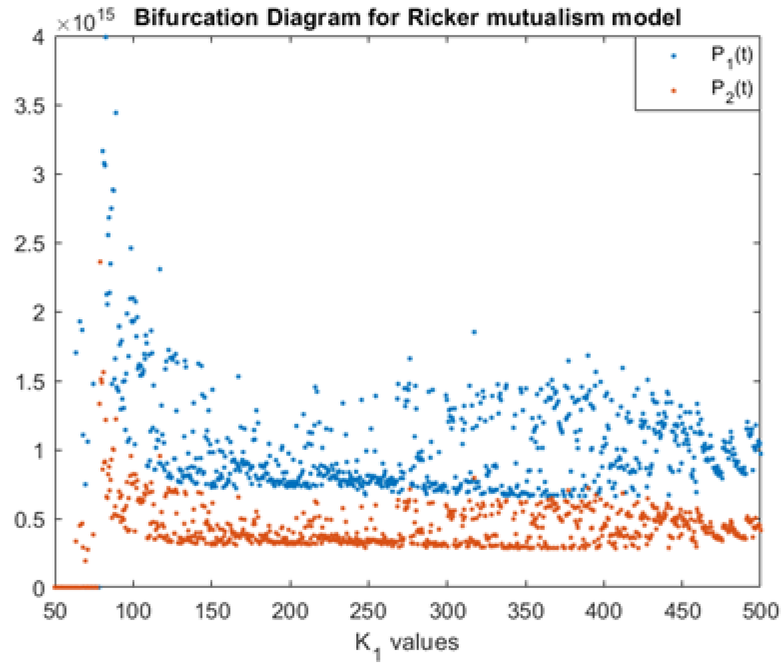

Figure 10. Bifurcation diagram of the Ricker mutualism model with the rate indicator function for varying values of K_1.

the understanding of both the importance of the rate indicator function and the role of almost periodicity in modeling seasonal dynamics.

Conflict of interest. The authors declare that there is no conflict of interests regarding the publication of this paper.

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