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SHARP CONDITIONS OF A NEW MATRIX SPLITTING ITERATION METHOD FOR GENERALIZED ABSOLUTE VALUE EQUATION

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Abstract For the generalized absolute value equation (denoted by GAVE), we develop a new matrix splitting iteration method, which is derived by reformulating equivalently GAVE as a three-by-three block non-linear equation. Convergence of the new proposal is obtained under certain assumptions imposed on the involved iteration parameters and splitting matrix. Moreover, sharp conditions of the iteration parameters are presented via the new analysis strategy and numerical experiments also confirm the achieved theoretical results. Compared with some well-known methods, the test results show the feasibility, robustness and effectiveness of the new matrix splitting iteration method with application to the linear complementarity problem.

Keywords Generalized absolute value equations, matrix splitting, convergence analysis.

MSC(2010) 65F10, 65H10.

1. Introduction

Consider the generalized absolute value equation (denoted by GAVE) of the following form

$$Ax - B|x| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. The symbol $|x|$ denotes the absolute value vector of $x \in \mathbb{R}^n$ by components. We remark that GAVE (1.1) is a non-smooth and non-linear equation due to the non-differentiability of absolute value function.

When $B = I$, GAVE reduces to the absolute value equation (denoted by AVE)

$$Ax - |x| = b, \tag{1.2}$$

which was first studied in [30].

For the general form (1.1), it was first introduced in [31]. GAVE subsumes many mathematical programming problems, for example, the linear complementarity problem, linear programming and convex quadratic programming. Due to involving absolute values, solving (1.1) is NP-hard [21, 22] and checking whether (1.1) has a unique or multiple solutions is also NP-hard [29].

In [32], Rohn proved that the singular value condition $\sigma_{\max}(|B|) < \sigma_{\min}(A)$ implies unique solvability of GAVE (1.1) for each right-hand side b, where $\sigma_{\max}$ and $\sigma_{\min}$, respectively, denote the maximal and minimal singular value of the matrix. In [34], Rohn et al. found that GAVE

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(1.1) for any b has a unique solution if $\rho(|A^{-1}B|) < 1$, where $\rho(\cdot)$ denotes the spectral radius of the matrix. Other sufficient conditions for the unique solution of GAVE (1.1), it can see [11–13, 24, 33, 36–38, 40] and references therein for more details.

In recent years, solving numerical solutions of absolute value equation has attracted much attention and has been investigated in the literature; see [2,4,5,8,14,15,20,23,25,26,28,33,35,39] and references therein. Most of those methods are based on the Newton method, with AVE in (1.2) being a weakly non-linear equation.

Recently, Ke and Ma in [19] proposed the following iteration scheme for solving AVE (1.2):

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(y^{(k)} + b), \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega|x^{(k+1)}|, \end{cases} \quad k = 0, 1, 2, \dots, \tag{1.3}$$

where the parameter $\omega > 0$. The scheme (1.3) has been widely extended as its simplicity and efficiency; see [5, 7, 9, 16–18, 39].

For the scheme (1.3), at each iteration step, it needs to solve the linear system $Av = y^{(k)} + b$ and will be costly to solve it when the coefficient matrix A is large-scale and dense. To overcome this problem, we consider to split the matrix A into $A = M - N$ with M being non-singular, and it is easy to solve the system $Mu = h$.

Inspired by the above facts, we propose a new matrix splitting iteration methods for solving GAVE (1.1). Firstly, we split the matrix A into $A = M - N$ with M being non-singular. Then, we introduce two intermediate variables $z = Nx$, $y = |x|$ and reformulate equivalently GAVE as a three-by-three block non-linear equation. Utilize the accelerated overrelaxation (AOR) method for linear system $Ax = b$ [10], we present a class of AOR-like matrix splitting iteration method for GAVE. It is worth mentioning that sharp conditions of the iteration parameters are presented via the new analysis strategy and numerical experiments also confirm the achieved theoretical results.

This paper is organized as follows. In Section 2, we propose a class of AOR-like matrix splitting iteration method for solving GAVE (1.1) and consider the convergence of the proposed iteration method. Moreover, we give the convergence domains of the iteration parameters under some conditions. In Section 3, we run some numerical experiments on the linear complementarity problem, which can be derived equivalently into GAVE (1.1). Experimental results verify our theoretical results. In addition, we test AVE (1.2) with the matrix A arising from University of Florida Sparse Matrix Collection [6]. The numerical results indicate that the proposed method has a good numerical performance. Finally, some conclusions are given in Section 4.

Notations. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. Let $|x| = (|x_1|, |x_2|, \dots, |x_n|)^{\mathrm{T}}$, where x_i is the i-th component of a vector $x \in \mathbb{R}^n$ for $i = 1, \dots, n$. For a real number a, the sign function is defined by

$$\operatorname{sgn}(a) = \begin{cases} -1, & a < 0, \\ 0, & a = 0, \\ 1, & a > 0. \end{cases}$$

A matrix $P = (p_{ij}) \in \mathbb{R}^{m\times n}$ is said to be non-negative (positive) if its entries satisfy $p_{ij} \geq 0$ ($p_{ij} > 0$) for all $1 \leq i \leq m$ and $1 \leq j \leq n$. For the vector $x \in \mathbb{R}^n$, the 1-norm, 2-norm and

Inf-norm are defined by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \quad \|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

For the matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|_p$ denotes the operator norm defined by $\|A\|_p := \max_{\|x\|_p=1} \|Ax\|_p$ for $p = 1, 2, \infty$. If it is not specified, the symbol $\|A\|(\|x\|)$ is represented to one of the 1-norm, 2-norm and Inf-norm. The symbol I denotes the identity matrix with suitable dimension.

2. Main results

In this section, we establish a novel iteration method for solving GAVE (1.1).

Let $A = M - N$ be a splitting of the matrix A, where the matrix M is non-singular. Then, we can get that

$$Mx - Nx - B|x| = b.$$

Denote $z := Nx$ and $y := |x|$, then it holds

$$\begin{cases} Mx - z - By = b, \\ Nx - z = 0, \\ |x| - y = 0, \end{cases} \tag{2.1}$$

that is

$$\mathbf{A}\mathbf{u} := \begin{pmatrix} M & -I & -B \\ N & -I & 0 \\ D(x) & 0 & -I \end{pmatrix} \begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} := \mathbf{c}, \tag{2.2}$$

where $D(x) := \mathrm{diag}(\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), \ldots, \mathrm{sgn}(x_n))$ is a diagonal matrix with $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Here, it has $D(x)x = |x|$.

Let $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$ with

$$\mathbf{D} = \begin{pmatrix} M & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & 0 & 0 \\ -N & 0 & 0 \\ -D(x) & 0 & 0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & I & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Utilize the AOR method for the linear system, we establish the following AOR-like matrix splitting iteration method for solving GAVE (1.1):

$$(\mathbf{D} - r\mathbf{L})\mathbf{u}^{(k+1)} = [(1-\omega)\mathbf{D} + (\omega - r)\mathbf{L} + \omega\mathbf{U}]\mathbf{u}^{(k)} + \omega\mathbf{c}, \tag{2.4}$$

where r and ω are positive constants. Method 2.1 presents the implementation of (2.4).

Method 2.1 (AOR-like matrix splitting iteration method for GAVE (1.1)).

Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Given the initial vectors $x^{(0)}, z^{(0)}, y^{(0)} \in \mathbb{R}^n$, for $k = 0, 1, 2, \ldots$, until the iteration sequence $\{(x^{(k)}, z^{(k)}, y^{(k)})\}_{k=0}^{+\infty}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega M^{-1}(z^{(k)} + By^{(k)} + b), \\ z^{(k+1)} = rNx^{(k+1)} + (\omega - r)Nx^{(k)} + (1-\omega)z^{(k)}, \\ y^{(k+1)} = r|x^{(k+1)}| + (\omega - r)|x^{(k)}| + (1-\omega)y^{(k)}. \end{cases} \tag{2.5}$$

Here, r and ω are given positive constants.

When $M = A$, the iteration scheme (2.5) reduces to

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(z^{(k)} + By^{(k)} + b), \\ z^{(k+1)} = (1-\omega)z^{(k)}, \\ y^{(k+1)} = r|x^{(k+1)}| + (\omega - r)|x^{(k)}| + (1-\omega)y^{(k)}. \end{cases} \tag{2.6}$$

Here, we called the iteration scheme (2.6) as the AOR-like iteration method.

When $r = \omega$, the iteration scheme (2.5) reduces to

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega M^{-1}(z^{(k)} + By^{(k)} + b), \\ z^{(k+1)} = \omega Nx^{(k+1)} + (1-\omega)z^{(k)}, \\ y^{(k+1)} = \omega|x^{(k+1)}| + (1-\omega)y^{(k)}. \end{cases} \tag{2.7}$$

Here, we called the iteration scheme (2.7) as the SOR-like matrix splitting iteration method.

When $M = A$, $B = I$, $r = \omega$, the iteration scheme (2.5) reduces to

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}(z^{(k)} + y^{(k)} + b), \\ z^{(k+1)} = (1-\omega)z^{(k)}, \\ y^{(k+1)} = \omega|x^{(k+1)}| + (1-\omega)y^{(k)}. \end{cases} \tag{2.8}$$

We remark that the iteration (2.8) is different from the iteration (1.3) for solving AVE (1.2).

Next, we recall some results that will be used in following analysis.

Lemma 2.1. ([3]) *For any vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following results hold:*
(1) $\| |x| - |y| \| \le \|x - y\|$;
(2) *if* $0 \le x \le y$, *then* $\|x\| \le \|y\|$;
(3) *if* $x \le y$ *and* P *is a non-negative matrix, then* $Px \le Py$.

Let (x^*, z^*, y^*) be the solution pair of the non-linear equation (2.1). Then it will satisfy

$$\begin{cases} x^* = (1-\omega)x^* + \omega M^{-1}(z^* + By^* + b), \\ z^* = rNx^* + (\omega - r)Nx^* + (1-\omega)z^*, \\ y^* = r|x^*| + (\omega - r)|x^*| + (1-\omega)y^*. \end{cases} \tag{2.9}$$

Let $(x^{(k)}, z^{(k)}, y^{(k)})$ be generated by the iteration method (2.5). We define the iteration errors as

$$e_x^{(k)} = x^* - x^{(k)}, \quad e_z^{(k)} = z^* - z^{(k)}, \quad e_y^{(k)} = y^* - y^{(k)}.$$

Now, we are ready to prove the main results of this paper.

Theorem 2.1. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with M being non-singular, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote*

$$T(\omega, r) := \begin{pmatrix} |1-\omega| & \omega\|M^{-1}\| & \omega\|M^{-1}B\| \\ (r|1-\omega| + |\omega - r|)\|N\| & \omega r\|M^{-1}\| \cdot \|N\| + |1-\omega| & \omega r\|M^{-1}B\| \cdot \|N\| \\ r|1-\omega| + |\omega - r| & \omega r\|M^{-1}\| & \omega r\|M^{-1}B\| + |1-\omega| \end{pmatrix}. \tag{2.10}$$

If $\rho(T(\omega, r)) < 1$, then the sequence $\{x^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.1 will converge to the solution of GAVE (1.1).

Proof. From (2.5) and (2.9), we have

$$\begin{cases} e_x^{(k+1)} = (1-\omega)e_x^{(k)} + \omega M^{-1}(e_z^{(k)} + Be_y^{(k)}), \\ e_z^{(k+1)} = rNe_x^{(k+1)} + (\omega - r)Ne_x^{(k)} + (1-\omega)e_z^{(k)}, \\ e_y^{(k+1)} = r(|x^*| - |x^{(k+1)}|) + (\omega - r)(|x^*| - |x^{(k)}|) + (1-\omega)e_y^{(k)}. \end{cases}$$

According to Lemma 2.1 (1), it follows that

$$\begin{cases} \|e_x^{(k+1)}\| \le |1-\omega| \cdot \|e_x^{(k)}\| + \omega\|M^{-1}\| \cdot \|e_z^{(k)}\| + \omega\|M^{-1}B\| \cdot \|e_y^{(k)}\|, \\ \|e_z^{(k+1)}\| \le r\|N\| \cdot \|e_x^{(k+1)}\| + |\omega - r| \cdot \|N\| \cdot \|e_x^{(k)}\| + |1-\omega| \cdot \|e_z^{(k)}\|, \\ \|e_y^{(k+1)}\| \le r\|e_x^{(k+1)}\| + |\omega - r| \cdot \|e_x^{(k)}\| + |1-\omega| \cdot \|e_y^{(k)}\|, \end{cases}$$

that is

$$\begin{pmatrix} 1 & 0 & 0 \\ -r\|N\| & 1 & 0 \\ -r & 0 & 1 \end{pmatrix} \begin{pmatrix} \|e_x^{(k+1)}\| \\ \|e_z^{(k+1)}\| \\ \|e_y^{(k+1)}\| \end{pmatrix} \le \begin{pmatrix} |1-\omega| & \omega\|M^{-1}\| & \omega\|M^{-1}B\| \\ |\omega - r| \cdot \|N\| & |1-\omega| & 0 \\ |\omega - r| & 0 & |1-\omega| \end{pmatrix} \begin{pmatrix} \|e_x^{(k)}\| \\ \|e_z^{(k)}\| \\ \|e_y^{(k)}\| \end{pmatrix}. \tag{2.11}$$

Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ r\|N\| & 1 & 0 \\ r & 0 & 1 \end{pmatrix} \ge 0.$$

Multiplying (2.11) from left by the non-negative matrix P and according to Lemma 2.1 (3), we have

$$\begin{pmatrix} \|e_x^{(k+1)}\| \\ \|e_z^{(k+1)}\| \\ \|e_y^{(k+1)}\| \end{pmatrix} \le T(\omega, r) \begin{pmatrix} \|e_x^{(k)}\| \\ \|e_z^{(k)}\| \\ \|e_y^{(k)}\| \end{pmatrix}, \tag{2.12}$$

where $T(\omega, r)$ is defined in (2.10). Denote $\mathrm{ERR}(k) := (\|e_x^{(k)}\|, \|e_z^{(k)}\|, \|e_y^{(k)}\|)^{\mathrm{T}}$, then

$$\mathrm{ERR}(k+1) \leq [T(\omega, r)]^{k+1}\mathrm{ERR}(0).$$

If $\rho(T(\omega, r)) < 1$, from (2.12), it holds that $\lim\limits_{k\to+\infty} \|\mathrm{ERR}(k)\| = 0$, that is

$$\lim_{k\to+\infty} \|e_x^{(k)}\| = 0, \quad \lim_{k\to+\infty} \|e_z^{(k)}\| = 0, \quad \lim_{k\to+\infty} \|e_y^{(k)}\| = 0.$$

Therefore, the sequence $\{x^{(k)}\}$ generated by Method 2.1 will converge to the solution of GAVE (1.1). This completes the proof. □

In particular, we consider the choice of the matrices M and N as well as the iteration parameters ω and r such that $\|T(\omega, r)\|_\infty < 1$.

Theorem 2.2. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with M being non-singular, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote*

$$\eta := \|M^{-1}\| + \|M^{-1}B\|, \quad \theta := \max\{\|N\|, 1\},$$
$$\widehat{\omega} := \frac{\theta - 1 + \sqrt{(1-\theta)^2 + 8\theta(1+\eta)}}{2\theta(1+\eta)}.$$

If $0 \leq \eta < 1$, $0 \leq \|N\| < \dfrac{1}{\eta}$ and the parameters ω, r satisfy one of the following conditions:

$$(1)\ \frac{\theta-1}{\theta(1-\eta)} < \omega \leq 1, \quad \frac{\theta-1}{\theta(1-\eta)} < r < \frac{\omega(1+\theta)}{\theta[2+\omega(\eta-1)]},$$
$$(2)\ 1 < \omega < \widehat{\omega}, \quad \frac{\omega(\theta+1)-2}{\theta[2-\omega(1+\eta)]} < r < \frac{2+\omega(\theta-1)}{\omega\theta(1+\eta)},$$

then the sequence $\{x^{(k)}\}_{k=0}^{+\infty}$ generated by Method 2.1 will converge to the solution of GAVE (1.1).

Proof. Let

$$\begin{aligned}
t_1 :=& |1-\omega| + \omega\|M^{-1}\| + \omega\|M^{-1}B\| \\
=& |1-\omega| + \omega\eta, \\
t_2 :=& (r|1-\omega| + |\omega - r|)\|N\| + \omega r\|M^{-1}\| \cdot \|N\| + |1-\omega| + \omega r\|M^{-1}B\| \cdot \|N\| \\
=& (r|1-\omega| + |\omega - r|)\|N\| + \omega r\eta\|N\| + |1-\omega|, \\
t_3 :=& r|1-\omega| + |\omega - r| + \omega r\|M^{-1}\| + \omega r\|M^{-1}B\| + |1-\omega| \\
=& r|1-\omega| + |\omega - r| + \omega r\eta + |1-\omega|.
\end{aligned}$$

If $|1-\omega| > 1$, then $t_i > 1$ for any $i = 1, 2, 3$. Note that $\|T(\omega, r)\|_\infty < 1$ if and only if $t_i < 1$ for any $i = 1, 2, 3$. Therefore, it is obvious that $0 < \omega < 2$.

For the parameters r and ω, we consider the following four Cases A-D. In each case, considering the value of $\|N\|$ can be further divided into two cases. Regardless of the value of $\|N\|$, to ensure $\|T(\omega, r)\|_\infty < 1$, one of the necessary conditions is $t_1 < 1$.

(i) If $0 < \omega \leq 1$, we have $1 - \omega + \omega\eta < 1$, which is from $t_1 < 1$. Consequently, it is essential to satisfy the condition $0 \leq \eta < 1$.

(ii) If $1 < \omega < 2$, we have $\omega - 1 + \omega\eta < 1$, which is from $t_1 < 1$. In order to make sure the existence of ω, it means that $\frac{2}{1+\eta} > 1$, that is $0 \leq \eta < 1$. Therefore, in the following discussion, regardless of the value range of $\|N\|$ and ω, the following condition must be satisfied: $0 \leq \eta < 1$.

Case A-1. when $0 < r < \omega \leq 1$ and $\|N\| \leq 1$. It follows that $t_2 \leq t_3$. Then $\|T(\omega, r)\|_\infty < 1$ if and only if

$$\begin{cases} 1 - \omega + \omega\eta < 1, \\ r(1-\omega) + \omega - r + \omega r\eta + 1 - \omega < 1. \end{cases}$$

Hence, when

$$0 \leq \eta < 1, \quad \|N\| \leq 1, \quad 0 < r < \omega \leq 1, \tag{2.13}$$

then $\|T(\omega, r)\|_\infty < 1$.

Case A-2. when $0 < r < \omega \leq 1$ and $\|N\| > 1$. It follows that $t_3 < t_2$. Then $\|T(\omega, r)\|_\infty < 1$ if and only if

$$\begin{cases} 1 - \omega + \omega\eta < 1, \\ \big[r(1-\omega) + (\omega - r)\big]\|N\| + \omega r\eta\|N\| + (1-\omega) < 1, \end{cases}$$

which is provided by

$$\begin{cases} 0 \leq \eta < 1, \\ r > \dfrac{\|N\| - 1}{\|N\|(1-\eta)}. \end{cases}$$

In order to make sure that the existence of r, it means that

$$\frac{\|N\| - 1}{\|N\|(1-\eta)} < 1,$$

which is provided by

$$\|N\| < \frac{1}{\eta}.$$

Hence, when

$$0 \leq \eta < 1, \quad 1 < \|N\| < \frac{1}{\eta}, \quad 0 < \frac{\|N\| - 1}{\|N\|(1-\eta)} < r < \omega \leq 1, \tag{2.14}$$

then $\|T(\omega, r)\|_\infty < 1$.

Case B-1. when $0 < \omega \leq 1$, $r \geq \omega$ and $\|N\| \leq 1$. It follows that $t_2 \leq t_3$. Then $\|T(\omega, r)\|_\infty < 1$ if and only if

$$\begin{cases} 1 - \omega + \omega\eta < 1, \\ r(1-\omega) + r - \omega + \omega r\eta + 1 - \omega < 1, \end{cases}$$

which is provided by

$$\begin{cases} 0 \leq \eta < 1, \\ r < \dfrac{2\omega}{2 + \omega(\eta - 1)}. \end{cases}$$

In order to make sure the existence of r, it means that

$$\frac{2\omega}{2+\omega(\eta-1)} > \omega.$$

This inequality holds while $0<\omega\le 1$ and $\eta<1$.

Hence, when

$$0\le\eta<1,\quad \|N\|\le 1,\quad 0<\omega\le 1,\quad \omega\le r<\frac{2\omega}{2+\omega(\eta-1)}, \tag{2.15}$$

then $\|T(\omega,r)\|_\infty<1$.

Case B-2. when $0<\omega\le 1$, $r\ge\omega$ and $\|N\|>1$. It follows that $t_3<t_2$. Then $\|T(\omega,r)\|_\infty<1$ if and only if

$$\begin{cases} 1-\omega+\omega\eta<1, \\ \big[r(1-\omega)+(r-\omega)\big]\|N\|+\omega r\eta\|N\|+(1-\omega)<1, \end{cases}$$

which is provided by

$$\begin{cases} 0\le\eta<1, \\ r<\dfrac{\omega(1+\|N\|)}{\|N\|[2+\omega(\eta-1)]}. \end{cases}$$

Firstly, in order to make sure the existence of r, it means that

$$\frac{\omega(1+\|N\|)}{\|N\|[2+\omega(\eta-1)]} > \omega.$$

In fact, we have

$$\begin{aligned} &\frac{\omega(1+\|N\|)}{\|N\|[2+\omega(\eta-1)]}-\omega>0 \\ \Leftrightarrow\ &\frac{\omega\big[1-\|N\|-\omega\|N\|(\eta-1)\big]}{\|N\|[2+\omega(\eta-1)]}>0 \\ \Leftarrow\ &1-\|N\|-\omega\|N\|(\eta-1)>0 \\ \Leftarrow\ &\omega>\frac{\|N\|-1}{\|N\|(1-\eta)}(>0). \end{aligned}$$

Secondly, in order to make sure the existence of ω, it means that

$$\frac{\|N\|-1}{\|N\|(1-\eta)}<1,$$

which is provided by

$$0\le\eta<1,\quad 1<\|N\|<\frac{1}{\eta}.$$

Hence, when

$$0\le\eta<1,\quad 1<\|N\|<\frac{1}{\eta},\quad 0<\frac{\|N\|-1}{\|N\|(1-\eta)}<\omega\le 1,\quad \omega\le r<\frac{\omega(1+\|N\|)}{\|N\|[2+\omega(\eta-1)]}, \tag{2.16}$$

then $\|T(\omega,r)\|_\infty < 1$.

Case C-1. when $1<\omega<2$, $0<r<\omega$ and $\|N\|\le 1$. It follows that $t_2 \le t_3$. Then $\|T(\omega,r)\|_\infty<1$ if and only if

$$\begin{cases} \omega-1+\omega\eta<1, \\ r(\omega-1)+(\omega-r)+\omega r\eta+\omega-1<1, \end{cases}$$

that is

$$\begin{cases} \omega<\dfrac{2}{1+\eta}, \\ r>\dfrac{2(\omega-1)}{2-\omega(1+\eta)}. \end{cases}$$

In order to make sure the existence of r, it means that

$$\frac{2(\omega-1)}{2-\omega(1+\eta)}<\omega,$$

that is

$$\omega^2(1+\eta)<2.$$

It can be provided by

$$\omega<\sqrt{\frac{2}{1+\eta}},$$

and

$$\sqrt{\frac{2}{1+\eta}}\le\frac{2}{1+\eta}.$$

Hence, when

$$0\le\eta<1,\quad \|N\|\le 1,\quad 1<\omega<\sqrt{\frac{2}{1+\eta}},\quad 0<\frac{2(\omega-1)}{2-\omega(1+\eta)}<r<\omega, \tag{2.17}$$

then $\|T(\omega,r)\|_\infty<1$.

Case C-2. when $1<\omega<2$, $0<r<\omega$ and $\|N\|>1$. It follows that $t_3<t_2$. Then $\|T(\omega,r)\|_\infty<1$ if and only if

$$\begin{cases} \omega-1+\omega\eta<1, \\ \big[r(\omega-1)+(\omega-r)\big]\|N\|+\omega r\eta\|N\|+\omega-1<1, \end{cases}$$

that is

$$\begin{cases} \omega<\dfrac{2}{1+\eta}, \\ r>\dfrac{\omega(\|N\|+1)-2}{\|N\|[2-\omega(1+\eta)]}. \end{cases}$$

In order to make sure the existence of r, it means that

$$\frac{\omega(\|N\|+1)-2}{\|N\|[2-\omega(1+\eta)]}<\omega,$$

which has

$$\|N\|(1+\eta)\omega^2 + (1-\|N\|)\omega - 2 < 0.$$

Denote

$$f(\omega) = \|N\|(1+\eta)\omega^2 + (1-\|N\|)\omega - 2.$$

It is obvious that the quadratic function f has two real distinct roots, that is

$$\omega_1^* = \frac{\|N\| - 1 - \sqrt{(1-\|N\|)^2 + 8\|N\|(1+\eta)}}{2\|N\|(1+\eta)} < 0$$

and

$$\omega_2^* = \frac{\|N\| - 1 + \sqrt{(1-\|N\|)^2 + 8\|N\|(1+\eta)}}{2\|N\|(1+\eta)} > 0.$$

Therefore, $f(\omega) < 0$ implies that $\omega \in (\omega_1^*, \omega_2^*)$. Note that $\omega \in (1,2)$. In order to make sure the existence of ω, it means that

$$f(1) = \|N\|(1+\eta) + (1-\|N\|) - 2 < 0,$$

that is

$$\|N\| < \frac{1}{\eta}.$$

In addition, it can verify that $w_2^* < \dfrac{2}{1+\eta} \leq 2$ while $0 \leq \eta < 1 < \|N\|$.

Hence, when

$$0 \leq \eta < 1, \quad 1 < \|N\| < \frac{1}{\eta}, \quad 1 < \omega < w_2^*, \quad \frac{\omega(\|N\|+1) - 2}{\|N\|[2 - \omega(1+\eta)]} < r < \omega, \tag{2.18}$$

then $\|T(\omega, r)\|_\infty < 1$.

Case D-1. when $1 < \omega < 2$, $r \geq \omega$ and $\|N\| \leq 1$. It follows that $t_2 \leq t_3$. Then $\|T(\omega, r)\|_\infty < 1$ if and only if

$$\begin{cases} \omega - 1 + \omega\eta < 1, \\ r(\omega - 1) + (r - \omega) + \omega r \eta + \omega - 1 < 1, \end{cases}$$

that is

$$\begin{cases} \omega < \dfrac{2}{1+\eta}, \\ r < \dfrac{2}{\omega(1+\eta)}. \end{cases}$$

In order to make sure the existence of r, it means that

$$\frac{2}{\omega(1+\eta)} > \omega.$$

It follows that

$$\omega < \sqrt{\frac{2}{1+\eta}},$$

and

$$\sqrt{\frac{2}{1+\eta}} \leq \frac{2}{1+\eta}.$$

Hence, when

$$0 \le \eta < 1, \quad \|N\| \le 1, \quad 1 < \omega < \sqrt{\frac{2}{1+\eta}}, \quad \omega \le r < \frac{2}{\omega(1+\eta)}, \tag{2.19}$$

then $\|T(\omega, r)\|_\infty < 1$.

Case D-2. when $1 < \omega < 2$, $r \ge \omega$ and $\|N\| > 1$. It follows that $t_3 < t_2$. Then $\|T(\omega, r)\|_\infty < 1$ if and only if

$$\begin{cases} \omega - 1 + \omega\eta < 1, \\ \big[r(\omega-1) + (r-\omega)\big]\|N\| + \omega r\eta\|N\| + \omega - 1 < 1, \end{cases}$$

that is

$$\begin{cases} \omega < \dfrac{2}{1+\eta}, \\ r < \dfrac{2+\omega(\|N\|-1)}{\omega\|N\|(1+\eta)}. \end{cases}$$

In order to make sure the existence of r, it means that

$$\frac{2+\omega(\|N\|-1)}{\omega\|N\|(1+\eta)} > \omega,$$

which has

$$\|N\|(1+\eta)\omega^2 + (1-\|N\|)\omega - 2 < 0.$$

It is the same with Case C-2, we get

$$\omega \in (\omega_1^*, \omega_2^*), \quad \|N\| < \frac{1}{\eta}.$$

Hence, when

$$0 \le \eta < 1, \quad 1 < \|N\| < \frac{1}{\eta}, \quad 1 < \omega < w_2^*, \quad \omega \le r < \frac{2+\omega(\|N\|-1)}{\omega\|N\|(1+\eta)}, \tag{2.20}$$

then $\|T(\omega, r)\|_\infty < 1$.

The above four classifications have been discussed in detail. Based on the results obtained from these classifications, we conducted a systematic integration process to extract a more concise and clear final conclusion.

Denote

$$\theta := \max\{\|N\|, 1\}.$$

From (2.13) and (2.15), we get

$$\begin{cases} 0 \le \eta < 1, \quad \|N\| \le 1, \quad 0 < \omega \le 1, \\ 0 < r < \dfrac{2\omega}{2+\omega(\eta-1)}, \end{cases} \tag{2.21}$$

then $\|T(\omega, r)\|_\infty < 1$. From (2.14) and (2.16), we get

$$\begin{cases} 0 \le \eta < 1, \quad 1 < \|N\| < \dfrac{1}{\eta}, \quad \dfrac{\|N\|-1}{\|N\|(1-\eta)} < \omega \le 1, \\ \dfrac{\|N\|-1}{\|N\|(1-\eta)} < r < \dfrac{\omega(1+\|N\|)}{\|N\|[2+\omega(\eta-1)]}, \end{cases} \tag{2.22}$$

then $\|T(\omega, r)\|_\infty < 1$. Based on (2.21) and (2.22), we can obtain that the parameters satisfy that

$$\begin{cases} 0 \le \eta < 1, \quad 0 \le \|N\| < \dfrac{1}{\eta}, \quad \dfrac{\theta - 1}{\theta(1-\eta)} < \omega \le 1, \\ \dfrac{\theta - 1}{\theta(1-\eta)} < r < \dfrac{\omega(1+\theta)}{\theta[2+\omega(\eta-1)]}. \end{cases} \tag{2.23}$$

From (2.17) and (2.19), we get

$$\begin{cases} 0 \le \eta < 1, \quad \|N\| \le 1, \quad 1 < \omega < \sqrt{\dfrac{2}{1+\eta}}, \\ \dfrac{2(\omega - 1)}{2 - \omega(1+\eta)} < r < \dfrac{2}{\omega(1+\eta)}, \end{cases} \tag{2.24}$$

then $\|T(\omega, r)\|_\infty < 1$. From (2.18) and (2.20), we get

$$\begin{cases} 0 \le \eta < 1, \quad 1 < \|N\| < \dfrac{1}{\eta}, \quad 1 < \omega < w_2^*, \\ \dfrac{\omega(\|N\|+1) - 2}{\|N\|[2 - \omega(1+\eta)]} < r < \dfrac{2 + \omega(\|N\| - 1)}{\omega\|N\|(1+\eta)}, \end{cases} \tag{2.25}$$

then $\|T(\omega, r)\|_\infty < 1$. Based on (2.24) and (2.25), we can obtain that the parameters satisfy that

$$\begin{cases} 0 \le \eta < 1, \quad 0 \le \|N\| < \dfrac{1}{\eta}, \quad 1 < \omega < \widehat{\omega}, \\ \dfrac{\omega(\theta+1) - 2}{\theta[2 - \omega(1+\eta)]} < r < \dfrac{2 + \omega(\theta - 1)}{\omega\theta(1+\eta)}, \end{cases} \tag{2.26}$$

where

$$\widehat{\omega} := \frac{\theta - 1 + \sqrt{(1-\theta)^2 + 8\theta(1+\eta)}}{2\theta(1+\eta)}.$$

We remark that if $\|N\| \le 1$, then $\theta = 1$ and

$$\widehat{\omega} = \sqrt{\frac{2}{1+\eta}}.$$

Therefore, from (2.23) and (2.26), we obtain the results of this theorem. This completes the proof. □

According to Theorem 2.2, we obtain the following corollaries for the iteration schemes (2.6), (2.7) and (2.8), respectively.

Corollary 2.1. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote $\eta := \|A^{-1}\| + \|A^{-1}B\|$. If $0 \le \eta < 1$ and the parameters ω, r satisfy one of the following conditions*:

$$(1)\; 0 < \omega \le 1, \quad 0 < r < \frac{2\omega}{2 + \omega(\eta - 1)},$$

$$(2)\; 1 < \omega < \sqrt{\frac{2}{1+\eta}}, \quad \frac{2\omega - 2}{2 - \omega(1+\eta)} < r < \frac{2}{\omega(1+\eta)},$$

then the sequence $\{x^{(k)}\}_{k=0}^{+\infty}$ generated by the iteration scheme (2.6) *will converge to the solution of GAVE* (1.1).

Corollary 2.2. *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$ with M being nonsingular, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Denote*

$$\eta := \|M^{-1}\| + \|M^{-1}B\|, \quad \theta := \max\{\|N\|, 1\},$$
$$\widehat{\omega} := \frac{\theta - 1 + \sqrt{(1-\theta)^2 + 8\theta(1+\eta)}}{2\theta(1+\eta)}.$$

If $0 \le \eta < 1$, $0 \le \|N\| < \frac{1}{\eta}$ and the parameter ω satisfies

$$\frac{\theta - 1}{\theta(1-\eta)} < \omega < \widehat{\omega},$$

then the sequence $\{x^{(k)}\}_{k=0}^{+\infty}$ generated by the iteration scheme (2.7) *will converge to the solution of GAVE* (1.1).

Corollary 2.3. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. If $\|A^{-1}\| < \frac{1}{2}$ and the parameter ω satisfies*

$$0 < \omega < \sqrt{\frac{2}{1 + 2\|A^{-1}\|}},$$

then the sequence $\{x^{(k)}\}_{k=0}^{+\infty}$ generated by the iteration scheme (2.8) *will converge to the solution of AVE* (1.2).

3. Numerical experiments

In this section, we use some test problems to examine the effectiveness of Method 2.1. All test problems are started from the initial zero vector, are terminated if the current iterations satisfy

$$\text{RES} := \|b + B|x^{(k)}| - Ax^{(k)}\| \le 10^{-6}$$

or if the number of the prescribed iteration steps $k_{\max} = 1000$ is exceeded, and are performed under MATLAB R2018b on a personal computer with 1.80 GHz central processing unit (Intel(R) Core(TM) i5-8265U), 8GB memory and Windows 10 operating system. In addition, 'IT' denotes the number of iteration steps and 'CPU' denotes the elapsed CPU time in seconds.

Example 3.1 (Linear complementarity problem)**.** The linear complementarity problem is to find a pair of real vectors w and $v \in \mathbb{R}^n$ such that

$$w := \widetilde{A}v + q \ge 0, \quad v \ge 0 \quad \text{and} \quad v^{\mathrm{T}}w = 0,$$

where $\widetilde{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a given large, sparse and real matrix and $q = (q_1, q_2, \ldots, q_n)^{\mathrm{T}} \in \mathbb{R}^n$ is a given real vector. It is abbreviated as LCP$(q, \widetilde{A})$. By utilizing the modulus, LCP$(q, \widetilde{A})$ can be reformulated into

$$(\widetilde{A}\Gamma + \Omega)x - (\Omega - \widetilde{A}\Gamma)|x| = -q, \tag{3.1}$$

where Ω and Γ are $n \times n$ positive diagonal matrices. For more details, it can see [1]. It is obvious that the equation (3.1) reduces to GAVE (1.1) with

$$A = \widetilde{A}\Gamma + \Omega, \quad B = \Omega - \widetilde{A}\Gamma, \quad b = -q.$$

Let m be a prescribed positive integer and $n = m^2$. Consider LCP$(q, \widetilde{A})$, in which $\widetilde{A} \in \mathbb{R}^{n\times n}$ is given by $\widetilde{A} = \widehat{A} + \mu I \in \mathbb{R}^{n\times n}$ with

$$\widehat{A} = \text{Tridiag}(-1.5I_m, \widehat{S}, -0.5I_m) \in \mathbb{R}^{n\times n}$$

being a block-tridiagonal matrix,

$$\widehat{S} = \text{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m\times m}$$

being a tridiagonal matrix and

$$q = (1, 2, \ldots, n)^{\mathrm{T}} \in \mathbb{R}^n.$$

Let D, $-L$ and $-U$ be the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix A, respectively. Let $H = \frac{1}{2}(A + A^{\mathrm{T}})$ and $S = \frac{1}{2}(A - A^{\mathrm{T}})$ be the Hermitian and skew-Hermitian parts of the matrix A. Let $D_{\widetilde{A}}$, $-L_{\widetilde{A}}$ and $-U_{\widetilde{A}}$ be the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix $\widetilde{A}$, respectively. Let $H_{\widetilde{A}} = \frac{1}{2}(\widetilde{A} + \widetilde{A}^{\mathrm{T}})$ and $S_{\widetilde{A}} = \frac{1}{2}(\widetilde{A} - \widetilde{A}^{\mathrm{T}})$ be the Hermitian and skew-Hermitian parts of the matrix $\widetilde{A}$. For Example 3.1, we choose $\Gamma = I_n$ and $\Omega = D_{\widetilde{A}}$. Abbreviations of testing methods for Example 3.1 are listed in Table 1.

Table 1. Abbreviations of testing methods for Example 3.1.

Method	Description	Iteration scheme
MM	The modified modulus method [27]	$(I + \widetilde{A})x^{(k+1)} = (I - \widetilde{A})\lvert x^{(k)}\rvert - q$
MGS	The modulus-based Gauss-Seidel method [1]	$(D_{\widetilde{A}} + \Omega - L_{\widetilde{A}})x^{(k+1)} = U_{\widetilde{A}}x^{(k)} + (\Omega - \widetilde{A})\lvert x^{(k)}\rvert - q$
MHS	The modulus-based HS method [1]	$(H_{\widetilde{A}} + \Omega)x^{(k+1)} = -S_{\widetilde{A}}x^{(k)} + (\Omega - \widetilde{A})\lvert x^{(k)}\rvert - q$
AOR	The AOR-like iteration method	Scheme (2.5) with $M = A$, $N = 0$
AOR-GS	The AOR-like Gauss-Seidel splitting iteration method	Scheme (2.5) with $M = D - L$, $N = U$
AOR-HS	The AOR-like HS splitting iteration method	Scheme (2.5) with $M = H$, $N = -S$

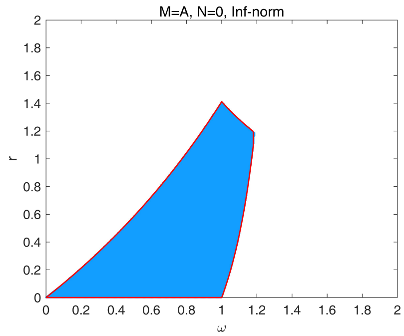
For Example 3.1, denote $\eta = \|M^{-1}\|_\infty + \|M^{-1}B\|_\infty$ and $\xi = \|N\|_\infty$. Figure 1 illustrates the domains of the parameters ω and r based on the conditions $\|T(\omega, r)\|_\infty < 1$ and Theorem 2.2, corresponding to the various splittings of matrix A. Here, blue points are obtained from the condition $\|T(\omega, r)\|_\infty < 1$ and inside of the red curve is the result according to Theorem 2.2.

In Table 2, we list IT, CPU and RES of different test methods as well as the parameters r and ω for AOR-like methods. From Table 2, we can see that the AOR-GS method costs much less number of iteration steps and lower CPU time than other methods. In addition, when the scale of the problem gradually expands, the AOR-HS method shows more significant advantages in the number of iterations and the required iteration time than the MM, MGS and MHS methods. Figure 2 plots RES curves of the test methods for Example 3.1 with $\mu = 4$ and $n = 400$. From this figure, we see that the residual norms of AOR-like methods decrease more sharply than the modulus-based methods for solving the linear complementarity problems.

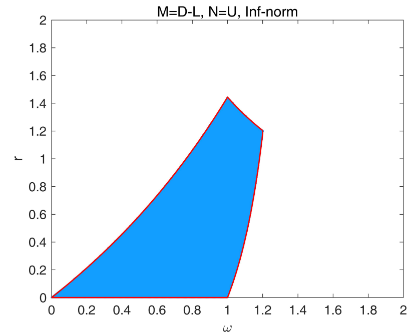
Example 3.2. Consider AVE (1.2), which the matrix $A \in \mathbb{R}^{n\times n}$ comes from six different test problems listed in Table 3. These test matrices A are sparse, symmetry and $\|A^{-1}\| < 1$. Let

$$x^* = (-1, 1, -1, 1, \cdots, -1, 1)^{\mathrm{T}} \in \mathbb{R}^n \quad \text{and} \quad b = Ax^* - |x^*|.$$

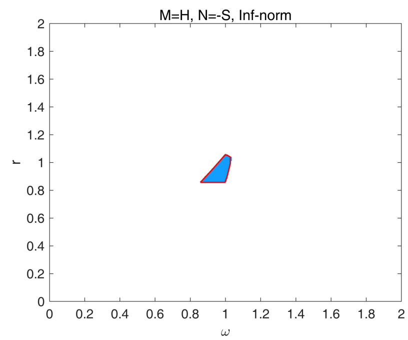
For more about these test problems, it can see [6].



(a) AOR: $\eta = 0.4167$, $\xi = 0$



(b) AOR-GS: $\eta = 0.3846$, $\xi = 1$



(c) AOR-HS: $\eta = 0.4167$, $\xi = 2$

Figure 1. The domains of the parameters ω and r for Example 3.1 with $\mu = 4$ and $n = 400$.

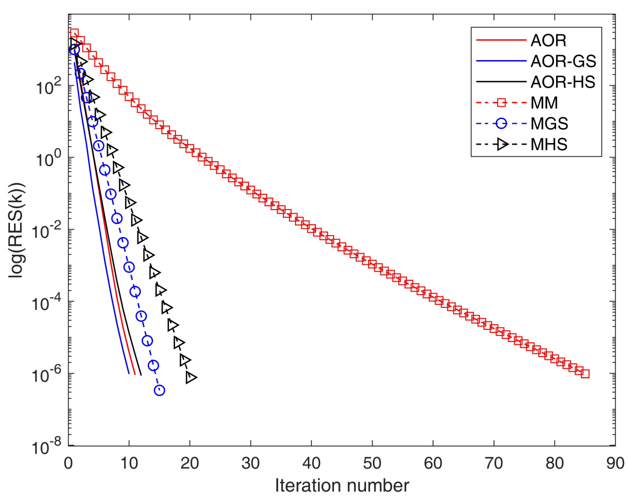


Figure 2. Curves of RES for Example 3.1 with $\mu = 4$ and $n = 400$.

For Example 3.2, we compare the proposed method with the SOR-like method (1.3). The numerical results are listed in Table 4 as well as the corresponding parameters. These parameters are obtained via minimize the number of iterations. From these results, it can see that AOR-GS method has a good numerical performance for the large scale problems.

Table 2. Numerical results for Example 3.1 with $\mu = 4$.

Method		$n = 400$	$n = 1600$	$n = 3600$	$n = 6400$	$n = 10000$
MM	IT	85	93	97	100	102
	CPU	0.1125	0.6131	1.7038	3.5265	7.9939
	RES	9.7684e-07	8.3783e-07	8.8300e-07	8.9255e-07	9.5837e-07
MGS	IT	15	17	18	18	19
	CPU	0.0046	0.0223	0.1304	0.4670	1.6172
	RES	3.3688e-07	2.9109e-07	2.8037e-07	7.4648e-07	3.5445e-07
MHS	IT	20	22	24	24	25
	CPU	0.0122	0.0453	0.1266	0.6890	1.1769
	RES	7.6949e-07	9.0595e-07	3.6937e-07	9.1737e-07	6.1224e-07
AOR	IT	11	12	13	13	14
$\omega = 0.94$	CPU	0.0236	0.1088	0.2976	1.2051	0.9449
$r = 0.93$	RES	9.0599e-07	8.6153e-07	4.7329e-07	8.6097e-07	3.2583e-07
AOR-GS	IT	10	11	11	12	12
$\omega = 0.90$	CPU	0.0034	0.0049	0.0074	0.0174	0.0594
$r = 0.98$	RES	9.5592e-07	4.6342e-07	7.5456e-07	2.2837e-07	3.0109e-07
AOR-HS	IT	12	13	14	14	15
$\omega = 0.92$	CPU	0.0064	0.0286	0.0962	0.2757	0.6835
$r = 0.94$	RES	8.6538e-07	9.1377e-07	5.4957e-07	9.7842e-07	4.1181e-07

Table 3. Test problems of Example 3.2.

Problem	n	Problem	n
mesh1e1	48	Trefethen_20b	19
mesh1em1	48	Trefethen_200b	199
mesh2e1	306	Trefethen_20000b	19999

4. Conclusions

We have presented a class of AOR-like matrix splitting iteration method to solve the NP-hard GAVE in (1.1), which is obtained by reformulating equivalently GAVE as a three-by-three block non-linear equation. Under suitable choices of the involved parameters and splitting matrix, we have presented the sufficient conditions for convergence of the proposed iteration method. Moreover, we have considered the convergence domains of the iteration parameters via the new analysis strategy, and numerical experiments also confirm the achieved theoretical results. From Figure 1, it can see that the results of Theorem 2.2 are sharp. Numerical examples have shown that the proposed iteration method is feasible and effective in computing.

Theorem 2.2 gives sufficient conditions for convergence based on $\|T(\omega, r)\|_\infty < 1$. However, it is still worth considering how to provide the convergence domains of the parameters ω and r such that $\|T(\omega, r)\|_1 < 1$ and $\|T(\omega, r)\|_2 < 1$ in theory. In addition, the choice of the optimal iteration parameters in theory also merits some consideration.

Table 4. Numerical results for Example 3.2.

Method		mesh1e1	mesh1em1	mesh2e1	Trefethen_20b	Trefethen_200b	Trefethen_20000b
SOR	ω	0.94	0.93	0.94	0.95	0.95	0.95
	IT	18	18	20	12	12	12
	CPU	0.0074	0.0056	0.0146	0.0004	0.0086	151.9966
AOR	ω	0.97	0.91	0.98	1.01	1.01	1.01
	r	0.94	0.95	0.93	0.90	0.91	0.91
	IT	17	18	19	11	11	11
	CPU	0.0086	0.0087	0.0146	0.0004	0.0066	133.0436
AOR-GS	ω	0.90	0.92	1.10	0.90	0.91	0.91
	r	1.15	1.20	1.20	1.10	1.10	1.10
	IT	19	51	402	11	11	13
	CPU	0.0033	0.0034	0.0138	0.0002	0.0003	0.1123
AOR-HS	ω	0.97	0.91	0.98	1.01	1.01	1.01
	r	0.94	0.95	0.93	0.90	0.91	0.91
	IT	17	18	19	11	11	11
	CPU	0.0015	0.0014	0.0086	0.0003	0.0083	134.2781

Declarations

Ethical approval. This manuscript does not contain any studies with human participants or animals performed by any of the authors.

Availability of supporting data. Enquiries about data availability should be directed to the authors.

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