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SOLVABILITY OF THE CAPUTO FRACTIONAL DIFFERENTIAL SYSTEM WITH RIEMANN-STIELTJES BOUNDARY CONDITIONS

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Abstract Fractional derivative is nonlocal which exhibit a long-term memory behavior. Having these advantages, fractional order systems are more accurate than integer order ones. In this article, our research focuses on the Caputo fractional differential system with Riemann-Stieltjes integral boundary conditions. Firstly, we convert the system to an integral operator. And then, based on the properties of the Green function, we have separately proven the existence of the unique solution and at least one solution for the system by applying the Banach contraction principle and the Leray-Schauder's alternative. Finally, The correctness of the results is verified through an example.

Keywords Solvability, uniqueness, existence, Caputo fractional differential system, Riemann-Stieltjes boundary conditions.

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1. Introduction

Compared to integer order calculus, fractional order calculus can handle differentiation and integration of any order, and has great advantages in describing the motion, intermediate processes, and memory features of some complex systems. Therefore, for long-term and large-scale biological and physical phenomena, fractional order models are more accurate than integer order, thanks to their ability to overcome the locality of integer order models. For instance, Arafa etc [1] studied the viral dynamic model under the definition of fractional derivatives $(\alpha_1, \alpha_2, \alpha_3 > 0)$:

$$\begin{cases} D^{\alpha_1}(x) = s - \mu x - \beta xz, \\ D^{\alpha_2}(y) = \beta xz - \varepsilon y, \\ D^{\alpha_3}(z) = cy - \gamma z, \end{cases}$$

subject to the initial values

$$x(0) = 200, y(0) = 0, z(0) = 1.$$

The Volterra's model for species population growth in closed systems can be represented by the fractional equation:

$${}^cD^{\alpha}y(t) = ay(t) - by^2(t) - cy(t)\int_0^t y(x)dx, \quad y(0) = y_0,$$

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where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $0 < \alpha \le 1$. Obviously, it is very meaningful to study the existence of positive solutions to equation models under appropriate initial and boundary conditions.

In this article, our main focus is the investigation of the following Caputo fractional differential system

$$\begin{cases} {}^cD^{\alpha_1}x(t) + \lambda_1 a_1(t) f_1(t, x(t), y(t)) = 0, \\ {}^cD^{\alpha_2}y(t) + \lambda_2 a_2(t) f_2(t, x(t), y(t)) = 0, \ 0 < t < 1, \\ x(0) = x''(0) = 0, \ x(1) = \mu_1 \int_0^1 b_1(s)x(s)dA_1(s), \\ y(0) = y''(0) = 0, \ y(1) = \mu_2 \int_0^1 b_2(s)y(s)dA_2(s), \end{cases} \tag{1.1}$$

where $\lambda_i > 0$ is a parameter, $2 < \alpha_i < 3$, ${}^cD^{\alpha_i}$ is the standard Caputo derivative. $\mu_i > 0$ is a constant, $A_i : [0,1] \to [0, +\infty)$ is the function of bounded variation, $\int_0^1 b_1(s)x(s)dA_1(s)$, $\int_0^1 b_2(s)y(s)dA_2(s)$ denote the Riemann-Stieltjes integral with a signed measure, $0 \le \mu_i \int_0^1 tb_i(t) dA_i(t) < 1$. $b_i : [0,1] \to [0, +\infty)$, $a_i : (0,1) \to [0, +\infty)$, $f_i : [0,1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous functions, $i = 1, 2$.

In the context of mathematics, profiting by the rapid development of nonlinear analysis theory in recent years, many practical tools and methods, such as the operator theory [11,12,16], fixed point theory [6,13–15], iterative methods [5,17,19], upper and lower solution methods [18], etc. are used to solve every kind of differential equations. Fractional differential equations involving different boundary conditions have attracted increasing interest from scholars. In [2], Cabada et al. consider the following nonlinear fractional differential equation boundary value problem

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t)) = 0, \ 0 < t < 1, \\ u'(0) = u''(0) = 0, \ u(1) = \mu \int_0^1 u(s)ds, \end{cases} \tag{1.2}$$

where $2 < \alpha \le 3$, $0 \le \mu < 1$, ${}^cD^\alpha$ is the Caputo differential operator of order α, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function. By using the Krasnoselskii's fixed point theorem, they deduce a general existence theorem for (1.2).

Apply the Krasnoselskii's fixed point theorem as well, Ma and Cui in [8] investigate the following Caputo fractional boundary value problem

$$\begin{cases} {}^cD^\alpha p(t) + \mu f(t, p(t)) = 0, \ 0 < t < 1, \\ p(0) = p''(0) = 0, \ p(1) = \int_0^1 p(s)dA(s), \end{cases}$$

where $2 < \alpha < 3$, $\mu > 0$ is a parameter, $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function.

Chen and Li [3] prove the existence of positive solutions for a system of nonlinear Caputo-type fractional differential equations

$$\begin{cases} {}^cD^{\alpha_1}u(t) + \lambda f_1(t, u(t), v(t)) = 0, \\ {}^cD^{\alpha_2}v(t) + \mu f_2(t, u(t), v(t)) = 0, \ 0 \le t \le 1, \\ u(0) = u''(0) = 0, \ u(1) = \int_0^1 u(t)dA_1(t), \\ v(0) = v''(0) = 0, \ v(1) = \int_0^1 v(t)dA_2(t), \end{cases} \tag{1.3}$$

where $2 < \alpha_i < 3$ $(i = 1, 2)$, ${}^cD^{\alpha_i}$ is the standard Caputo derivative, $\lambda,\ \mu > 0$ are parameters, $f_i : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function, A_i is a bounded variation function, $i = 1, 2$. In [7], by the virtue of the fixed point index theory, the authors obtain the two positive solutions for the equations (1.3) with parameters $\lambda = \mu = 1$.

The objective of this study is to obtain the existence and uniqueness of solutions for the system (1.1), we study it according to the Banach contraction principle and the Leray-Schauder's alternative. Compared to the papers we mentioned earlier, the system in this paper is more complex, and the Riemann-Stieltjes integral boundary conditions can be converted into ordinary integral boundary conditions, two-point, multi-point boundary conditions as well. What's more, in the system (1.1) we are discussing, $\lambda_1 > 0$ and $\lambda_2 > 0$ can be unequal, when the order of the fractional derivative is not the same in the fractional differential equation, the system is called incommensurate order.

The plan for this article is as follows. In Section 2, we provide the preliminaries and necessary definitions, lemmas that are to be used to prove our main results. The main results are given in Section 3, we show the proof of the existence and uniqueness of positive solutions for the system (1.1). In Section 4, an example is given to demonstrate the application of our theoretical results. In Section 5, we give the conclusions of this article.

2. Preliminaries and lemmas

In an effort to the system (1.1), in this section, we provide some preliminaries and lemmas to be used in the rest of this article. Firstly, we present here some necessary definitions and lemma about fractional calculus theory for convenience of the readers.

Definition 2.1. [9, 10] The Caputo fractional order derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$${}^cD^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds,$$

where $u \in C^n(J,\ \mathbb{R})$, $J = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ denotes the natural number set, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Definition 2.2. [9, 10] Let $\alpha > 0$ and let u be piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $[0, +\infty)$. Then for $t > 0$, we call

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds,$$

the Riemann-Liouville fractional integral of u of order α.

Lemma 2.1. [9,10] *Let $n-1<\alpha\le n$, $u\in C^n[0,1]$. Then*

$$I^{\alpha}({}^cD^{\alpha}u)(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in(-\infty,+\infty)$ $(i=1,2,\cdots,n-1)$, n is the smallest integer greater than or equal to α.

In the following Lemmas 2.2 and 2.3, we consider the following linear Caputo fractional differential system (2.1), prove that system (2.1) has a unique integral representation and illustrate the properties of the Green function $G_i(t,s)$ $(i=1,2)$.

Lemma 2.2. *Let $h_i\in C(0,1)\cap L(0,1)$ $(i=1,2)$, then the system*

$$\begin{cases} {}^cD^{\alpha_1}x(t)+h_1(t)=0,\\ {}^cD^{\alpha_2}y(t)+h_2(t)=0,\ 0<t<1,\\ x(0)=x''(0)=0,\ x(1)=\mu_1\displaystyle\int_0^1 b_1(s)x(s)dA_1(s),\\ y(0)=y''(0)=0,\ y(1)=\mu_2\displaystyle\int_0^1 b_2(s)y(s)dA_2(s), \end{cases}\tag{2.1}$$

has a unique integral representation

$$\begin{cases} x(t)=\displaystyle\int_0^1 G_1(t,s)h_1(s)ds,\\ y(t)=\displaystyle\int_0^1 G_2(t,s)h_2(s)ds, \end{cases}\tag{2.2}$$

where

$$\begin{aligned} &G_i(t,s) \qquad (2.3)\\ &=\frac{1}{\Gamma(\alpha_i)}\begin{cases} \dfrac{t}{(1-\chi_i)}\left((1-s)^{\alpha_i-1}-\mu_i\displaystyle\int_s^1 b_i(t)(t-s)^{\alpha_i-1}dA_i(t)\right)-(t-s)^{\alpha_i-1},\\ 0\le s\le t\le 1,\\ \dfrac{t}{(1-\chi_i)}\left((1-s)^{\alpha_i-1}-\mu_i\displaystyle\int_s^1 b_i(t)(t-s)^{\alpha_i-1}dA_i(t)\right),\ 0\le t\le s\le 1, \end{cases}\\ &\chi_i=\mu_i\int_0^1 tb_i(t)dA_i(t),\quad i=1,2. \qquad (2.4) \end{aligned}$$

Proof. By Lemma 2.1 and the conditions

$$x(0)=x''(0)=0,\ y(0)=y''(0)=0.$$

System (2.1) is equivalent to the following integral equations:

$$x(t)=-\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)ds+ct,\tag{2.5}$$

$$y(t)=-\int_0^t\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}h_2(s)ds+\overline{c}t.\tag{2.6}$$

Combining the conditions

$$x(1)=\mu_1\int_0^1 b_1(s)x(s)dA_1(s),\ y(1)=\mu_2\int_0^1 b_2(s)y(s)dA_2(s),$$

we get

$$\begin{aligned}c&=x(1)+\int_0^1\frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)ds,\\ \overline{c}&=y(1)+\int_0^1\frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}h_2(s)ds.\end{aligned}$$

Then, we have

$$\begin{aligned}c=&\frac{1}{\Gamma(\alpha_1)\,(1-\chi_1)}\left(\int_0^1(1-s)^{\alpha_1-1}h_1(s)ds\right.\\ &\left.-\mu_1\int_0^1\int_0^t b_1(t)(t-s)^{\alpha_1-1}h_1(s)dsdA_1(t)\right), \qquad (2.7)\\ \overline{c}=&\frac{1}{\Gamma(\alpha_2)\,(1-\chi_2)}\left(\int_0^1(1-s)^{\alpha_2-1}h_2(s)ds\right.\\ &\left.-\mu_2\int_0^1\int_0^t b_2(t)(t-s)^{\alpha_2-1}h_2(s)dsdA_2(t)\right). \qquad (2.8)\end{aligned}$$

Substituting (2.7) and (2.8) into (2.5) and (2.6) separately, we can get

$$\begin{aligned}x(t)=&\frac{t}{\Gamma(\alpha_1)\,(1-\chi_1)}\left(\int_0^1(1-s)^{\alpha_1-1}h_1(s)ds-\mu_1\int_0^1\int_s^1 b_1(t)(t-s)^{\alpha_1-1}h_1(s)dA_1(t)ds\right)\\ &-\frac{1}{\Gamma(\alpha_1)}\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)ds, \qquad (2.9)\\ y(t)=&\frac{t}{\Gamma(\alpha_2)\,(1-\chi_2)}\left(\int_0^1(1-s)^{\alpha_2-1}h_2(s)ds-\mu_2\int_0^1\int_s^1 b_2(t)(t-s)^{\alpha_2-1}h_2(s)dA_2(t)ds\right)\\ &-\frac{1}{\Gamma(\alpha_1)}\int_0^t\frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}h_2(s)ds. \qquad (2.10)\end{aligned}$$

Comprehensive equations (2.3) and (2.4), we can obtain (2.2). The proof is completed. □

From the expression of Green Function $G_i(t,s)$, we can proof the following Lemma 2.3 holds.

Lemma 2.3. *The Green Function $G_i(t,s)$ $(i=1,2)$ defined by (2.3) has the following properties:*

(1)

$$G_i(t,s)\geq 0 \text{ and } G_i(t,s) \text{ is continuous on } [0,1]\times[0,1]. \qquad (2.11)$$

(2)

$$G_i(t,s)\leq\frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)(1-\chi_i)},\quad t,\ s\in[0,1]. \qquad (2.12)$$

In the following proof, for the sake of convenience, denote $\Phi_i(s) = \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)(1-\chi_i)}$. Let $X = C[0,1] \times C[0,1]$, clearly X is a Banach space with the norm

$$\|(x,y)\| = \|x\| + \|y\|,\ \|x\| = \max_{t\in[0,1]} |x(t)|,\ \|y\| = \max_{t\in[0,1]} |y(t)|.$$

For any $(x,y) \in X$, defining an integral operator $T : X \to X$

$$\begin{aligned} &T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t)),\ 0 \le t \le 1, \\ &T_1(x,y)(t) = \lambda_1 \int_0^1 G_1(t,s)a_1(s)f_1(s,x(s),y(s)ds, \quad 0 \le t \le 1, \\ &T_2(x,y)(t) = \lambda_2 \int_0^1 G_2(t,s)a_2(s)f_2(s,x(s),y(s)ds, \quad 0 \le t \le 1. \end{aligned} \tag{2.13}$$

(x,y) is a positive solutions of system (1.1) if and only if (x,y) is a fixed point of T.

Lemma 2.4. [4] *Let E be a Banach space. Assume that $T : E \to E$ be a completely continuous operator. Let $V = \{x \in E | x = \mu Tx,\ 0 < \mu < 1\}$. Then either the set V is unbounded, or T has at least one fixed point.*

3. Main results

In what follows, we list the conditions to be used later:

($\mathbf{H}_0$) $\int_0^1 \Phi_i(s)a_i(s) < +\infty,\ i = 1,2.$

($\mathbf{H}_1$) There exist constants $\zeta_i,\ \eta_i \ge 0$, such that

$$|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)| \le \zeta_i|u_1 - u_2| + \eta_i|v_1 - v_2|,\ t \in [0,1],\ u_i,\ v_i \in [0,+\infty),\ i = 1,2.$$

($\mathbf{H}_2$) There exist constants $\tau_i > 0,\ \rho_i,\ \varrho_i \ge 0$, such that $\forall t \in [0,1],\ x,y \in [0,+\infty)$,

$$f_i(t,u,v) \le \tau_i + \rho_i|u| + \varrho_i|v|,\ t \in [0,1],\ u,\ v \in [0,+\infty),\ i = 1,2.$$

According to the Ascoli-Arzela theorem, by a routine discussion, we can prove that the following Lemma 3.1 holds.

Lemma 3.1. *Assume that ($\boldsymbol{H}_0$) hold. Then $T : X \to X$ is a completely continuous operator.*

Theorem 3.1 is about the uniqueness theorem of solutions. In the following proof, it mainly shows that T is a contraction operator, that is $\|T(x_2,y_2) - T(x_1,y_1)\| < \|x_2 - x_1\| + \|x_2 - y_1\|$.

Theorem 3.1. *Assume that ($\boldsymbol{H}_0$)($\boldsymbol{H}_1$) hold, $\Delta_1(\zeta_1 + \eta_1) + \Delta_2(\zeta_2 + \eta_2) < 1$, where*

$$\Delta_i = \lambda_i \int_0^1 \Phi_i(s)a_i(s)ds. \tag{3.1}$$

Then the system (1.1) has a unique solution.

Proof. Suppose

$$\sup f_i(t,0,0) = \varpi < +\infty,$$

by ($\mathbf{H}_1$), we have

$$f_i(t,u,v) \le \varpi_i + \zeta_i|u| + \eta_i|v|,\ i = 1,2.$$

Let

$$r=\frac{\Delta_1\varpi_1+\Delta_2\varpi_2}{1-\Delta_1(\zeta_1+\eta_1)-\Delta_2(\zeta_2+\eta_2)},\ P_r=\{(x,y)\in X:\|(x,y)\|<r\}.$$

Nest, we prove that $TP_r\subset P_r$. For any $(x,y)\in P_r$,

$$\begin{aligned}|T_1(x,y)(t)|&\le\max_{t\in[0,1]}\left|\lambda_1\int_0^1 G_1(t,s)a_1(s)f_1(s,x(s),y(s))ds\right|\\&\le\lambda_1\int_0^1\Phi_i(s)a_1(s)f_1(s,x(s),y(s)ds\\&\le\lambda_1\int_0^1\Phi_i(s)a_1(s)(\varpi_1+\zeta_1|x|+\eta_1|y|)ds\\&\le\Delta_1\left(\varpi_1+\zeta_1\|x\|+\eta_1\|y\|\right),\end{aligned}$$

therefore

$$\|T_1(x,y)\|\le\Delta_1\left(\varpi_1+\zeta_1\|x\|+\eta_1\|y\|\right). \tag{3.2}$$

By the similar proofing as (3.2), for any $(x,y)\in P_r$, we can obtain

$$\|T_2(x,y)\|\le\Delta_2\left(\varpi_2+\zeta_2\|x\|+\eta_2\|y\|\right). \tag{3.3}$$

Combining (3.2) and (3.3), we know

$$\begin{aligned}\|T(x,y)\|&=\|T_1(x,y)\|+\|T_2(x,y)\|\\&\le\Delta_1\left(\varpi_1+\zeta_1\|x\|+\eta_1\|y\|\right)+\Delta_2\left(\varpi_2+\zeta_2\|x\|+\eta_2\|y\|\right)\\&\le r.\end{aligned}$$

For any $(x_1,y_1),(x_2,y_2)\in X$, $t\in[0,1]$, by Lemma 2.3 and $(\mathbf{H}_1)$, we can get

$$\begin{aligned}&|T_1(x_2,y_2)(t)-T_1(x_1,y_1)(t)|\\&\le\lambda_1\int_0^1 G_1(t,s)a_1(s)\left|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))\right|ds\\&\le\lambda_1\int_0^1\Phi_1(s)a_1(s)\left|f_1(s,x_2(s),y_2(s))-f_1(s,x_1(s),y_1(s))\right|ds\\&\le\Delta_1\left(\zeta_1\|x_2-x_1\|+\eta_1\|y_2-y_1\|\right)\\&\le\Delta_1(\zeta_1+\eta_1)(\|x_2-x_1\|+\|y_2-y_1\|).\end{aligned}$$

Therefore, for $(x_1,y_1),(x_2,y_2)\in X$, we observe

$$\|T_1(x_2,y_2)-T_1(x_1,y_1)\|\le\Delta_1(\zeta_1+\eta_1)(\|x_2-x_1\|+\|y_2-y_1\|). \tag{3.4}$$

Similar proof to (3.4), for $(x_1,y_1),(x_2,y_2)\in X$, we achieve

$$\|T_2(x_2,y_2)-T_2(x_1,y_1)\|\le\Delta_2(\zeta_2+\eta_2)(\|x_2-x_1\|+\|y_2-y_1\|). \tag{3.5}$$

As can be seen from (3.4) and (3.5),

$$\|T(x_2,y_2)-T(x_1,y_1)\|\le\left(\Delta_1(\zeta_1+\eta_1)+\Delta_2(\zeta_2+\eta_2)\right)(\|x_2-x_1\|+\|x_2-y_1\|).$$

Owing to $\Delta_1(\zeta_1+\eta_1)+\Delta_2(\zeta_2+\eta_2)<1$, T is a contraction operator, thus T has a unique fixed point basing on the fixed point theorem of the contraction mapping principle, the system (1.1) has a unique solution. The proof is complete. □

Theorem 3.2 is about the existence of solutions. We prove that the set P in E is bounded, and then obtain the conclusion based on the Lemma 2.4.

Theorem 3.2. *Assume that* $(\boldsymbol{H}_0)(\boldsymbol{H}_2)$ *hold,* $(\Delta_1\rho_1+\Delta_2\rho_2)+(\Delta_1\varrho_1+\Delta_2\varrho_2)<1$*, where* Δ_1 *and* Δ_2 *are defined as (3.1). Then the system (1.1) has at least one solution.*

Proof. Suppose

$$P=\{(u,v)\in X:(u,v)=\chi T(u,v),\ 0\le\chi\le 1\}.$$

For $(u,v)\in V$, $(u,v)=\chi T(u,v)$, so we have

$$u=\chi T_1(u,v),\ v=\chi T_2(u,v).$$

Thus, for any $t\in[0,1]$, by Lemma (2.3) and $(\mathbf{H}_2)$, we obtain

$$\begin{aligned}|u(t)|&\le\left|\lambda_1\int_0^1 G_1(t,s)a_1(s)f_1(s,x(s),y(s))ds\right|\\&\le\lambda_1\int_0^1\Phi_1(s)a_1(s)f_1(s,x(s),y(s)ds\\&\le\Delta_1(\tau_1+\rho_1\|u\|+\varrho_1\|v\|),\end{aligned}\tag{3.6}$$

$$\begin{aligned}|v(t)|&\le\left|\lambda_2\int_0^1 G_2(t,s)a_2(s)f_2(s,x(s),y(s))ds\right|\\&\le\lambda_2\int_0^1\Phi_2(s)a_2(s)f_2(s,x(s),y(s)ds\\&\le\Delta_2(\tau_2+\rho_2\|u\|+\varrho_2\|v\|).\end{aligned}\tag{3.7}$$

As can be seen from (3.6) and (3.7),

$$\begin{aligned}\|u\|+\|v\|&\le\Delta_1(\tau_1+\rho_1\|u\|+\varrho_1\|v\|)+\Delta_2(\tau_2+\rho_2\|u\|+\varrho_2\|v\|)\\&=\Delta_1\tau_1+\Delta_2\tau_2+(\Delta_1\rho_1+\Delta_2\rho_2)\|u\|+(\Delta_1\varrho_1+\Delta_2\varrho_2)\|v\|.\end{aligned}$$

Thus

$$\|(u,v)\|=\|u\|+\|v\|\le\frac{\Delta_1\tau_1+\Delta_2\tau_2}{1-(\Delta_1\rho_1+\Delta_2\rho_2)-(\Delta_1\varrho_1+\Delta_2\varrho_2)}.$$

Therefore, P is bounded. By Lemma 2.4, the operator T has at least one fixed point. That is to say, the system (1.1) has at least one solution. The proof is complete. □

4. An example

An example is given to illustrate our main results. Consider the following problem:

$$\begin{cases}{}^cD^{\frac{8}{3}}x(t)+\dfrac{1}{5}t^{-\frac{1}{3}}(1-t)^{-\frac{5}{3}}f_1(t,x(t),y(t))=0,\\{}^cD^{\frac{9}{4}}y(t)+\dfrac{3}{7}t(1-t)^{-\frac{5}{4}}f_2(t,x(t),y(t))=0,\ 0<t<1,\\x(0)=x''(0)=0,\ x(1)=\displaystyle\int_0^1 x(s)ds,\\y'(0)=y''(0)=0,\ y(1)=\dfrac{1}{3}\displaystyle\int_0^1 y(s)ds^2.\end{cases}\tag{4.1}$$

Let

$$\alpha_1 = \frac{8}{3},\ \alpha_2 = \frac{9}{4},\ \lambda_1 = \frac{1}{5},\ \lambda_2 = \frac{3}{7},\ \mu_1 = 1,\ \mu_2 = \frac{1}{3},$$
$$a_1(t) = t^{-\frac{1}{3}}(1-t)^{-\frac{5}{3}},\ a_2(t) = t(1-t)^{-\frac{5}{4}},$$
$$b_1(s) = b_2(s) = 1,\ A_1(s) = s,\ A_2(s) = s^2.$$

In the following, we verify that conditions $(\mathbf{H}_0)$ and $(\mathbf{H}_2)$ are met, through calculation, we get

$$\chi_1 = \mu_1 \int_0^1 tb_1(t)dA_1(t) = \int_0^1 tdt = \frac{1}{2},\quad \chi_2 = \mu_2 \int_0^1 tb_2(t)dA_2(t) = \frac{2}{3}\int_0^1 t^2 dt = \frac{2}{9},$$
$$\int_0^1 \Phi_1(s)a_1(s) = \frac{2}{\Gamma\left(\frac{8}{3}\right)}\int_0^1 t^{-\frac{1}{3}}dt < +\infty,\quad \int_0^1 \Phi_2(s)a_2(s) = \frac{9}{7\Gamma\left(\frac{9}{4}\right)}\int_0^1 tdt < +\infty,$$

the condition $(\mathbf{H}_0)$ holds. For $t \in [0,1]$, $x, y \in [0, +\infty)$, take

$$f_1(t,x,y) = \frac{t}{2+e^t}\left(1 + \frac{1}{4}\sin x + \frac{1}{8}y\right),$$
$$f_2(t,x,y) = \frac{t}{(1+t)^3}\left(1 + 2x + 3y\right).$$

Notice that

$$|f_1(t,x,y)| = \left|\frac{t}{2+e^t}\left(1 + \frac{1}{4}\sin x + \frac{1}{8}y\right)\right| \le \frac{1}{2} + \frac{1}{8}|x| + \frac{1}{16}|y|,$$
$$|f_2(t,x,y)| = \left|\frac{t}{(1+t)^3}\left(1 + 2x + 3y\right)\right| \le \frac{1}{8} + \frac{1}{4}|x| + \frac{3}{8}|y|,$$

the condition $(\mathbf{H}_2)$ holds. What's more

$$\Delta_1 = \lambda_1 \int_0^1 \Phi_1(t)a_1(t)dt = \frac{1}{5}\cdot\frac{2}{\Gamma\left(\frac{8}{3}\right)}\int_0^1 t^{-\frac{1}{3}}dt \approx 0.3976,$$
$$\Delta_2 = \lambda_2 \int_0^1 \Phi_2(t)a_2(t)dt = \frac{3}{7}\cdot\frac{9}{7\Gamma\left(\frac{9}{4}\right)}\int_0^1 tdt \approx 0.2432,$$
$$(\Delta_1\rho_1 + \Delta_2\rho_2) + (\Delta_1\varrho_1 + \Delta_2\varrho_2) \approx 0.2266 < 1.$$

Therefore, all conditions of Theorem 3.2 are satisfied, by Theorem 3.2, the system (4.1) has at least one solution.

5. Conclusions

This article, we investigate positive solutions for the Caputo differential system involving Riemann-Stieltjes integral boundary conditions. By using the Leray-Schauder's nonlinear alternative theorem and the Banach contraction principle, we present the existence and uniqueness results of positive solutions (Theorems 3.1 and 3.2). Since f_i $(i = 1,\ 2)$ is the abstract function, in real world, there are a large class of functions that satisfying the conditions given in the article, which proves the effectiveness and feasibility of these theorems.

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