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DELAY-INDUCED HOPF BIFURCATION OF A FRACTIONAL-ORDER NEURAL NETWORK WITH BOTH NEUTRAL AND INERTIAL TERMS*

Xueying Shi[1], Xiaoping Chen[1,†] and Chengdai Huang[2]

Abstract This paper focuses on the stability and bifurcation in a fractional-order neutral-type inertial neural network with time delay. We mainly analyze the new system with time delay by using Cramer's rule to derive precise bifurcation conditions. The accuracy of the theoretical findings is ultimately confirmed through two numerical experiments. Moreover, the remarkable advantages of the fractional-order model are found in delaying the occurrence of inherent bifurcations and enhancing the stability performance.

Keywords Fractional-order, neural networks, stability, Hopf bifurcation, neutral and inertial terms.

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1. Introduction

Currently, research on fractional-order neural networks is garnering significant attention, as their qualitative analysis and containment research hold substantial value. This is largely due to the fact that Neural networks have attained extraordinary success across numerous domains, including image processing [20], combinatorial optimization [15], associative memory [21], pattern recognition [14] and so on. As we know, fractional calculus serves as a highly effective mathematical tool for depicting the inherent non-locality and hereditary characteristics of real materials and processes. Besides, it can accurately capture the memory characteristics of neural networks, thereby enhancing their realism [2, 25, 28]. Consequently, the research of fractional-order neural networks aligns more closely with reality. Time delay is a prevalent phenomenon that often leads to instability and oscillations within the neural network systems [7, 19, 24]. Given the existence of various types of time delays, each type manifests distinct dynamic characteristics. Hopf bifurcation is a significant dynamic issue that typically arises in a category of differential equation systems featuring nonlinear coupling terms. Within these systems, as a particular parameter undergoes variation, the stability of the systems will be altered. This induces remarkable changes in the behavior of the systems [8, 16, 18]. Thus, it is of paramount importance to investigate the bifurcation issues in different systems. We have conducted extensive research on

†The corresponding author.
[1]Department of Mathematics, Taizhou University, Jichuan Road, 225300 Taizhou, China
[2]School of Mathematics and Statistics, Xinyang Normal University, Nanhu Road, 464000 Xinyang, China
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Email: shixueying26@163.com(X. Shi), cxpnuaa@163.com(X. Chen), huangchengdai@163.com(C. Huang)

the asymptotic stability analysis of various types of neural networks with time-varying delays. The Hopfield neural network is the most prevalent type. Regarding the Hopfield neural network, relevant research has yielded significant findings, covering scenarios ranging from integer-order to fractional-order, from one-dimensional and two-dimensional to high-dimensional, and even from single-delay to multi-delay [9, 10, 23].

In 1997, Wheeler and Schieve pioneered the introduction of a second-order inertial neural network model [26]. They examined the stability, bifurcation, and chaos phenomena at the system's equilibrium point. It has been demonstrated that the dynamic behavior becomes more sophisticated when inertial terms are included in neural networks, which has sparked scholars' interest in studying inertial neural networks [1]. The issue of anti-synchronization in switched inertial neural networks with constant leakage delay and time-varying mixed delay was explored in [29]. In [22], the Lyapunov comparison principle was utilized to investigate the dynamic behavior of fractional-order generalized reaction-diffusion inertial neural networks with time delay. In the practical scenarios, the dynamic behavior of some systems is influenced not only by delays in their state variables but also by delays in the derivatives of these state variables. Such systems are specifically categorized as neutral systems. Consequently, the authors initiated research on neural networks featuring neutral time-varying delays and introduced delays in the neutral case [6]. When set against their integer-order counterparts in neural networks, neutral fractional-order neural networks exhibit superior stability performance. In contrast to traditional delayed neural networks, they can achieve shorter convergence times by selecting appropriate parameters. For instance, in [12], the authors concentrated on the stability analysis of fractional-order neural networks with neutral delay and effectively created bifurcation diagrams for various delays. It was demonstrated that both time delay and fractional order significantly affect the stability of the developed fractional-order neural networks. It was also observed that fractional-order neural networks with neutral delay can delay the occurrence of bifurcation. In [5], the Lyapunov functional method was employed to study the global exponential stability of neutral Cohen-Grossberg neural networks incorporating multiple discrete neutral delays with time-varying characteristics.

Building upon previous research, a new research domain has emerged, known as neutral inertial neural networks. In [4], the authors developed novel neutral-type differential inequalities and Lyapunov methods to investigate the global exponential dissipation rate of neutral-type BAM inertial neural networks with mixed time-varying delays. Furthermore, in [27], the authors explored the global dissipation issue of memristive neutral inertial neural networks with distributed and discrete time-varying delays. A review of existing literature reveals that research in this area is rather limited, with only a handful of papers available. In comparison with previous models, neutral inertial neural networks possess more intricate dynamic characteristics and hold greater research value.

The aim of this paper is to explore the dynamic characteristics of a fractional-order neutral-type inertial neural network with time delay. The key innovations of this study are outlined as follows.

(1) Currently, research on fractional-order neural networks that incorporate time delay, inertia term and neutral delay is quite limited. This paper extends the model in [13] by introducing neutral delay, thereby making it more aligned with the characteristics of realistic complicated networks.

(2) This paper explores the dynamic properties of the system under two conditions. We investigate the dynamic characteristics of the system without time delay and also analyze the

system that incorporates time delay. Subsequently, we derive precise criteria for identifying the bifurcation point.

(3) In our study, we find that fluctuations in time delay or fractional order can shift the bifurcation point of the system. By manipulating the variations of two orders, we analyze the changes in fractional order at the system's bifurcation point.

(4) It is observed that fractional-order neural networks can enhance the stability of the system more effectively compared to integer-order neural networks.

The primary structure of this paper is summarized as below. Section 2 introduces the definition of Caputo fractional derivative along with its associated Laplace transform. Section 3 constructs the main model studied in this paper. In Section 4, the conditions for the existence of Hopf bifurcation and the accurate expression of bifurcation points are obtained by selecting time delay as bifurcation parameter. In Section 5, the validity of the key findings is illustrated by two examples. Finally, a summary of this paper is provided in Section 6.

2. Principal conceptions

In this section, we introduce Caputo fractional derivative along with its corresponding Laplace transform.

Definition 2.1. [17] The definition of Caputo fractional derivative is given below

$$D^h f(t) = \frac{1}{\Gamma(\mu - h)} \int_0^t (t-s)^{\mu-h-1} f^{(\mu)}(s)ds,$$

where $\mu - 1 < h \leq \mu \in Z^+$ and $\Gamma(\cdot)$ is the Gamma function.

The corresponding Laplace transform of Caputo fractional derivative is

$$L\{D^h f(t); s\} = s^h F(s) - \sum_{k=0}^{\mu-1} s^{h-k-1} f^{(k)}(0), \quad \mu - 1 < h \leq \mu \in Z^+.$$

If $f^{(k)}(0) = 0$ with $k = 1, 2, \ldots, \mu$, then $L\{D^h f(t); s\} = s^h F(s)$.

3. Model formulation

In this section, the system is established to discuss the dynamical behavior. The model analyzed in this paper is given below.

$$\begin{cases} D^{h_1}\chi_1(t) = -m_1 D^{h_2}\chi_1(t) - p_1\chi_1(t) + \mu_{11}F_{11}(\chi_3(t-\varsigma)) + \mu_{12}F_{12}(\chi_4(t-\varsigma)) \\ \qquad\qquad + \alpha_{11}D^{h_3}\chi_3(t-\varsigma) + \alpha_{12}D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_2(t) = -m_2 D^{h_2}\chi_2(t) - p_2\chi_2(t) + \mu_{21}F_{21}(\chi_3(t-\varsigma)) + \mu_{22}F_{22}(\chi_4(t-\varsigma)) \\ \qquad\qquad + \alpha_{21}D^{h_3}\chi_3(t-\varsigma) + \alpha_{22}D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_3(t) = -m_3 D^{h_2}\chi_3(t) - p_3\chi_3(t) + \nu_{11}G_{11}(\chi_1(t-\varsigma)) + \nu_{12}G_{12}(\chi_2(t-\varsigma)) \\ \qquad\qquad + \beta_{11}D^{h_3}\chi_1(t-\varsigma) + \beta_{12}D^{h_3}\chi_2(t-\varsigma), \\ D^{h_1}\chi_4(t) = -m_4 D^{h_2}\chi_4(t) - p_4\chi_4(t) + \nu_{21}G_{21}(\chi_1(t-\varsigma)) + \nu_{22}G_{22}(\chi_2(t-\varsigma)) \\ \qquad\qquad + \beta_{21}D^{h_3}\chi_1(t-\varsigma) + \beta_{22}D^{h_3}\chi_2(t-\varsigma), \end{cases} \tag{3.1}$$

where $D^{h_1}\chi_\imath(t)$ symbolizes the inertial terms with $\imath = 1,2,3,4$, $\chi_\imath(t)$ denotes state variables with $\imath = 1,2,3,4$, $h_\imath$ stands for the fractional orders and $h_\imath \in (0,1]$ with $\imath = 1,2,3$, $m_\imath > 0$ is the coefficient of damping terms with $\imath = 1,2,3,4$, $p_\imath > 0$ represents self-regulating parameters with $\imath = 1,2,3,4$, $\mu_{\imath\jmath}$ and $\nu_{\imath\jmath}$ signifies connection weights with $\imath,\jmath = 1,2$, $\alpha_{\imath\jmath}$ and $\beta_{\imath\jmath}$ mean neutral connection weights with $\imath,\jmath = 1,2$, $F_{\imath\jmath}(\cdot)$ and $G_{\imath\jmath}(\cdot)$ indicate activation functions with $\imath,\jmath = 1,2$, and ς is communicate delay.

Remark 3.1. The system considered in this paper is revised from [13], which investigated the subject in the matter of bifurcations in a fractional-order neural network incorporating time delay and inertia terms. This paper adds neutral delay to the model in [13].

Remark 3.2. It is worth noting that if $\alpha_{\imath\jmath} = \beta_{\imath\jmath} = 0$ with $\imath,\jmath = 1,2$, system (3.1) becomes an inertial neural network. In [13], Huang et al. carefully studied this type. In addition, when $m_\imath = 0$ with $\imath = 1,2,3,4$, system (3.1) converts to the neutral type neural network. In [11], the fractional-order neutral bidirectional associative memory neural network under this type was considered.

Remark 3.3. It is noteworthy that system (3.1) can be transformed into the retarded fractional-order neural network when both $m_\imath = 0$ and $\alpha_{\imath\jmath} = \beta_{\imath\jmath} = 0$ are satisfied. It should be pointed out that system (3.1) is more complex and has higher research value.

Remark 3.4. In system (3.1), there is no need to sustain a fixed size relationship between h_1 and h_2. Whether $h_1 > h_2$ or $h_2 > h_1$, system (3.1) can also be turned into the form of a generalized fractional-order neutral-type inertial neural network by implementing some measures.

4. Key results

In this section, we will analyze the dynamic properties in two circumstances. The first scenario involves system (3.1) without time delay, while the second scenario focuses on investigating the system that incorporates time delay.

In order to continue the derivation, we are required to put forward the following premise.

(**H1**) $F_{\imath\jmath}(\cdot), G_{\imath\jmath}(\cdot) \in C(R,R)$, and $F_{\imath\jmath}(0) = G_{\imath\jmath}(0) = 0$ where $\imath,\jmath = 1,2$.

According to the hypothesis (**H1**), we notice that there is an equilibrium point $E^* = (0,0,0,0)$ for system (3.1). By the Laplace transformation, the linearization modality of system (3.1) can be denoted as

$$\begin{cases} D^{h_1}\chi_1(t) = -m_1 D^{h_2}\chi_1(t) - p_1\chi_1(t) + k_{11}\chi_3(t-\varsigma) + k_{12}\chi_4(t-\varsigma) \\ \qquad\qquad + \alpha_{11}D^{h_3}\chi_3(t-\varsigma) + \alpha_{12}D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_2(t) = -m_2 D^{h_2}\chi_2(t) - p_2\chi_2(t) + k_{21}\chi_3(t-\varsigma) + k_{22}\chi_4(t-\varsigma) \\ \qquad\qquad + \alpha_{21}D^{h_3}\chi_3(t-\varsigma) + \alpha_{22}D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_3(t) = -m_3 D^{h_2}\chi_3(t) - p_3\chi_3(t) + \lambda_{11}\chi_1(t-\varsigma) + \lambda_{12}\chi_2(t-\varsigma) \\ \qquad\qquad + \beta_{11}D^{h_3}\chi_1(t-\varsigma) + \beta_{12}D^{h_3}\chi_2(t-\varsigma), \\ D^{h_1}\chi_4(t) = -m_4 D^{h_2}\chi_4(t) - p_4\chi_4(t) + \lambda_{21}\chi_1(t-\varsigma) + \lambda_{22}\chi_2(t-\varsigma) \\ \qquad\qquad + \beta_{21}D^{h_3}\chi_1(t-\varsigma) + \beta_{22}D^{h_3}\chi_2(t-\varsigma), \end{cases} \tag{4.1}$$

where $k_{\imath\jmath} = \mu_{\imath\jmath}F'_{\imath\jmath}(0)$ and $\lambda_{\imath\jmath} = \nu_{\imath\jmath}G'_{\imath\jmath}(0)$ with $\imath,\jmath = 1,2$.

4.1. Case 1. $\varsigma = 0$

In this section, we analyze the dynamic characteristics of system (3.1) in the absence of time delay. Under this condition, system (4.1) transforms into

$$\begin{cases} D^{h_1}\chi_1(t) = -m_1 D^{h_2}\chi_1(t) - p_1\chi_1(t) + k_{11}\chi_3(t) + k_{12}\chi_4(t) + \alpha_{11}D^{h_3}\chi_3(t) + \alpha_{12}D^{h_3}\chi_4(t), \\ D^{h_1}\chi_2(t) = -m_2 D^{h_2}\chi_2(t) - p_2\chi_2(t) + k_{21}\chi_3(t) + k_{22}\chi_4(t) + \alpha_{21}D^{h_3}\chi_3(t) + \alpha_{22}D^{h_3}\chi_4(t), \\ D^{h_1}\chi_3(t) = -m_3 D^{h_2}\chi_3(t) - p_3\chi_3(t) + \lambda_{11}\chi_1(t) + \lambda_{12}\chi_2(t) + \beta_{11}D^{h_3}\chi_1(t) + \beta_{12}D^{h_3}\chi_2(t), \\ D^{h_1}\chi_4(t) = -m_4 D^{h_2}\chi_4(t) - p_4\chi_4(t) + \lambda_{21}\chi_1(t) + \lambda_{22}\chi_2(t) + \beta_{21}D^{h_3}\chi_1(t) + \beta_{22}D^{h_3}\chi_2(t). \end{cases} \tag{4.2}$$

The above system's characteristic equation (4.2) can be attained as

$$\begin{aligned} &\aleph_0 + \aleph_1 s^{h_1} + \aleph_2 s^{2h_1} + \aleph_3 s^{3h_1} + \aleph_4 s^{4h_1} + \aleph_5 s^{h_2} + \aleph_6 s^{2h_2} + \aleph_7 s^{3h_2} + \aleph_8 s^{4h_2} + \aleph_9 s^{h_3} + \aleph_{10} s^{2h_3} \\ &+ \aleph_{11} s^{3h_3} + \aleph_{12} s^{4h_3} + \aleph_{13} s^{h_1+h_2} + \aleph_{14} s^{h_1+h_3} + \aleph_{15} s^{h_2+h_3} + \aleph_{16} s^{h_1+h_2+h_3} + \aleph_{17} s^{h_1+2h_2} \\ &+ \aleph_{18} s^{h_1+2h_3} + \aleph_{19} s^{h_2+2h_3} + \aleph_{20} s^{2h_2+h_3} + \aleph_{21} s^{h_1+3h_2} + \aleph_{22} s^{2h_1+h_2} + \aleph_{23} s^{2h_1+h_3} \\ &+ \aleph_{24} s^{2h_1+2h_2} + \aleph_{25} s^{2h_1+2h_3} + \aleph_{26} s^{3h_1+h_2} + \aleph_{27} s^{h_1+h_2+2h_3} + \aleph_{28} s^{2h_2+2h_3} = 0, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \aleph_0 &= -k_{21}\lambda_{12}p_1p_4 - k_{22}\lambda_{22}p_1p_3 + p_1p_2p_3p_4 + k_{11}k_{22}\lambda_{11}\lambda_{22} - k_{11}\lambda_{11}p_2p_4 \\ &\quad - k_{12}k_{21}\lambda_{11}\lambda_{22} - k_{11}k_{22}\lambda_{12}\lambda_{21} + k_{12}k_{21}\lambda_{12}\lambda_{21} - k_{12}\lambda_{21}p_2p_3, \\ \aleph_1 &= -(k_{21}\lambda_{12}p_1 + k_{21}\lambda_{12}p_4 + k_{22}\lambda_{22}p_1 + k_{22}\lambda_{22}p_3 + k_{11}\lambda_{11}p_2 + k_{11}\lambda_{11}p_4 \\ &\quad + k_{12}\lambda_{21}p_2 + k_{12}\lambda_{21}p_3 - p_1p_2p_3 - p_1p_2p_4 - p_1p_3p_4 - p_2p_3p_4), \\ \aleph_2 &= -(k_{21}\lambda_{12} + k_{22}\lambda_{22} + k_{11}\lambda_{11} + k_{12}\lambda_{21} - p_1p_2 - p_1p_3 - p_1p_4 - p_2p_3 \\ &\quad - p_2p_4 - p_3p_4), \\ \aleph_3 &= p_1 + p_2 + p_3 + p_4, \\ \aleph_4 &= 1, \\ \aleph_5 &= -(k_{21}\lambda_{12}m_1p_4 + k_{21}\lambda_{12}m_4p_1 + k_{22}\lambda_{22}m_1p_3 + k_{22}\lambda_{22}m_3p_1 + k_{11}\lambda_{11}m_2p_4 \\ &\quad + k_{11}\lambda_{11}m_4p_2 + k_{12}\lambda_{21}m_2p_3 + k_{12}\lambda_{21}m_3p_2 - m_1p_2p_3p_4 - m_2p_1p_3p_4 \\ &\quad - m_3p_1p_2p_4 - m_4p_1p_2p_3), \\ \aleph_6 &= -(k_{21}\lambda_{12}m_1m_4 + k_{22}\lambda_{22}m_1m_3 + k_{11}\lambda_{11}m_2m_4 + k_{12}\lambda_{21}m_2m_3 - m_1m_2p_3p_4 \\ &\quad - m_1m_3p_2p_4 - m_1m_4p_2p_3 - m_2m_3p_1p_4 - m_2m_4p_1p_3 - m_3m_4p_1p_2), \\ \aleph_7 &= m_1m_2m_3p_4 + m_1m_2m_4p_3 + m_1m_3m_4p_2 + m_2m_3m_4p_1, \\ \aleph_8 &= m_1m_2m_3m_4, \\ \aleph_9 &= -(\alpha_{21}\lambda_{12}p_1p_4 + \alpha_{22}\lambda_{22}p_1p_3 + \beta_{12}k_{21}p_1p_4 + \beta_{22}k_{22}p_1p_3 - \alpha_{11}k_{22}\lambda_{11}\lambda_{22} \\ &\quad + \alpha_{12}k_{21}\lambda_{11}\lambda_{22} + \alpha_{21}k_{12}\lambda_{11}\lambda_{22} - \alpha_{22}k_{11}\lambda_{11}\lambda_{22} - \beta_{11}k_{11}k_{22}\lambda_{22} + \beta_{11}k_{11}p_2p_4 \\ &\quad + \beta_{11}k_{12}k_{21}\lambda_{22} - \beta_{22}k_{11}k_{22}\lambda_{11} + \beta_{22}k_{12}k_{21}\lambda_{11} + \alpha_{11}k_{22}\lambda_{12}\lambda_{21} - \alpha_{12}k_{21}\lambda_{12}\lambda_{21} \\ &\quad + \alpha_{12}\lambda_{21}p_2p_3 - \alpha_{21}k_{12}\lambda_{12}\lambda_{21} + \alpha_{22}k_{11}\lambda_{12}\lambda_{21} + \beta_{12}k_{11}k_{22}\lambda_{21} - \beta_{12}k_{12}k_{21}\lambda_{21} \\ &\quad + \alpha_{11}\lambda_{11}p_2p_4 + \beta_{21}k_{11}k_{22}\lambda_{12} - \beta_{21}k_{12}k_{21}\lambda_{12} + \beta_{21}k_{12}p_2p_3), \\ \aleph_{10} &= -(\alpha_{21}\beta_{12}p_1p_4 + \alpha_{22}\beta_{22}p_1p_3 - \alpha_{11}\alpha_{22}\lambda_{11}\lambda_{22} - \alpha_{11}\beta_{11}k_{22}\lambda_{22} + \alpha_{11}\beta_{11}p_2p_4 \\ &\quad - \alpha_{11}\beta_{22}k_{22}\lambda_{11} + \alpha_{12}\alpha_{21}\lambda_{11}\lambda_{22} + \alpha_{12}\beta_{11}k_{21}\lambda_{22} + \alpha_{12}\beta_{22}k_{21}\lambda_{11} + \alpha_{21}\beta_{11}k_{12}\lambda_{22} \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_{21}\beta_{22}k_{12}\lambda_{11} - \alpha_{22}\beta_{11}k_{11}\lambda_{22} - \alpha_{22}\beta_{22}k_{11}\lambda_{11} - \beta_{11}\beta_{22}k_{11}k_{22} + \beta_{11}\beta_{22}k_{12}k_{21}\\
&+ \alpha_{11}\alpha_{22}\lambda_{12}\lambda_{21} + \alpha_{11}\beta_{12}k_{22}\lambda_{21} + \alpha_{11}\beta_{21}k_{22}\lambda_{12} - \alpha_{12}\alpha_{21}\lambda_{12}\lambda_{21} - \alpha_{12}\beta_{12}k_{21}\lambda_{21}\\
&- \alpha_{12}\beta_{21}k_{21}\lambda_{12} + \alpha_{12}\beta_{21}p_2p_3 - \alpha_{21}\beta_{12}k_{12}\lambda_{21} - \alpha_{21}\beta_{21}k_{12}\lambda_{12} + \alpha_{22}\beta_{12}k_{11}\lambda_{21}\\
&+ \alpha_{22}\beta_{21}k_{11}\lambda_{12} + \beta_{12}\beta_{21}k_{11}k_{22} - \beta_{12}\beta_{21}k_{12}k_{21}),\\
\aleph_{11} &= \alpha_{11}\alpha_{22}\beta_{11}\lambda_{22} + \alpha_{11}\alpha_{22}\beta_{22}\lambda_{11} + \alpha_{11}\beta_{11}\beta_{22}k_{22} - \alpha_{12}\alpha_{21}\beta_{11}\lambda_{22} - \alpha_{12}\alpha_{21}\beta_{22}\lambda_{11}\\
&- \alpha_{12}\beta_{11}\beta_{22}k_{21} - \alpha_{21}\beta_{11}\beta_{22}k_{12} + \alpha_{22}\beta_{11}\beta_{22}k_{11} - \alpha_{11}\alpha_{22}\beta_{12}\lambda_{21} - \alpha_{11}\alpha_{22}\beta_{21}\lambda_{12}\\
&- \alpha_{11}\beta_{12}\beta_{21}k_{22} + \alpha_{12}\alpha_{21}\beta_{12}\lambda_{21} + \alpha_{12}\alpha_{21}\beta_{21}\lambda_{12} + \alpha_{12}\beta_{12}\beta_{21}k_{21} + \alpha_{21}\beta_{12}\beta_{21}k_{12}\\
&- \alpha_{22}\beta_{12}\beta_{21}k_{11},\\
\aleph_{12} &= \alpha_{11}\alpha_{22}\beta_{11}\beta_{22} - \alpha_{12}\alpha_{21}\beta_{11}\beta_{22} - \alpha_{11}\alpha_{22}\beta_{12}\beta_{21} + \alpha_{12}\alpha_{21}\beta_{12}\beta_{21},\\
\aleph_{13} &= -(k_{21}\lambda_{12}m_1 + k_{21}\lambda_{12}m_4 + k_{22}\lambda_{22}m_1 + k_{22}\lambda_{22}m_3 + k_{11}\lambda_{11}m_2 + k_{11}\lambda_{11}m_4\\
&+ k_{12}\lambda_{21}m_2 + k_{12}\lambda_{21}m_3 - m_1p_2p_3 - m_1p_2p_4 - m_1p_3p_4 - m_2p_1p_3 - m_2p_1p_4\\
&- m_2p_3p_4 - m_3p_1p_2 - m_3p_1p_4 - m_3p_2p_4 - m_4p_1p_2 - m_4p_1p_3 - m_4p_2p_3),\\
\aleph_{14} &= -(\alpha_{21}\lambda_{12}p_1 + \alpha_{21}\lambda_{12}p_4 + \alpha_{22}\lambda_{22}p_1 + \alpha_{22}\lambda_{22}p_3 + \beta_{12}k_{21}p_1 + \beta_{12}k_{21}p_4\\
&+ \beta_{22}k_{22}p_1 + \beta_{22}k_{22}p_3 + \beta_{11}k_{11}p_2 + \beta_{11}k_{11}p_4 + \beta_{21}k_{12}p_2 + \beta_{21}k_{12}p_3 + \alpha_{11}\lambda_{11}p_2\\
&+ \alpha_{11}\lambda_{11}p_4 + \alpha_{12}\lambda_{21}p_2 + \alpha_{12}\lambda_{21}p_3),\\
\aleph_{15} &= -(\alpha_{12}\lambda_{21}m_2p_3 + \alpha_{12}\lambda_{21}m_3p_2 + \alpha_{21}\lambda_{12}m_1p_4 + \alpha_{21}\lambda_{12}m_4p_1 + \alpha_{22}\lambda_{22}m_1p_3\\
&+ \alpha_{22}\lambda_{22}m_3p_1 + \alpha_{11}\lambda_{11}m_2p_4 + \alpha_{11}\lambda_{11}m_4p_2 + \beta_{21}k_{12}m_2p_3 + \beta_{21}k_{12}m_3p_2\\
&+ \beta_{12}k_{21}m_1p_4 + \beta_{12}k_{21}m_4p_1 + \beta_{22}k_{22}m_1p_3 + \beta_{22}k_{22}m_3p_1 + \beta_{11}k_{11}m_2p_4\\
&+ \beta_{11}k_{11}m_4p_2),\\
\aleph_{16} &= -(\beta_{21}k_{12}m_2 + \beta_{21}k_{12}m_3 + \beta_{12}k_{21}m_1 + \beta_{12}k_{21}m_4 + \beta_{22}k_{22}m_1 + \beta_{22}k_{22}m_3\\
&+ \beta_{11}k_{11}m_2 + \beta_{11}k_{11}m_4 + \alpha_{21}\lambda_{12}m_1 + \alpha_{21}\lambda_{12}m_4 + \alpha_{22}\lambda_{22}m_1 + \alpha_{22}\lambda_{22}m_3\\
&+ \alpha_{11}\lambda_{11}m_2 + \alpha_{11}\lambda_{11}m_4 + \alpha_{12}\lambda_{21}m_2 + \alpha_{12}\lambda_{21}m_3),\\
\aleph_{17} &= m_1m_2p_3 + m_1m_2p_4 + m_1m_3p_2 + m_1m_3p_4 + m_1m_4p_2 + m_1m_4p_3 + m_2m_3p_1\\
&+ m_2m_3p_4 + m_2m_4p_1 + m_2m_4p_3 + m_3m_4p_1 + m_3m_4p_2,\\
\aleph_{18} &= -(\alpha_{21}\beta_{12}p_1 + \alpha_{21}\beta_{12}p_4 + \alpha_{22}\beta_{22}p_1 + \alpha_{22}\beta_{22}p_3 + \alpha_{11}\beta_{11}p_2 + \alpha_{11}\beta_{11}p_4\\
&+ \alpha_{12}\beta_{21}p_2 + \alpha_{12}\beta_{21}p_3),\\
\aleph_{19} &= -(\alpha_{21}\beta_{12}m_1p_4 + \alpha_{21}\beta_{12}m_4p_1 + \alpha_{22}\beta_{22}m_1p_3 + \alpha_{22}\beta_{22}m_3p_1 + \alpha_{11}\beta_{11}m_2p_4\\
&+ \alpha_{11}\beta_{11}m_4p_2 + \alpha_{12}\beta_{21}m_2p_3 + \alpha_{12}\beta_{21}m_3p_2),\\
\aleph_{20} &= -(\alpha_{21}\lambda_{12}m_1m_4 + \alpha_{22}\lambda_{22}m_1m_3 + \alpha_{11}\lambda_{11}m_2m_4 + \alpha_{12}\lambda_{21}m_2m_3\\
&+ \beta_{12}k_{21}m_1m_4 + \beta_{22}k_{22}m_1m_3 + \beta_{11}k_{11}m_2m_4 + \beta_{21}k_{12}m_2m_3),\\
\aleph_{21} &= m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 + m_2m_3m_4,\\
\aleph_{22} &= m_1p_2 + m_1p_3 + m_1p_4 + m_2p_1 + m_2p_3 + m_2p_4 + m_3p_1 + m_3p_2 + m_3p_4\\
&+ m_4p_1 + m_4p_2 + m_4p_3,\\
\aleph_{23} &= -(\alpha_{21}\lambda_{12} + \alpha_{22}\lambda_{22} + \alpha_{11}\lambda_{11} + \alpha_{12}\lambda_{21} + \beta_{12}k_{21} + \beta_{22}k_{22} + \beta_{11}k_{11} + \beta_{21}k_{12}),\\
\aleph_{24} &= m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4,\\
\aleph_{25} &= -(\alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} + \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21}),\\
\aleph_{26} &= m_1 + m_2 + m_3 + m_4,\\
\aleph_{27} &= -(\alpha_{21}\beta_{12}m_1 + \alpha_{21}\beta_{12}m_4 + \alpha_{22}\beta_{22}m_1 + \alpha_{22}\beta_{22}m_3 + \alpha_{11}\beta_{11}m_2
\end{aligned}
$$

$$+ \alpha_{11}\beta_{11}m_4 + \alpha_{12}\beta_{21}m_2 + \alpha_{12}\beta_{21}m_3),$$
$$\aleph_{28} = -(\alpha_{21}\beta_{12}m_1m_4 + \alpha_{22}\beta_{22}m_1m_3 + \alpha_{11}\beta_{11}m_2m_4 + \alpha_{12}\beta_{21}m_2m_3).$$

Assuming that all roots s of Eq. (4.3) satisfy $|arg(s)| > h_\imath\pi/2$ with $\imath = 1, 2, 3$. According to Corollary 1 of [3], we can obtain that system (4.2) is asymptotically stable when $\varsigma = 0$.

4.2. Case 2. $\varsigma \neq 0$

This part delves into the bifurcation analysis of system (3.1) incorporating time delay. By applying the theoretical results presented in [3], the characteristic equation of system (4.1) can be expressed as the equation below

$$\Re_0(s) + \Re_1(s)e^{-2s\varsigma} + \Re_2(s)e^{-4s\varsigma} = 0, \tag{4.4}$$

where

$$\begin{aligned}
\Re_0 = {} & p_1p_2p_3p_4 + (p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4)s^{h_1} \\
& + (p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4)s^{2h_1} + (p_1 + p_2 + p_3 + p_4)s^{3h_1} + s^{4h_1} \\
& + (m_1p_2p_3p_4 + m_2p_1p_3p_4 + m_3p_1p_2p_4 + m_4p_1p_2p_3)s^{h_2} \\
& + (m_1m_2p_3p_4 + m_1m_3p_2p_4 + m_1m_4p_2p_3 + m_2m_3p_1p_4 + m_2m_4p_1p_3 + m_3m_4p_1p_2)s^{2h_2} \\
& + (m_1m_2m_3p_4 + m_1m_2m_4p_3 + m_1m_3m_4p_2 + m_2m_3m_4p_1)s^{3h_2} + m_1m_2m_3m_4s^{4h_2} \\
& + (m_1p_2p_3 + m_1p_2p_4 + m_1p_3p_4 + m_2p_1p_3 + m_2p_1p_4 + m_2p_3p_4 + m_3p_1p_2 + m_3p_1p_4 \\
& + m_3p_2p_4 + m_4p_1p_2 + m_4p_1p_3 + m_4p_2p_3)s^{h_1+h_2} + (m_1m_2p_3 + m_1m_2p_4 \\
& + m_1m_3p_2 + m_1m_3p_4 + m_1m_4p_2 + m_1m_4p_3 + m_2m_3p_1 + m_2m_3p_4 + m_2m_4p_1 \\
& + m_2m_4p_3 + m_3m_4p_1 + m_3m_4p_2)s^{h_1+2h_2} + (m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 \\
& + m_2m_3m_4)s^{h_1+3h_2} + (m_1p_2 + m_1p_3 + m_1p_4 + m_2p_1 + m_2p_3 + m_2p_4 + m_3p_1 \\
& + m_3p_2 + m_3p_4 + m_4p_1 + m_4p_2 + m_4p_3)s^{2h_1+h_2} + (m_1 + m_2 + m_3 + m_4)s^{3h_1+h_2} \\
& + (m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4)s^{2h_1+2h_2}, \\
\Re_1 = {} & -k_{11}\lambda_{11}p_2p_4 - k_{12}\lambda_{21}p_2p_3 - k_{21}\lambda_{12}p_1p_4 - k_{22}\lambda_{22}p_1p_3 - (k_{11}\lambda_{11}p_2 \\
& + k_{11}\lambda_{11}p_4 + k_{12}\lambda_{21}p_2 + k_{12}\lambda_{21}p_3 + k_{21}\lambda_{12}p_1 + k_{21}\lambda_{12}p_4 + k_{22}\lambda_{22}p_1 \\
& + k_{22}\lambda_{22}p_3)s^{h_1} - (k_{11}\lambda_{11}m_2p_4 + k_{11}\lambda_{11}m_4p_2 + k_{12}\lambda_{21}m_2p_3 + k_{12}\lambda_{21}m_3p_2 \\
& + k_{21}\lambda_{12}m_1p_4 + k_{21}\lambda_{12}m_4p_1 + k_{22}\lambda_{22}m_1p_3 + k_{22}\lambda_{22}m_3p_1)s^{h_2} - (\alpha_{11}\lambda_{11}p_2p_4 \\
& + \alpha_{12}\lambda_{21}p_2p_3 + \alpha_{21}\lambda_{12}p_1p_4 + \alpha_{22}\lambda_{22}p_1p_3 + \beta_{11}k_{11}p_2p_4 + \beta_{12}k_{21}p_1p_4 \\
& + \beta_{21}k_{12}p_2p_3 + \beta_{22}k_{22}p_1p_3)s^{h_3} - (k_{11}\lambda_{11} + k_{12}\lambda_{21} + k_{21}\lambda_{12} + k_{22}\lambda_{22})s^{2h_1} \\
& - (k_{11}\lambda_{11}m_2m_4 + k_{12}\lambda_{21}m_2m_3 + k_{21}\lambda_{12}m_1m_4 + k_{22}\lambda_{22}m_1m_3)s^{2h_2} \\
& - (\alpha_{11}\beta_{11}p_2p_4 + \alpha_{12}\beta_{21}p_2p_3 + \alpha_{21}\beta_{12}p_1p_4 + \alpha_{22}\beta_{22}p_1p_3)s^{2h_3} - (k_{11}\lambda_{11}m_2 \\
& + k_{11}\lambda_{11}m_4 + k_{12}\lambda_{21}m_2 + k_{12}\lambda_{21}m_3 + k_{21}\lambda_{12}m_1 + k_{21}\lambda_{12}m_4 + k_{22}\lambda_{22}m_1 \\
& + k_{22}\lambda_{22}m_3)s^{h_1+h_2} - (\alpha_{11}\lambda_{11}p_2 + \alpha_{11}\lambda_{11}p_4 + \alpha_{12}\lambda_{21}p_2 + \alpha_{12}\lambda_{21}p_3 \\
& + \alpha_{21}\lambda_{12}p_1 + \alpha_{21}\lambda_{12}p_4 + \alpha_{22}\lambda_{22}p_1 + \alpha_{22}\lambda_{22}p_3 + \beta_{11}k_{11}p_2 + \beta_{11}k_{11}p_4 \\
& + \beta_{12}k_{21}p_1 + \beta_{12}k_{21}p_4 + \beta_{21}k_{12}p_2 + \beta_{21}k_{12}p_3 + \beta_{22}k_{22}p_1 + \beta_{22}k_{22}p_3)s^{h_1+h_3} \\
& - (\alpha_{11}\lambda_{11}m_2p_4 + \alpha_{11}\lambda_{11}m_4p_2 + \alpha_{12}\lambda_{21}m_2p_3 + \alpha_{12}\lambda_{21}m_3p_2 + \alpha_{21}\lambda_{12}m_1p_4
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_{21}\lambda_{12}m_4p_1+\alpha_{22}\lambda_{22}m_1p_3+\alpha_{22}\lambda_{22}m_3p_1+\beta_{11}k_{11}m_2p_4+\beta_{11}k_{11}m_4p_2\\
&+\beta_{12}k_{21}m_1p_4+\beta_{12}k_{21}m_4p_1+\beta_{21}k_{12}m_2p_3+\beta_{21}k_{12}m_3p_2+\beta_{22}k_{22}m_1p_3\\
&+\beta_{22}k_{22}m_3p_1)s^{h_2+h_3}-(\alpha_{11}\lambda_{11}m_2+\alpha_{11}\lambda_{11}m_4+\alpha_{12}\lambda_{21}m_2+\alpha_{12}\lambda_{21}m_3\\
&+\alpha_{21}\lambda_{12}m_1+\alpha_{21}\lambda_{12}m_4+\alpha_{22}\lambda_{22}m_1+\alpha_{22}\lambda_{22}m_3+\beta_{11}k_{11}m_2+\beta_{11}k_{11}m_4\\
&+\beta_{12}k_{21}m_1+\beta_{12}k_{21}m_4+\beta_{21}k_{12}m_2+\beta_{21}k_{12}m_3+\beta_{22}k_{22}m_1+\beta_{22}k_{22}m_3)s^{h_1+h_2+h_3}\\
&-(\alpha_{11}\beta_{11}p_2+\alpha_{11}\beta_{11}p_4+\alpha_{12}\beta_{21}p_2+\alpha_{12}\beta_{21}p_3+\alpha_{21}\beta_{12}p_1+\alpha_{21}\beta_{12}p_4+\alpha_{22}\beta_{22}p_1\\
&+\alpha_{22}\beta_{22}p_3)s^{h_1+2h_3}-(\alpha_{11}\beta_{11}m_2p_4+\alpha_{11}\beta_{11}m_4p_2+\alpha_{12}\beta_{21}m_2p_3+\alpha_{12}\beta_{21}m_3p_2\\
&+\alpha_{21}\beta_{12}m_1p_4+\alpha_{21}\beta_{12}m_4p_1+\alpha_{22}\beta_{22}m_1p_3+\alpha_{22}\beta_{22}m_3p_1)s^{h_2+2h_3}\\
&-(\alpha_{11}\lambda_{11}+\alpha_{12}\lambda_{21}+\alpha_{21}\lambda_{12}+\alpha_{22}\lambda_{22}+\beta_{11}k_{11}+\beta_{12}k_{21}+\beta_{21}k_{12}+\beta_{22}k_{22})s^{2h_1+h_3}\\
&-(\alpha_{11}\lambda_{11}m_2m_4+\alpha_{12}\lambda_{21}m_2m_3+\alpha_{21}\lambda_{12}m_1m_4+\alpha_{22}\lambda_{22}m_1m_3\\
&+\beta_{11}k_{11}m_2m_4+\beta_{12}k_{21}m_1m_4+\beta_{21}k_{12}m_2m_3+\beta_{22}k_{22}m_1m_3)s^{2h_2+h_3}\\
&-(\alpha_{11}\beta_{11}+\alpha_{12}\beta_{21}+\alpha_{21}\beta_{12}+\alpha_{22}\beta_{22})s^{2h_1+2h_3}-(\alpha_{11}\beta_{11}m_2m_4+\alpha_{12}\beta_{21}m_2m_3\\
&+\alpha_{21}\beta_{12}m_1m_4+\alpha_{22}\beta_{22}m_1m_3)s^{2h_2+2h_3}-(\alpha_{11}\beta_{11}m_2+\alpha_{11}\beta_{11}m_4+\alpha_{12}\beta_{21}m_2\\
&+\alpha_{12}\beta_{21}m_3+\alpha_{21}\beta_{12}m_1+\alpha_{21}\beta_{12}m_4+\alpha_{22}\beta_{22}m_1+\alpha_{22}\beta_{22}m_3)s^{h_1+h_2+2h_3},
\end{aligned}
$$

$$
\begin{aligned}
\Re_2=\;&k_{11}k_{22}\lambda_{11}\lambda_{22}-k_{12}k_{21}\lambda_{11}\lambda_{22}-k_{11}k_{22}\lambda_{12}\lambda_{21}+k_{12}k_{21}\lambda_{12}\lambda_{21}+(\alpha_{11}k_{22}\lambda_{11}\lambda_{22}\\
&-\alpha_{11}k_{22}\lambda_{12}\lambda_{21}-\alpha_{12}k_{21}\lambda_{11}\lambda_{22}+\alpha_{12}k_{21}\lambda_{12}\lambda_{21}-\alpha_{21}k_{12}\lambda_{11}\lambda_{22}+\alpha_{21}k_{12}\lambda_{12}\lambda_{21}\\
&+\alpha_{22}k_{11}\lambda_{11}\lambda_{22}-\alpha_{22}k_{11}\lambda_{12}\lambda_{21}+\beta_{11}k_{11}k_{22}\lambda_{22}-\beta_{11}k_{12}k_{21}\lambda_{22}-\beta_{12}k_{11}k_{22}\lambda_{21}\\
&+\beta_{12}k_{12}k_{21}\lambda_{21}-\beta_{21}k_{11}k_{22}\lambda_{12}+\beta_{21}k_{12}k_{21}\lambda_{12}+\beta_{22}k_{11}k_{22}\lambda_{11}-\beta_{22}k_{12}k_{21}\lambda_{11})s^{h_3}\\
&+(\alpha_{11}\alpha_{22}\lambda_{11}\lambda_{22}-\alpha_{11}\alpha_{22}\lambda_{12}\lambda_{21}+\alpha_{11}\beta_{11}k_{22}\lambda_{22}-\alpha_{11}\beta_{12}k_{22}\lambda_{21}-\alpha_{11}\beta_{21}k_{22}\lambda_{12}\\
&+\alpha_{11}\beta_{22}k_{22}\lambda_{11}-\alpha_{12}\alpha_{21}\lambda_{11}\lambda_{22}+\alpha_{12}\alpha_{21}\lambda_{12}\lambda_{21}-\alpha_{12}\beta_{11}k_{21}\lambda_{22}+\alpha_{12}\beta_{12}k_{21}\lambda_{21}\\
&+\alpha_{12}\beta_{21}k_{21}\lambda_{12}-\alpha_{12}\beta_{22}k_{21}\lambda_{11}-\alpha_{21}\beta_{11}k_{12}\lambda_{22}+\alpha_{21}\beta_{12}k_{12}\lambda_{21}+\alpha_{21}\beta_{21}k_{12}\lambda_{12}\\
&-\alpha_{21}\beta_{22}k_{12}\lambda_{11}+\alpha_{22}\beta_{11}k_{11}\lambda_{22}-\alpha_{22}\beta_{12}k_{11}\lambda_{21}-\alpha_{22}\beta_{21}k_{11}\lambda_{12}+\alpha_{22}\beta_{22}k_{11}\lambda_{11}\\
&+\beta_{11}\beta_{22}k_{11}k_{22}-\beta_{11}\beta_{22}k_{12}k_{21}-\beta_{12}\beta_{21}k_{11}k_{22}+\beta_{12}\beta_{21}k_{12}k_{21})s^{2h_3}-(\alpha_{12}\beta_{11}\beta_{22}k_{21}\\
&-\alpha_{12}\beta_{12}\beta_{21}k_{21}+\alpha_{21}\beta_{11}\beta_{22}k_{12}-\alpha_{21}\beta_{12}\beta_{21}k_{12}-\alpha_{22}\beta_{11}\beta_{22}k_{11}+\alpha_{22}\beta_{12}\beta_{21}k_{11}\\
&-\alpha_{11}\alpha_{22}\beta_{11}\lambda_{22}+\alpha_{11}\alpha_{22}\beta_{12}\lambda_{21}+\alpha_{11}\alpha_{22}\beta_{21}\lambda_{12}-\alpha_{11}\alpha_{22}\beta_{22}\lambda_{11}-\alpha_{11}\beta_{11}\beta_{22}k_{22}\\
&+\alpha_{11}\beta_{12}\beta_{21}k_{22}+\alpha_{12}\alpha_{21}\beta_{11}\lambda_{22}-\alpha_{12}\alpha_{21}\beta_{12}\lambda_{21}-\alpha_{12}\alpha_{21}\beta_{21}\lambda_{12}+\alpha_{12}\alpha_{21}\beta_{22}\lambda_{11})s^{3h_3}\\
&+(\alpha_{11}\alpha_{22}\beta_{11}\beta_{22}-\alpha_{11}\alpha_{22}\beta_{12}\beta_{21}-\alpha_{12}\alpha_{21}\beta_{11}\beta_{22}+\alpha_{12}\alpha_{21}\beta_{12}\beta_{21})s^{4h_3}.
\end{aligned}
$$

Multiplying Eq. (4.4) by $e^{2s\varsigma}$ and $e^{4s\varsigma}$ respectively, yields

$$
\begin{cases}
\Re_0(s)e^{2s\varsigma}+\Re_1(s)+\Re_2(s)e^{-2s\varsigma}=0,\\
\Re_0(s)e^{4s\varsigma}+\Re_1(s)e^{2s\varsigma}+\Re_2(s)=0.
\end{cases}
\tag{4.5}
$$

We postulate that Eq. (4.4) has one imaginary root, which is $s=w(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})(w>0)$. For the convenience of calculation, we label the real part and imaginary part of $\Re_\imath(s)(\imath=0,1,2)$ as $\Re_\imath^R$ and $\Re_\imath^I$, respectively. Then the following results can be derived

$$
\begin{cases}
\kappa_{11}\cos 4w\varsigma+\kappa_{12}\sin 4w\varsigma+\kappa_{13}\cos 2w\varsigma+\kappa_{14}\sin 2w\varsigma=\kappa_{15},\\
\kappa_{21}\cos 4w\varsigma+\kappa_{22}\sin 4w\varsigma+\kappa_{23}\cos 2w\varsigma+\kappa_{24}\sin 2w\varsigma=\kappa_{25},\\
\kappa_{31}\cos 4w\varsigma+\kappa_{32}\sin 4w\varsigma+\kappa_{33}\cos 2w\varsigma+\kappa_{34}\sin 2w\varsigma=\kappa_{35},\\
\kappa_{41}\cos 4w\varsigma+\kappa_{42}\sin 4w\varsigma+\kappa_{43}\cos 2w\varsigma+\kappa_{44}\sin 2w\varsigma=\kappa_{45},
\end{cases}
\tag{4.6}
$$

where

$$\begin{aligned}
&\kappa_{11}=0, &&\kappa_{12}=0, &&\kappa_{13}=\Re_0^R+\Re_2^R, &&\kappa_{14}=\Re_2^I-\Re_0^I, &&\kappa_{15}=-\Re_1^R,\\
&\kappa_{21}=0, &&\kappa_{22}=0, &&\kappa_{23}=\Re_0^I+\Re_2^I, &&\kappa_{24}=\Re_0^R-\Re_2^R, &&\kappa_{25}=-\Re_1^I,\\
&\kappa_{31}=\Re_0^R, &&\kappa_{32}=-\Re_0^I, &&\kappa_{33}=\Re_1^R, &&\kappa_{34}=-\Re_1^I, &&\kappa_{35}=-\Re_2^R,\\
&\kappa_{41}=\Re_0^I, &&\kappa_{42}=\Re_0^R, &&\kappa_{43}=\Re_1^I, &&\kappa_{44}=\Re_1^R, &&\kappa_{45}=-\Re_2^I.
\end{aligned}$$

After the above substitutions, the determinant of the coefficient matrix of Eq. (4.6) can be abbreviated as

$$\varrho_1=\begin{vmatrix}\kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14}\\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{24}\\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{34}\\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{44}\end{vmatrix}.$$

To ensure the existence and uniqueness of solutions to Eq. (4.6), we assume that the coefficient matrix associated with Eq. (4.6) is invertible. Then we exploit Cramer's rule to obtain the following system:

$$\begin{cases}\cos 2w\varsigma=\dfrac{\varrho_2}{\varrho_1}=\Xi_1(w),\\ \sin 2w\varsigma=\dfrac{\varrho_3}{\varrho_1}=\Xi_2(w),\end{cases}\tag{4.7}$$

where

$$\varrho_2=\begin{vmatrix}\kappa_{11} & \kappa_{12} & \kappa_{15} & \kappa_{14}\\ \kappa_{21} & \kappa_{22} & \kappa_{25} & \kappa_{24}\\ \kappa_{31} & \kappa_{32} & \kappa_{35} & \kappa_{34}\\ \kappa_{41} & \kappa_{42} & \kappa_{45} & \kappa_{44}\end{vmatrix},$$

$$\varrho_3=\begin{vmatrix}\kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{15}\\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{25}\\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{35}\\ \kappa_{41} & \kappa_{42} & \kappa_{43} & \kappa_{45}\end{vmatrix}.$$

It follows that

$$\Xi_1^2(w)+\Xi_2^2(w)=1,\tag{4.8}$$

$$\cos 2w\varsigma=\Xi_1(w).\tag{4.9}$$

In order to make sure the fidelity of this conclusion, we commit the following assumption.

(**H2**) Eq. (4.8) possesses one positive real root.

On the basis of Eq. (4.9), we can derive the expression of $\varsigma^{(\iota)}$ as

$$\varsigma_1^{(\iota)}=\frac{1}{2w}\Big[\arccos\Xi_1(w)+2\iota\pi\Big],\quad \iota=0,1,2,\dots.\tag{4.10}$$

Then the bifurcation point can be signalled as

$$\varsigma_0 = \min\{\varsigma_1^{(\iota)}\}, \quad \iota = 0, 1, 2, \dots,$$

where $\varsigma_1^{(\iota)}$ is defined by Eq. (4.10).

Remark 4.1. What we need to know is that the method in [13] can still be used in this paper, but we adopt analytical methods of characteristic equations and Cramer's rule to derive Hopf bifurcation criteria.

To establish the conditions for determining the bifurcation, we further propose the following assumption.

(**H3**) $\frac{\wp_1\Im_1+\wp_2\Im_2}{\Im_1^2+\Im_2^2} \neq 0$, where $\wp_1$,$\wp_2$, $\Im_1$ and $\Im_2$ are characterized in Eq. (4.14).

Lemma 4.1. *Let $s(\varsigma) = \zeta(\varsigma) + iw(\varsigma)$ be the root of Eq. (4.4) near $\varsigma = \varsigma_0$ complying with $\zeta(\varsigma_0) = 0$ and $w(\varsigma_0) = w_0$. Then the transversality condition $\mathrm{Re}\Big[\frac{ds}{d\varsigma}\Big]\Big|_{(\varsigma=\varsigma_0,w=w_0)} \neq 0$ is tenable.*

Proof. Taking the derivative of both sides of Eq. (4.4) with respect to ς by the implicit function theorem, we can derive

$$\begin{aligned} &\Re_0'(s)\frac{ds}{d\varsigma} + \Big[\Re_1'(s)\frac{ds}{d\varsigma}e^{-2s\varsigma} + \Re_1(s)e^{-2s\varsigma}\Big(-2\varsigma\frac{ds}{d\varsigma} - 2s\Big)\Big] \\ &+ \Big[\Re_2'(s)\frac{ds}{d\varsigma}e^{-4s\varsigma} + \Re_2(s)e^{-4s\varsigma}\Big(-4\varsigma\frac{ds}{d\varsigma} - 4s\Big)\Big] = 0. \end{aligned} \tag{4.11}$$

Then we have

$$\frac{ds}{d\varsigma} = \frac{\wp(s)}{\Im(s)}, \tag{4.12}$$

where

$$\begin{aligned} \wp(s) &= s[2\Re_1(s)e^{-2s\varsigma} + 4\Re_2(s)e^{-4s\varsigma}], \\ \Im(s) &= \Re_0'(s) + [\Re_1'(s) - 2\varsigma\Re_1(s)]e^{-2s\varsigma} + [\Re_2'(s) - 4\varsigma\Re_2(s)]e^{-4s\varsigma}. \end{aligned} \tag{4.13}$$

It follows that

$$\mathrm{Re}\Big[\frac{ds}{d\varsigma}\Big]\Big|_{(\varsigma=\varsigma_0,w=w_0)} = \frac{\wp_1\Im_1 + \wp_2\Im_2}{\Im_1^2 + \Im_2^2}, \tag{4.14}$$

where

$$\begin{aligned} \wp_1 &= w_0(2\Re_1^R \sin 2w_0\varsigma_0 - 2\Re_1^I \cos 2w_0\varsigma_0 + 4\Re_2^R \sin 4w_0\varsigma_0 - 4\Re_2^I \cos 4w_0\varsigma_0), \\ \wp_2 &= w_0(2\Re_1^R \cos 2w_0\varsigma_0 + 2\Re_1^I \sin 2w_0\varsigma_0 + 4\Re_2^R \cos 4w_0\varsigma_0 + 4\Re_2^I \sin 4w_0\varsigma_0), \\ \Im_1 &= \Re_0'^R + (\Re_1'^R - 2\varsigma_0\Re_1^R)\cos 2w_0\varsigma_0 + (\Re_1'^I - 2\varsigma_0\Re_1^I)\sin 2w_0\varsigma_0 \\ &\quad + (\Re_2'^R - 4\varsigma_0\Re_2^R)\cos 4w_0\varsigma_0 + (\Re_2'^I - 4\varsigma_0\Re_2^I)\sin 4w_0\varsigma_0, \\ \Im_2 &= \Re_0'^I + (Q_1'^I - 2\varsigma_0\Re_1^I)\cos 2w_0\varsigma_0 - (\Re_1'^R - 2\varsigma_0\Re_1^R)\sin 2w_0\varsigma_0 \\ &\quad + (\Re_2'^I - 4\varsigma_0\Re_2^I)\cos 4w_0\varsigma_0 - (\Re_2'^R - 4\varsigma_0\Re_2^R)\sin 4w_0\varsigma_0. \end{aligned}$$

Under the condition that (**H3**) is constructed, Lemma 4.1 holds true. □

The above process indicates that the following theorem is correct.

Theorem 4.1. *By using hypotheses **H1** − **H3**, the following main results are achieved.*

(1) If the $\varsigma \in [0, \varsigma_0)$ is satisfied, the equilibrium point E^* of system (3.1) is asymptotically stable.

(2) When $\varsigma = \varsigma_0$, system (3.1) undergoes Hopf bifurcation at the equilibrium point E^*.

Remark 4.2. We study the factors that affect system stability through two numerical simulations. The results indicate that except time delay, changes in fractional order can also influence the stability of system (3.1). We change one fractional order and control the other two fractional orders to further analyze the spillover effects of fractional order changes at the bifurcation points of the system. For system (5.1) and system (5.2), the impact of sequential changes on ς are shown in Figure 5 and Figure 14. With the increase of h_1 and h_2, the value of system bifurcation can be reduced, and the increase of h_3 will delay the bifurcation time. Hence, the variation of fractional order may lead to premature or delayed bifurcations of the system.

Remark 4.3. In Section 5, we compare the simulation results of the fractional-order neural networks and corresponding integer-order neural networks. The continuous state diagrams of the two systems are shown in the Figures 6-9 and Figures 15-18. It can be observed that the bifurcation of integer-order neural networks emerges ahead of time compared with fractional-order neural networks.

5. Experimental verifications

This section verifies the correctness of the previous theoretical results through two numerical examples.

5.1. Example 1

In this example, we take the following system into consideration.

$$\begin{cases} D^{h_1}\chi_1(t) = -0.2D^{h_2}\chi_1(t) - 0.5\chi_1(t) - 0.4F_{11}(\chi_3(t-\varsigma)) - 0.4F_{12}(\chi_4(t-\varsigma)) \\ \qquad\qquad - 0.9D^{h_3}\chi_3(t-\varsigma) - 0.4D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_2(t) = -0.3D^{h_2}\chi_2(t) - 0.5\chi_2(t) + 0.2F_{21}(\chi_3(t-\varsigma)) - 0.2F_{22}(\chi_4(t-\varsigma)) \\ \qquad\qquad - 0.8D^{h_3}\chi_3(t-\varsigma) - 0.3D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_3(t) = -0.7D^{h_2}\chi_3(t) - 0.6\chi_3(t) + 0.6G_{11}(\chi_1(t-\varsigma)) - 0.2G_{12}(\chi_2(t-\varsigma)) \\ \qquad\qquad - 0.1D^{h_3}\chi_1(t-\varsigma) - 0.4D^{h_3}\chi_2(t-\varsigma), \\ D^{h_1}\chi_4(t) = -0.3D^{h_2}\chi_4(t) - 0.8\chi_4(t) + 1.8G_{21}(\chi_1(t-\varsigma)) - 0.7G_{22}(\chi_2(t-\varsigma)) \\ \qquad\qquad - 0.9D^{h_3}\chi_1(t-\varsigma) - 0.6D^{h_3}\chi_2(t-\varsigma). \end{cases} \tag{5.1}$$

The activation functions are set $F_{\imath\jmath}(\cdot) = G_{\imath\jmath}(\cdot) = \tanh(\cdot)$, $\imath, \jmath = 1, 2$. The incipient values are prescribed as $(\chi_1(0), \chi_2(0), \chi_3(0), \chi_4(0)) = (0.2, 0.3, -0.3, 0.2)$, $h_1 = 0.95, h_2 = 0.96$ and $h_3 = 0.97$. When $\varsigma \neq 0$, we specify ς as a bifurcation parameter. By the above deduction, it can be calculated that $w_0 = 0.726$ and $\varsigma_0 = 0.9733$.

The waveform and phase diagrams of $\varsigma = 0.85 < \varsigma_0 = 0.9733$ under stable conditions are simulated in Figures 1-2 and the correctness of the first result in Theorem 4.1 is verified. Similarly, regarding the instability of $\varsigma = 1.1 > \varsigma_0 = 0.9733$ as shown in Figures 3-4, it follows the description of the second result in Theorem 4.1.

Figure 5 displays the difference in ς_0 caused by fractional-order changes. It can be seen that as h_1 and h_2 increase, the value of system bifurcation will reduce, and the enhancement of h_3 will delay the bifurcation time. Meanwhile, it shows that when $h_1 = h_2 = h_3 = 1$, the bifurcation point of system (5.1) is calculated to be $\varsigma_0{}^* = 0.7774 < \varsigma_0 = 0.9733$. The results as shown in Figures 6-9 reveal that fractional-order neural networks have better stability compared to integer-order neural networks.

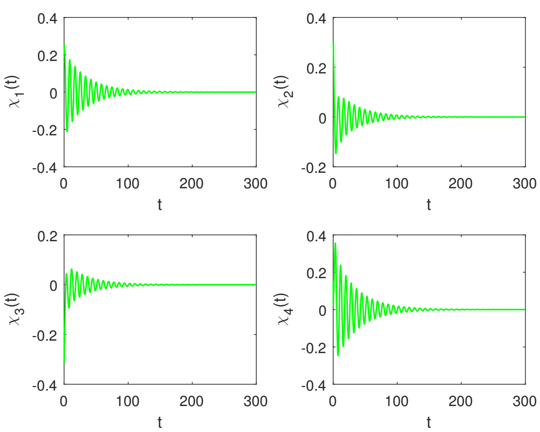

Figure 1. The waveform diagrams of system (5.1) with $\varsigma = 0.85 < \varsigma_0 = 0.9733$.

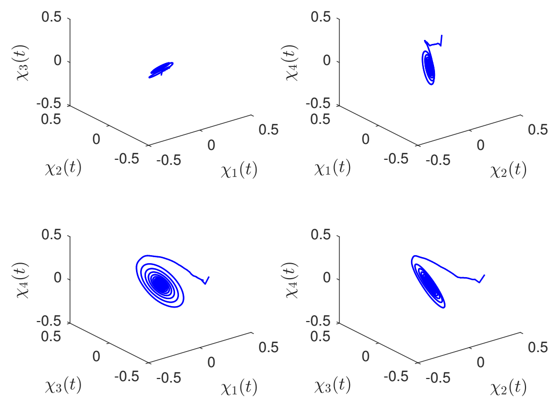

Figure 2. The phase diagrams of system (5.1) with $\varsigma = 0.85 < \varsigma_0 = 0.9733$.

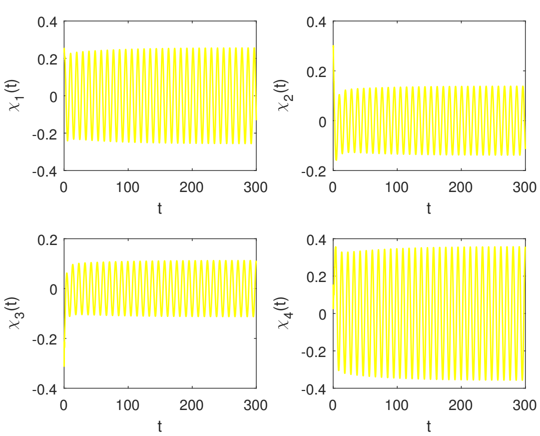

Figure 3. The waveform diagrams of system (5.1) with $\varsigma = 1.1 > \varsigma_0 = 0.9733$.

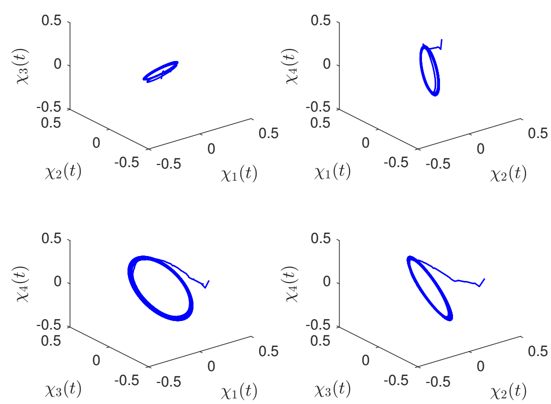

Figure 4. The phase diagrams of system (5.1) with $\varsigma = 1.1 > \varsigma_0 = 0.9733$.

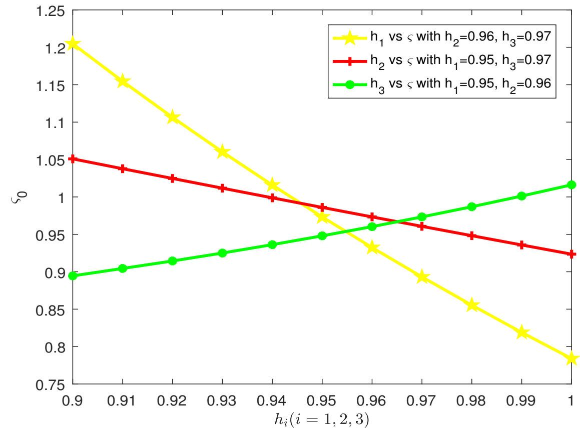

Figure 5. Influence of h_i on ς_0 of system (5.1).

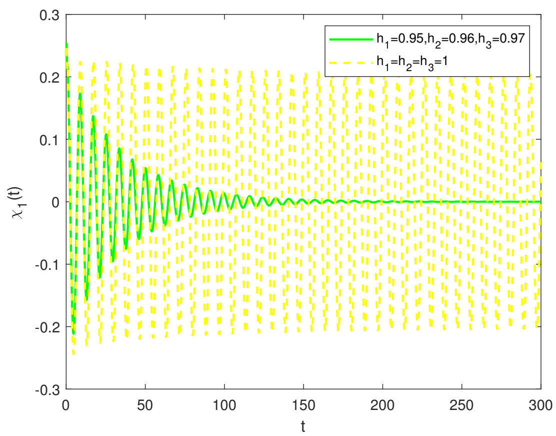

Figure 6. Comparison on the stability of system (5.1) in integer-order and fractional-order.

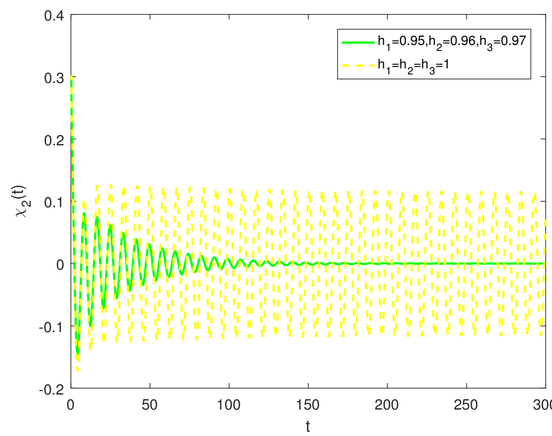

Figure 7. Comparison on the stability of system (5.1) in integer-order and fractional-order.

5.2. Example 2

In this example, the following system is studied.

$$\begin{cases} D^{h_1}\chi_1(t) = -0.3D^{h_2}\chi_1(t) - 0.3\chi_1(t) - 0.5F_{11}(\chi_3(t-\varsigma)) - 0.4F_{12}(\chi_4(t-\varsigma)) \\ \qquad\qquad + 0.8D^{h_3}\chi_3(t-\varsigma) - 0.5D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_2(t) = -0.4D^{h_2}\chi_2(t) - 0.5\chi_2(t) + 0.3F_{21}(\chi_3(t-\varsigma)) - 0.5F_{22}(\chi_4(t-\varsigma)) \\ \qquad\qquad - 0.6D^{h_3}\chi_3(t-\varsigma) - 0.4D^{h_3}\chi_4(t-\varsigma), \\ D^{h_1}\chi_3(t) = -0.6D^{h_2}\chi_3(t) - 0.4\chi_3(t) + 0.5G_{11}(\chi_1(t-\varsigma)) - 0.2G_{12}(\chi_2(t-\varsigma)) \\ \qquad\qquad - 0.5D^{h_3}\chi_1(t-\varsigma) - 0.4D^{h_3}\chi_2(t-\varsigma), \\ D^{h_1}\chi_4(t) = -0.4D^{h_2}\chi_4(t) - 0.7\chi_4(t) + 1.5G_{21}(\chi_1(t-\varsigma)) - 0.3G_{22}(\chi_2(t-\varsigma)) \\ \qquad\qquad - 0.6D^{h_3}\chi_1(t-\varsigma) - 0.5D^{h_3}\chi_2(t-\varsigma). \end{cases} \tag{5.2}$$

The activation functions are set $F_{\imath\jmath}(\cdot) = G_{\imath\jmath}(\cdot) = \tanh(\cdot)$, $\imath, \jmath = 1, 2$. The incipient values are

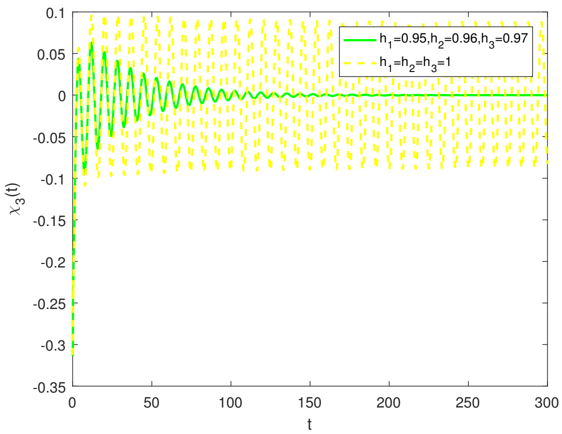

Figure 8. Comparison on the stability of system (5.1) in integer-order and fractional-order.

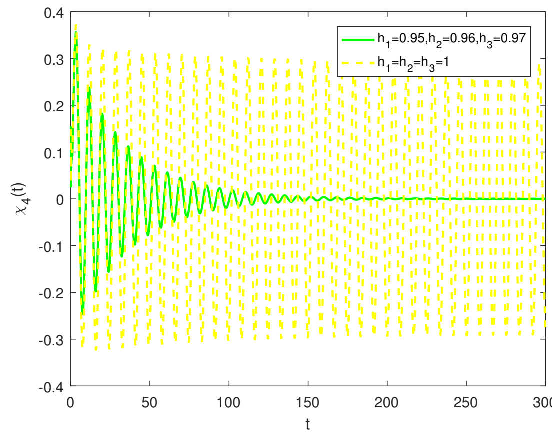

Figure 9. Comparison on the stability of system (5.1) in integer-order and fractional-order.

prescribed as $(\chi_1(0), \chi_2(0), \chi_3(0), \chi_4(0)) = (0.2, 0.2, 0.3, 0.4)$, $h_1 = 0.91, h_2 = 0.94$ and $h_3 = 0.96$. When $\varsigma \neq 0$, we specify ς as a bifurcation parameter. By the above deduction, it can be calculated that $w_0 = 0.4894$ and $\varsigma_0 = 0.8025$.

The waveform and phase diagrams of $\varsigma = 0.65 < \varsigma_0 = 0.8025$ under stable conditions are simulated in Figures 10-11 and the correctness of the first clause in Theorem 4.1 is verified. Similarly, regarding the instability of $\varsigma = 0.9 > \varsigma_0 = 0.8025$ as shown in Figures 12-13, it follows the description of the second clause in Theorem 4.1.

Figure 14 displays the difference in ς_0 caused by the changes of fractional order. We can see that as h_1 and h_2 increase, the value of system bifurcation will reduce, and the enhancement of h_3 will delay the bifurcation time. Meanwhile, we find that the bifurcation point of system (5.2) is calculated to be $\varsigma_0{}^* = 0.4338 < \varsigma_0 = 0.8025$ when $h_1 = h_2 = h_3 = 1$. The results as shown in Figures 15-18 reveal that fractional-order neural networks have better stability compared to integer-order neural networks.

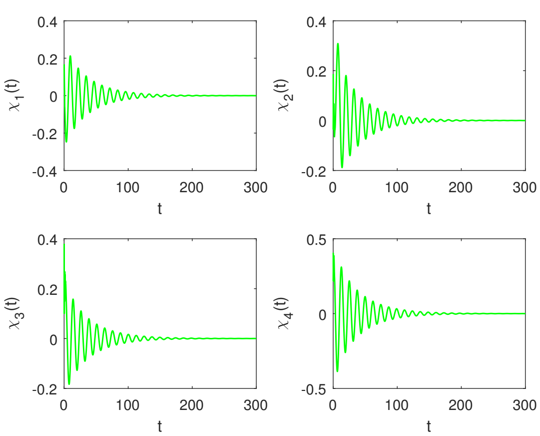

Figure 10. The waveform diagrams of system (5.2) with $\varsigma = 0.65 < \varsigma_0 = 0.8025$.

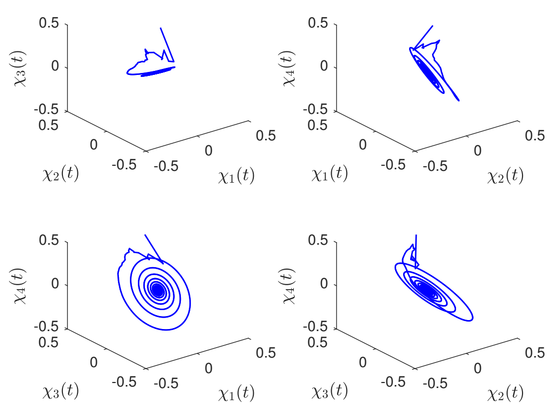

Figure 11. The phase diagrams of system (5.2) with $\varsigma = 0.65 < \varsigma_0 = 0.8025$.

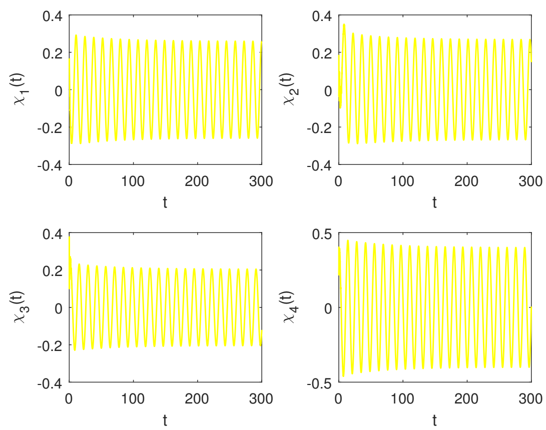

Figure 12. The waveform diagrams of system (5.2) with $\varsigma = 0.9 > \varsigma_0 = 0.8025$.

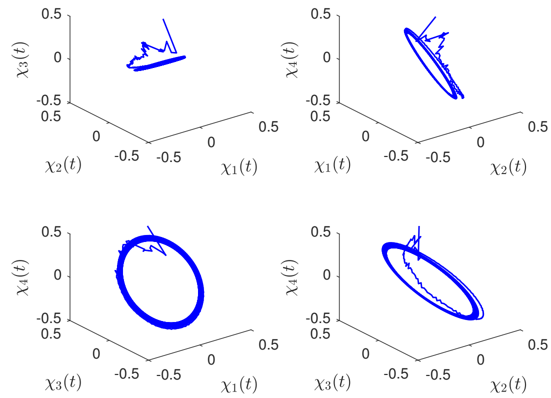

Figure 13. The phase diagrams of system (5.2) with $\varsigma = 0.9 > \varsigma_0 = 0.8025$.

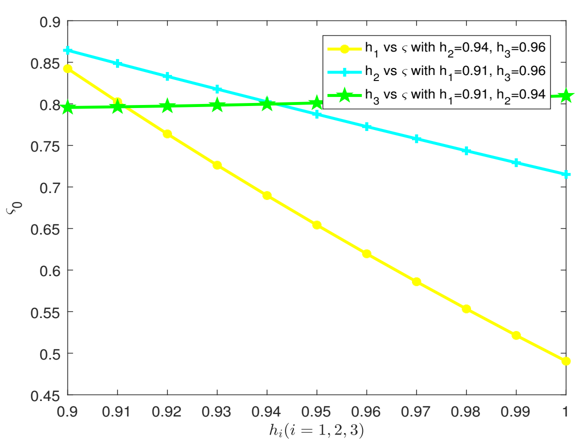

Figure 14. Influence of h_i on ς_0 of system (5.2).

6. Conclusion

This paper has examined the dynamic properties of an inertial neutral fractional-order neural network. The primary findings of this paper encompass the following aspects. By applying Cramer's rule, the characteristic equation has been analyzed. And the stability results and bifurcation conditions for the above system have been obtained. We further analyzed the spillover effects of fractional-order changes at system's bifurcation points by changing one fractional order and controlling the other two fractional orders. We also have compared the bifurcations of fractional-order neural networks and corresponding integer-order neural networks, and found that fractional-order neural networks can better improve the stability of the system compared to integer-order neural networks.

According to this paper, there are latent and worthwhile research directions that can be explored. (1) Based on previous studies, leakage delay has a significant impact on the stability of neural networks. In the future models, the influence of leakage delay on bifurcations can be considered. (2) Time delays are inherently present in neural networks. Although this paper focused on equal time delay, models with unequal time delays are equally important. We will

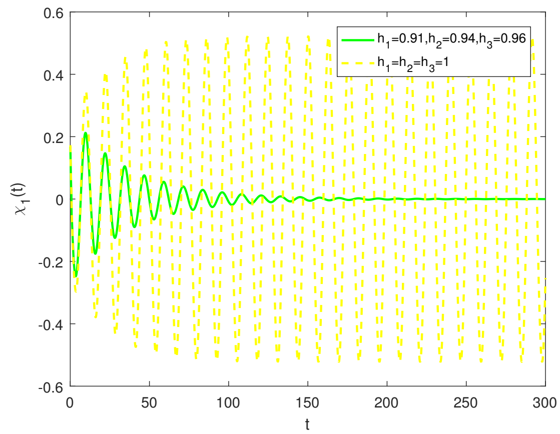

Figure 15. Comparison on the stability of system (5.2) in integer-order and fractional-order.

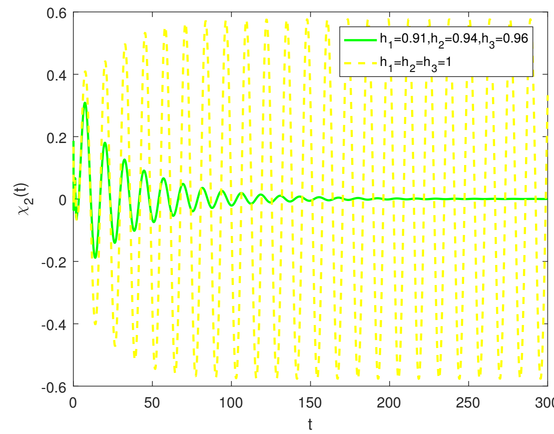

Figure 16. Comparison on the stability of system (5.2) in integer-order and fractional-order.

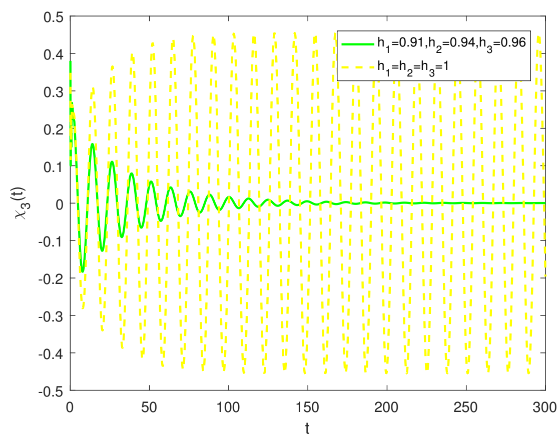

Figure 17. Comparison on the stability of system (5.2) in integer-order and fractional-order.

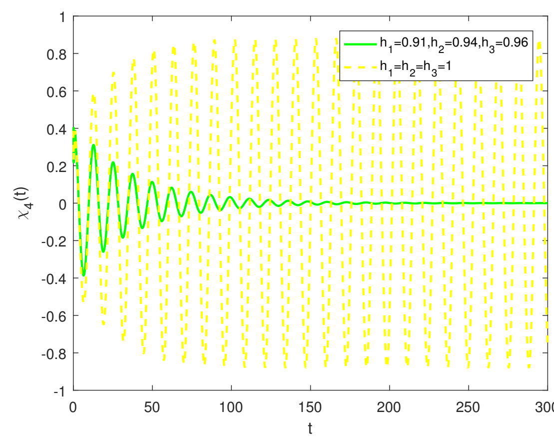

Figure 18. Comparison on the stability of system (5.2) in integer-order and fractional-order.

further investigate this more complex model.

Conflicts of interest. The authors declare no conflict of interest.

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