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THE ASYMPTOTIC BOUNDS OF SOLUTIONS OF A GENERALIZED PANTOGRAPH EQUATION

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Abstract This paper studies the asymptotic behavior of solutions of the generalized pantograph equation $y'(t) = Ay(qt) + By(t) + Cy'(qt)$, where A, B, C are $n \times n$ complex matrices. By considering two cases of the coefficient matrix B: Diagonalizable and non-diagonalizable, the asymptotic boundaries of solutions are discussed, respectively. When B is diagonalizable, the asymptotic boundary of solutions is dominated by the largest positive real part of the eigenvalues: If the smallest positive real part of eigenvalues exceeds the product of the delay parameter and the largest positive real part, then the components of solutions grow exponentially according to the corresponding eigenvalues, otherwise, all solutions are constrained by the largest positive real part of eigenvalues. When B cannot be diagonalized, the asymptotic boundary of solutions depends on the distribution of eigenvalues: If B has a unique multiple eigenvalue, then the real part of this eigenvalue determines the growth rate of solutions, otherwise, the components of solutions grow exponentially according to the corresponding eigenvalues in the Jordan blocks. Hence, every solution has an exponential asymptotic boundary, which depends on the eigenvalues of the coefficient matrix B.

Keywords Asymptotic bounds, generalized pantograph equation, diagonalizable and non-diagonalizable, exponential form.

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1. Introduction

In 1971, Ockendon et al. [19] firstly put forward the pantograph equation $y'(t) = ay(qt) + by(t)$, where a is a complex constant, b is a real constant, and q is a non-negative constant. This equation refers to an industrial problem involving wave motion in the overhead supply line of the railway system. The term "pantograph" comes from Ockendon and Tayler [19] and Iserles [12,13], reflecting its connection between the trolley and the overhead trolley wire. Fox et al. [7] investigated the properties of solutions to the pantograph equation by analytical and numerical methods, the theoretical results provide strict mathematical support for the dynamic modeling, parameter design, stability analysis and numerical simulation of pantograph system, and ensure the controllability and robustness of the system dynamics in practical applications. Let $t = e^s$, $q = e^c$, $y(t) = z(s)$, we get the equation $e^{-s}z'(s) = az(s + c) + bz(s)$, i.e., the pantograph equation is also a type of functional differential equation. When $0 < q < 1$, $c = \ln(q) < 0$, the equation belongs to the retarded functional differential equation; when $q > 1$, $c > 0$, the equation belongs to the advanced functional differential equation. Such functional differential

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systems have been extensively studied, which mainly focus on equation solving and numerical methods [1,3,4,18,20], stability analysis [11,22], and asymptotic behaviors [5,8,15,23].

If $y(t)$ is a vector in the pantograph equation, then the equation becomes $y'(t) = Ay(qt) + By(t)$, where A, B are $n \times n$ matrices. Many scholars studied this kind of generalized pantograph equation and obtained a lot of results. In 1976, Lim studied the asymptotic behavior of solutions to this equation with $0 < q < 1$. He proved that: If matrix B can be diagonalized and the real parts of its eigenvalues are all negative, then there is a constant α such that every solution is $O(t^{\alpha})$ as $t \to \infty$; if matrix B can be diagonalized with eigenvalues b_i such that $0 < \mathrm{Re}b_1 \le \mathrm{Re}b_2 \le \cdots \le \mathrm{Re}b_n$ and $q\mathrm{Re}b_n > \mathrm{Re}b_1$, then every solution is $O(e^{b_n t})$ as $t \to \infty$ [16]. When $n = 2$, the pantograph equation with matrix coefficients transforms into a system of coupled pantograph equations. This form of equation has wide applications in various fields of science and engineering, including complex networks [17] and delay-coupled semiconductor lasers [14]. Additionally, the corresponding fractional differential equations play a crucial role in mathematical modeling, particularly in areas such as seismic nonlinear oscillations, nanotechnology, and materials [2,9,10,21].

In 1993, Iserles gave the condition for the well-posedness of the following pantograph equation

$$y'(t) = Ay(qt) + By(t) + Cy'(qt), \tag{1.1}$$

and derived the conditions for $\lim_{t\to\infty} y(t) = 0$ [12]. For the discretized form of equation (1.1), Buhmann et al. analyzed the stability of its numerical solutions [6].

Motivated by the previous work [12,16], particularly the insights from Lim [16] regarding how the spectral properties of matrix B delineate solution boundaries through eigenvalues, in this study, we concentrate on estimating the boundaries of the solution to equation (1.1), especially on the dynamic structural alterations induced by incorporating the term $Cy'(qt)$, where $0 < q < 1$, $y(t)$ is the n-dimensional column vector, and A, B, C are $n \times n$ complex constant matrices. Our research is divided into two parts: (i) The matrix B is diagonalizable; (ii) The matrix B is non-diagonalizable. In case (i), we make a discussion when the real parts of characteristic roots are greater than zero and less than or equal to zero. In case (ii), the equation can be regarded as a non-homogenous system, which is more difficult.

The paper is organized as follows. In Section 2, some definitions are introduced for convenience. In Section 3 and Section 4, the asymptotic behavior of solutions of equation (1.1) is studied with matrix B being diagonlizable and non-diagonlizable, respectively.

2. Some preliminary notations

For convenience, we introduce several definitions from [16] and [12].

Definition 2.1. [16] Let $y(t)$ be a column vector in $\mathbb{C}^n$ and $f(t)$ be a complex-valued function defined on $[0, \infty)$. We say that $y(t)$ is $O(f(t))$ as $t \to \infty$ if there are constants $K > 0$ and $N > 0$ such that $|y(t)| \le K|f(t)|$ for $t \ge N$.

Definition 2.2. [16] Let $y(t)$ be defined in Definition 2.1. We say that $y(t)$ is $o(f(t))$ as $t \to \infty$ if for any $\epsilon > 0$, there is a constant $N > 0$ such that $|y(t)| \le \epsilon|f(t)|$ for $t > N$.

Definition 2.3. [12] The ordered pair $\{P, Q\}$ is said to be "q-canonical" if, given the set of eigenvalues of matrix P is $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ and the set of eigenvalues of matrix Q is $\{\mu_1, \mu_2, \cdots, \mu_n\}$, it is true that $\mu_k \ne q^l \lambda_j$ for all $k, j \in \{1, 2, \cdots, n\}$ and $l = 1, 2, \cdots$.

In [12], Iserles gives the theorem about existence and uniqueness for solutions $Y(t)$ (a $n \times n$ matrix) of the equation $Y'(t) = AY(qt) + BY(t) + CY'(qt)$ with the initial-value condition $Y(0) = I$, which is shown as follows.

Lemma 2.1. *The initial-value problem* (1.1) *with initial-value condition* $Y(0) = I$ *is well-posed if and only if the pair* $\{C, q^{-1}I\}$ *is q-canonical.*

To analyze the asymptotic behavior of solutions $y(t)$ of equation (1.1), it is necessary to give a sufficient condition for the existence of solutions. According to the above lemma, and the transformation from matrix to column vector: $y(t) = Y(t)y_0$ with y_0 being an n-dimensional unit column vector, we suppose the matrix C in equation (1.1) satisfies that the ordered pair $\{C, q^{-1}I\}$ is q-canonical in the whole paper.

3. Diagonalized matrix B

If B can be diagonalized, then there is a non-singular matrix P, such that $P^{-1}BP = \mathrm{diag}(b_1, b_2, \cdots, b_n)$. By replacing $y(t)$ with $Py(t)$, equation (1.1) becomes $y'(t) = \hat{A}y(qt) + \hat{B}y(t) + \hat{C}y'(qt)$, where $\hat{A} = P^{-1}AP$, $\hat{B} = P^{-1}BP$, $\hat{C} = P^{-1}CP$. Let $\hat{A} = (\hat{a}_{ij}), \hat{C} = (\hat{c}_{ij})$, then

$$y_i'(t) = \sum_{j=1}^{n} \hat{a}_{ij} y_j(qt) + b_i y_i(t) + \sum_{j=1}^{n} \hat{c}_{ij} y_j'(qt). \tag{3.1}$$

Denote $\beta_i = \mathrm{Re}(b_i)$. In general, if B can be diagonalized with eigenvalues b_i, then there exists an integer $r \in [0, n]$, such that $\mathrm{Re}b_1, \mathrm{Re}b_2, \cdots, \mathrm{Re}b_r > 0$ and $\mathrm{Re}b_{r+1}, \cdots, \mathrm{Re}b_n \leq 0$.

Theorem 3.1. *If* B *can be diagonalized to* $\mathrm{diag}(b_1, b_2, \cdots, b_n)$ *with*

$$\mathrm{Re}b_1 \geq \mathrm{Re}b_2 \geq \cdots \geq \mathrm{Re}b_r > 0 \geq \mathrm{Re}b_{r+1} \geq \mathrm{Re}b_{r+2} \geq \cdots \geq \mathrm{Re}b_n,$$

and $\mathrm{Re}b_r > q\mathrm{Re}b_1$, *then*

(i) $y_i(t)$ *are* $O(e^{b_i t})$ *as* $t \to \infty$, *for* $i = 1, 2, \cdots, r$.

(ii) $y_i(t)$ *are* $O(e^{b_r t})$ *as* $t \to \infty$, *for* $i = r+1, r+2, \cdots, n$.

Proof. Equation (3.1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}[y_i(t)e^{-b_i t}] = \sum_{j=1}^{n} \hat{a}_{ij} y_j(qt) e^{-b_i t} + \sum_{j=1}^{n} \hat{c}_{ij} y_j'(qt) e^{-b_i t}. \tag{3.2}$$

For $t \in [\frac{1}{q^{m-1}}, \frac{1}{q^m}]$, we let

$$\begin{aligned}
M_m &= \sup\Big\{|y_i(t)e^{-b_i t}| \big| i = 1, 2, \cdots, r\Big\}, \\
\hat{M}_m &= \sup\Big\{|y_i(t)e^{-b_r t}| \big| i = r+1, r+2, \cdots, n\Big\}, \\
B_m &= \max\{M_m, \hat{M}_m\}, \\
\alpha &= \max\Big\{\sum_{i=1}^{n}\sum_{j=1}^{n}(|\hat{a}_{ij}| + \frac{|b_i||\hat{c}_{ij}|}{q}), \frac{1}{q}\sum_{i=1}^{n}\sum_{j=1}^{n}|\hat{c}_{ij}|\Big\}.
\end{aligned}$$

Define $s \in \left[\frac{1}{q^m}, \frac{1}{q^{m+1}}\right]$. Integrating (3.2) from $\frac{1}{q^m}$ to s and taking modulus, we have

$$\left| \left[y_i(t)e^{-b_i t}\right]_{\frac{1}{q^m}}^{s} \right| \le \sum_{j=1}^{n} |\hat{a}_{ij}| \int_{\frac{1}{q^m}}^{s} \left|y_j(qt)e^{-b_i t}\right| \mathrm{d}t + \sum_{j=1}^{n} |\hat{c}_{ij}| \left| \int_{\frac{1}{q^m}}^{s} y_j'(qt)e^{-b_i t} \mathrm{d}t \right|.$$

The last integral of the above equation is equal to $\int_{\frac{1}{q^m}}^{s} \frac{1}{q} e^{-b_i t} \mathrm{d}y_j(qt)$. Using the method of integration by parts, the above equation becomes

$$\begin{aligned} &\left|y_i(s)e^{-b_i s}\right| - \left|y_i(\frac{1}{q^m})e^{-b_i \frac{1}{q^m}}\right| \\ \le& \sum_{j=1}^{n} \left(|\hat{a}_{ij}| + \frac{|b_i||\hat{c}_{ij}|}{q}\right) \int_{\frac{1}{q^m}}^{s} |y_j(qt)e^{-b_i t}| \mathrm{d}t + \frac{1}{q} \sum_{j=1}^{n} |\hat{c}_{ij}| \left| \left[y_j(qt)e^{-b_i t}\right]_{\frac{1}{q^m}}^{s} \right|. \end{aligned} \tag{3.3}$$

Dividing $\sum_{j=1}^{n}$ into $\sum_{j=1}^{r} + \sum_{j=r+1}^{n}$, and rewriting the exponential term: $e^{-b_i t} = e^{-b_j qt} e^{(qb_j - b_i)t}$ as $j = 1, \cdots, r$; $e^{-b_i t} = e^{-b_r qt} e^{(qb_r - b_i)t}$ as $j = r+1, \cdots, n$, we have

$$\begin{aligned} &|y_i(s)e^{-b_i s}| \\ \le& \left|y_i(\frac{1}{q^m})e^{-b_i \frac{1}{q^m}}\right| + \sum_{j=1}^{r} \left(|\hat{a}_{ij}| + \frac{|b_i||\hat{c}_{ij}|}{q}\right) \int_{\frac{1}{q^m}}^{s} \left|y_j(qt)e^{-b_j qt}\right| e^{(q\beta_j - \beta_i)t} \mathrm{d}t \\ &+ \sum_{j=r+1}^{n} \left(|\hat{a}_{ij}| + \frac{|b_i||\hat{c}_{ij}|}{q}\right) \int_{\frac{1}{q^m}}^{s} \left|y_j(qt)e^{-b_r qt}\right| e^{(q\beta_r - \beta_i)t} \mathrm{d}t \\ &+ \frac{1}{q} \sum_{j=1}^{r} |\hat{c}_{ij}| \left| \left[y_j(qt)e^{-b_j qt} e^{(qb_j - b_i)t}\right]_{\frac{1}{q^m}}^{s} \right| \\ &+ \frac{1}{q} \sum_{j=r+1}^{n} |\hat{c}_{ij}| \left| \left[y_j(qt)e^{-b_r qt} e^{(qb_r - b_i)t}\right]_{\frac{1}{q^m}}^{s} \right|. \end{aligned} \tag{3.4}$$

(i) For $i = 1, 2, \cdots, r$, inequality (3.4) is reduced to

$$\begin{aligned} &|y_i(s)e^{-b_i s}| \\ \le& M_m + \alpha M_m \int_{\frac{1}{q^m}}^{s} e^{(q\beta_1 - \beta_r)t} \mathrm{d}t + \alpha \hat{M}_m \int_{\frac{1}{q^m}}^{s} e^{(q\beta_r - \beta_r)t} \mathrm{d}t \\ &+ \frac{1}{q} \sum_{j=1}^{r} |\hat{c}_{ij}| \left|y_j(qs)e^{-b_j qs}\right| e^{(q\beta_j - \beta_i)s} + \frac{1}{q} \sum_{j=1}^{r} |\hat{c}_{ij}| \left|y_j(\frac{1}{q^{m-1}})e^{-\frac{b_j}{q^{m-1}}}\right| e^{(q\beta_j - \beta_i)\frac{1}{q^m}} \\ &+ \frac{1}{q} \sum_{j=r+1}^{n} |\hat{c}_{ij}| \left|y_j(qs)e^{-b_r qs}\right| e^{(q\beta_r - \beta_i)s} + \frac{1}{q} \sum_{j=r+1}^{n} |\hat{c}_{ij}| \left|y_j(\frac{1}{q^{m-1}})e^{-\frac{b_r}{q^{m-1}}}\right| e^{(q\beta_r - \beta_i)\frac{1}{q^m}} \\ \le& B_m + \alpha B_m \frac{1}{\beta_r - q\beta_1} e^{-(\beta_r - q\beta_1)\frac{1}{q^m}} + \alpha B_m \frac{1}{(1-q)\beta_r} e^{-(1-q)\beta_r \frac{1}{q^m}} \\ &+ 2\alpha B_m e^{-(\beta_r - q\beta_1)\frac{1}{q^m}} + 2\alpha B_m e^{-(1-q)\beta_r \frac{1}{q^m}}. \end{aligned}$$

From $\beta_r \le \beta_1$, we have $-(1-q)\beta_r \le -(\beta_r - q\beta_1)$. Then

$$M_{m+1} \le B_m \Big[1 + O\big(e^{-(\beta_r - q\beta_1)\frac{1}{q^m}}\big)\Big]. \tag{3.5}$$

(ii) For $i = r+1, r+2, \cdots, n$, it is clear that $\beta_i = \mathrm{Re} b_i < 0$. Inequality (3.4) becomes

$$\begin{aligned}|y_i(s)e^{-b_i s}| \le & \big|y_i(\frac{1}{q^m})e^{-b_r \frac{1}{q^m}}\big|e^{(\beta_r-\beta_i)\frac{1}{q^m}} + \alpha M_m \int_{\frac{1}{q^m}}^{s} e^{(q\beta_1-\beta_i)t}\mathrm{d}t \\ & + \alpha\hat{M}_m \int_{\frac{1}{q^m}}^{s} e^{(q\beta_r-\beta_i)t}\mathrm{d}t + 2\alpha M_m e^{(q\beta_1-\beta_i)s} + 2\alpha\hat{M}_m e^{(q\beta_r-\beta_i)s} \\ \le & \hat{M}_m e^{(\beta_r-\beta_i)\frac{1}{q^m}} + \alpha M_m \frac{e^{(q\beta_1-\beta_i)s}}{q\beta_1-\beta_i} + \alpha\hat{M}_m \frac{e^{(q\beta_r-\beta_i)s}}{q\beta_r-\beta_i} \\ & + 2\alpha M_m e^{(q\beta_1-\beta_i)s} + 2\alpha\hat{M}_m e^{(q\beta_r-\beta_i)s}.\end{aligned}$$

Since $|y_i(s)e^{-b_i s}| = |y_i(s)e^{-b_r s}|e^{(\beta_r-\beta_i)s}$ and $-(1-q)\beta_r s \le -(\beta_r - q\beta_1)s$, we obtain that

$$\begin{aligned}|y_i(s)e^{-b_r s}| \le & \hat{M}_m e^{(\beta_r-\beta_i)(\frac{1}{q^m}-s)} + \alpha M_m \frac{e^{-(\beta_r-q\beta_1)s}}{q\beta_1-\beta_{r+1}} + \alpha\hat{M}_m \frac{e^{-(1-q)\beta_r s}}{q\beta_r-\beta_{r+1}} \\ & + 2\alpha M_m e^{-(\beta_r-q\beta_1)s} + 2\alpha\hat{M}_m e^{-(1-q)\beta_r s} \\ \le & B_m\Big[1 + \big(\frac{2}{q\beta_1-\beta_{r+1}} + 4\alpha\big)e^{-(\beta_r-q\beta_1)\frac{1}{q^m}}\Big].\end{aligned}$$

Thus

$$\hat{M}_{m+1} \le B_m\Big[1 + O\big(e^{-(\beta_r-q\beta_1)\frac{1}{q^m}}\big)\Big]. \tag{3.6}$$

From (3.5) and (3.6), we have that $B_{m+1} \le B_m\big[1 + O(e^{-(\beta_r-q\beta_1)\frac{1}{q^m}})\big]$, then

$$B_m \le B_1 \prod_{k=1}^{m-1} \big[1 + O(e^{-(\beta_r-q\beta_1)\frac{1}{q^k}})\big].$$

Since the infinite product $\prod_{k=1}^{\infty}\big[1 + O(e^{-(\beta_r-q\beta_1)\frac{1}{q^k}})\big]$ converges, B_m is bounded for all m. Therefore, $M_m, \hat{M}_m$ are bounded. The proof is completed. □

Theorem 3.2. *For* $r \in \{1, 2, \cdots, n-1\}$*. If* B *can be diagonalized to* $\mathrm{diag}(b_1, \cdots, b_n)$ *with* $\mathrm{Re} b_1 \ge \mathrm{Re} b_2 \ge \cdots \ge \mathrm{Re} b_r > 0 \ge \mathrm{Re} b_{r+1} \ge \mathrm{Re} b_{r+2} \ge \cdots \ge \mathrm{Re} b_n$*, then every solution of* (1.1) *is* $O(e^{b_1 t})$ *as* $t \to \infty$*.*

Proof. From Theorem 3.1, we only need to analyze equation (3.3). For $t \in \big[\frac{1}{q^{m-1}}, \frac{1}{q^m}\big]$, we set $M_{i,m} = \sup\big\{|y_i(t)e^{-b_1 t}|\big\}$, $M_m = \max_{1\le i\le n}\{M_{i,m}\}$. For $i = 1, 2, \cdots, r$, we need to discuss the relationship between $q\beta_1$ and β_i.

(i) If $q\beta_1 < \beta_1, \beta_2, \cdots, \beta_r$, then

$$\big|y_i(s)e^{-b_i s}\big| \le \Big|y_i(\frac{1}{q^m})e^{-b_i \frac{1}{q^m}}\Big| + M_m(\frac{\alpha}{\beta_i - q\beta_1} + 2\alpha)e^{-(\beta_i-q\beta_1)\frac{1}{q^m}}.$$

Since

$$|y_i(s)e^{-b_i s}| = |y_i(s)e^{-b_1 s}|e^{(\beta_1-\beta_i)s} \ge |y_i(s)e^{-b_1 s}|e^{(\beta_1-\beta_i)\frac{1}{q^m}},$$

we have

$$\begin{aligned}|y_i(s)e^{-b_1 s}| & \le \Big|y_i(\frac{1}{q^m})e^{-b_1 \frac{1}{q^m}}\Big| + M_m\big(\frac{\alpha}{\beta_i - q\beta_1} + 2\alpha\big)e^{-(1-q)\beta_1 \frac{1}{q^m}} \\ & \le M_m\big[1 + (\frac{\alpha}{\beta_n - q\beta_1} + 2\alpha)e^{-(1-q)\beta_1\frac{1}{q^m}}\big]\end{aligned}$$

for $i = 1, 2, \cdots, r$.

(ii) If $q\beta_1 < \beta_1, \beta_2, \cdots, \beta_h$ and $q\beta_1 \geq \beta_{h+1}, \beta_{h+2}, \cdots, \beta_r$, then it can be divided into two cases: $q\beta_1 > \beta_{h+1}$ and $q\beta_1 = \beta_{h+1}$. In the case of $q\beta_1 > \beta_{h+1}$, for $i = 1, 2, \cdots, h$, we have

$$M_{i,m} \leq M_1 \prod_{k=1}^{m-1} \big[1 + \big(\frac{\alpha}{\beta_h - q\beta_1} + 2\alpha\big)e^{-(1-q)\beta_1 \frac{1}{q^k}}\big], \quad m = 2, 3, 4, \cdots.$$

For $i = h+1, h+2, \cdots, r$, we have

$$|y_i(s)e^{-b_i s}| \leq \Big|y_i(\frac{1}{q^m})e^{-b_i \frac{1}{q^m}}\Big| + M_m\big(\frac{\alpha}{q\beta_1 - \beta_i} + 2\alpha\big)e^{(q\beta_1 - \beta_i)s}.$$

Substituting $|y_i(s)e^{-b_i s}| = |y_i(s)e^{-b_1 s}|e^{(\beta_1 - \beta_i)s}$ into the above equation, yields

$$\begin{aligned} |y_i(s)e^{-b_1 s}| &\leq \Big|y_i(\frac{1}{q^m})e^{-b_1 \frac{1}{q^m}}\Big|e^{-(\beta_1 - \beta_i)(s - \frac{1}{q^m})} + M_m\big(\frac{\alpha}{q\beta_1 - \beta_i} + 2\alpha\big)e^{-(1-q)\beta_1 s} \\ &\leq M_m\Big[1 + \big(\frac{\alpha}{q\beta_1 - \beta_{h+1}} + 2\alpha\big)e^{-(1-q)\beta_1 \frac{1}{q^m}}\Big] \end{aligned}$$

for $i = h+1, h+2, \cdots, r$.

To sum up, for $i = 1, 2, \cdots, r$, it follows that

$$M_{i,m} \leq M_1 \prod_{k=1}^{m-1} \big[1 + O\big(e^{-(1-q)\beta_1 \frac{1}{q^k}}\big)\big]$$

for $m = 2, 3, 4, \cdots$. Then $M_m \leq M_1 \prod_{k=1}^{m-1} \big[1 + O\big(e^{-(1-q)\beta_1 \frac{1}{q^k}}\big)\big]$ for $m = 2, 3, 4, \cdots$. Therefore, M_m is bounded for all m.

In the case of $q\beta_1 = \beta_{h+1}$, we have $q\beta_1 < \beta_1, \beta_2, \cdots, \beta_h$ and $q\beta_1 > \beta_{h+2}, \cdots, \beta_r$. From the above analysis, it follows that

$$M_m \leq M_1 \prod_{k=1}^{m-1} \big[1 + O\big(e^{-(1-q)\beta_1 \frac{1}{q^k}}\big)\big], \quad m = 2, 3, 4, \cdots$$

for $i \in \{1, 2, \cdots, r\} \setminus \{h+1\}$. For $i = h+1$, equation (3.3) becomes

$$|y_{h+1}(s)e^{-b_{h+1}s}| \leq \big|y_{h+1}(\frac{1}{q^m})e^{-b_{h+1}\frac{1}{q^m}}\big| + 2\alpha M_m s.$$

Then we have

$$\begin{aligned} |y_{h+1}(s)e^{-b_1 s}| &\leq \Big|y_{h+1}(\frac{1}{q^m})e^{-b_1 \frac{1}{q^m}}\Big|e^{-(\beta_1 - \beta_{h+1})(s - \frac{1}{q^m})} + 2\alpha M_m s e^{-(\beta_1 - \beta_{h+1})s} \\ &\leq M_m\big[1 + \frac{2\alpha}{q^{m+1}}e^{-(\beta_1 - \beta_{h+1})\frac{1}{q^m}}\big]. \end{aligned}$$

We have

$$M_{h+1,m+1} \leq M_m\big[1 + \frac{2\alpha}{q^{m+1}}e^{-(\beta_1 - \beta_{h+1})\frac{1}{q^m}}\big].$$

Let $\bar{h} = \min\{\beta_h - q\beta_1, q\beta_1 - \beta_{h+2}\}$. By $\beta_{h+1} = q\beta_1$, we have

$$M_{i,m+1} \leq \max\Big\{M_m\big[1 + \big(\frac{\alpha}{\bar{h}} + 2\alpha\big)e^{-(1-q)\beta_1 \frac{1}{q^m}}\big], M_m\big[1 + \frac{2\alpha}{q^{m+1}}e^{-(1-q)\beta_1 \frac{1}{q^m}}\big]\Big\} \tag{3.7}$$

for $i = 1, 2, \cdots, r$, where $\bar{h} = \min\{\beta_h - q\beta_1, q\beta_1 - \beta_{h+2}\}$. Equation (3.7) is always true regardless of the relationship between $q\beta_1$ and β_i, for $i = 1, 2, \cdots, r$.

For $i = r+1, r+2, \cdots, n$, we have $\beta_i \leq 0$. Then $q\beta_1 > \beta_{r+1}, \cdots, \beta_n$, equation (3.3) becomes

$$|y_i(s)e^{-b_i s}| \leq \Big|y_i(\frac{1}{q^m})e^{-b_i \frac{1}{q^m}}\Big| + M_m\big(\frac{\alpha}{q\beta_1 - \beta_i} + 2\alpha\big)e^{(q\beta_1 - \beta_i)s}.$$

Similar to the above process, based on $|y_i(s)e^{-b_i s}| = |y_i(s)e^{-b_1 s}|e^{(\beta_1 - \beta_i)s}$, we have

$$\begin{aligned}|y_i(s)e^{-b_1 s}| &\leq \Big|y_i(\frac{1}{q^m})e^{-b_1 \frac{1}{q^m}}\Big|e^{-(\beta_1-\beta_i)(s-\frac{1}{q^m})} + M_m\big(\frac{\alpha}{q\beta_1 - \beta_i} + 2\alpha\big)e^{-(1-q)\beta_1 s} \\ &\leq M_m\big[1 + \big(\frac{\alpha}{q\beta_1} + 2\alpha\big)e^{-(1-q)\beta_1 \frac{1}{q^m}}\big]\end{aligned}$$

for $i = r+1, r+2, \cdots, n$. Together with (3.7), we have

$$M_{m+1} \leq \max\Big\{M_m\big[1 + \frac{2\alpha}{q^{m+1}}e^{-(1-q)\beta_1\frac{1}{q^m}}\big], M_m\big[1 + (\frac{\alpha}{\tilde{h}} + 2\alpha)e^{-(1-q)\beta_1\frac{1}{q^m}}\big]\Big\},$$

where $\tilde{h} = \min\big\{\bar{h}, q\beta_1\big\}$. The convergence of infinite products

$$\prod_{j=1}^{\infty}\big[1 + \frac{2\alpha}{q^{j+1}}e^{-(1-q)\beta_1\frac{1}{q^j}}\big]$$

and

$$\prod_{j=1}^{\infty}\big[1 + (\frac{\alpha}{\tilde{h}} + 2\alpha)e^{-(1-q)\beta_1\frac{1}{q^j}}\big]$$

implies M_m is bounded for all m. □

4. Non-diagonalized matrix B

If B can not be diagonalized, then we consider the following two cases: (i). Matrix B has an n-multiple eigenvalue b; (ii). Matrix B has eigenvalues $b_1, b_2, \cdots, b_l$ (where l is a positive constant and $l < n$).

4.1. Matrix B with a unique multiple eigenvalue b

If B has a unique multiple eigenvalue b, then there is a non-singular $n \times n$ matrix Q, such that

$$\tilde{B} = Q^{-1}BQ = \begin{pmatrix} b & \eta_1 & 0 & \cdots & 0 \\ & b & \eta_2 & \cdots & 0 \\ & & b & \ddots & \vdots \\ & & & \ddots & \eta_{n-1} \\ \text{\Large 0} & & & & b \end{pmatrix}_n$$

with $\eta_1, \eta_2, \cdots, \eta_{n-1}$ are equal to 0 or 1. By replacing $y(t)$ with $Qy(t)$, equation (1.1) is reduced to $y'(t) = \tilde{A}y(qt) + \tilde{B}y(t) + \tilde{C}y'(qt)$, where $\tilde{A} = Q^{-1}AQ$, $\tilde{B} = Q^{-1}BQ$, $\tilde{C} = Q^{-1}CQ$. Then

$$y_1'(t) = \sum_{j=1}^{n} \tilde{a}_{1,j} y_j(qt) + \sum_{j=1}^{n} \tilde{c}_{1,j} y_j'(qt) + by_1(t) + \eta_1 y_2(t), \tag{4.1}$$

$$\vdots$$

$$y_{n-1}'(t) = \sum_{j=1}^{n} \tilde{a}_{n-1,j} y_j(qt) + \sum_{j=1}^{n} \tilde{c}_{n-1,j} y_j'(qt) + by_{n-1}(t) + \eta_{n-1} y_n(t), \tag{4.2}$$

$$y_n'(t) = \sum_{j=1}^{n} \tilde{a}_{n,j} y_j(qt) + \sum_{j=1}^{n} \tilde{c}_{n,j} y_j'(qt) + by_n(t). \tag{4.3}$$

Theorem 4.1. *If B can not be diagonalized and has an n-multiple eigenvalue b, then for any real number $\hat{\beta}$ with $\mathrm{Re}b < \hat{\beta} < \frac{1}{q}\mathrm{Re}b$, the solutions of (1.1) satisfy $y_1(t), y_2(t), \cdots, y_{n-1}(t) = O(e^{\hat{\beta}t})$ and $y_n(t) = O(e^{bt})$, as $t \to \infty$.*

Proof. For $t \in [q^{-(m-1)}, q^{-m}]$, we define

$$M_{i,m} = \sup_{i=1,2,\cdots,n-1} \left\{ |y_i(t)e^{-\hat{\beta}t}| \right\},\ M_{n,m} = \sup \left\{ |y_n(t)e^{-bt}| \right\},\ M_m = \max_{1 \le i \le n} \{M_{i,m}\}.$$

Equation (4.3) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}[y_n(t)e^{-bt}] = \sum_{j=1}^{n} \tilde{a}_{n,j} y_j(qt) e^{-bt} + \sum_{j=1}^{n} \tilde{c}_{n,j} y_j'(qt) e^{-bt}.$$

Let $s \in [q^{-m}, q^{-(m+1)}]$, integrating the above equation from q^{-m} to s and taking modulus, we have

$$\begin{aligned}
&|y_n(s)e^{-bs}| - \left| y_n(\frac{1}{q^m}) e^{-b\frac{1}{q^m}} \right| \\
\le& \sum_{j=1}^{n} \left(|\tilde{a}_{n,j}| + \frac{|b||\tilde{c}_{n,j}|}{q} \right) \int_{\frac{1}{q^m}}^{s} |y_j(qt)e^{-bt}| \mathrm{d}t + \frac{1}{q} \sum_{j=1}^{n} |\tilde{c}_{n,j}| \left| \left[y_j(qt)e^{-bt} \right]_{\frac{1}{q^m}}^{s} \right| \\
=& \sum_{j=1}^{n-1} \left(|\tilde{a}_{n,j}| + \frac{|b||\tilde{c}_{n,j}|}{q} \right) \int_{\frac{1}{q^m}}^{s} |y_j(qt)e^{-q\hat{\beta}t}| e^{-(\beta - q\hat{\beta})t} \mathrm{d}t \\
&+ \left(|\tilde{a}_{n,n}| + \frac{|b||\tilde{c}_{n,n}|}{q} \right) \int_{\frac{1}{q^m}}^{s} |y_n(qt)e^{-q\beta t}| e^{-(1-q)\beta t} \mathrm{d}t \\
&+ \frac{1}{q} \sum_{j=1}^{n-1} |\tilde{c}_{n,j}| \left| \left[y_j(qt) e^{-q\hat{\beta}t} e^{-(b-q\hat{\beta})t} \right]_{\frac{1}{q^m}}^{s} \right| + |\tilde{c}_{n,n}| \left| \left[y_n(qt) e^{-qbt} e^{-(1-q)bt} \right]_{\frac{1}{q^m}}^{s} \right| \\
\le& \alpha M_m \frac{1}{\beta - q\hat{\beta}} e^{-(\beta - q\hat{\beta})\frac{1}{q^m}} + \alpha M_m \frac{1}{(1-q)\beta} e^{-(1-q)\beta \frac{1}{q^m}} + 2\alpha M_m \frac{1}{\beta - q\hat{\beta}} e^{-(\beta - q\hat{\beta})\frac{1}{q^m}} \\
&+ 2\alpha M_m \frac{1}{(1-q)\beta} e^{-(1-q)\beta \frac{1}{q^m}}.
\end{aligned}$$

Then

$$M_{n,m+1} \le M_m\big[1+\frac{6\alpha}{\beta-q\hat{\beta}}e^{-(\beta-q\hat{\beta})\frac{1}{q^m}}\big].$$

For equation (4.2), if $\eta_{n-1}=0$, then it is equal to (4.3); we can easily get the same estimate: $M_{n-1,m+1} \le M_m\big[1+\frac{6\alpha}{\beta-q\hat{\beta}}e^{-(\beta-q\hat{\beta})\frac{1}{q^m}}\big]$. If $\eta_{n-1}=1$, then equation (4.2) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[y_{n-1}(t)e^{-bt}] = \sum_{j=1}^{n}\tilde{a}_{n-1,j}y_j(qt)e^{-bt} + \sum_{j=1}^{n}\tilde{c}_{n-1,j}y_j'(qt)e^{-bt} + y_n(t)e^{-bt},$$

then

$$\begin{aligned}|y_{n-1}(s)e^{-bs}| - \Big|y_{n-1}(\frac{1}{q^m})e^{-b\frac{1}{q^m}}\Big| &\le M_m\frac{6\alpha}{\beta-q\hat{\beta}}e^{-(\beta-q\hat{\beta})\frac{1}{q^m}} + M_{n,m}\\ &\le M_m\Big[1+\frac{6\alpha}{\beta-q\hat{\beta}}e^{-(\beta-q\hat{\beta})\frac{1}{q^m}}\Big].\end{aligned}$$

According to $|y_{n-1}(s)e^{-bs}| = |y_{n-1}(s)e^{-\hat{\beta}s}|e^{(\hat{\beta}-\beta)s}$, we have

$$M_{n-1,m+1} \le M_m\Big[1+e^{-(\hat{\beta}-\beta)\frac{1}{q^m}} + \frac{6\alpha}{\beta-q\hat{\beta}}e^{-(1-q)\hat{\beta})\frac{1}{q^m}}\Big].$$

To sum up, the above formula is true whether $\eta=0$ or $\eta=1$. The other equations follow the same steps. Then

$$M_{i,m+1} \le M_m\Big[1+e^{-(\hat{\beta}-\beta)\frac{1}{q^m}} + \frac{6\alpha}{\beta-q\hat{\beta}}e^{-(1-q)\hat{\beta})\frac{1}{q^m}}\Big]$$

for $i=1,2,\cdots,n$. Together with $q\hat{\beta}<\beta$, we have

$$M_m \le M_1\prod_{k=1}^{m-1}\big[1+e^{-(\hat{\beta}-\beta)\frac{1}{q^k}} + \frac{6\alpha}{\beta-q\hat{\beta}}e^{-(\hat{\beta}-\beta)\frac{1}{q^k}}\big].$$

The convergence of the infinite product $\prod_{k=1}^{\infty}\Big[1+O\big(e^{-(\hat{\beta}-\beta)\frac{1}{q^k}}\big)\Big]$ implies that M_m is bounded for all m. ☐

Corollary 4.1. *If B can not be diagonalized with eigenvalue b, then for any real number $\hat{\beta} >$ Reb, every solution of* (1.1) *satisfies $y_i(t)=O(e^{\hat{\beta}t})$ for $i=1,2,\cdots,n$ as $t\to\infty$.*

4.2. Matrix B with eigenvalues b_1, $b_2,...,b_l$

If B can not be diagonalized, and possess eigenvalues $b_1,b_2,\cdots,b_l$, then there is a non-singular $n\times n$ matrix R, such that

$$\bar{B}=R^{-1}BR=\begin{pmatrix}J_1 & & & 0\\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_l\end{pmatrix},\ \text{where } J_k=\begin{pmatrix}b_k & 1 & \cdots & 0\\ & b_k & \ddots & \vdots\\ & & \ddots & 1\\ 0 & & & b_k\end{pmatrix}_{n_k}$$

for $k=1,2,\cdots,l$, and $n_1+n_2+\cdots+n_l=n$. By replacing $y(t)$ with $Ry(t)$, equation (1.1) becomes

$$y'(t)=\bar{A}y(qt)+\bar{B}y(t)+\bar{C}y'(qt), \tag{4.4}$$

where $\bar{A}=R^{-1}AR$, $\bar{C}=R^{-1}CR$ are $n\times n$ matrices.

Theorem 4.2. *If B can not be diagonalized, its eigenvalues $b_1,b_2,\cdots,b_l$ satisfy $\mathrm{Re}b_1\geq\mathrm{Re}b_2\geq\cdots\geq\mathrm{Re}b_r>0\geq\mathrm{Re}b_{r+1}\geq\mathrm{Re}b_{r+2}\geq\cdots\geq\mathrm{Re}b_l$, then for any real number $\hat{\beta}_1$ with $\mathrm{Re}b_1<\hat{\beta}_1<\frac{1}{q}\mathrm{Re}b_r$ and any $\hat{\beta}_k$ with $\hat{\beta}_k>\mathrm{Re}b_k$ $(k=2,\cdots,r)$, solutions of (1.1) satisfy*

(i) $y_i(t)=O(e^{\hat{\beta}_1 t})$ for $i=1,2,\cdots,n_1-1$ as $t\to\infty$.

(ii) $y_i(t)=O(e^{\hat{\beta}_k t})$ for $i=n_1+\cdots+n_{k-1}+1,\cdots,n_1+\cdots+n_k-1$ as $t\to\infty$.

(iii) $y_{n_1}(t)=O(e^{b_1 t}), y_{n_1+n_2}(t)=O(e^{b_2 t}),\cdots,y_{n_1+n_2+\cdots+n_r}(t)=O(e^{b_r t})$ as $t\to\infty$.

(iv) $y_i(t)=O(e^{b_r t})$ as $t\to\infty$, for integer $i\in[n_1+n_2+\cdots+n_r+1,n]$.

Proof. We just need to analyze equation (4.4). From $\hat{\beta}_k>\beta_k$ $(k=1,2,\cdots,r)$, there are some constants such that

$$\begin{aligned}
&\hat{\beta}_1>\beta_{1,1}>\beta_{1,2}>\cdots>\beta_{1,n_1-1}>\beta_1,\\
&\vdots\\
&\hat{\beta}_r>\beta_{r,n_1+n_2+\cdots+n_{r-1}+1}>\cdots>\beta_{r,n_1+n_2+\cdots+n_r-1}>\beta_r.
\end{aligned}$$

For $t\in[\frac{1}{q^{m-1}},\frac{1}{q^m}]$, we set $M_m=\max_{k=1,\cdots,r}\{M_{i,m,n_k},\hat{M}_{i,m},M_{n_k,m}\}$ with

$$\begin{aligned}
M_{i,m,n_1}&=\sup\big\{|y_i(t)e^{-\beta_{1,i}t}|\big\},\ \text{for } i=1,2,\cdots,n_1-1,\\
M_{i,m,n_k}&=\sup\big\{|y_i(t)e^{-\beta_{k,i}t}|\big\},\\
&\qquad\text{for } i=n_1+n_2+\cdots+n_{k-1}+1,\cdots,n_1+n_2+\cdots+n_k-1,\\
\hat{M}_{i,m}&=\sup\big\{|y_i(t)e^{-b_r t}|\big\},\\
&\qquad\text{for } i=n_1+n_2+\cdots+n_r+1,n_1+n_2+\cdots+n_r+2,\cdots,n,
\end{aligned}$$

and $M_{n_1,m}=\sup\big\{|y_{n_1}(t)e^{-b_1 t}|\big\}$, $M_{n_1+\cdots+n_k,m}=\sup\big\{|y_n(t)e^{-b_k t}|\big\}$. Similar to the proof of Theorem 4.1, we easily get that

$$M_{i,m+1}\leq M_m\big[1+O\big(e^{-(\beta_r-q\hat{\beta}_1)\frac{1}{q^m}}\big)+K_m e^{-\delta\frac{1}{q^m}}+K_m O\big(e^{-(\beta_r-q\hat{\beta}_1)\frac{1}{q^m}}\big)\big] \tag{4.5}$$

for $i=1,\cdots,n_1+\cdots+n_r$, where

$$K_m=\max_{k=1,2,\cdots,r}\Big\{\frac{1}{q^{m+1}}+\frac{1}{q^{2(m+1)}}+\cdots+\frac{1}{q^{(n_k-1)(m+1)}}\Big\},\ \delta=\min_{k=1,2,\cdots,r}\{\beta_{k,n-1}-\beta_k\}.$$

For $i=n_1+n_2+\cdots+n_r+1,\cdots,n$, we have $\beta_i\leq 0$. Equation (4.4) is equal to

$$\begin{aligned}
y'_{n_1+\cdots+n_r+1}(t)=&\sum_{j=1}^{n}\bar{a}_{n_1+n_2+\cdots+n_r+1,j}y_j(qt)+\sum_{j=1}^{n}\bar{c}_{n_1+n_2+\cdots+n_r+1,j}y'_j(qt)\\
&+b_{r+1}y_{n_1+n_2+\cdots+n_r+1}(t)+y_{n_1+n_2+\cdots+n_r+2}(t),
\end{aligned} \tag{4.6}$$

$$\vdots$$

$$y'_{n-1}(t) = \sum_{j=1}^{n} \bar{a}_{n-1,j} y_j(qt) + \sum_{j=1}^{n} \bar{c}_{n-1,j} y'_j(qt) + b_l y_{n-1}(t) + y_n(t), \tag{4.7}$$

$$y'_n(t) = \sum_{j=1}^{n} \bar{a}_{n,j} y_j(qt) + \sum_{j=1}^{n} \bar{c}_{n,j} y'_j(qt) + b_l y_n(t). \tag{4.8}$$

For equation (4.8), we have

$$\begin{aligned}
&|y_n(s)e^{-b_l s}| - \Big|y_n(\frac{1}{q^m})e^{-b_l \frac{1}{q^m}}\Big| \\
\le& \sum_{j=1}^{n} \big(|\bar{a}_{nj}| + \frac{|b||\bar{c}_{nj}|}{q}\big) \int_{\frac{1}{q^m}}^{s} |y_j(qt)e^{-b_l t}|\mathrm{d}t + \frac{1}{q}\sum_{j=1}^{n} |\bar{c}_{nj}| \Big|\big[y_j(qt)e^{-b_l t}\big]_{\frac{1}{q^m}}^{s}\Big| \\
=& \sum_{j=1}^{n} \big(|\bar{a}_{nj}| + \frac{|b||\bar{c}_{nj}|}{q}\big) \int_{\frac{1}{q^m}}^{s} |y_j(qt)e^{-\hat{\beta}_1 qt}| e^{(q\hat{\beta}_1-\beta_l)t}\mathrm{d}t \\
&+ \frac{1}{q}\sum_{j=1}^{n} |\bar{c}_{nj}| \Big|\big[y_1(qt)e^{-\hat{\beta}_1 qt} e^{(q\hat{\beta}_1 - b_l)t}\big]_{\frac{1}{q^m}}^{s}\Big| \\
\le& M_m\big(\frac{\alpha}{q\hat{\beta}_1 - \beta_l} + 2\alpha\big)e^{(q\hat{\beta}_1-\beta_l)s}.
\end{aligned} \tag{4.9}$$

Substituting $|y_n(s)e^{-b_l s}| = |y_n(s)e^{-b_r s}|e^{(\beta_r - \beta_l)s}$ into (4.9), we obtain

$$|y_n(s)e^{-b_r s}| \le M_m\Big[1 + \big(\frac{\alpha}{q\hat{\beta}_1 - \beta_l} + 2\alpha\big)e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^m}}\Big].$$

Similar to the above analysis, we can easily get that

$$M_{i,m+1} \le M_m\Big[1 + \big(\frac{\alpha}{q\hat{\beta}_1 - \beta_l} + 2\alpha\big)e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^m}}\Big]$$

for $i = n_1 + n_2 + \cdots + n_r + 1, \cdots, n$. Together with (4.5), we have

$$\begin{aligned}
M_{m+1} \le& \max\{M_m[1 + O(e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^m}}) + K_m e^{-\delta\frac{1}{q^m}} + K_m O(e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^m}})], \\
& M_m[1 + (\frac{\alpha}{q\hat{\beta}_1 - \beta_{r+1}} + 2\alpha)e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^m}}]\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_m \le& M_1 \max\{\prod_{j=1}^{m-1}[1 + O(e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^j}}) + K_j e^{-\delta\frac{1}{q^j}} + K_j O(e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^j}})], \\
& \prod_{j=1}^{m-1}[1 + (\frac{\alpha}{q\hat{\beta}_1 - \beta_{r+1}} + 2\alpha)e^{-(\beta_r - q\hat{\beta}_1)\frac{1}{q^j}}]\}.
\end{aligned}$$

The infinite products corresponding to the above products are convergent, which implies that M_m is bounded for all m, and the proof is completed. □

Corollary 4.2. *If B can not be diagonalized, its eigenvalues $b_1, b_2, \cdots, b_l$ satisfy $\mathrm{Re}b_1 \ge \mathrm{Re}b_2 \ge \cdots \ge \mathrm{Re}b_r > 0 \ge \mathrm{Re}b_{r+1} \ge \mathrm{Re}b_{r+2} \ge \cdots \ge \mathrm{Re}b_l$ for $r = 1, 2, \cdots, l-1$, then for any real number β_0 that satisfies $\beta_0 > \mathrm{Re}b_1$, every solution of (1.1) is $O(e^{\beta_0 t})$ as $t \to \infty$.*

5. Numerical simulations

We give some numerical examples to verify the theoretical results. For the convergence of solutions in [12], we choose the matrices that satisfy $\rho(B^{-1}A) < 1$.

Example 5.1. Let $n = 3$ and

$$A = 0.05(2+\mathrm{i})I, \quad B = \begin{bmatrix} 1.2+0.1\mathrm{i} & 0 & 0 \\ 0 & 0.8+0.5\mathrm{i} & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \quad C = O.$$

Equation (1.1) becomes

$$\begin{aligned} y_1'(t) &= 0.05(2+\mathrm{i})y_1(qt) + (1.2+0.1\mathrm{i})y_1(t), \\ y_2'(t) &= 0.05(2+\mathrm{i})y_2(qt) + (0.8+0.5\mathrm{i})y_2(t), \\ y_3'(t) &= 0.05(2+\mathrm{i})y_3(qt) - 0.5y_3(t). \end{aligned}$$

Let $q = 0.5$ and the initial value

$$Y_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

We obtain that the matrix B has eigenvalues $b_1 = 1.2+0.1\mathrm{i}$, $b_2 = 0.8+0.5\mathrm{i}$, $b_3 = -0.5$, with $\mathrm{Re}(b_1) > \mathrm{Re}(b_2) > 0 > \mathrm{Re}(b_3)$ and $\mathrm{Re}(b_2) > q\mathrm{Re}(b_1)$.

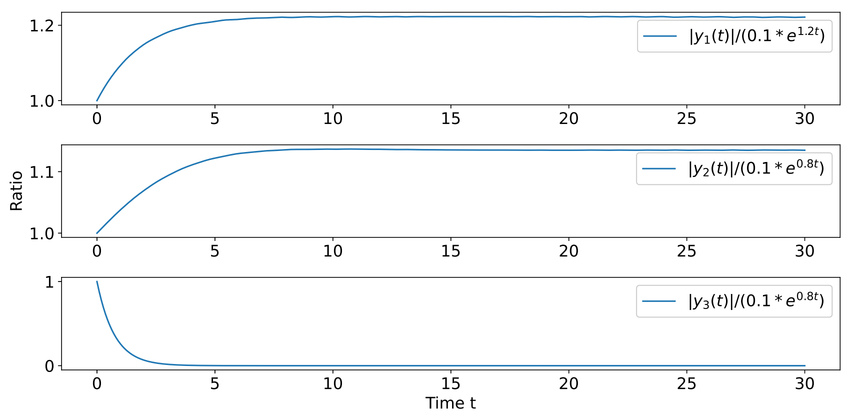

Figure 1. The ratio of $|y_1(t)|/(0.1\mathrm{e}^{1.2t})$, $|y_2(t)|/(0.1\mathrm{e}^{0.8t})$, and $|y_3(t)|/(0.1\mathrm{e}^{0.8t})$ (Example 5.1).

By separating the real and imaginary parts of the matrix, a python program is used to draw the graph of numerical solution using the implicit Runge-Kutta (RK) method. The comparison with the exponential function is shown in Figure 1, which shows that the ratios of $|y_1(t)|/(0.1\mathrm{e}^{1.2t})$, $|y_2(t)|/(0.1\mathrm{e}^{0.8t})$, and $|y_3(t)|/(0.1\mathrm{e}^{0.8t})$ are bounded. As time t increases (simulating $t \to \infty$), the graph is similar and the ratios are bounded. We get that $|y_1(t)| \le C_1 e^{\mathrm{Re}(b_1)t}$ and $|y_2(t)|, |y_3(t)| \le C_2 e^{\mathrm{Re}(b_2)t}$ as $t \to \infty$ for some constants $C_1 \le 0.125$, $C_2 \le 0.12$. This is consistent with the results of Theorem 3.1.

Example 5.2. Let $n = 3$ and

$$A = 0.1(1+\mathrm{i})I, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = 0.5I.$$

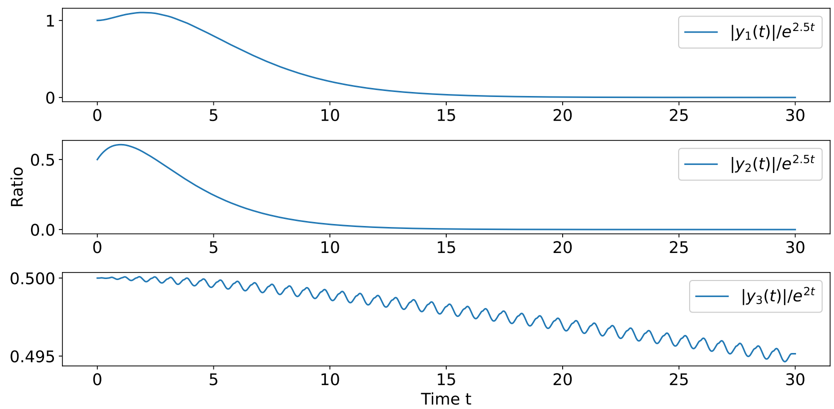

Figure 2. The ratio of $|y_1(t)|/\mathrm{e}^{2.5t}$, $|y_2(t)|/\mathrm{e}^{2.5t}$, and $|y_3(t)|/\mathrm{e}^{2t}$ (Example 5.2).

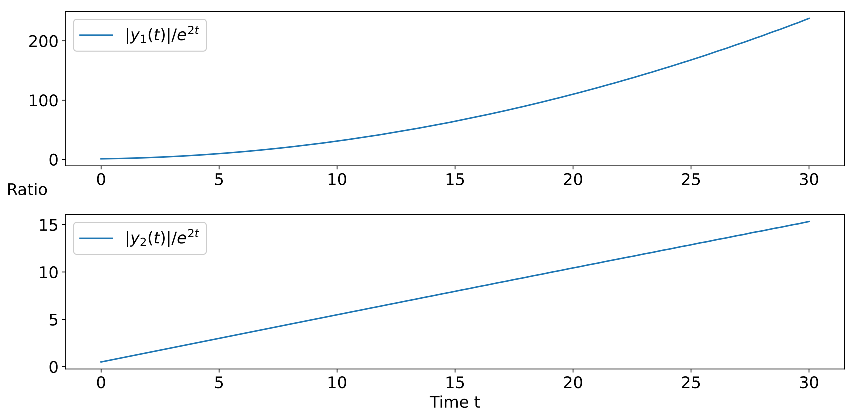

Figure 3. The ratio of $|y_1(t)|/\mathrm{e}^{2t}$ and $|y_2(t)|/\mathrm{e}^{2t}$ (Example 5.2).

Equation (1.1) becomes

$$\begin{aligned} y_1'(t) &= 0.1(1+\mathrm{i})y_1(qt) + 2y_1(t) + y_2(t) + 0.5y_1'(qt), \\ y_2'(t) &= 0.1(1+\mathrm{i})y_2(qt) + 2y_2(t) + y_3(t) + 0.5y_2'(qt), \\ y_3'(t) &= 0.1(1+\mathrm{i})y_3(qt) + 2y_3(t) + 0.5y_3'(qt). \end{aligned}$$

Let $q = 0.8$ and the initial value $Y_0 = [1, 0.5, 0.5]^T$. We obtain that the matrix B has a 3-multiple eigenvalue $b = 2$, and $\frac{\mathrm{Re(b)}}{q} = 2.5$. The comparisons with exponential functions are shown in Figures 2-4. From Figure 2, we get that $|y_1(t)|, |y_2(t)| \leq C_3 e^{2.5t}$ and $|y_3(t)| \leq C_4 e^{2t}$ as $t \to \infty$

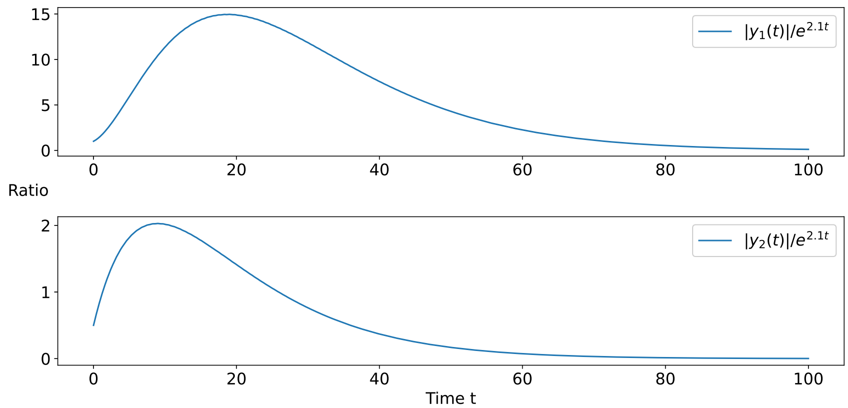

Figure 4. The ratio of $|y_1(t)|/\mathrm{e}^{2.1t}$ and $|y_2(t)|/\mathrm{e}^{2.1t}$ (Example 5.2).

for some constants $C_3 \leq 1.5,\ C_4 \leq 1$. From Figures 3 and 4, we get that $|y_1(t)|, |y_2(t)| \leq C_5 e^{\hat{\beta} t}$ as $t \to \infty$ for some constant $C_5 \leq 15$ and $2 < \hat{\beta} < 2.5$ (in Figure 4, we choose $\hat{\beta} = 2.1$). This is consistent with the results of Theorem 4.1.

6. Conclusion

In this paper, we have investigated the asymptotic bounds of solutions of the generalized pantograph equation

$$y'(t) = Ay(qt) + By(t) + Cy'(qt).$$

By discussing whether the coefficient matrix B can be diagonalized, we know it plays an important role in the boundary of solutions. When B can be diagonalized to $\mathrm{diag}(b_1, b_2, \cdots, b_n)$, we assume that

$$\mathrm{Re}b_1 \geq \mathrm{Re}b_2 \geq \cdots \geq \mathrm{Re}b_r > 0 \geq \mathrm{Re}b_{r+1} \geq \mathrm{Re}b_{r+2} \geq \cdots \geq \mathrm{Re}b_n.$$

According to whether these eigenvalues satisfy $\mathrm{Re}b_r > q\mathrm{Re}b_1$, there are two cases:

(1) Every solution is $O(e^{b_1 t})$ as $t \to \infty$.

(2) If $\mathrm{Re}b_r > q\mathrm{Re}b_1$, then $y_i(t)$ are $O(e^{b_i t})$ as $t \to \infty$ for $i = 1, 2, \cdots, r$; $y_i(t)$ are $O(e^{b_r t})$ as $t \to \infty$ for $i = r+1, r+2, \cdots, n$.

When B cannot be diagonalized. According to the distribution of eigenvalues, there are two cases:

(1) If B has a unique multiple eigenvalue b, then $y_1(t), \cdots, y_{n-1}(t) = O(e^{\hat{\beta} t})$ and $y_n(t) = O(e^{bt})$, as $t \to \infty$, for any real number $\hat{\beta}$ with $\mathrm{Re}b < \hat{\beta} < \frac{1}{q}\mathrm{Re}b$.

(2) If B has eigenvalues $b_1, b_2, \cdots, b_l$ and

$$\mathrm{Re}b_1 \geq \mathrm{Re}b_2 \geq \cdots \geq \mathrm{Re}b_r > 0 \geq \mathrm{Re}b_{r+1} \geq \mathrm{Re}b_{r+2} \geq \cdots \geq \mathrm{Re}b_l,$$

then for any real number $\hat{\beta}_1$ with $\mathrm{Re}b_1 < \hat{\beta}_1 < \frac{1}{q}\mathrm{Re}b_r$ and any $\hat{\beta}_k$ with $\hat{\beta}_k > \mathrm{Re}b_k$ $(k = 2, \cdots, r)$, we have

(i) $y_i(t) = O(e^{\hat{\beta}_1 t})$ for $i = 1, 2, \cdots, n_1 - 1$ as $t \to \infty$.

(ii) $y_i(t) = O(e^{\hat{\beta}_k t})$ for $i = n_1 + \cdots + n_{k-1} + 1, \cdots, n_1 + \cdots + n_k - 1$ as $t \to \infty$.

(iii) $y_{n_1}(t) = O(e^{b_1 t}), y_{n_1+n_2}(t) = O(e^{b_2 t}), \cdots, y_{n_1+\cdots+n_r}(t) = O(e^{b_r t})$ as $t \to \infty$.

(iv) $y_i(t) = O(e^{b_r t})$ for integer $i \in [n_1 + \cdots + n_r + 1, n]$ as $t \to \infty$.

The results show that all solutions have exponential boundaries, and their asymptotic boundaries are closely related to whether the coefficient matrix B can be diagonalized and the real part of its eigenvalues. In the future, we intend to explore the asymptotic behaviors of solutions when the parameter $q > 1$, possessing new complexities in the system's dynamics. Additionally, extending the current framework to fractional-order pantograph equations is another promising direction, as fractional-order models can capture memory effects and anomalous dynamics in the real world. These directions not only broaden the applicability of our findings, but also align with the growing interest in fractional calculus in the scientific community.

References

[1] I. Ahmad, R. Amin, T. Abdeljawad and K. Shah, *A numerical method for fractional pantograph delay integro-differential equations on haar wavelet*, Int. J. Appl. Comput. Math., 2021, 7, 1–13.

[2] H. Alrabaiah, I. Ahmad, K. Shah and G. Rahman, *Qualitative analysis of nonlinear coupled pantograph differential equations of fractional order with integral boundary conditions*, Bound. Value Probl., 2020, 2020, 1–13.

[3] R. Alrebdi and H. Al-Jeaid, *Two different analytical approaches for solving the pantograph delay equation with variable coefficient of exponential order*, Axioms, 2024, 13(4), 229.

[4] N. Barrouk and S. Mesbahi, *Existence of global solutions of a reaction-diffusion system with a cross-diffusion matrix and fractional derivatives*, Palest. J. Math., 2024, 13(3), 340–353.

[5] M. Bohner, J. R. Graef and I. Jadlovská, *Asymptotic properties of Kneser solutions to third-order delay differential equations*, J. Appl. Anal. Comput., 2022, 12(5), 2024–2032.

[6] M. Buhmann and A. Iserles, *Stability of the discretized pantograph differential equation*, Math. Comput., 1993, 60(202), 575–589.

[7] L. Fox, D. Mayers, J. Ockendon and A. Tayler, *On a functional differential equation*, IMA J. Appl. Math., 1971, 8(3), 271–307.

[8] J. Graef, I. Jadlovská and E. Tunç, *Sharp asymptotic results for third-order linear delay differential equations*, J. Appl. Anal. Comput., 2021, 11(5), 2459–2472.

[9] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[10] P. Höfer and A. Lion, *Modelling of frequency-and amplitude-dependent material properties of filler-reinforced rubber*, J. Mech. Phys. Solids, 2009, 57(3), 500–520.

[11] M. Houas, K. Kaushik, A. Kumar, A. Khan and T. Abdeljawad, *Existence and stability results of pantograph equation with three sequential fractional derivatives*, AIMS Math., 2023, 8(3), 5216–5232.

[12] A. Iserles, *On the generalized pantograph functional-differential equation*, Eur. J. Appl. Math., 1993, 4(1), 1–38.

[13] A. Iserles and Y. Liu, *On pantograph integro-differential equations*, J. Integral Equ. Appl., 1994, 213–237.

[14] T. Jüngling, X. Porte, N. Oliver, M. Soriano and I. Fischer, *A unifying analysis of chaos synchronization and consistency in delay-coupled semiconductor lasers*, IEEE J. Sel. Top. Quant., 2019, 25(6), 1–9.

[15] T. Kato and J. B. Mcleod, *The functional-differential equation* $y'(x) = ay(\lambda x) + by(x)$, B. Am. Math. Soc., 1971, 77(4), 21–22.

[16] E. Lim, *Asymptotic behavior of solutions of the functional differential equation* $x'(t) = Ax(\lambda t) + Bx(t)$, $\lambda > 0$, J. Math. Anal. Appl., 1976, 55(3), 794–806.

[17] W. Lu, T. Chen and G. Chen, *Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay*, Physica D, 2006, 221(2), 118–134.

[18] T. Nabil, *Solvability of nonlinear coupled system of urysohn-volterra quadratic integral equations in generalized banach algebras*, J. Fract. Calc. Nonlin. Syst., 2024, 5(2), 16–32.

[19] J. Ockendon and A. Tayler, *The dynamics of a current collection system for an electric locomotive*, Proc. R. Soc. Lond. A, 1971, 322(1551), 447–468.

[20] F. Rihan, *Continuous Runge-Kutta schemes for pantograph type delay differential equations*, J. Partial Differ. Equ. Appl. Math., 2024, 11, 100797.

[21] K. Shah, I. Ahmad, J. Nieto, G. Rahman and T. Abdeljawad, *Qualitative investigation of nonlinear fractional coupled pantograph impulsive differential equations*, Qual. Theor. Dyn. Syst., 2022, 21(4), 131.

[22] N. Sriwastav, A. Barnwal, A. Wazwaz and M. Singh, *A novel numerical approach and stability analysis for a class of pantograph delay differential equation*, J. Comput. Sci-Neth., 2023, 67, 101976.

[23] C. Zhang, *Analytical study of the pantograph equation using Jacobi theta functions*, J. Approx. Theory, 2023, 296, 105974.

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